Fine Analysis on Lineally Convex Domains of Finite Type

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Fine analysis on lineally convex domains of finite type
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Abstract
A discussion of methods of nonisotropic fine quantitative complex analysis on lineally convex domains of finite type is given. The needed support functions with best possible estimates are considered together with the estimation of their corresponding Leray sections with respect to nonisotropic pseudodistances. The most recent developments in this subject are studied and open questions listed.

1 Motivation and state of the art

A general main topic of Complex Analysis consists in investigating the relation between suitable geometric properties of complex manifolds and fine analytic properties of them, in particular, quantitative behavior of analytic objects on them. Of course, each time a question concerning the quantitative behavior of an analytic object is asked, the correct notion of “geometric object” has to be found which dominates its behavior.

There are two main branches of this research:

1) The case of compact manifolds: The analytic objects here are mostly holomorphic line bundles or, more generally, holomorphic vector bundles, and their tensor products. Certain norm conditions with respect to suitable metrics on the original manifold or the bundle itself might be put on their sections. One of the main goals of the analysis on them are vanishing theorems for analytic cohomology, existence theorems for sections, asymptotic behavior for tensor powers. The main geometric informations needed deal with curvature conditions (positivity resp. negativity) or with the metrics themselves. The non-degenerate case (concerning the metrics and their curvature) is mostly quite well understood. All kinds of allowable degeneracies are

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the main topic in recent research in this field. The theory of multipliers and the corresponding multiplier ideal sheaves which originally was introduced into Complex Analysis by J. J. Kohn in [48] and then carried over to the case of complex manifolds by the work of A. Nadel (see [53]) is the most important tool for this study (see [22]).

2) The case of (relatively compact) domains in \( \mathbb{C}^n \) with more or less smooth boundaries. Here the fundamental goals are the existence questions for holomorphic functions with all kinds of growth conditions and the study of their boundary behavior. The basic tool is, of course, the study of the \( \overline{\partial} \)-Neumann problem. The necessary geometric information sits in the complex differential geometry of the boundary, in particular, its Levi geometry, sometimes together with topological properties of the domains and their boundaries.

Of course, there also is a mixed case (between 1) and 2)), namely the case of (relatively compact) domains with boundaries in open complex manifolds. Here quite new interesting phenomena occur. However, this case is not the subject of this talk.

Project 2) has been largely realized for the case of domains with non-degenerate Levi forms. However, a lot of questions on both sides, the geometric and the analytic aspect, remain open. In many cases the suitable geometric invariants have not yet been found and important analytic questions are still open, since the analytic tools are missing (see, for instance, [31]). The situation is described in more detail in the following subsection.

1.1 The case of degenerate Levi forms

1.1.1 Geometry

Besides the Levi geometry of the boundaries of the domains plurisubharmonic exhaustion functions and suitable complete metrics with their curvatures are important geometric input in this case. However, if the boundaries are sufficiently smooth, their existence is implied by the boundary geometry.

Linked to the degeneracies of the Levi geometry new geometric phenomena appear and have to be studied:

- the non-diagonalizability of the Levi form.
- The notion of type and its jumping from point to point (see [19], [21], [10]).
- The failure of semi-continuity of types (see [20]).
- The jumping non-isotropic behavior of the geometry.

With respect to the degree of degeneracy there is the following scale of severity:

a) Relatively open Levi flat pieces exist in the boundary,

b) relatively open pieces of the boundary are foliated by complex manifolds of codimension > 1,
c) germs of complex analytic varieties of positive dimension exist in the boundary,

d) boundary points of infinite type exist,

e) the boundary is of finite type.

It follows from [26] that all (relative compact) domains $D$ with smooth $C^\omega$-boundary are of finite type.

1.1.2 Analysis: $L^2$-theory.

The analytic theory of domains with degenerate Levi form so-far is to some extent only understood in the pseudoconvex case, to which the talk restricts its attention from now on.

The $\overline{\partial}$-Neumann problem is qualitatively understood in the finite type case. For domains with $C^\omega$-smooth boundaries its subellipticity follows from [48] together with [26]. However, the estimates for the gain of subellipticity, so-far, are very rough and it is not even understood which geometric invariant attached to the hypersurfaces really determines its exact size (see [31]). Moreover, one might ask whether the nonisotropic nature of the geometry of these domains has been taken into account sufficiently enough in the $\overline{\partial}$-Neumann problem.

Multipliers have been used first in [48] in treating these domains. It seems to be hopeful to also use multiplier ideal theory to get more precise information on the gain of regularity.

The theory is even less developed for pseudoconvex domains with $C^\omega$-smooth boundary of finite type. For them the question of subellipticity has qualitatively been clarified by the work of D. Catlin (see [10] and [11]). However, the estimates on the gain of regularity are extremely rough and it seems, that so-far good methods are missing in order to improve them for general such domains.

In addition to the finite type case which, from the point of view of analysis just is the subelliptic case, new cases also appear on the analytic side among weakly pseudoconvex domains with $C^\infty$-smooth boundaries. We will list them in the following.

1. Even if subellipticity breaks down, the Neumann-operator $N$ might still be compact, still implying at least global regularity of the $\overline{\partial}$-Neumann problem. This feature has attained a lot of attention in the last years (for a general treatment see [42], special results are, for instance, in [54] and already in [12]).

Local hypoellipticity at the boundary for the $\overline{\partial}$-problem breaks down already in the presence of a germ of a complex analytic variety in the boundary (see [36] and [9]). However, the question, whether the $\overline{\partial}$-Neumann problem is at least globally hypoelliptic on all bounded weakly pseudoconvex domains with smooth boundary was much more difficult to decide. In fact, for a long time, this was conjectured. It was only in 1996, when M. Christ (see [13]) showed that for certain worm domains as constructed in [29] the $\overline{\partial}$-Neumann problem fails to be globally hypoelliptic (see also work of D. Barrett [4], [5] and Chr. Kiselman
It, however, should be pointed out, that it does not become clear from
M. Christ’s proof (and still has not been clarified) for which particular turning
numbers the corresponding worm domain has this property. This indicates al-
ready, how little is known about global hypoellipticity. It becomes even more
clear from the fact, that the answer to the following question is not known:
Let $D \in \mathbb{C}^2$ be a worm domain. Then the set

$$A := \overline{D} \cap \{(z, w) \in \mathbb{C}^2 : w = 0\}$$

is a closed annulus and $\partial D$ is strictly pseudoconvex at all other points. Now,
let $z_0 \in \partial D \setminus A$ be arbitrary and let $V$ be a small neighborhood of $z_0$. Consider
any small perturbation $\tilde{D}$ of $D$, such that on $\mathbb{C}^2 \setminus V$ the domains $\tilde{D}$ and $D$ are
equal. It is an open question, whether global hypoellipticity of the $\overline{\partial}$-Neumann
problem also necessarily fails on $\tilde{D}$.

**Remark 1.1** Notice, that locally and semilocally around $A$ the character of $\partial D$
and $\partial \tilde{D}$ is the same and also the topologies agree.

More generally, the following conjecture should hold true:

**Conjecture 1.2** Let $D$ be a worm domain such that its $\overline{\partial}$-Neumann problem
is not globally hypoelliptic. Let $V \in \mathbb{C}^2$ be an arbitrarily small open neighbour-
hood of its boundary annulus $A$ and let $\tilde{D} \subset \mathbb{C}^2$ be another bounded smooth
pseudoconvex domain such that $\tilde{D} \cap V = D \cap V$ and such that $\partial \tilde{D} \setminus V$ consists
of points of finite type only. Then the $\overline{\partial}$-Neumann problem on $\tilde{D}$ also is not
globally hypoelliptic.

**1.1.3 Analysis: Other norms.**

For treating $\overline{\partial}$-problems in norms different from $L^2$, like $L^p, p \neq 2$, Hölder
norms or $C^k$-norms, different methods are required. There are, essentially, two
approaches

1. A passage from $L^2$ to Hölder using techniques of microlocal analysis (see
   [37], [38]).

2. $\overline{\partial}$-solving integral kernels.

Both methods do not work on all bounded pseudoconvex domains with smooth
boundary (even not on those of finite type).

For 1. the class of domains, for which this method, so-far, has been applied
successfully, is quite restricted.

For the construction of the integral kernels in 2. suitable Leray sections have
to be produced. They, essentially, require the existence of holomorphic support
functions depending nicely on the boundary points and satisfying good estimates
there. Unfortunately, those do not always exist, even not in $\mathbb{C}^2$ on finite type
domains, as was first shown by J.J. Kohn and L. Nirenberg in [49] (see also [41]).
(It has been tried by J.J. Fornæss, [40] and others to overcome this difficulty by
introducing some extra techniques for the treatment in neighborhoods of the
eXceptional boundary points. But also such techniques exist so-far only in very
special cases.)

In this talk we want to consider a class of finite type domains which - on the one
hand - allows the construction of suitable families of support functions and - on
the other hand - is general enough to allow a large variety of degeneracies of the
Levi form, namely, lineally convex smooth domains of finite type. According
to the results of H.P. Boas, E. Straube and Yu, they do not allow failure of
semicontinuity of the type, since their general type agrees with their linear type
(see [6]) and [55]. However, under many other aspects, their degeneracies can
be very bad, in particular, they are not always diagonalizable.

2 The $\bar{\partial}$-theory on lineally convex domains of finite
type

2.1 Support functions

There are essentially two different methods for the construction of smooth fam-
ilies of support functions with good estimates for these domains:

1. A method developed by A. Cumenge in [15] and [16] based on sharp esti-
mates on the boundary behavior of the Bergman kernel of E. Stein and J.
McNeal (see [52]) using $L^2$-methods. (It only works in the lineally convex
case.)

2. A direct construction of a smooth family of holomorphic support functions
with best possible estimates (see [90] and [28])

One of the essential difficulties in the construction of the support functions
lies in the following: Although they might satisfy the right estimates in each
complex tangential direction, there might be exceptional real lines where the
estimates become much worse. For the necessary estimates of the $\bar{\partial}$-solving
integral kernels the appearance of such exceptional real lines is deadly. Support
functions without failure of the optimal estimates in some real directions are
needed.

The appearance of such lines, already, becomes clear in the following simple
example:
We consider in $\mathbb{C}^3 = \{z = (z_1, z_2, z_3) : z_j = x_j + iy_j\}$ the defining function
\[ r(z) := y_1 + x_2^4 + x_3^6 + y_3^{10} \]
and the smooth hypersurface $S = \{r = 0\}$ at the point 0. $S$ is convex, so
the real tangent space $T = \{y_1 = 0\}$ to $S$ at 0 is supporting. But even in
its maximally complex subspace $T^C$ it contains with respect to the order of
contact with $S$ two exceptional real lines. Namely, consider the complex linear
subspaces $T_2^C := \{z_1 = z_3 = 0\}$ and $T_3^C := \{z_1 = z_2 = 0\}$ together with their
real subspaces $T_2 : \{z_1 = z_3 = x_2 = 0\}$ resp. $T_3 : \{z_1 = z_2 = x_3 = 0\}$. $T_2^C$ 

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has order of contact with \( S \) at \( 0 \) equal to 4, whereas the order of contact of \( T_2 \) is \( \infty \), the order of contact of \( T_3 \) is 6, whereas that of \( T_3 \) is 10. The difficulty disappears if we replace the supporting real hypersurface \( T \) by
\[
\hat{T} := \{ y_1 - \varepsilon \text{Re} z_3^4 + \varepsilon z_3^6 = 0 \} \tag{1}
\]
for \( \varepsilon > 0 \) small enough.

Obviously, this defining function is pluriharmonic and the hypersurface defined by it has the required order of contact with \( S \) in all real directions of the tangent space.

A correction analogous to (1) by a suitable small perturbation of the tangent plane can be achieved at a fixed boundary point of any smooth convex hypersurface of finite type as was shown in [35] using an idea from [27]. However, such a construction does not seem to suffice for the construction of the desired \( \overline{\partial} \)-solving integral kernels, since it does not give differentiability of the supporting surfaces in the base point. Namely, the direction of the exceptional real lines might jump from point to point, such that the correction terms analogous to the ones in (1) would not depend differentiably on the base point.

This difficulty has been overcome by a construction by K. Diederich and J. E. Fornæss at first for the linearly convex case of finite type considered in [30] and afterwards generalized to linearly convex domains of finite type in [28]. The constructions are based on a new kind of analysis of the Taylor series of convex functions which also might be interesting for different purposes.

We only state here the main result of [28]. For this we denote by \( D = \{ r(z) < 0 \} \) the given linearly convex domain of finite type \( m \). For a point \( \zeta \in \partial D \) and a vector \( t \in T_{\partial D}^1(\zeta) \) with euclidean norm \( ||t|| = 1 \) we define for any \( w = (w_1, w_2) \in \mathbb{C}^2 \)
\[
z_{\zeta, t}(w) := \zeta - iw_1 n_{\zeta} + w_2 t
\]
where \( n_{\zeta} \) is the real unit normal vector to \( \partial D \) at \( \zeta \).

Furthermore, we choose a small enough open neighborhood \( W_0 \) of \( \partial D \) and put for any point \( \zeta \in W_0 \)
\[
D_{\zeta, t} := \{ w \in \mathbb{C}^2 : z_{\zeta, t}(w) \in W_0 : r_{\zeta, t}(w) := r(z_{\zeta, t}(w)) - r(\zeta) < 0 \} \tag{2}
\]
We put for \( j = 2, \ldots, 2m \)
\[
P_{\zeta, t}^j(w) := \sum_{k+l=j} \frac{1}{k! l!} \frac{\partial^j r_{\zeta, t}(0)}{\partial w_1^k \partial w_2^l} w_1^k w_2^l \tag{3}
\]
Notice, that the coefficients of \( P_{\zeta, t}^j \) are \( \mathcal{C}^\infty \) in \((\zeta, t)\).

In order to be able to formulate our main result, we need the following notation:

**Definition 2.1** For any polynomial \( \sum_{j=0}^N \sum_{|\alpha|+|\beta|=j} a_{\alpha \beta} z^{\alpha} \zeta^{\beta} \) on any \( \mathbb{C}^k \) we put
\[
\| P \| := \sum_{j=0}^N \sum_{|\alpha|+|\beta|=j} | a_{\alpha \beta} | \tag{4}
\]
We then have

**Theorem 2.2 (Di/Fo 2004)** \( \exists \hat{S}(z, \zeta) \in C^\infty(\mathbb{C}^n \times W_0), \text{ a holomorphic polynomial of degree } 2m \text{ in } z \forall \zeta \in W_0, \text{ such that} \)

i) \( \hat{S}(\zeta, \zeta) = 0; \)

ii) For any given \( \varepsilon > 0 \) the function \( \hat{S} \) can be chosen in such a way, that the restriction \( S_{\zeta,t} := \hat{S}(z_{\zeta,t}(w), \zeta) \) satisfies

\[
\text{Re } S_{\zeta,t}(w) \leq r_{\zeta,t}(w) - \varepsilon \sum_{j=2}^{2m} \| P_{j,t}^j \| \| w_2 \|^j \tag{5}
\]

**Remark 2.3** It should be stressed, that the function \( \hat{S} \) is given by a formula which is explicit except for the choice of two constants. Furthermore, inequality (5) is the best possible estimate which can be reached on the intersection of \( D \) with all the \( \mathbb{C}^2 \)'s as spanned by all \( n \zeta \) and \( t \).

### 2.2 The pseudodistance

#### 2.2.1 The definition of the pseudodistance

Following ideas of E. Stein and from [8], J. McNeal introduced in [51] for linearly convex domains of finite type a pseudometric which reflects very precisely the non-isotropic nature of the geometry of these domains. It has been generalized to linearly convex domains of finite type in [14] (using ideas from [43]). It is defined in the following way:

Let \( g \) be a defining function of the linearly convex domain \( D \subset \subset \mathbb{C}^n \) of finite type. Put for any \( \varepsilon > 0 \), any point \( \zeta \) close enough to \( \partial D \) and any vector \( \gamma \in \mathbb{C}^n \)

\[
\tau(\zeta, \gamma, \varepsilon) := \max \{ c : |g(\zeta + \lambda \gamma) - g(\zeta)| < \varepsilon \forall \lambda \in \mathbb{C} : |\lambda| < c \}
\]

With this we next choose what we call an \( \varepsilon \)-extremal basis of \( \mathbb{C}^n \) at \( \zeta \) in the following way:

We choose as \( v_1(\zeta, \varepsilon) \) the unit vector orthogonal to the level set of \( g \) passing through \( \zeta \). Then we restrict attention to the linear subspace \( H_1 \) orthogonal to \( v_1(\zeta, \varepsilon) \) and in it we choose a unit vector \( v_2 \) pointing in a direction \( \gamma \in H_1 \) such that \( g(\zeta + \gamma) = \pm \varepsilon \) and \( \tau(\zeta, \gamma, \varepsilon) \) is maximal among such \( \gamma \). This procedure is repeated until the orthonormal basis \( (v_1(\zeta, \varepsilon), \ldots, v_n(\zeta, \varepsilon)) \) is complete.

**Remark 2.4** Notice, that, in general, the dependence of the \( \tau(\zeta, \gamma, \varepsilon) \) and the \( \varepsilon \)-extremal basis on the point \( \zeta \) is not even continuous.

The \( \varepsilon \)-distinguished polydiscs and their versions scaled by \( A > 0 \) are defined as

\[
AP_\varepsilon(\zeta) := \Big\{ z = \zeta + \sum \lambda_k v_k(\zeta, \varepsilon) : |\lambda_k| \leq A\tau_k(\zeta, \varepsilon) \text{ for } k = 1, \ldots, n \Big\}
\]

Finally, the non-isotropic pseudodistance is defined as

\[
d(\zeta, z) := \inf \{ \varepsilon : z \in P_\varepsilon(\zeta) \} \]
2.2.2 Properties of the pseudodistance

Although the pseudodistance is not continuous, it can be shown to satisfy certain uniform estimates making it, nevertheless, possible to do analysis with it. One has:

i) \( \exists c > 0 : c P_{\|z\|} \subset D \forall \zeta \)

ii) If \( \gamma = \sum_{j=1}^{n} a_j v_j(\zeta, \varepsilon) \Rightarrow \)

\[
\frac{1}{\tau(\zeta, \gamma, \varepsilon)} \approx \sum_{j=1}^{n} \frac{|a_j|}{\tau_j(\zeta, \varepsilon)}
\]

iii) \( \forall k > 0 \exists \) constants \( c(k), C(k) \) such that

\( c_k P_{\varepsilon}(\zeta) \subset P_{k\varepsilon}(\zeta) \subset C(k) P_{\varepsilon}(\zeta) \) \( (6) \)

iv) \( \forall z \in P_{\varepsilon}(\zeta) \Rightarrow \tau(\zeta, \gamma, \varepsilon) \approx \tau(z, \gamma, \varepsilon) \)

v) \( d(z, \zeta) \approx d(\zeta, z) \)

\( d(z, \zeta) \lesssim d(z, w) + d(w, \zeta) \)

Here the relation \( \approx \) respectively \( \lesssim \) stand for the corresponding strict relations = resp. \( \leq \) up to constants uniform in the choice of \( \zeta \) chosen from a small enough neighborhood of the boundary \( \partial D \) and \( \varepsilon > 0 \).

2.2.3 Estimates relative to the pseudodistance

The pseudodistance introduced above reflects exactly the non-isotropic geometry of the corresponding domain \( D \). It, therefore, is not astonishing that the estimates required for the quantitative solutions of \( \bar{\partial} \), become simple and easy to use - just as the estimates with respect to the euclidean metric in the strictly pseudoconvex case. We will show this by giving a few examples.

**Lemma 2.5** Define \( \zeta \in D \cap W \) and \( \varepsilon > 0 \) the set \( P_{\varepsilon, \varepsilon}^0(\zeta) \) := \( CP_{\varepsilon}(\zeta) \backslash \frac{1}{2}P_{\varepsilon}(\zeta) \) \( C = C(1) \) being the constant of (6)) and let \( \pi \) be the orthogonal projection to \( \partial D \). Then one has \( \forall z \in D \cap U \)

\[
|S(z, \zeta)| \gtrsim \varepsilon \forall \zeta \in \partial D \cap P_{\varepsilon, \varepsilon}^0(\pi(z))
\]

\[
|S(z, \zeta)| \gtrsim |\varrho(z)| \forall \zeta \in \partial D \cap P_{|\varrho(z)|}(\pi(z))
\]

> From the explicit formula for the family of support functions \( S(z, \zeta) \) one easily gets a corresponding **Leary decomposition**, for which one has

\[
S(z, \zeta) = \sum_{j=1}^{n} Q_j(z, \zeta)(z_j - \zeta_j)
\]

(7)
(for details see [28] and [24]). In order to get the estimates required for the corresponding \(\bar{\partial}\)-solving Cauchy-Fantappiè kernels, derivatives of the Leray section \((Q_1(z, \zeta), \ldots, Q_n(z, \zeta))\) have to be estimated. Using the pseudometric in a consequent way, this calculus becomes very natural:

For a point \(\zeta_0 \in \partial D\) and arbitrary \(z, \zeta \in P_\varepsilon(\zeta_0)\) we choose a linear transformation \(\Phi\) giving the \(\varepsilon\)-extremal coordinates at \(\zeta_0\) and put

\[
\begin{align*}
    w &:= \Phi(z - \zeta_0) \text{ and } \eta := \Phi(\zeta - \zeta_0) \\
(\Rightarrow |\eta_k| &\leq C\tau_k(\zeta_0, \varepsilon); |w_1| \leq C; |w_k| \leq C\tau_k(\zeta_0, \varepsilon))
\end{align*}
\]

We denote the Leray section in the new coordinates by \(Q_1^*(w, \eta), \ldots, Q_n^*(w, \eta)\) and get

Lemma 2.6

\[
\begin{align*}
    |Q_k^*(w, \eta)| &\leq \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon)} \\
    \left| \frac{\partial}{\partial w_i} Q_k^*(w, \eta) \right| &\leq \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon) \tau_i(\zeta_0, \varepsilon)} \\
    \left| \frac{\partial}{\partial \eta_j} Q_k^*(w, \eta) \right| &\leq \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon) \tau_j(\zeta_0, \varepsilon)} \\
    \left| \frac{\partial^2}{\partial w_i \partial \eta_j} Q_k^*(w, \eta) \right| &\leq \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon) \tau_i(\zeta_0, \varepsilon) \tau_j(\zeta_0, \varepsilon)}
\end{align*}
\]

2.3 A first result on solving \(\bar{\partial}\) with nonisotropic estimates

At first, using the above-mentioned smooth family of holomorphic support functions, quantitative results on solving \(\bar{\partial}\) where given only with respect to isotropic Hölder norms. We mention as examples [25, 44, 15, 16]. However, it is natural, that also the norms measuring the solutions to the \(\bar{\partial}\)-equation have to respect the nonisotropic nature of the geometry. As a typical example we define new Hölder norms by

Definition 2.7 For any \(\mu > 0\) and small \(\varepsilon > 0\) we define

\[
\tilde{\Lambda}^{\mu,\varepsilon}(D) := \left\{ h \in C^0(D) : |h(z_0) - h(z_1)| \leq C_h \max \left\{ d(z_0, z_1)^\mu, |z_0 - z_1|^{1-\varepsilon} \right\} \right\}
\]

Remark 2.8 For any given \(h\) the smallest constant \(C_h\) possible in this inequality is called the corresponding Hölder norm \(\|h\|_{\mu,\varepsilon}\). The term \(|z_0 - z_1|^{1-\varepsilon}\) only appears in this definition only for technical reasons, namely, in order to take care of the case, when the points \(z_0\) and \(z_1\) are far apart. For the understanding of the results, it suffices to neglect this term and, hence, the role of \(\varepsilon\) in the definition of the Hölder norm.

One has with respect to these nonisotropic Hölder norms (see also [18] for a slightly weaker result)

Theorem 2.9 (Di/Fischer 2004) For any \(1 \leq q \leq n\) \exists a continuous linear operator \(T_q : L^\infty \rightarrow \tilde{\Lambda}^{1,\varepsilon}_{(0, q-1)}\) such that \(\bar{\partial} T_q f = f \forall f : \bar{\partial} f = 0\).
Remark 2.10  
a) The result of the theorem only is a test case for a whole series of possible similar results. For instance, correct nonisotropic H"older norms have to be defined for $\Theta$-closed $(0,q)$-forms and the corresponding $(0,q-1)$-forms solving $\Theta$ have to be found solving them (see also below).

b) The proof of Theorem 2.9 is quite natural once one has the support functions of Theorem 2.2, the pseudodistance and the estimates of Lemmas 2.5 and 2.6. In fact, it seems, that it might be possible to formalize the theory by proving a technical general proposition which says, that, on a given domain, one always has such a theorem if for it support functions and a pseudodistance exist such that the corresponding estimates of the Lemmas 2.5 and 2.6 hold (i.e. other special properties of the domain do not enter into the proof).

2.4 $\Theta$ and differentiability up to the boundary

A general result of J. J. Kohn (see [47]) guarantees that on any weakly pseudoconvex domain with smooth boundary any $\Theta$-closed $(0,1)$-form which is $C^\infty$ up to the boundary can be solved by a function $C^\infty$ up to the boundary. However, there is no good $C^k$-estimate for the solution.

For strictly pseudoconvex smooth domains this problem has first been solved by G. Henkin (see [45]). Recently, W. Alexandre (see [2, 3]) showed a first result in this direction on smoothly bounded, bounded convex domains of finite type. He uses as a main tool again the above mentioned smooth family of holomorphic support functions.

Theorem 2.11 Let $D \subset \subset \mathbb{C}^n$ be a linearly convex domain with $C^\infty$-smooth boundary of finite type $\leq m$. Then there is for any $1 \leq q \leq n$ and any $k = 0,1,2,\ldots$ a bounded linear operator

$$T_q : C^{(0,q)}_{0,q}(\overline{D}) \cap \ker \Theta \rightarrow C^{k,\frac{1}{m}}_{0,q-1}(\overline{D})$$

such that

$$\Theta T_q f = f$$

Remark 2.12  
a) In the Theorem the spaces $C^{k,\frac{1}{m}}_{(0,q)}(\overline{D})$ are provided with the sup-norm on $\overline{D}$ over all partial derivatives up to the total order $k$ of all coefficients and the norm on the space $C^{k,\frac{1}{m}}_{(0,q)}(\overline{D})$ is the sum of the norm on $C^{k}_{(0,q)}(\overline{D})$ and the isotropic H"older norms of order $\frac{1}{m}$ on $\overline{D}$ of all partial derivatives of order $k$ of all the coefficients of the forms.

b) Recently, K. Di. and B. Fischer have carried over this result to linearly convex domains of finite type and, more importantly, to the analogous nonisotropic H"older norms on $\text{Im} T_q$. The corresponding preprint will appear soon and will contain some further research on the subject.

2.5 The extension problem

In [50] E. Mazzilli gave an example of a pseudoellipsoid $D$ together with a complex-analytic algebraic variety $X$, smooth in an open neighborhood of $\overline{D}$.
and intersecting $\partial D$ transversally, such that, nevertheless, there is a bounded holomorphic function $f$ on $X \cap D$ which does not extend to a bounded holomorphic function on $D$. This strange phenomenon was, afterwards, more closely studied in [32]. Later, K. Di and E. Mazzilli showed in [33], that, in contrast to this, every bounded holomorphic function on $X \cap D$ extends to a bounded holomorphic function if $X$ is affine linear and if $D$ is just a bounded linearly convex domain with smooth boundary of finite type.

The question which submanifolds allow a bounded holomorphic extension of all bounded holomorphic functions has first been studied by E. Mazzilli. He gave a sufficient condition using different type notions for the intersection $\partial D \cap X$. However, his condition is too strong to be also necessary. Recently, W. Alexandre used in [1] the nonisotropic pseudometric as introduced above to give a sufficient condition on $X \cap D$ for bounded extendibility (for linearly convex domains of finite type) with respect to the sup-norm which is very close to also being necessary. Since the formulation of the condition is rather technical we refer the interested reader to the original paper.

2.6 Some open problems

Besides the work mentioned so-far, many more questions of quantitative complex analysis have been studied on linearly (or even linearly) convex domains of finite type. In addition to the above, we mention the following selection:

- Boundary behavior of $H^p$-functions, see [23].
- Bounded solvability of $\overline{\partial}$ with respect to $L^p$-norms with $p \neq 2$, see [39].
- Characterization of zero sets of the Nevanlinna class, see [7, 17, 34]
- Dependence of isotropic Hölder and $L^p$-norms on Catlin's multitype, see [43].
- $(0, 1)$-forms with finite Bruna-Charpentier-Dupain-norm can be solved in $L^1(\partial D)$, see [7, 17, 34].

As closing remarks of this survey we mention several important open questions of the subject:

**Question 2.13**

1. Generalize all results known only for linearly convex domains of finite type to linearly convex domains of finite type.

2. Investigate for $\mu > 1$ the relation between the spaces $C^{1, \mu}(\overline{D})$ and $C^{1+k(\mu), \mu-k}(\overline{D})$, where $k(\mu)$ is the largest integer $\leq \mu$ and all Hölder norms are taken with respect to the pseudodistance on $\overline{D}$.

3. Solve the $\overline{\partial}$-equation with best possible estimates for forms with coefficients in $\Lambda^\mu$ respectively in $C^{k, \mu}$.

4. Study more general classes of domains to which the construction of a smooth family of holomorphic support functions satisfying best possible estimates of Di-Fornæss can be carried over.
Added in proof: Recently, after finishing this survey, the author has been informed, that a preprint of J. Michel appeared, in which he generalizes the Diederich-Fornæss construction of support functions to so-called K-convex domains.

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