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Abstract

We show that the Proper Forcing Axiom implies the Singular Cardinal Hypothesis. The proof uses ideas of Moore from [11] and the notion of a relativized trace function on pairs of ordinals.

Introduction

In one of the first applications of the forcing techniques, Easton [3] showed that the exponential function $\kappa \mapsto 2^\kappa$ on regular cardinals can be arbitrary modulo some mild restrictions. The situation for singular cardinals is much more subtle. For instance, Silver [14] showed that the Singular Cardinal Hypothesis SCH cannot first fail at a singular cardinal of uncountable cofinality. Recall that SCH states that $2^\kappa = \kappa^+$, for all singular strong limit cardinals κ. However, it is known that SCH can first fail even at $\aleph_\omega$.

The role of large cardinals in this context is twofold. On the one hand they are necessary for the construction of models of the negation of SCH (see [5] for a survey of Prikry type forcings and applications to SCH). On the other hand it is a theorem of Solovay [15] that SCH holds above a strongly compact cardinal. Forcing axioms imply reflection principles similar to the one used in Solovay's proof, thus it was reasonable to expect that they would also settle SCH. Indeed, in [4], Foreman, Magidor and Shelah showed that the strongest forcing axiom, Martin's Maximum MM, implies SCH. This was later improved by Veličković [17] who also showed that SCH follows from PFA^+. In fact, what is shown in [17] is that if $\theta > \aleph_1$ is regular and stationary subsets of $[\theta]^\omega$ reflect to an internally closed and unbounded set, then $\theta^\omega = \theta$. This, combined with Silver's theorem, implies SCH. At this point, it was left open whether SCH is a consequence of PFA.

Very little progress was made on this problem for over fifteen years. Then, in 2003, Moore [10] introduced a new reflection principle, the *Mapping Reflection Principle* MRP and deduced it from PFA. He showed that MRP implies the continuum is equal to $\aleph_2$ and the failure of $\square(\kappa)$, for all $\kappa > \aleph_1$. MRP has

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many features in common with the reflection principles which follow from MM, so it should be expected that MRP could affect the behaviour of the exponential function also on higher cardinals. In fact, Moore showed in [11] that if MRP holds and $\kappa > \omega_1$ is a regular cardinal with a nonreflecting stationary set consisting of points of countable cofinality, then $\kappa^{\omega_1} = \kappa$. This, combined with the above result of Veličković, strongly suggests that PFA implies SCH.

In this paper we confirm this conjecture. The new technical tool we introduce is the notion of a relativized trace function defined on pairs of ordinals. This allows us to get rid of the nonreflecting stationary set in Moore's argument.

The paper is organized as follows. In section §1 we introduce the relativized trace, which is a generalization of the functions considered in [16], and we outline the properties of this function which are relevant for our purposes. In section §2 we prove that if MRP holds then $\kappa^\omega = \kappa$ for all regular cardinals $\kappa > \omega_1$. This, together with Silver's theorem, implies that SCH follows from PFA. In the final section we show that the techniques introduced in this paper can be applied to study another interesting problem in the area of forcing axioms, i.e. to investigate what kind of forcing notions can preserve this type of axioms.

Our notation is standard and follows [7] and [11]. For a regular cardinal θ, we use $H(\theta)$ to denote the structure $(H(\theta), \in, <)$ whose domain is the collection of sets whose transitive closure is of size less than θ and where $<$ is a predicate for a fixed well ordering of $H(\theta)$. If X is an uncountable set, $C \subseteq [X]^\omega$ is closed and unbounded (club) if there is $f : [X]^{<\omega} \to X$ such that C is the set of all $Y \in [X]^\omega$ such that $f[Y]^{<\omega} \subseteq Y$. $S \subseteq [X]^\omega$ is stationary if it intersects all club subsets of $[X]^\omega$. The f-closure of X is the smallest Y containing X such that $f[Y]^{<\omega} \subseteq Y$. If X is a set of ordinals then $\overline{X}$ denotes the topological closure of X in the order topology.

1 The relativized trace function

In this section we introduce a relativized version of the trace function considered by Todorčević (see [16]) and outline some of its relevant properties. This function describes a descending walk from a larger ordinal β to a smaller ordinal α similar to the standard trace function. The rules of the walk are determined by a parameter X. This relativized trace function is designed in order to control very well its behavior on the points of the walk belonging to X.

Given κ regular cardinal, fix a sequence $(C_\alpha : \alpha \in \kappa \,\&\, cof\,\alpha = \omega)$ such that C_α is a cofinal subset of α of order type ω for all $\alpha < \kappa$ of cofinality ω.

Definition 1 *Let $X \in [\kappa]^{\leq\omega}$, then for all $\alpha, \beta \in \kappa$, $tr_X(\alpha, \beta)$ is defined as follows:*

(i) *if $\beta < \alpha$ then $tr_X(\alpha, \beta) = \emptyset$*
(ii) $tr_X(\alpha, \alpha) = \{\alpha\}$
(iii) $tr_X(\alpha, \beta + 1) = tr_X(\alpha, \beta) \cup \{\beta + 1\}$
(iv) *if $\beta > \alpha$ is a limit ordinal then:*

if $cof\,\beta > \omega$ then $tr_X(\alpha, \beta) = tr_X(\alpha, sup(\beta \cap X)) \cup \{\beta\}$, *if* $cof\,\beta = \omega$ *then set* $tr_X(\alpha, \beta) = tr_X(\alpha, min(C_\beta \setminus \alpha)) \cup \{\beta\}$.

So, walking from β to α we treat points which are either successors or of countable cofinality as in the usual walk. However, points of uncountable cofinality are treated as successors in the sense that we go from β to $sup(\beta \cap X)$. We now state the key properties of the function tr_X which we will use.

Fact 1 *For* $\alpha, \beta < \kappa$ *we have the following.*

(i) *If* $\alpha \in X$ *and* $\beta \geq \alpha$, *then* $\alpha \in tr_X(\alpha, \beta)$

(ii) *For all* $\alpha \leq \beta$ *there is a* $\gamma < \alpha$ *such that* $tr_X(\alpha, \beta) \subseteq tr_X(\eta, \beta)$, *for all* $\eta \in (\gamma, \alpha]$.

Proof:
(i) If $\alpha \in X$ then $sup(\beta \cap X) \geq \alpha$ for any $\beta > \alpha$. We proceed by induction. The base case is trivial by *(ii)* in the definition of tr_X. If the claim holds for all γ such that $\alpha \leq \gamma < \beta$, we have three cases, β successor, β of countable cofinality, β of uncountable cofinality.

(a) If $\beta = \gamma + 1$, the induction hypothesis applies to $tr_X(\alpha, \gamma)$. Since $tr_X(\alpha, \beta) = tr_X(\alpha, \gamma) \cup \{\beta\}$ we are done.

(b) If $cof(\beta) = \omega$, then the induction hypothesis applies to $tr_X(\alpha, min(C_\beta \setminus \alpha))$, since $\alpha \leq min(C_\beta \setminus \alpha) < \beta$. Now $tr_X(\alpha, \beta) = tr_X(\alpha, sup(C_\beta \setminus \alpha)) \cup \{\beta\}$, so we are done.

(c) If $cof(\beta) > \omega$ then $\alpha \leq sup(\beta \cap X) < \beta$ and

$$tr_X(\alpha, \beta) = tr_X(\alpha, sup(\beta \cap X)) \cup \{\beta\}$$

now the induction hypothesis applies to $tr_X(\alpha, sup(\beta \cap X))$.

This concludes the proof of *(i)*.

(ii) We may assume that α is a limit ordinal. Then there are two cases:

(a) $sup(\alpha \cap X) = \alpha$. Then let

$$\gamma_\beta = max\{max(C_\eta \cap \alpha) \,:\, \eta \in tr_X(\alpha, \beta) \cap cof(\omega),\, \eta \neq \alpha\}.$$

It is easy to check by induction on $\beta \geq \alpha$ that if $\rho \in (\gamma_\beta, \alpha]$ then $tr_X(\alpha, \beta) \subseteq tr_X(\rho, \beta)$, using the fact that $\gamma_\beta \geq \gamma_\eta$ and that $sup(\eta \cap X) \geq \alpha$, for all $\eta \in tr_X(\alpha, \beta)$.

(b) $sup(\alpha \cap X) < \alpha$. Then proceed as above taking

$$\gamma_\beta = max(\{sup(\alpha \cap X)\} \cup \{max(C_\eta \cap \alpha) \,:\, \eta \in tr_X(\alpha, \beta) \cap cof(\omega),\, \eta \neq \alpha\}).$$

□

2 The main result

The purpose of this section is to show that MRP implies that $\lambda^\omega = \lambda$, for every cardinal $\lambda > \omega_1$ with $cof(\lambda) > \omega$. We start by recalling the relevant definitions from [10].

Definition 2 *Let θ be a regular cardinal, let X be uncountable, and let $M \prec H(\theta)$ be countable such that $[X]^\omega \in M$. A subset Σ of $[X]^\omega$ is M-stationary if for all $E \in M$ such that $E \subseteq [X]^\omega$ is club, $\Sigma \cap E \cap M \neq \emptyset$.*

Recall that the Ellentuck topology on $[X]^\omega$ is obtained by declaring a set open if it is the union of sets of the form

$$[x, N] = \{Y \in [X]^\omega : x \subseteq Y \subseteq N\}$$

where $N \in [X]^\omega$ and $x \subseteq N$ is finite. When we say 'open' in this paper we refer to this topology.

Definition 3 *Σ is an open stationary set mapping if there is an uncountable set X and a regular cardinal θ such that $[X]^\omega \in H(\theta)$, the domain of Σ is a club in $[H(\theta)]^\omega$ of countable elementary submodels M such that $X \in M$ and for all M, $\Sigma(M) \subseteq [X]^\omega$ is open and M-stationary.*

The Mapping Reflection Principle (MRP) asserts that:

> If Σ is an open stationary set mapping, there is a continuous $\in$-chain $\vec{N} = (N_\xi : \xi < \omega_1)$ of elements in the domain of Σ such that for all limit ordinals $\xi < \omega_1$ there is $\nu < \xi$ such that $N_\eta \cap X \in \Sigma(N_\xi)$ for all η such that $\nu < \eta < \xi$.

If $(N_\xi : \xi < \omega_1)$ satisfies the conclusion of MRP for Σ then it is said to be a reflecting sequence for Σ. It is shown in [10] that MRP is a consequence of PFA. We are now ready to prove the following theorem.

Theorem 1 *Assume* MRP*. Then $\lambda^{\aleph_0} = \lambda$, for every $\lambda \geq \omega_2$ of uncountable cofinality.*

Proof: We will prove the theorem by induction. The base case $\lambda = \aleph_2$ is handled by Moore's result [10] that MRP implies $2^{\aleph_0} = \aleph_2$. If $\lambda = \kappa^+$ with $cof(\kappa) > \omega$ then $\lambda^{\aleph_0} = \lambda \cdot \kappa^{\aleph_0}$, so the result holds by the inductive hypothesis. If λ is a limit cardinal and $cof(\lambda) > \omega$ then $\lambda^{\aleph_0} = sup\{\mu^{\aleph_0} : \mu < \lambda\}$, so the result also follows by the inductive hypothesis. Thus, the only interesting case is when $\lambda = \kappa^+$, with κ singular of countable cofinality. In this case we will show, using MRP, that $\kappa^{\aleph_0} = \kappa^+$.

Now, let κ be singular of countable cofinality and assume the theorem holds below κ. Fix a sequence $(C_\delta : \delta \in \kappa^+)$ such that C_δ is a club in δ of minimal order type. In fact, we will be interested only in ordinals δ of cofinality $\leq \omega_1$. For every pair of ordinals $\delta, \beta < \kappa^+$, we fix a decomposition $\delta = \bigcup_n K(n, \delta, \beta)$ such that:

(i) $|K(n,\delta,\beta)| < \kappa$, for all n

(ii) $K(n,\delta,\beta) \subseteq K(m,\delta,\beta)$, for $n < m$

(iii) if $\eta < \beta$ is of cofinality ω_1 then there is n such that $C_\eta \cap \delta \subseteq K(n,\delta,\beta)$

(iv) $K(n,\delta,\beta)$ is a closed subset of δ, for all n.

This is easily achieved, for example, as follows. First of all, fix an increasing sequence $(\kappa_n : n \in \omega)$ of regular cardinals converging to κ. For all $\eta < \kappa^+$ let $\phi_\eta : \kappa \to \eta$ be a surjection. Now set:

$$K(n,\delta,\beta) = \delta \cap \overline{\phi_\delta[\kappa_n] \cup \bigcup\{C_\eta \cap \delta \ : \eta \in \phi_\beta[\kappa_n] \,\&\, cof\,\eta = \omega_1\}}.$$

Fix also a partition $\{A_s : s \in \kappa^{<\omega}\}$ of $\{\delta < \kappa^+ : cof(\delta) = \omega\}$ into disjoint stationary sets. Let $D(n,\delta,\beta)$ be the set of all $g \in \kappa^\omega$ such that there are infinitely many j such that $K(n,\delta,\beta) \cap A_{g\restriction j} \neq \emptyset$. Using the fact that $K(n,\delta,\beta)$ is of size $< \kappa$ and the inductive hypothesis we immediately have the following.

Fact 2 *$D(n,\delta,\beta)$ is of size smaller than κ, for all n, δ and β.* □

We will be done once we show the following.

Lemma 1 *Assume* MRP*. Then $\bigcup\{D(n,\delta,\beta) : n < \omega$ and $\delta,\beta < \kappa^+\} = \kappa^\omega$.*

Proof: Fix $g \in \kappa^\omega$. We have to find some (n,δ,β) such that $g \in D(n,\delta,\beta)$. We are going to define an open stationary set mapping Σ_g and apply MRP. We first fix some notation. Given a countable set X, we let $\delta_X = sup(X \cap \kappa^+)$ and $\alpha_X = sup(X \cap \omega_1)$. If $\alpha < \gamma < \omega_1$, let the *height* of α in γ be defined by $ht_\gamma(\alpha) = |C_\gamma \cap \alpha|$. Fix a sufficiently large regular cardinal θ. Suppose M is a countable elementary submodel of $H(\theta)$ containing all the relevant information. Fix $\beta_M < \kappa^+$ large enough such that for every $\gamma < \kappa^+$ of cofinality ω_1 there is $\eta < \beta_M$ of cofinality ω_1 such that $C_\gamma \cap M = C_\eta \cap M$. Now let, for all n,

$$M_n = \overline{M \cap K(n,\delta_M,\beta_M)}.$$

If $\gamma \in M \cap \kappa^+$ let n_γ be the smallest integer l such that $\gamma \in M_l$. Now, let $\Sigma_g(M)$ be the set of all $X \in [M \cap \kappa^+]^\omega$ such that either X has a largest element, or X does not have a largest element, $\alpha_X < \alpha_M$, $\delta_X < \delta_M$ and letting $m = ht_{\alpha_M}(\alpha_X)$, there is an $l \leq n_{\delta_X}$ with the property that:

$$tr_{M_l}(\delta_X,\delta_M) \cap A_{g\restriction m} \cap M_l \neq \emptyset.$$

We will show that $\Sigma_g(M)$ is open and M-stationary, for all M.

Claim 1 *$\Sigma_g(M)$ is open.*

Proof: Suppose $X \in \Sigma_g(M)$. Suppose first X has a largest element, i.e. $\delta_X \in X$. Then clearly every subset of X containing δ_X also has δ_X as its largest element and, thus, $[\{\delta_X\}, X] \subseteq \Sigma_g(M)$. Now, suppose $\delta_X \notin X$. First find $\alpha \in X \cap \omega_1$ such that $ht_{\alpha_M}(\alpha) = ht_{\alpha_M}(\alpha_X) = m$. Let $n = n_{\delta_X}$. By the definition of n_{δ_X} we have that $\delta_X \notin M_{n-1}$. Since M_{n-1} is a closed set of ordinals, there is a γ below δ_X such that $(\gamma, \delta_X] \cap M_{n-1} = \emptyset$. Moreover, by Fact 1*(ii)*, there is, for all $l \leq n$, a $\gamma_l < \delta_X$ such that $tr_{M_l}(\delta_X, \delta_M) \subseteq tr_{M_l}(\eta, \delta_M)$, for all $\eta \in (\gamma_l, \delta_X]$. Pick $\delta \in X$ larger than γ and all γ_i, for $i \leq n$. Then we have the following.

Subclaim 1 *$[\{\alpha, \delta\}, X] \subseteq \Sigma_g(M)$.*

Proof: If $Y \in [\{\alpha, \delta\}, X]$, clearly, we have that $ht_{\alpha_M}(\alpha_Y) = m$. Since $\delta \in Y$ and $Y \subseteq X$ we have that $\delta \leq \delta_Y \leq \delta_X$. By the above remarks we can conclude that $n = n_{\delta_X} \leq n_{\delta_Y}$ and that $tr_{M_l}(\delta_X, \delta_M)$ is a subset of $tr_{M_l}(\delta_Y, \delta_M)$, for all $l \leq n$. Now, since $X \in \Sigma_g(M)$, there is an $l \leq n = n_{\delta_X} \leq n_{\delta_Y}$ such that:

$$tr_{M_l}(\delta_X, \delta_M) \cap A_{g \restriction m} \cap M_l \neq \emptyset.$$

By the above considerations, for this l also $tr_{M_l}(\delta_Y, \delta_M) \cap A_{g\restriction m} \cap M_l$ is nonempty. So $ht_{\alpha_M}(\alpha_Y) = m$ and there is an $l \leq n_{\delta_Y}$ such that $tr_{M_l}(\delta_Y, \delta_M) \cap A_{g\restriction m} \cap M_l$ is nonempty, i.e. $Y \in \Sigma_g(M)$. □

Claim 2 *$\Sigma_g(M)$ is M-stationary.*

Proof: Given $f : [\kappa^+]^{<\omega} \to \kappa^+$ belonging to M, we must find $X \in M \cap \Sigma_g(M)$ which is closed under f. First, find $N \in M$, a countable elementary submodel of $H(\kappa^{++})$ containing all the relevant objects for the argument below. Let $ht_{\alpha_M}(\alpha_N) = m$ and find $\alpha \in N$ with the same height in α_M. Now let C be the set of $\delta < \kappa^+$ such that $f[\delta]^{<\omega} \subseteq \delta$. Then C is a club subset of κ^+ and $C \in N$. Since $A_{g\restriction m}$ is stationary in κ^+ and, by our assumption it belongs to N, we can find $\delta \in C \cap A_{g\restriction m} \cap N$. Then $\delta \in tr_{M_{n_\delta}}(\delta, \delta_M)$, by Fact 1 *(i)*. Finally, let $Z \in N$ be a countable set cofinal in δ and let X be the f-closure of $\{\alpha\} \cup Z$. Then $\delta_X = \delta$ and

$$m = ht_{\alpha_M}(\alpha) \leq ht_{\alpha_M}(\alpha_X) \leq ht_{\alpha_M}(\alpha_N) = m.$$

Since $tr_{n_{\delta_X}}(\delta_X, \delta_M) \cap A_{g\restriction m} \cap M_{n_{\delta_X}}$ is nonempty, $X \in \Sigma_g(M) \cap M$ and X is closed under f. □

Let $(M_\eta : \eta < \omega_1)$ be a reflecting sequence for Σ_g provided by MRP. Let $N = \bigcup_\eta M_\eta$ and $\delta = sup(N \cap \kappa^+)$. Let $\delta_\eta = sup(M_\eta \cap \kappa^+)$, for every $\eta < \omega_1$. We find a club $E \subseteq \omega_1$ such that $\{\delta_\eta : \eta \in E\} \subseteq C_\delta$ and $M_\eta \cap \omega_1 = \eta$, for all $\eta \in E$. Let α be a limit point of E. For the rest of this proof let $M = M_\alpha$. Now $C_\delta \cap M = C_\gamma \cap M$ for some $\gamma < \beta_M$, by the choice of β_M. By *(iii)* of the definition of $K(i, \delta_M, \beta_M)$ there is an n such that $C_\gamma \cap \delta_M$ is a subset of $K(n, \delta_M, \beta_M)$. Since $C_\gamma \cap M \subseteq C_\gamma \cap \delta_M$, we can conclude that $C_\delta \cap M$ is a subset of $K(n, \delta_M, \beta_M)$.

Let $\nu < \alpha$ be such that $M_\eta \in \Sigma_g(M)$, for all η such that $\nu < \eta < \alpha$. For any such $\eta \in E$, $M_\eta \in M$, so $\delta_\eta \in C_\delta \cap M \subseteq K(n, \delta_M, \beta_M)$. If $\eta \in E$ and $ht_{\alpha_M}(\eta) = j$, there is an $l \leq n_{\delta_\eta} \leq n$ such that $A_{g \restriction j} \cap tr_{M_l}(\delta_\eta, \delta_M) \cap M_l \neq \emptyset$. Now, for any i we can find an $\eta \in E$ such that $\nu < \eta < \alpha$ and $ht_{\alpha_M}(\eta) \geq i$, so there are infinitely many j such that $A_{g \restriction j} \cap K(n, \delta_M, \beta_M) \neq \emptyset$, i.e. $g \in D(n, \delta_M, \beta_M)$, as desired. ■

Now we have the following immediate corollary.

Corollary 1 PFA *implies* SCH.

Proof: This follows by induction. By Silver's theorem [14] the first cardinal violating SCH cannot be singular strong limit of uncountable cofinality. On the other hand, if κ is a singular strong limit cardinal of countable cofinality then, by Theorem 1, $2^\kappa = \kappa^{\aleph_0} = \kappa^+$. ■

3 Final remarks and side results

The techniques presented in the previous sections can be applied to investigate another interesting problem in the area of forcing axioms. Since forcing axioms have been able to settle many of the classical problems of set theory, we can expect that the models of a forcing axiom are in some sense canonical. There are many ways in which one can give a precise formulation to this concept. For example, one can study what kind of forcings can preserve PFA, or else if a model V of a forcing axiom can have an interesting inner model M of the same forcing axiom. There are many results in this area, some of them very recent. For instance, König and Yoshinobu [8, Theorem 6.1] showed that PFA is preserved by ω_2-closed forcing. The same holds for BPFA. In fact, BPFA is preserved by any proper forcing that does not add subsets of ω_1. In the other direction, in [17] Veličković showed that if MM holds and M is an inner model such that $\omega_2^M = \omega_2$, then $\mathcal{P}(\omega_1) \subseteq M$ and in a very recent paper Caicedo and Veličković [1] showed that if $M \subseteq V$ are models of BPFA and $\omega_2^M = \omega_2$ then $\mathcal{P}(\omega_1) \subseteq M$. Their argument also shows that if $M \subseteq V$ are models of MRP and $\omega_2^M = \omega_2$, then $\mathcal{P}\omega \subseteq M$. We can use the result of the previous section combined with this last result to show that PFA is destroyed by many of the cardinal preserving notions of forcing which add new ω-sequences. A result of this sort has been obtained by Moore in [11].

Theorem 2 *Let V and W be two models of set theory with the same cardinals with $V \subseteq W$. Assume V and W are both models of* MRP*. and that, moreover, for every cardinal κ, stationary subsets in V of $\{\alpha \in \kappa^+ : cof\, \alpha = \omega\}$ are also stationary in W. Then V and W have the same ω-sequences of ordinals.*

Proof: Assume otherwise. We proceed by induction on the least cardinal κ such that there in an ω-sequence of elements of κ which is in W, but not in V. The base case ω is handled by the above result of Caicedo and Veličković. We now run into two cases: either the least such κ has countable cofinality in V or it doesn't. The more involved case appears when $cof^V \kappa > \omega$. We present in some detail how to prove the induction step in this situation. With minor modifications the reader can supply the proof for the case that κ is of countable cofinality. The idea is to redo the proof of the previous section using a $g \in \kappa^\omega \setminus V$. However some extra care has to be paid in the definition of the sets $K(n, \delta, \beta)$. Let $\{A_s : s \in \kappa^{<\omega}\} \in V$ be a partition of the points of countable V-cofinality of κ^+ into disjoint stationary sets. By the assumptions each A_s is still stationary in W. Fix $(E_\delta : \delta < \kappa^+ \,\&\, cof^V \delta \geq \omega_1) \in V$ such that for all δ in its domain, E_δ is a club in δ of minimal V-order-type. So for each δ, E_δ has order type at most κ. Define in V, for all $\alpha < \kappa$ and $\delta, \beta < \kappa^+$, sets $K(\alpha, \delta, \beta)$ such that $\delta = \bigcup_\alpha K(\alpha, \delta, \beta)$ and:

(i) $|K(\alpha, \delta, \beta)| < \kappa$

(ii) $K(\alpha, \delta, \beta) \subseteq K(\gamma, \delta, \beta)$ for $\alpha < \gamma$

(iii) if $\eta < \beta$ and $cof^V \eta \geq \omega_1$ then there is α such that $E_\eta \cap \delta \subseteq K(\alpha, \delta, \beta)$

(iv) $K(\alpha, \delta, \beta)$ is a closed subset of δ.

This is easily achieved, for example, as follows. For all $\eta \in [\kappa, \kappa^+)$ let $\phi_\eta : \kappa \to \eta$ be a bijection. Now set:

$$K(\alpha, \delta, \beta) = \delta \cap \overline{\phi_\delta[\alpha] \cup \bigcup \{E_\eta \cap \delta \, : \eta \in \phi_\beta[\alpha] \,\&\, |E_\eta \cap \delta| \leq |\alpha|\}}.$$

Define $D(\alpha, \delta, \beta)$ to be the set of all $g \in \kappa^\omega$ such that there are infinitely many j such that $A_{g\restriction j} \cap K(\alpha, \delta, \beta) \neq \emptyset$ and use the inductive hypothesis to get that $D(\alpha, \delta, \beta)^V = D(\alpha, \delta, \beta)^W$. Since κ is the least cardinal with a new ω-sequence, it follows that $cof^W(\kappa) = \omega$. Let $g = (\alpha_n : n \in \omega) \in W$ be cofinal in κ. From now on work in W. Let $K(n, \delta, \beta) := K(\alpha_n, \delta, \beta)$. Fix a sequence $(C_\alpha : \alpha \in \kappa^+ \,\&\, cof^W(\alpha) = \omega) \in W$ such that, for all α, C_α is cofinal in α and of order type ω. Use this sequence to define in W the relativized trace functions tr_X. Now as in the previous section use the parameters

$$\{K(n, \delta, \beta) : n < \omega \,\&\, \delta, \beta < \kappa^+\},\ \{A_s : s \in \kappa^{<\omega}\},\ (C_\alpha : \alpha \in \kappa^+ \,\&\, cof^W(\alpha) = \omega)$$

to define Σ_g and show that it is an open stationary set mapping. We also refer to the previous section for the notation.

Now, apply MRP to Σ_g and let $(M_\eta : \eta < \omega_1)$ be a reflecting sequence provided by MRP. Let $N = \bigcup_\eta M_\eta$ and $\delta = sup(N \cap \kappa^+)$.* Let $\delta_\eta = sup(M_\eta \cap \kappa^+)$, for all η. Find a club $C \subseteq \omega_1$ such that $\{\delta_\eta : \eta \in C\} \subseteq E_\delta$ and $M_\eta \cap \omega_1 = \eta$, for all $\eta \in C$. Let α be a limit point of C. For the rest of this proof let $M = M_\alpha$. Now $E_\delta \cap M = E_\gamma \cap M$ for some $\gamma < \beta_M$, by the choice of β_M. By *(iii)* of the definition of $K(i, \delta_M, \beta_M) := K(\alpha_i, \delta_M, \beta_M)$ there is an n such that $E_\gamma \cap \delta_M$ is a subset of $K(n, \delta_M, \beta_M)$. Since $E_\gamma \cap M \subseteq E_\gamma \cap \delta_M$, we can conclude that $E_\delta \cap M$ is a subset of $K(n, \delta_M, \beta_M)$.

* Notice that δ may have a larger cofinality in V, however this will not be a problem.

Let $\nu < \alpha$ be such that $M_\eta \in \Sigma_g(M)$, for all $\eta \in (\nu, \alpha)$. For any such $\eta \in C$, $M_\eta \in M$, so $\delta_\eta \in E_\delta \cap M \subseteq K(n, \delta_M, \beta_M)$. If $\eta \in C$ and $ht_{\alpha_M}(\eta) = j$, there is an $l \leq n_{\delta_\eta} \leq n$ such that $A_{g\restriction j} \cap tr_{M_l}(\delta_\eta, \delta_M) \cap M_l \neq \emptyset$. Now, for any i we can find an $\eta \in C$ such that $\nu < \eta < \alpha$ and $ht_{\alpha_M}(\eta) \geq i$, so there are infinitely many j such that $A_{g\restriction j} \cap K(n, \delta_M, \beta_M) \neq \emptyset$, i.e. $g \in D(\alpha_n, \delta_M, \beta_M)$, which is a contradiction. ■

In fact the theorem can be proved under the milder assumptions that V and W have the same cardinals, the same reals and, for every cardinal κ, there is in V a partition $\{A_s : s \in \kappa^{<\omega}\}$ of the points of κ^+ of countable W-cofinality into disjoint stationary sets. By a recent result of Larsson developping on ideas of Todorčević it is known that such partitions can be found in ZFC for $\kappa = \omega$ just assuming that $\omega_1^V = \omega_1^W$. It is open whether for higher cardinals such partitions exists in ZFC. A positive answer to this question would entail that if $V \subseteq W$ are models with the same reals and cardinals and $W \models$ MRP then $ORD^\omega \cap V = ORD^\omega \cap W$.

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