A Discrete Model for the Efficient Analysis of Time–Varying Narrowband Communication Channels

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Abstract

We derive an efficient numerical algorithm for the analysis of certain classes
of Hilbert–Schmidt operators that naturally occur in models of wireless radio
and sonar communications channels.

A common short-time model of these channels writes the channel output as
a weighted superposition of time- and frequency shifted copies of the transmit-
ted signal, where the weight function is usually called the spreading function
of the channel operator.

It is often believed that a good channel model must allow for spreading
functions containing Dirac delta distributions. However, we show that many
narrowband finite lifelength systems such as wireless radio communications
can be well modelled by smooth and compactly supported spreading functions.

Further, we exploit this fact to derive a fast algorithm for computing the
matrix representation of such operators with respect to well time-frequency
localized Gabor bases (such as pulsedhaped OFDM bases). Hereby we use a
minimum of approximations, simplifications, and assumptions on the channel.
Finally, we provide and discuss some sample plots from a MATLAB implement-
ation which is fast enough for channels and communication systems of sizes
typically in use today.

The derived algorithm and software can be used, for example, for compar-
ing how different system settings and pulse shapes affect the diagonalization
properties of an OFDM system acting on a given channel.

Keywords: Communication channel model, spreading function, discretiza-
tion, Hilbert–Schmidt operators, Gabor systems, diagonalization, BEM (Ba-
sis Expansion Model), OFDM, DMT, mobile phone communications, satellite
communications, underwater sonar communications.

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1 Introduction

Channel-dependent customization is expected to provide considerable performance improvements in time-varying systems such as future generations of wireless communications systems. Consequently, the idea of shaping the transmission pulses in order to minimize the InterCarrier and InterSymbol Interference (ISI and ICI) in Orthogonal Frequency Division Multiplexing (OFDM) communications is an active research area in the applied harmonic analysis and signal processing communities (see [BS01, MSG+05] and references therein). Even though some insights can be gained from careful mathematical modelling and analysis, there remains a need for fast algorithms and implementations aimed at the numerical evaluation of performance improvements through pulseshaping. In this paper, we discuss two closely connected topics that we regard of vital importance to fulfill this demand.

1. We review the most important physical properties of wireless channels and show how these lead naturally to a model of the short-time behavior of a channel as an operator $H$ that maps an input signal $s$ to a weighted superposition of time and frequency shifts of $s$, that is,

$$Hs(\cdot) = \int_{K \times [A, \infty)} S_H(\nu, t) e^{i2\pi\nu(\cdot - t_0)} s(\cdot - t) \, d(\nu, t), \quad K \text{ compact.} \quad (1.1)$$

This model is well-known and the coefficient function $S_H$ is usually called the spreading function of $H$. The model given by (1.1) is mostly used either under the assumption that $S_H$ is square-integrable or that $S_H$ is a tempered distribution. The latter, weaker assumption suggests that the input signal $s$ should be a Schwartz class function and requires the use of distribution theory in the analysis of $H$, both of which we shall try to avoid. Therefore, we derive (1.1) using some refinements of the standard multipath propagation model of the channel. Our analysis implies that the short-time behaviour in many communications applications can be completely described by a smooth $S_H$ with rapid decay ensuring “essentially compact” support. This model has the big advantage that it allows for both Fourier analysis and numerical evaluation of the performance of OFDM procedures without the need of deviating into distribution theory.

2. We employ the channel model described above to derive an efficient algorithm for the numerical evaluation of ISI and ICI in pulseshaped OFDM systems.

We shall now motivate and describe the principles of our discretization in some detail:

For multicarrier modulation systems in general, the aim is the joint diagonalization of a class of possible channel operators in a given environment. That is, we try to find a transmission basis $(g_i)$ and a receiver basis (filters) $(\tilde{\gamma}_j)$ with the property that all coefficient mappings that correspond to channels in the environment have

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1With essentially compact we mean that the function decays fast enough to assure that in any practical application, the function values outside some “reasonably small” compact set are very small compared to the overall noise level and therefore negligible (see also Section 6.1).
matrix representations $G_{i,j} = \langle Hg_i, \tilde{\gamma}_j \rangle$ that are as close to diagonal as possible, that is, $|G_{i,j}|$ decays fast with $|i - j|$. In general, an easily computable inverse of this coefficient mapping would allow us to regain the transmitted coefficients $(c_i)$ in the input signal $s = \sum c_i g_i$, and, therefore, the information embedded in these coefficients, from the inner products $\langle Hs, \tilde{\gamma}_i \rangle$ which are calculated on the receiver side.

In wireline communications, the problem described above has a well accepted solution, namely OFDM (also called Discrete MultiTone or DMT) with cyclic prefix. Here, the transmission basis $(g_i)$ and the receiver basis $(\gamma_i)$ are so-called Gabor bases, that is, each basis consists of time and frequency shifts of a single prototype function which is often referred to as window function. Diagonalization of the channel operator using Gabor bases with rectangular prototype function is then possible since wired channels are assumed to be time-invariant. This allows us to model such channel operators as convolution operators with complex exponentials $e^{i2\pi\omega t}$ as “eigenfunctions”. This cyclic prefix procedure applies if the channel has finite lifelength and is explained in more detail in Section 4.1 and with further references in [Gri02]. The superiority of Gabor bases in comparison to Wavelet and Wilson bases for wireline communications is examined in detail in [KPZ02].

Wireless channels are inherently time-varying. The generality of time-varying channel operators and, in particular, the fact that they do not commute in general, implies that joint diagonalization of classes of such channels cannot be achieved as in the general case, so approximate diagonalization becomes our goal. In many cases, for example in mobile telephony, the channel varies only “slowly” with time. Hence, we use the results for time-invariant channels as a starting point and consider in this paper only the use of Gabor bases as transmitter and receiver bases.

For such slowly time-varying systems, Matz, Schafhuber, Gröchenig, Hartmann and Hlawatsch conclude that excellent joint time-frequency concentration of the windows $g$ and $\gamma$ is the most important requirement for low ISI and ICI [MSG+05]. There, it is shown how to compute a $\gamma$ (or an orthogonalization of the basis $(g_i)$) that diagonalizes the coefficient mapping in the idealized borderline case when the channel is the identity operator ($(H)_{i,j} = \delta_{i,j}$). They show that both $\gamma$ and the corresponding orthogonalized basis inherit certain polynomial or subexponential time-frequency decay properties from $g$. They also derive exact and approximate expressions for the ISI and ICI and present an efficient FFT-based modulator and demodulator implementation.

For multicarrier systems with excellent joint time-frequency localization of $g$ and $\gamma$, we derive, starting from our channel model, a procedure for the numerical computation of the matrix entries $G_{i,j} = \langle Hg_i, \tilde{\gamma}_j \rangle$ under a minimum of assumptions, simplifications, or approximations. We derive our algorithm in a multivariate setting for potential use in other theoretical or practical applications that use a time-variant impulse response model (such as the condition monitoring applications in cf. [CBWB99, CBB01, CB00, CBUB+02, MH99]). These properties make our approach different from and complementary to a number of papers that use discrete Gabor bases (sometimes under the name BEM or Basis Expansion Model) for time-varying channels and statistical applications, such as [BLM04, BLM05, LM04, LZG03, MG02, MG03b, MG03a, MLG05, SA99, TL04].
The paper is organized as follows: The Notation and some mathematical preliminaries are described in Section 2. We derive a channel model in Section 3, and use it to derive formulas for the matrix elements in Section 4. In Section 5 we describe a MATLAB implementation of these formulas and give suggestions on how to do the necessary parameter and window/pulse shape choices. In Section 6 we provide typical system-dependent parameters for and demonstrate our software on some example mobile phone communications, satellite communications and underwater sonar communications applications. Finally, our conclusions follow in Section 7.

2 Preliminaries

For completeness and easy availability we collect our notation in Section 2.1 and give an overview of the mathematical tools that we shall use. In Section 2.2 we shall discuss the availability of functions that are compactly supported and “essentially bandlimited”, in particular, we explain how compactly supported functions can be designed to have subexponential decay. Section 2.3 covers the Gabor system expansions which are used to obtain diagonal dominant coefficient mappings of channel operators. Finally, in Section 2.4 we discuss the Hilbert–Schmidt operator theory and the integral representation of channel operators in terms of system functions such as the spreading function and the time-varying impulse response.

2.1 Notation

We assume the reader to know some basic tools and notation from functional analysis and measure theory, which otherwise can be found in [Fol99, Rud87].

The conjugate of a complex number $z$ is denoted $\overline{z}$. We use boldface font for elements in $\mathbb{R}^d$, write $\mathbb{R}_+^d \eqdef (0, \infty)^d \eqdef \mathbb{R}_+ \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+$ and $\mathbb{Z}_+^d \eqdef \mathbb{Z}^d \cap \mathbb{R}_+^d$. The Fourier transform of a function $f$ is formally given by $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(t) e^{-i2\pi \langle \xi, t \rangle} \, dt$ for $\xi \in \mathbb{R}^d$ and $l^2 \eqdef l^2(\mathbb{Z}^d \times \mathbb{Z}^d)$ is the Hilbert space of sequences $(c_{q,r})$ for which the $l^2$-norm is given by

$$\|(c_{q,r})\|_2 \eqdef \left( \sum_{q,r \in \mathbb{Z}^d} |c_{q,r}|^2 \right)^{1/2} < \infty.$$

Throughout the paper we use Roman and Greek letters for variables that have a physical interpretation as time or spatial variable and frequency, respectively. For
Here, sinc is extended continuously to $\mathbb{R}^d$ and we shall frequently use that
$$
\int_{I_{C,B}} e^{-i2\pi(\xi,x)} \, d\xi = e^{-i2\pi(C,x)} \text{sinc}_B(x).
$$

For $\epsilon > 0$ we define the $\epsilon$-essential support of a bounded function $f: \mathbb{R}^d \to \mathbb{C}$ to be the closure of the set $\{x : \epsilon \leq |f(x)|/\sup_x |f(x)|\}$. For an almost everywhere defined function $f$, $\text{supp} f$ denotes the intersection of the supports of all representatives of $f$ (and similarly for $\epsilon$-essential support). For any set $I$, $\chi_I$ is the characteristic function $\chi_I(x) = 1$ if $x \in I$ and $\chi_I(x) = 0$ otherwise. The sets of $n$ times, respectively infinitely many, times continuously differentiable functions are denoted $C^n$ and $C^\infty$, respectively.

We denote by $L^p = L^p(\mathbb{R}^d)$ the Banach space of complex-valued measurable functions $f$ with norm
$$
\|f\|_p \overset{\text{def}}{=} \left( \int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{1/p} < \infty.
$$

$L^2(\mathbb{R}^d)$ is a Hilbert space with inner product $\langle f, g \rangle \overset{\text{def}}{=} \int_{\mathbb{R}^d} f(x)\overline{g(x)} \, dx$. We say that two sequences $(f_n)$ and $(g_n)$ of functions are biorthogonal if $\langle f_m, g_n \rangle = 0$ whenever $m \neq n$ and $\langle f_m, g_n \rangle = 1$ for all $n$. The Wiener amalgam space $W(A, l^1) = S_0(\mathbb{R}^d)$ (also named the Feichtinger algebra) consists of the set of all continuous $f: \mathbb{R}^d \to \mathbb{C}$ for which
$$
\sum_{n \in \mathbb{Z}^d} \|\langle f(\cdot - n) \rangle\|_1 < \infty
$$
for some compactly supported $\psi$ such that $\hat{\psi} \in L^1(\mathbb{R}^d)$ and $\sum_{n \in \mathbb{Z}^d} \psi(x - n) = 1$. We write $S_0'$ for the space of linear bounded functionals on $S_0$. $S_0$ is also a so-called modulation space, described at more depth and with notation $S_0 = M^{1,1} = M^1$ and $S_0' = M^{\infty,\infty} = M^\infty$ in [Grö00, FZ98].

A real-valued, measurable and locally bounded function $w$ on $\mathbb{R}^d$ is said to be a weight function if for all $x, y \in \mathbb{R}^d$,
$$
w(x) \geq 1 \quad \text{and} \quad w(x + y) \leq w(x)w(y).
$$
For weight functions $w$ we define $L^1_w = L^1_w(\mathbb{R}^d)$ to be the family of functions $f \in L^1(\mathbb{R}^d)$ such that
$$\|f\|_{1,w} \overset{\text{def}}{=} \|fw\|_1 < \infty.$$ 

### 2.2 Frequency localization of compactly supported functions

The Gabor window $g$ in the introduction needs to be compactly supported in a time interval short enough to satisfy typical maximum delay restrictions, such as 25 ms for voice communications. Moreover, its Fourier transform $\hat{g}$ has to decay fast enough to allow for reasonably high transmission power (which determines the signal-to-noise ratio) without exceeding standard regulations on the allowed power leakage into other frequency bands. In other words, $g$ should have good, joint time-frequency localization, which also is of great importance for achieving low ISI and ICI [MSG+05]. For this reason, we seek to know to what extent compact support of a function can be combined with good decay of its Fourier transform. This classical question was first answered by Beurling [Beu38, Theorem V B] and generalized from functions on $\mathbb{R}$ to functions on locally compact abelian groups, such as $\mathbb{R}^d$, by Domar [Dom56, Theorem 2.11]. Domar’s results are explained in much more detail in [RS00, Ch. 6.3 + appendices].

One way to measure the speed of decay of a Fourier transform is to check for how fast growing weight functions $w$ it belongs to $L^1_w(\mathbb{R})$. To describe a function’s asymptotic decay, we only need to consider continuous $w$ such that $w(\xi)$ and $w(-\xi)$ are nondecreasing for positive $\xi$. The following theorem can be obtained from a combination of a similar result [RS00, Theorem A.1.13] for locally compact abelian groups with the here added continuity and decay assumptions on $w$ (see [GP05] for a proof).

**Theorem 1.** Let $w$ be a continuous weight function such that $w(\xi)$ and $w(-\xi)$ are nondecreasing for positive $\xi$. Suppose that there is a non-zero compactly supported function $f \in L^2(\mathbb{R})$ such that $\hat{f} \in L^1_w(\mathbb{R})$. Then

$$\int_{\mathbb{R}} \frac{\log(w(\xi))}{1 + \xi^2} \, d\xi < \infty. \quad (2.3)$$

The so-called logarithmic integral condition (2.3) limits the decay of both the amplitude and “the area under the tail” of $\hat{f}$. For example, the Fourier transform of a compactly supported function $f$ cannot be either $O(e^{-\alpha|\xi|})$ nor $f(\xi) = \sum_{n \in \mathbb{Z}} \phi(e^{\alpha|n|}(\xi - n))$ for any $\phi \in C^\infty$ with support supp $\phi \in [0,1]$, because in both cases, $\hat{f} \in L^1_w(\mathbb{R})$ for $w(\xi) = e^{a|\xi|}$ and $a < \alpha$ but $w$ does not satisfy (2.3). This fact rules out the existence of compactly supported functions $f$ with exponentially decaying $\hat{f}$. However, Dziubański and Hernández [DH98] have shown how to use a construction by Hörmander [Hör03, Theorem 1.3.5] to construct a compactly supported function $f$ whose Fourier transform is subexponentially decaying. That is, they construct $f$ such that for every $0 < \varepsilon < 1$ there exists $C_\varepsilon > 0$ such that

$$|\hat{f}(\xi)| \leq C_\varepsilon e^{-|\xi|^{1-\varepsilon}}, \quad \forall \xi \in \mathbb{R}.$$ 

From their example and standard techniques such as convolution with a characteristic function, it is then easy to design for any compact set $K$ a compactly supported function $f$ such that $f(x) = 1$ for $x \in K$, and $\hat{f}$ is subexponentially decaying.
Note however, that subexponential decay is not everything. For example, the function \( f(x) = e^{-|x^2|} \chi_{[-1,1]}(x) \) is a compactly supported \( C^\infty \) function, so that \( \hat{f}(\xi) = O((1+|\xi|)^{-n}) \) for all \( n \in \mathbb{N} \), whereas the function \( g(x) = (1+\cos(\pi x))^4 \chi_{[-1,1]}(x) \) is only four times continuously differentiable, so \( \hat{g} = O((1+|x|)^{-n}) \) only for \( 0 \leq n \leq 4 \). However, Figure 1 shows that \( \hat{g} \) decays much faster down to amplitude thresholds such as the power leakage restrictions described above (see also Figure 7, page 33). Thus it can be an important design issue to choose functions and forms of decay that are optimal for a given application.

However, for simplicity and a clear presentation in this paper, we shall consistently claim subexponential decay although also other forms of decay are rapid enough for all of our results to hold.

### 2.3 Gabor analysis

Here, we give a brief review of some basic Gabor frame theory that is needed to understand the relevance of the coefficient mappings that we introduce in (2.6) below. For a more complete and general coverage of this subject, see, for example, [Chr02, Gri02, Grö00].

A Gabor (or Weyl-Heisenberg) system with window \( g \) and lattice constants \( a \) and \( b \) is the sequence \( (g_{q,r})_{q,r \in \mathbb{Z}^d} \) of translated and modulated functions

\[
g_{q,r} \overset{\text{def}}{=} T_{ra} M_{qb} g = e^{i2\pi (qb, x - ra)} g(x - ra).
\]

The corresponding synthesis or reconstruction operator

\[
R_g : l^2 \to L^2, \quad R_g c \overset{\text{def}}{=} \sum_{q,r \in \mathbb{Z}^d} c_{q,r} g_{q,r}
\]

Figure 1: The Fourier transform decay after normalizing the following positive functions to have integral 1: (a) \( f(x) = e^{-|x^2|} \chi_{[-1,1]}(x) \). (b) \( g(x) = (1+\cos(\pi x))^4 \chi_{[-1,1]}(x) \).
is defined with convergence in the $L^2$-norm if and only if its adjoint, the so-called analysis operator

$$R_g^*: L^2 \to l^2, \quad R_g^* f = \langle f, g_{q,r} \rangle_{q,r \in \mathbb{Z}^d},$$

is bounded, i.e., if and only if $\sum_{q,r \in \mathbb{Z}^d} |\langle f, g_{q,r} \rangle|^2 \leq B \|f\|^2_2$ for some $B \in \mathbb{R}_+$ and all $f \in L^2(\mathbb{R}^d)$ [Gri02, p. 14].

We call $(g_{q,r})_{q,r \in \mathbb{Z}^d}$ a Gabor frame for $L^2(\mathbb{R}^d)$ if there are frame bounds $A, B \in \mathbb{R}_+$ such that for all $f \in L^2(\mathbb{R}^d),$

$$A \|f\|^2_2 \leq \|R_g^* f\|^2_2 \leq B \|f\|^2_2.$$  \hspace{1cm} (2.4)

It follows from (2.4) that the frame operator $S_g \overset{\text{def}}{=} R_g R_g^*$ is invertible. We call a frame with elements $\bar{g}_{q,r} \overset{\text{def}}{=} T_{ra} M_q b \tilde{g}$ a dual Gabor frame if for every $f \in L^2,$

$$f = \sum_{q,r \in \mathbb{Z}^d} \langle f, \bar{g}_{q,r} \rangle g_{q,r} = \sum_{q,r \in \mathbb{Z}^d} \langle f, g_{q,r} \rangle \bar{g}_{q,r}$$  \hspace{1cm} (2.5)

with $L^2$-norm convergence of both series. There may exist (infinitely) many different dual windows $\bar{g}$ for $g.$ However, we shall always consider the canonical dual window, which is the minimum $L^2$-norm dual window [Jan98, p. 51]. The dual frame has frame bounds $A^{-1}, B^{-1}$ and the coefficients in (2.5) are not unique in $l^2,$ but they are the unique minimum $l^2$-norm coefficients. It follows also from (2.4) and (2.5) that $R_g^*$ picks coefficients from $C_{\bar{g}} \overset{\text{def}}{=} R_{\bar{g}}^{-1}(L^2(\mathbb{R}^d)) \subseteq l^2$ and that $R_g$ is a bounded invertible mapping of $C_{\bar{g}}$ onto $L^2$ with bounded inverse $R_g^{-1} = R_{\bar{g}}^*.$ By this isomorphism and the usual definition of operator norms we can use two Gabor frames $(g_{q,r})$ and $(\gamma_{q,r})$ (possibly with different lattice constants) to obtain an isomorphism of the family of linear bounded operators $H: L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ with the coefficient mappings $G = R_g^* HR_g,$ as illustrated in the following commutative diagram.

$$\begin{align*}
L^2(\mathbb{R}^d) \xrightarrow{H} L^2(\mathbb{R}^d) \\
\uparrow R_g \\
C_{\bar{g}} \xrightarrow{G=R_{\bar{g}}^* HR_g} C_{\gamma}
\end{align*}$$  \hspace{1cm} (2.6)

We will provide an explicit expression for $G$ in Section 4.

The frame $(g_{q,r})_{q,r \in \mathbb{Z}^d}$ is called a Riesz basis if $C_{\bar{g}} = l^2.$ Then, the coefficients in (2.5) are truly unique and, as a consequence, $(g_{q,r})$ and $(\bar{g}_{q,r})$ are biorthogonal.

### 2.4 Hilbert–Schmidt operators

The mathematical framework for the use of Hilbert–Schmidt operators acting on functions defined on locally compact abelian groups has been developed in great generality in harmonic and functional analysis [FK98]. For the basic theory, see, for example, [Con00, RS80] or [Fol95, Appendix 2].

We will use the following classification of Hilbert-Schmidt operators, which is equivalent to the classical definition (see [GP05] or [RS80, Theorem VI.23] for details).
**Theorem 2.** A linear bounded operator \( H : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) is Hilbert–Schmidt if and only if there is a function \( S_H \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \) such that for all \( s \in L^2(\mathbb{R}^d) \),

\[
(Hs)(t_0) = \int_{\mathbb{R}^d} S_H(\nu, t)s(t_0 - t)e^{i2\pi(\nu, t_0 - t)} \, dt.
\]

The integral in (2.7) is defined in a weak sense. In fact, for \( s, g \in L^2(\mathbb{R}^d) \), we have \( g(\cdot)s(\cdot - t) \in L^1 \), so that the short-time Fourier transform of \( g \) with window \( s \) is well-defined as the function

\[
V_sg(\nu, t) \overset{\text{def}}{=} \int_{\mathbb{R}^d} g(t_0)\overline{s(t_0 - t)}e^{-i2\pi(\nu, t_0 - t)} \, dt_0
\]

on \( L^2(\mathbb{R}^d \times \mathbb{R}^d) \). Hence, \( Hs \) is defined to be the unique \( L^2(\mathbb{R}^d) \)-function with

\[
(Hs, g)_{L^2(\mathbb{R}^d)} = (S_H, V_sg)_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}.
\]

There are many similar versions of Theorem 2, some of which can be obtained by applying partial Fourier transforms to \( S_H \) and replacing (2.7) with corresponding mappings relating \( s \) or \( \widehat{s} \) to either \( Hs \) or \( \widetilde{H}s \) as done in (2.9) below. Many so obtained system functions are known under a rich plethora of different names in the literature, ranging back to a first systematic study by Zadeh and Bello [Zad50, Bel63, Bel64] (see also [Ric03] for an overview). The integral representations of importance in this text describe \( H \) in terms of the spreading function \( S_H \), the kernel \( \kappa_H \), the time-varying impulse response \( h \), the Kohn-Nirenberg symbol \( \sigma_H \) and the bifrequency function \( B_H \). These system functions are related via the following partial Fourier transforms:

\[
\kappa_H(t_0, t_0 - t) = h(t_0, t) \overset{\mathcal{F}-t}{\longrightarrow} \sigma_H(t_0, \xi) \overset{\mathcal{F}-t}{\longrightarrow} B_H(\nu, \xi)
\]

For \( \kappa_H \) being smooth and compactly supported, we apply the Fubini–Tonelli theorem, (2.9a) and Plancherel’s theorem to (2.7) to get

\[
(Hs)(t_0) = \int \kappa_H(t_0, t)s(t) \, d\mu(t)
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} S_H(\nu, t)e^{i2\pi(\nu, x - t)}d\nu s(t_0 - t) \, dt
= \int_{\mathbb{R}^d} h(t_0, t)s(t_0 - t) \, dt
= \int_{\mathbb{R}^d} \sigma_H(t_0, \xi)\widehat{s}(\xi)e^{i2\pi(t_0, \xi)} \, d\xi
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} B_H(\nu - \xi, \xi)\widehat{s}(\xi) \, d\xi e^{i2\pi(t_0, \nu)} \, d\nu.
\]

Note that the validity above extends to general Hilbert–Schmidt operators via a density argument. Certainly, the convergence of the integrals is considered in the