Isotopy Problems for Saddle Surfaces

G. Panina


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Abstract. Four mutually dependent facts are proven.

• Each hyperbolic hérisson $H$ generates an arrangement of disjoint oriented great semicircles on the unite sphere $S^2$. On one hand, the semicircles correspond to the horns of the hérisson and on the other hand, to the inflection arches of the graph of the support function of $H$.

• The arrangement contains at least one of two basic 4-arrangements.

• A new type of a hyperbolic polytope with 4 horns is constructed.

• There exist two non-isotopic smooth hérissons (and two non-isotopic hyperbolic virtual polytopes), both with 4 horns.

This is important because of the obvious relationship with the problems of extrinsic geometry of saddle surfaces in the Euclidean space, of the non-obvious relationship with A.D. Alexandrov uniqueness problem and of a possible relationship with V. Arnold’s hypothesis about saddle surfaces in projective space.

1. Introduction.

The paper proceeds the study of hyperbolic virtual polytopes, hyperbolic hérissons, and associated saddle surfaces. These notions arose originally as a tool for constructing counterexamples to the following uniqueness conjecture, proved by A.D. Alexandrov for analytic surfaces.

Uniqueness conjecture for smooth convex surfaces [1].

Let $K \subset \mathbb{R}^3$ be a smooth convex body. If for a constant $C$, at every point of $\partial K$, we have $R_1 \leq C \leq R_2$, then $K$ is a ball. ($R_1$ and $R_2$ stand for the principal curvature radii of $\partial K$).

Roughly speaking, given a counterexample $K$ to the conjecture, the Minkovski difference of $K$ and the ball of radiuse $C$ is a hyperbolic hérisson. Conversely, given a hyperbolic hérisson, after adding by Minkovski a sufficiently big ball, we get a counterexample to the conjecture (see [6]).

It is also worthy to study a dual object - the spherical graph of support function. It is a sphere-homeomorphic closed saddle surface embedded in the 3D-sphere.

A hyperbolic polytope is a nice discrete version of a hyperbolic hérisson.

In the paper, we prove some combinatorial and isotopy properties of hyperbolic objects and related saddle surfaces.
• (Theorem 3.3) Each hyperbolic hérisson $H$ generates an object with non-trivial combinatorics: an arrangement of disjoint oriented great semicircles on the unite sphere $S^2$. More precisely, there is a natural one-to-one correspondence

"semicircles of the arrangement $\leftrightarrow$ horns of the hérisson $\leftrightarrow$ inflection arches of the graph of the support function $h_H$".

This phenomenon recalls the Möbius theorem on inflection points of a curve in the projective plane.

To prove this, we used the technique developed by A.V. Pogorelov in [11]. In the paper, he erroneously asserts that the above A.D. Alexanrov's conjecture is true. We indicate his gap and demonstrate that his methods lead to the existence of inflection arches.

• The generated by a hyperbolic hérisson (or by a hyperbolic polytope) arrangement contains at least one of two isotopy types of 4-arrangements $A_1$ and $A_2$ (see Fig. 4.2).

• (Theorem 4.3.) There exist two non-isotopic smooth hérissons (and non-isotopic hyperbolic virtual polytopes), both with 4 horns. One of them is already known - it is the hérisson presented by Martinez-Maure [6], see Fig. 2.3. This hérisson generates the arrangement $A_1$. To construct the second one (generating $A_2$, see Section 4), we use the developed by the author (see [9]) technique. Namely, we construct a saddle surface spanned by some special linkage on the 3-dimensional sphere.

Still some natural questions remain open. Here they are, and here are my intuitive impressions about the answers.

1. Does the Theorem 4.3 remain valid, if we claim the strict saddle property of the surface? (I guess, it does.)

2. What is the number of connected components of the set of hyperbolic hérissons? (I guess, it is greater than 2).

3. Is the problem solved somehow related to V.I. Arnold’s conjectures on saddle surfaces in projective space (see [2] for details)?

2. Preliminaries.

In the section, we give a brief recollection of all necessary notations, referring the reader to [8], [9], or [10] for details. The theory of virtual polytopes ([4], [8], [10]) and hérissons ([6], [7], [12]) gives a geometric interpretation of the Minkowski difference of convex polytopes and smooth convex bodies.

Here we sketch briefly the part of this theory to be used in the paper.

Let $h : \mathbb{R}^3 \to \mathbb{R}$ be a continuous positively homogeneous function which is either piecewise linear or $C^2$-smooth.

If $h$ is convex, it is the support function of some convex polytope or a smooth body.

Omitting the convexity properties, we get a group (instead of a semigroup).
Two cases are of particular interest: when \( h \) is smooth and when \( h \) is piecewise linear. Then \( h \) is the difference of two convex functions, either piecewise linear or smooth, so it makes sense to interpret \( h \) as the Minkowski difference of two polytopes (or two smooth bodies).

We associate below with the function \( h \) two mutually dual objects: a surface in \( \mathbb{R}^3 \), which generalizes the correspondence ”support function ↔ convex body”, and the spherical graph of \( h \).

\[ \square \]

Virtual polytopes and hérissons as surfaces in \( \mathbb{R}^3 \)

- Hérissons.

Let \( h \) be smooth. By the hérésson \( H \) with the support function \( h \) we mean the envelope of the family of planes \( \{ e_h(\xi) \}_{\xi \in S^2} \), where \( e_h(\xi) = h(\xi) \). It is an oriented sphere homeomorphic surface with possible self-intersections and self-overlapings.

As a set of points, a hérésson \( H \) coincides with the image of the mapping

\[ \phi : S^2 \longrightarrow \mathbb{R}^3, \]
\[ (x, y, z) \longrightarrow (h'_x(x, y, z), h'_y(x, y, z), h'_z(x, y, z)). \]

- Virtual polytopes.

Let \( h \) be piecewise linear. Similarly to the convex case, the fan \( \Sigma_h \) of the function \( h \) (and of the associated below virtual polytope \( H \)) can be defined as the minimal splitting of \( \mathbb{R}^3 \) for which \( h \) is linear on each cell.

As usual, we use for convenience the spherical fan, i.e., the intersection of \( \Sigma_h \) with the unite sphere centered at \( O \).

The associated surface \( H \) is piecewise linear. Its combinatorial structure is dual to that of the fan \( \Sigma_h \), and the coordinates of the vertices can be easily read off from the function \( h \) (exactly as it holds for the convex case):

Each 3-cell \( \sigma \) of \( \Sigma_h \) corresponds to the vertex with coordinates

\[ (h|_\sigma)'_x(x, y, z), (h|_\sigma)'_y(x, y, z), (h|_\sigma)'_z(x, y, z)). \]

\( (h|_\sigma) \) stands for the restriction of \( h \) on the cell \( \sigma \). Since it is a linear function, the expression does not depend on the choice of \( (x, y, z) \).

\[ \square \]

Graphs of support function.

- Spherical graph of \( h \).

It makes sense to draw the graph of the function \( h \) on the 3-dimensional sphere. Fix an embedding of the 3-dimensional real space \( \mathbb{R}^3 \) in \( \mathbb{R}^4 \). The unite sphere around \( O \) in \( \mathbb{R}^3 \) (respectively, \( \mathbb{R}^4 \)) is denoted by \( S^2 \) (respectively, by \( S^3 \)). Let \( h : \mathbb{R}^3 \rightarrow \mathbb{R} \) be a positively homogeneous continuous function. Denote by \( \Gamma \) its graph. The intersection of \( \Gamma \) with \( S^3 \)

\[ \Gamma_{sph}(h) = \Gamma(h) \cap S^3 \]
is called the spherical graph of the function $h$.

It is a 2-dimensional submanifold of $S^3$. The spherical central projection \( \pi : S^3 \setminus \{(0, 0, 0, 1), (0, 0, 0, -1)\} \to S^2 \) maps $\Gamma_{sph}(h)$ one-to-one on $S^2$.

The spherical graph of has the same convexity properties as $h$:

$h$ is convex, if and only if $\Gamma_{sph}$ is (spherically) convex.

This fact motivates the following definition.

**Definition 2.1.** $h$ is called hyperbolic, if the surface $\Gamma_{sph}$ is (spherically) saddle.

A hérisson (or a virtual polytope) is called hyperbolic if its support function is hyperbolic.

In the sequel, we call hyperbolic virtual polytopes simply hyperbolic polytopes.

(The latter means that for any great sphere $S$, no closure of a connected component of $\Gamma_{sph} \setminus S$ lies in an open semisphere. This definition is consistent with the generic notion of saddle surfaces in the Euclidean space: a surface $F$ is called saddle if there is no plane cutting off a bounded connected component of $F$ (see [3]).)

- **Collection of affine graphs of $h$.**

Sometimes it is reasonable to ”affinize” the spherical graph and to consider instead the described below collection of affine graphs. In fact, the affine graphs were used in the first papers on hyperbolic virtual polytopes and hyperbolic hérissons.

For each $\xi \in S^2$, denote by $e(\xi)$ the 2-plane in $\mathbb{R}^3$ tangent to $S^2$ at the point $\xi$. Denote by $h|_e$ the restriction of $h$ on the plane $e = e(\xi)$.

Consider the affine graph of the restriction $h|_e$, namely,

$$\Gamma_{aff}(h, e) := \{(x, y, z, t) \in \mathbb{R}^4 \mid (x, y, z) \in e; t = h(x, y, z)\}.$$ 

The union of all images of affine graphs $\Gamma_{sph}(h, e)$ on $S^3$ under the central projection $\phi$ with the center $O$ (see Fig. 2.2) is the spherical graph of the function $h$. 
Obviously, a hérisson (or a virtual polytope) is hyperbolic if and only if $\Gamma_{aff}(h,e(\xi))$ is a saddle surface for any $\xi \in S^2$.

The most remarkable fact known about hyperbolic objects is that they exist. The first $C^2$-smooth hyperbolic hérisson was constructed by Martinez-Maure [6], see Fig. 2.3. It was followed by series of $C^\infty$-smooth hyperbolic hérissons and hyperbolic polytopes presented by Panina [8], [10].


\[\Box\text{Horns of hyperbolic objects.}\]

**Definition 2.4.** Let \( H \) be a hyperbolic polytope or a hyperbolic hérisson, considered as a surface in \( \mathbb{R}^3 \). A point \( P \in H \) is called a horn if its arbitrary small neighbourhood can be cut off the surface by a plane. Or, equivalently, if there exists a plane \( e \) passing through \( P \) and intersecting the surface locally just in one point \( P \).

The outward normal vector \( n \) of such a plane is called an outward vector of the horn \( P \). Denote the set of all outward vectors of the horn \( P \) by \( N(P) \).

**Definition 2.5.** A vector \( d \) is called a direction vector of the horn \( P \) if \((d, n) > 0\) for each \( n \in N(P) \). The set of direction vectors of a horn is always non-empty.

**Lemma 2.6.** Let \( \{d_i\} \) be a collection of direction vectors of the horns for a hyperbolic hérisson (or a hyperbolic polytope) \( H \), (one direction vector for each horn).

Then the set \( \{d_i\} \) spans positively \( \mathbb{R}^3 \). (I.e. each vector from \( \mathbb{R}^3 \) is a linear combination of \( \{d_i\} \) with positive coefficients.) \[\Box\]

For a vector \( d \), denote \( S^+(d) = \{x \in S^2 : (x, d) > 0\} \).

**Corollary 2.7.** In the assumption of the lemma,

\[\bigcup_i S^+(d) = S^2. \] \[\Box\]

\[3.\text{Arches of inflection: a version of Möbius theorem for two-dimensional saddle surfaces on } S^3\]

**Definition 3.1.** Let \( \Gamma \) be a smooth saddle surface in \( S^3 \).

An arch of inflection of the surface \( \Gamma \) is a great semicircle \( S \subset S^3 \) such that

- \( S \subset \Gamma \)
- for each great 2-dimensional sphere \( e \subset S^3 \) which intersects \( S \) transversely, the point \( e \cap S \) is an inflection point of the curve \( e \cap \Gamma \).
Figure 3.2.

**Theorem 3.3.** Let $\Gamma$ be a two-dimensional closed smooth saddle surface lying in $S^3$ and admitting the bijective orthogonal projection on some great sphere $S^2$. Assume that $\Gamma$ is non-degenerate, i.e. it does not coincide with a great sphere.

Then

1. $\Gamma$ contains at least 4 arches of inflection. Each of them carries a natural orientation (see Fig. 3.2).
2. Their projections of all arches of inflection form an arrangement of disjoint oriented great semicircles $\{A_i\}$ such that
   \[ \bigcup_i S^+(A_i) = S^2, \]
   where $S^+(A_i)$ is a semisphere such that $A_i$ lies on its boundary which is consistent with the orientation of $A_i$.
3. The arrangement $\{A_i\}$ contains at least one subarrangement of four great semicircles which is isotopic to one of the arrangements $\mathcal{A}_1, \mathcal{A}_2$ presented on Fig.

By this reason, the arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ are called basic arrangements.

**Remark 3.4.** The assertion (1) is a smooth version of Theorem from [10]. Just note that a cell of the fan of a hyperbolic simplicial polytope contains an arch of inflection if and only if the colour changes twice when going around the cell.

Proof of the theorem.

(1) By the assumption, $\Gamma$ is the spherical graph of the
support function $h$ of some hyperbolic hérisson $H$.

The map $\phi : \mathbb{R}^3 \setminus O \rightarrow \mathbb{R}^3$, given by the formula

$$\phi(\xi) = \nabla h = (h'_x, h'_y, h'_z)$$

maps the sphere $S^2$ to the surface $\Phi = \phi(S^2)$.

In the paper [11] it is proven that $\dim(\text{aff}(\Phi)) = 3$. Therefore the surface $\Phi$ has at least 4 horns, which we denote by $P_1, ..., P_4$.

We prove that each preimage $\phi^{-1}(P_i) \cap S^2$ contains an arch of inflection.

Fix a Cartesian coordinate system $(x, y, z)$ in $\mathbb{R}^3$ such that $O = P_1$, and the vector $(-1, 0, 0)$ is the outward vector of the horn. Thus all $x$-coordinates of $\Phi$ are positive. Therefore, $h'_x > 0$ everywhere on $\Phi$ except for the horn $P_1$.

Let $\xi \in S^2 \cap \phi^{-1}(P_i)$. Let the plane $E = E_\xi$ be such that $\xi \in E$ and $E$ contains a line $l$ parallel to the axes $(x)$. Denote by $pr_\xi = pr : E \rightarrow S^2$ the projection on the unite sphere.

Consider the restriction of the function $h$ to the plane $E$. Denote by $F = F_\xi$ the graph of the restriction. Let the Cartesian coordinate system $(u, v, w)$ be such that $E = (u, v), (0, 0) = \xi$ and the axes $u$ is parallel to the axes $x$. Let the surface $F$ be given by the equation $w = f(u, v)$.

**Figure 3.5.**

For the function $f$ and its graph $F$, we have:

- $f'_u(u, v) \geq 0$.
- $f'_u(u, v) = 0 \Rightarrow (u, v) \in \phi^{-1}(P)$
- $f'_u(0, 0) = 0$
$f(\xi) = f(0,0) = 0.

The surface $F$ is saddle.

For a small $\alpha > 0$, there exists a plane $E(\alpha)$ such that

- $\xi \in E(\alpha)$,
- The angle between $E$ and $E(\alpha)$ equals $\alpha$,
- The intersection $l(\alpha) = E \cap E(\alpha)$ tends to $l$ as $\alpha \to 0$.
- The connected component $C(\alpha)$ of the set $F \setminus E(\alpha)$, which lies behind the point $(0,0)$ is infinite in both sides along the axes $v$.

Indeed, let us take a plane $E(\alpha)$ passing through the points $(0,0,0)$ and $(0,\varepsilon, f(\varepsilon))$.

One of such planes satisfies the required condition: consider all rotations of the plane around the line passing through these points. Denote the connected component of $F \setminus E(\alpha)$ lying above the segment $[(0,0,0), (0,\varepsilon, f(\varepsilon))]$ by $C$. Since $F$ is saddle, $C$ is not bounded. For some rotation of the plane $E(\alpha)$ about the line $((0,0,0), (0,\varepsilon, f(\varepsilon)))$, the component $C$ is infinite to the left along the axes $v$. Similarly, for some rotation, $C$ is infinite to the right along the axes $v$. Therefore, for some rotation, it is infinite in both directions.

Denote by $(\alpha)$ the orthogonal projection of $C(\alpha)$ on the plane $E$, and following A.V. Pogorelov, study its behaviour, as $\alpha \to 0$.

We obviously have

$$
\lim_{\alpha \to 0} P(\alpha) = P(0) = E \cap F.
$$

Let $P_{\text{tan}} = \{(u_0, v_0) \in P(0)\}$ for all small $\alpha \geq 0$, the line $v = v_0$ intersects the boundary of $P(\alpha)$ at least twice. We have $P_{\text{tan}} \subseteq \{(u, v) \in E \mid F$ tangents $E$ at the point $(u, v)\}$. Therefore $P_{\text{tan}} \subset \phi^{-1}(P)$.

The set $P_{\text{tan}}$ is an unbounded connected subset of $E$. At this point A.V. Pogorelov came to an erroneous conclusion that $P_{\text{tan}}$ coincides with $P(0)$, and splits the plane $E$ into two parts (See Case 1 in Fig. 3.6.).

He missed the Case 2 (It leads to the conclusion that the original hérisson is degenerate, i.e. point).
Figure 3.6.

Case 1. $\mathcal{P}(0) \subset \mathcal{P}_{\text{tan}}$.

Case 2. $\mathcal{P}(0) \neq \mathcal{P}_{\text{tan}}$.

Study this set more detailed.

**Lemma 3.7.** The set $\mathcal{P}_{\text{tan}} \subset E = (u, v)$ is bounded by graphs of two functions, say, $p_1$ and $p_2$. These functions are defined on a ray, say on the ray $[a, \infty)$. Besides, $p_1(a) = p_2(a)$, and the function $p_1$ (respectively, $p_2$) is concave down (respectively, up).
Figure 3.8.

The lemma implies that the intersection of $\phi^{-1}(P)$ with the plane $E(\xi)$ contains a ray. Passing to the spherical graph, we conclude that $\Gamma$ contains a great semicircle which corresponds to the horn $P_1$.

The other horns give at least three more semicircles. They are disjoint because they are contained in preimages of different points.

(2) follows directly from Corollary 2.7.

(3) Put $S_i^- = S^2 \setminus S_i^+$. (2) is equivalent to the identity $\bigcap S_i^- = \emptyset$. Show that the same identity is valid for some 4 semispheres from the arrangement. Assume the contrary, i.e. that each 4 semispheres have a common point. By Helly’s Theorem, all semispheres have a non-empty intersection. It remains to observe that there exist just two isotopy types of arrangements of four oriented great semicircles satisfying the equation

$$\bigcup_i S^+(d) = S^2.$$

\[ \square \]

4. TWO NON-ISOTOPIC HYPERBOLIC POLYTOPES WITH 4 HORNs

We present two hyperbolic polytopes with non-isotopic arrangements of great semicircles $\mathfrak{A}(H_{1,2})$.

Lemma 4.1. The two arrangements $\mathfrak{A}_1$ and $\mathfrak{A}_2$ of four oriented great semicircles presented on Fig. 4.2 are non-isotopic.