Multiresolution Expansions and Approximation Order of Tempered Distributions

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ABSTRACT

Properties of projections $h_j$ of a tempered distribution $h$ to the corresponding spaces $V_j$, $j \in \mathbb{Z}$, in a regular multiresolution approximation of $L^2(\mathbb{R})$ are studied. It is shown that the derivatives of $h_j$ multiplied by a polynomial weight converge in sup norm, i.e. $h_j \to h$ in $S_\ell(\mathbb{R})$, $h$ being smooth enough and of appropriate decay. Results related to tempered distributions are obtained by duality. The analysis of the approximation order of the projection operator within the framework of the theory of shift-invariant spaces gives a further refinement of the results. As an application, we give Abelian and Tauberian type theorems concerning the quasiasymptotic behavior of a tempered distribution at infinity.

Key Words: multiresolution expansions, shift-invariant spaces, approximation order, tempered distributions, quasiasymptotic behavior

Subject Classification: 41A35, 42C15, 42C40, 46F12
1. INTRODUCTION

Wavelet expansions are usually defined through the multiresolution approximations of $L^2(\mathbb{R}^n)$ ([23, 25]). Related convergence questions in various spaces of functions and distributions are discussed by many authors, see [1, 8, 15, 16, 25, 32]. It is known that, under rather weak hypotheses about the scaling function (see Definition 1), the wavelet expansion of an $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, converges to $f$ pointwise almost everywhere, see [15]. We refer to [1] for related results in weighted $L^p(\mathbb{R}^n)$ spaces. The analysis of regularity properties of functions through the wavelet type approximations is usually done within Besov and Sobolev type spaces, [15, 16, 18, 25]. We note that [25, Chapter 6] is devoted to the wavelet expansions of various spaces of functions and distributions through.

A more general approach to approximation in function spaces is based on approximations from shift-invariant subspaces of $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, or of Sobolev spaces generated by smooth refinable functions with a prescribed decay (see [2, 3, 4, 9, 14, 10, 11, 12, 13, 19, 21, 29] and references therein). Usually, spaces encountered in approximation theory and in finite element analysis, are generated by shifts of a finite number of functions on $\mathbb{R}^n$. In each of these applications, one is interested in approximation and density orders, that is, at what rate a general function $f$, element of a certain class of spaces (a potential type space, for example), can be approximated by the elements of the scaled spaces $S^h := \{s(c/h) : h > 0, s \in S\}$. Here $S$ denotes a principle shift-invariant space defined as the closure of the set of linear combinations of the shifts of a given function $s$ (see Section 4).

The emphasis is put on the properties of $s$ which govern the rates of the approximation procedure.

The main goal of our paper is the analysis of a class of tempered distributions. Thus it is necessary to assume that the scaling function fulfills some regularity (decay and smoothness) conditions. That is, an $r$-regular multiresolution approximation should be used (see [25, 31]). In the present paper we study approximation properties of wavelet expansions in $S_r(\mathbb{R})$ and, by duality, in $S'_r(\mathbb{R})$. Convergence in $S_r(\mathbb{R})$ involves almost uniform convergence (convergence on compact sets) of derivatives of a function as well as polynomial decay properties of the convergence.

Our approach is related to [9, 19, 21] where the approximation schemes associated to shift-invariant spaces are considered through cardinal interpolation, quasi-interpolation, projection and convolution with appropriate kernels. Approximation procedures of the above mentioned papers are valid in (weighted) Lebesgue spaces and Sobolev type spaces. We point out that the techniques given in [9, 19, 21] can not be used in the analysis of approximation procedures of the present paper.

More precisely, in our Theorem 2, functions and their derivatives up to the order $r \in \mathbb{N}$ are bounded by polynomials of order $r$ and the approximation procedure is done for derivatives up to the order $\alpha$, $0 \leq \alpha \leq r$, ...
on compact sets. Moreover, in Theorem 3 we have the convergence of the approximations in the sense of \( S_c(R) \) (see Section 2). It is completely different from the results based on the estimates of the type

\[ \| E_j f - f \|_{L^p} \leq C 2^{-j\beta} \| f \|_{W^s_p} \]

(see, for example, [21]). \( W^s_p \) is a Sobolev space given by

\[ W^s_p = \{ f \in L^p 
\mid \sum_{|\alpha| \leq s} \| f^{(\alpha)} \|_{L^p} \}^{1/p} < \infty, 1 \leq p \leq \infty, \]

and \( E_j \) is given by (12). On can say, in a rather unprecise manner, that in this case we should assume \( f \in S_{2r+\epsilon}(R) \) with appropriate \( k > 0 \) in order to obtain information on the rate of convergence of \( |E_j f|^{(\alpha)} \), \( 0 \leq \alpha \leq r \), see Theorem 5.

We have chosen the multiresolution approximation framework (instead of the principle shift-invariant spaces approach) in order to simplify the exposition. Some additional remarks are given whenever similarities or differences of different approaches should be emphasized. Moreover, Section 4 provides a principal shift-invariant spaces reformulation of the main results.

In order to give an illustration, we show in the second part of the paper an application of the obtained results in qualitative analysis of tempered distributions. We prove Abelian and Tauberian type theorems on the behavior of distributions. Recall, in general, a distribution does not have a value at a point, including infinity (see [22]). This motivates generalized asymptotic analysis of distributions [7, 26, 27, 28, 30, 34]. Relation between the quasiaymptotic behavior of a tempered distribution at a given point and its multiresolution expansion is studied in [28]. In contrast to quasiaymptotic behavior at a point, quasiaymptotic behavior at infinity is global property of a distribution. Therefore the present paper can be considered as a continuation and completion of [28].

Relation between the quasiaymptotic behavior of a tempered distribution at a given point and its multiresolution expansion is studied in [28]. In the present paper we study the quasiaymptotic behavior at infinity within the multiresolution analysis framework. In contrast to quasiaymptotic behavior at a point, quasiaymptotic behavior at infinity is global property of a distribution. Therefore the present paper can be considered as a continuation and completion of [28].

The paper is organized as follows. We establish in Section 2 notation and recall the definition of an \( r \)-regular multiresolution analysis, the scaling function and the corresponding wavelet. Then in Section 3 we prove the uniform convergence on compact sets of the orthogonal projections and its derivatives (Theorem 2). Furthermore, we prove that if \( h \in S_c(R) \) is of order \( r_0 \), then its multiresolution expansion \( h_j \), defined by an \( r \)-regular multiresolution analysis, \( r \geq r_0 \), converges to \( h \) in \( S_c(R) \). In Section 4 we recall the definition of principal shift-invariant spaces and introduce the
order of approximation of tempered distributions. We then present our main results as a part of the theory of shift-invariant spaces. The proof of Theorem 4 is given in Section 5. In Section 6 we apply obtained results to the analysis of the quasiasymptotic behavior of a distribution at infinity (Theorems 7 and 8).

Let us briefly explain why our results related to functions and distributions from \( \mathcal{S}_r \) and \( \mathcal{S}_r^\prime \) respectively, could not be obtained in the framework of commonly used spaces of functions or distributions. Since distributions are derivatives of slowly increasing functions, results for Besov type spaces can not be used in Theorem 2. Moreover, concerning the main result of the first part of the paper (Theorems 3 and 5), one can not obtain information about \(((1 + |x|^2)^{\gamma/2}(E_j h)^{(\alpha)}(x))_{j \in \mathbb{N}}, \quad 0 \leq |\alpha| \leq r, \) by means of known results for Besov type spaces. Concerning Sobolev type spaces \( W_p^s \) (see [25, Section 6.2]) note that \((1 + |x|^2)^{\gamma/2} f^{(\alpha)}(x) \in L^\infty(\mathbb{R}^n), 0 \leq |\alpha| \leq r, r > 0, \) implies \( f^{(\alpha)} \in L^p, \) for \( pr > n, p \in [1, \infty), |\alpha| \leq r, \) and thus \( f \in W_p^s, \) \( 0 \leq s \leq r. \) This implies that with our results we may recover [25, Theorem 2, page 172]. However, the converse is not possible since from \( E_j h \in W_p^s \) we can not obtain information about the sequence \(((1 + |x|^2)^{\gamma/2}(E_j h)^{(\alpha)}(x))_{j \in \mathbb{N}}, \) which is the subject of Theorems 3 and 5. With the given arguments it is clear that assertions of Theorems 7 and 8 can not be treated within the known techniques used for Besov and Sobolev type spaces.

Remark 1. Definitions and assertions of Sections 2-6 can be transferred to dimensions \( n > 1. \) This brings necessarily more complex calculations. Moreover, Section 4 should be carefully rewritten for the case \( n > 1, \) since there is a significant difference between shift-invariant spaces in dimension \( n = 1 \) and \( n > 1, \) see [2, 4, 9]. We present all our results in dimension \( n = 1 \) in order to make them more transparent. Another reason is their use in the applications (Section 5) which are done only in the case \( n = 1. \)

2. NOTIONS AND NOTATION

The domain of observed spaces is always \( \mathbb{R}. \) Therefore we omit the suffix and write \( L^1 \) instead of \( L^1(\mathbb{R}), \) \( \mathcal{S}_r \) instead of \( \mathcal{S}_r(\mathbb{R}), \) and so on. Throughout the paper the letter \( C \) will denote a positive constant, not necessarily the same every time when it occurs, and the integration is taken over \( \mathbb{R} \) unless otherwise indicated. The set of nonnegative integers is denoted by \( \mathbb{N}_0. \)

The space of tempered distributions of order \( r, r \in \mathbb{N}_0, \) \( \mathcal{S}_r^\prime \) is the dual space of the space \( \mathcal{S}_r = \{ f \in C^\infty ; |f^{(\alpha)}(x)| \leq C_\alpha (1 + |x|^2)^{-r/2}, \quad x \in \mathbb{R}, \quad 0 \leq \alpha \leq r \}. \) A sequence \( \{ f_k \}_{k \in \mathbb{N}}, f_k \in \mathcal{S}_r, \) converges to 0 in \( \mathcal{S}_r \) as \( k \to \infty \) if and only if for every \( \alpha, 0 \leq \alpha \leq r, \)

\[
\lim_{k \to \infty} \sup_{x \in \mathbb{R}} (1 + |x|^2)^{\gamma/2} |f_k^{(\alpha)}(x)| = 0.
\]

The Schwartz space of rapidly decreasing functions \( \mathcal{S} \) is the projective limit of \( \mathcal{S}_r \) as \( r \) tends to infinity. Its strong dual is called the space of tempered
distributions $\mathcal{S}'$. It holds $\mathcal{S}' = \bigcup_{r \in \mathbb{N}_0} \mathcal{S}'_r$, and the smallest $r \in \mathbb{N}_0$ such that $f \in \mathcal{S}'_r$ is the order of $f$. The Fourier transform of $f \in L^1$ is defined by $\mathcal{F}f(\xi) = \hat{f}(\xi) = \int e^{-2\pi i \xi x} f(x) dx, \quad \xi \in \mathbb{R}$. It defines a topological isomorphism between $\mathcal{S}$ and $\mathcal{S}$ and extends to isometric transform from the space of square integrable functions $L^2$ onto itself. The inverse Fourier transform is given by $\mathcal{F}^{-1}f(\xi) = \int e^{2\pi i \xi x} f(x) dx, \quad \xi \in \mathbb{R}$.

**Definition 1.** [25] A multiresolution approximation of $L^2$ (shortly MRA) is an increasing sequence of closed linear subspaces $V_j, j \in \mathbb{Z}$, of $L^2$, with the following properties:

\[
\bigcap_{-\infty}^{\infty} V_j = \{0\} \quad \text{and} \quad \bigcup_{-\infty}^{\infty} V_j \text{ is dense in } L^2, \quad (1)
\]

for all $f \in L^2$ and all $j \in \mathbb{Z}$, \( f(x) \in V_j \iff f(2x) \in V_{j+1}; \quad (2) \)

for all $f \in L^2$ and all $k \in \mathbb{Z}$, \( f(x) \in V_0 \iff f(x - k) \in V_0; \quad (3) \)

there exists a function $\phi \in V_0$, such that the sequence \( \{\phi(x - k), k \in \mathbb{Z}\} \) is an orthonormal basis for $V_0$.

The function $\phi$ given by (4) is called scaling function.

We say that a multiresolution approximation, $V_j, j \in \mathbb{Z}$, is $r$-regular ($r \in \mathbb{N}_0$), if and only if for every $m \in \mathbb{N}$ there exists $C_m > 0$ such that

\[
|\phi^{(q)}(x)| \leq C_m (1 + |x|^2)^{-m/2}, \quad x \in \mathbb{R}, \quad 0 \leq q \leq r, \quad (5)
\]

The set of functions which satisfy (5) will be denoted by $\hat{\mathcal{S}}_r$ for short. Obviously $\hat{\mathcal{S}}_r \subset \mathcal{S}_r$. It is well known that for every $r \in \mathbb{N}$ there exists an $r$-regular MRA, i.e., a function $\phi$ which satisfies conditions (4) and (5). This fact enables the analysis of the space of tempered distributions $\mathcal{S}'$.

**Remark 2.**

1. The conditions of Definition 1 are not mutually independent. Also, condition (4) is usually replaced by weaker assumption that the sequence \( \{\phi(x - k), k \in \mathbb{Z}\} \) is a Riesz basis for $V_0$. We refer to [8, Section 2.1] for a comprehensive discussion on the matter.

2. It is well known that the regularity of the scaling function $\phi$ in a given MRA of $L^2$ is limited by the size of its support; if $\phi \in C^\infty$ is the scaling function of a MRA of $L^2$, then it cannot be compactly supported, [6, 25]. An example of MRA of $L^2$ with $\phi \in \mathcal{S}$ is the Littlewood-Paley MRA, see [25, pages 22-23].

3. At a more abstract level, instead of introducing an MRA of $L^2$ we could start with a function (generator) $\phi \in \hat{\mathcal{S}}_r$, or $\phi \in C^\infty_0$ (a compactly supported $r$-times continuously differentiable function), and
define $V_0$ to be the closure of the linear span of $\{\phi(x - k), k \in \mathbb{Z}\}$, assuming the orthogonality of shifts, (4). $V_j$ is then defined as the closure of the span of $\{\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k), k \in \mathbb{Z}\}$. With these notation and assumptions, we can consider the approximation procedure within the theory principle shift-invariant spaces with the refinable generator function. In this sense Theorems 2 and 3 can be reformulated, see Section 4.

Let $V_j, j \in \mathbb{Z}$, be an $r$-regular MRA of $L^2$. Orthogonal complement of $V_j$ in $V_{j+1}$ is denoted by $W_j$. Consequently, $L^2$ can be written as a direct sum of subspaces $W_j, j \in \mathbb{Z}$. It is well known that using the scaling function $\phi \in \mathcal{S}$, satisfying (4) and (5) one can construct a function $\psi \in W_0$ with the following properties:

a) for every $m \in \mathbb{N}$ there exists $C_m > 0$ such

\[ |\psi^{(q)}(x)| \leq C_m (1 + |x|^2)^{-m/2}, \quad x \in \mathbb{R}, \quad 0 \leq q \leq r; \tag{6} \]

and the sequence $\psi(\cdot - k), k \in \mathbb{Z}$, is an orthonormal basis of $W_0$ \tag{7} (see [25, pp. 7281] for details);

b) family $\psi_{j,k}(\cdot) = 2^{j/2}\psi(2^j \cdot - k), j, k \in \mathbb{Z}$, is an orthonormal basis of $L^2$.

Functions $\psi_{j,k}, j, k \in \mathbb{Z}$, are called wavelets of class $r$, associated to the given $r$-regular MRA.

It can be shown that for $\hat{\phi}$ and $\hat{\psi}$, the Fourier transforms of a scaling function $\phi$ and its corresponding wavelet $\psi$, respectively, the following relation holds

\[ \hat{\psi}(\xi) = ((\hat{\phi}(\frac{\xi}{2}))^2 - (\hat{\phi}(\xi))^2)^{1/2}e^{-i\xi}, \quad \xi \in \mathbb{R}. \tag{8} \]

Note that it is possible to find a function $\psi \in L^2$ such that the family $\psi_{j,k}(\cdot) = 2^{j/2}\psi(2^j \cdot - k), j, k \in \mathbb{Z}$, is an orthonormal basis of $L^2$ but the corresponding scaling function $\phi$ does not exist. We refer to [8, Chapter 7] for a necessary and sufficient condition on a wavelet $\psi \in L^2$ to be an MRA wavelet.

Let there be given an $r$-regular MRA of $L^2 (r \in \mathbb{N}_0)$ and let $\phi$ be a scaling function with properties (4) and (5). Operator $E_0$, the orthogonal projection of $L^2$ onto $V_0$, is defined by the kernel

\[ E(x, y) = \sum_{k \in \mathbb{Z}} \phi(x - k)\phi(y - k), \quad x, y \in \mathbb{R}, \tag{9} \]

as follows: If $h \in L^2$, then

\[ E_0 h(x) = \langle h(y), E_0(x, y) \rangle = \int E(x, y)h(y)dy, \quad x \in \mathbb{R}. \tag{10} \]
Let $j \in \mathbb{Z}$. Then the kernel of the projection operator onto $V_j$ is

$$E_j(x, y) = 2^j E(2^j x, 2^j y), \quad x, y \in \mathbb{R},$$  \hspace{1cm} (11)$$

and the projection of $h \in L^2$ onto $V_j$ is given by

$$E_j h(x) = \int h(y) E_j(x, y) dy, \quad x \in \mathbb{R}.$$  \hspace{1cm} (12)$$

Functions $E_j(x, y), \ x, y \in \mathbb{R}$, are the reproducing kernels for $V_j, \ j \in \mathbb{Z}$, i.e., $E_j h = h$, if $h \in V_j$. Since we are interested in the properties of the kernel of integral transform (12) we may leave the MRA framework and allow $j$ to be a real number, not necessarily an integer. Furthermore, the assumption $j \geq 0$ is not a restriction for the results of the paper, and we shall use it for the sake of simplicity.

The definition of the kernel $E$ and the properties of $\phi \in \mathcal{S}_r$ imply that for every $m \in \mathbb{N}$ there exists $C_m > 0$ such that

$$\sup_{x, y \in \mathbb{R}} \sup_{0 \leq \alpha, \beta \leq r} (1 + |x - y|)^m \frac{d^{\alpha} d^{\beta}}{dx^{\alpha} dy^{\beta}} E(x, y) \leq C_m,$$  \hspace{1cm} (13)$$

Also, we have the symmetry, $E(x, y) = E(y, x)$. As indicated in [25, pp. 33-38] and [32, pp. 46-43], it holds

$$\int E(x, y) x^{\alpha} dy = y^{\alpha}, \quad x \in \mathbb{R}, \quad 0 \leq \alpha \leq r.$$  \hspace{1cm} (14)$$

Thus it is possible to observe more general MRA’s than the MRA of $L^2$. In the sequel we will consider MRA in this setting. Actually, (14) implies that, if the scaling function $\phi$ is $r$–regular, all polynomials up to the order $r$ locally belong to $V_0$ and, therefore, to $V_j, \ j \geq 0$. The proof of Theorem 1 is based on that property.

**Remark 3.** 1. From (5) and (9) it follows that the integral operator $E_\delta$ given by (10) belongs to a wide class of operators studied in [21]. By [21, Theorem 2.1] we immediately have

$$\|E_j f - f\|_{L^p} \leq C 2^{-j r} \|f\|_{W^p}, \quad f \in W^p, \quad 1 \leq p \leq \infty.$$ $$Note that, due to weak assumptions on kernels $E$ of integral operators observed in [21] the reproducing property (14) is assumed there. In our case the properties of an $r$–regular MRA imply (14). This fact is, however, far from being obvious, see [25, Theorem 4, page 33].

2. From the definition of MRA follows that a scaling function satisfies the refinement equation

$$\frac{1}{2} \phi \left( \frac{x}{2} \right) = \sum_{k \in \mathbb{Z}} a_k \phi(x + k), \quad a_k = \frac{1}{2} \int \phi \left( \frac{x}{2} \right) \phi(x + k) dx.$$  \hspace{1cm} (15)$$
This, together with (4) implies the so called accuracy of \( \phi \), that is the fact that the set \( \text{span}\{\phi(\cdot-k), \ k \in \mathbb{Z}\} \) contains polynomials up to a certain order. In order to obtain accuracy in the general context of principle (resp. finitely-generated) shift-invariant spaces it is usually assumed that the set of generators \( \Phi \) (usually containing one or finitely many functions) consists of compactly supported functions with linearly independent integer translates. For example, if \( \Phi = \{\phi\}, \ \phi \in C^1 \), satisfies the assumptions, then the accuracy of order \( n \) is equivalent to the Strang-Fix conditions of the same order, [4, Theorem 1]. More details on the subject and related applications can be found in [4, 5, 10, 12, 13, 29].

3. MAIN RESULTS

As noted in the introduction, the theory of principal shift-invariant spaces generated by a refinable function \( \phi \) is developed for (weighted) \( L^p \) \( (1 \leq p \leq \infty) \) and Sobolev type spaces. The corresponding norms measuring the order of approximation do not include estimates on the derivatives of projections. In our case the analysis must involve such estimates. The following Theorem implies that differentiation operators commute with the projection operators up to a term which can be controlled. This fact appears to be crucial in what follows.

**Theorem 1.** [25, Theorem 5, page 39] Let \( V_j, \ j \in \mathbb{Z}, \) be an \( r \)-regular MRA of \( L^2 \). Then, for every \( a, \ 0 \leq a \leq r \), there exists \( R_\alpha \in L^\infty(\mathbb{R} \times \mathbb{R}) \), satisfying

\[
|R_\alpha(x, y)| \leq C_m(1 + |x-y|)^{-m}, \ x, y \in \mathbb{R}, \ m \in \mathbb{N},
\]

\[
\int R_\alpha(x, y) dy = 0, \ x \in \mathbb{R},
\]

and if \( f \in C^r \), then

\[
\frac{d^\alpha}{dx} E_j f(x) = 2^j \int g(2^j (x - y)) \frac{d^\alpha}{dy} f(y) dy
\]

\[
+ 2^j \int R_\alpha(2^j x, 2^j y) \frac{d^\alpha}{dy} f(y) dy, \ x \in \mathbb{R},
\]

where \( g \in \mathcal{D}, \ \int g = 1 \).

Recall, the space \( \mathcal{D} \) consists of compactly supported infinitely differentiable functions. Its strong dual is denoted by \( \mathcal{D}' \). A sequence \( \{g_j\}_{j \in \mathbb{Z}} \) with the properties described in Theorem 1 is in the literature usually called delta sequence. For a delta sequence \( \{g_j\}_{j \in \mathbb{Z}} \) and any function \( h \in C^\infty \) we have

\[
\lim_{j \to \infty} \langle g_j(x-y), h^{(\alpha)}(y) \rangle = h^{(\alpha)}(x), \ x \in \mathbb{R},
\]

(15)
uniformly on compact sets, for every $a \geq 0$.

The proof of Theorem 1 is based on the property 14 and a representation of $f \in \tilde{S}_r$ such that $\int f(x)x^a \, dx = 0, 0 \leq a \leq r, [25, \text{Lemma } 12]$.\[\text{Theorem 2. Let } h \in C^r, \text{ such that } h^{(a)} \text{ is of polynomial growth for every } a, 0 \leq a \leq r \text{ and let } h_j(\cdot) = \langle h(y), E_j(\cdot, y) \rangle, j \in \mathbb{N}. \text{ Then the sequence of derivatives } \{h_j^{(a)}\}_{j \in \mathbb{N}} \text{ converges pointwise and, moreover, uniformly on compact sets to } h^{(a)} \text{ as } j \to \infty, \text{ for any } a, 0 \leq a \leq r.\]

\textbf{Proof.} By Theorem 1 we have
\[h_j^{(a)}(x) = \langle \partial_x^a E_j(x, y), h(y) \rangle \]
\[= \langle g_j(x - y), h^{(a)}(y) \rangle + 2^j \int R_a(2^j, 2^j y)h^{(a)}(y)dy, \quad x \in \mathbb{R}.\]

Since (15) holds, we have to prove that
\[\lim_{j \to \infty} 2^j \int R_a(2^j x, 2^j y)h^{(a)}(y)dy = 0,\]
uniformly on compact sets. Let there be given compact set $K \subset \mathbb{R}$. Note that $h \in C^r$ implies the uniform continuity of $h^{(a)}$ on compact sets, $0 \leq a \leq r$. Therefore, if $|y - x| \leq \epsilon$ then $|h^{(a)}(y) - h^{(a)}(x)| \leq d(y - x)$, where $d$ is a continuous function and $d(0) = 0$. We use this fact below.

\[2^j \int R_a(2^j x, 2^j y)h^{(a)}(y)dy \leq 2^j \int |R_a(2^j, 2^j y)(h^{(a)}(y) - h^{(a)}(x))|dy \]
\[\leq 2^j \int \frac{C_m}{(1 + 2^j |y|)^m} |h^{(a)}(y) - h^{(a)}(x)|dy \]
After the change of variables $r = 2^j(y - x)$ the last integral becomes
\[\int \frac{C_m}{(1 + r)^m} |h^{(a)}(r2^{-j} + x) - h^{(a)}(x)|dr \leq \int \frac{C_m}{(1 + |r|)^m} |d(r2^{-j})|dr \]
for $j$ large enough and $x \in K$. Since $m$ can be chosen arbitrarily large by dominated convergence we obtain
\[\lim_{j \to \infty} 2^j \int R_a(2^j x, 2^j y)h^{(a)}(y)dy \leq \lim_{j \to \infty} \int \frac{C_m}{(1 + |r|)^m} |d(r2^{-j})|dr = 0 \]
uniformly for $x \in K$. \[\]

Theorem 2 can be compared to [32, Corollary 8.3]: Let $h$ be continuous on $(a, b)$ and let $h_j$ be the projection of $h$ onto $V_j$, $j \in \mathbb{N}$. Then $h_j \to h$ as $j \to \infty$ uniformly on compact subsets of $(a, b)$.

If the scaling function $\phi$ is bounded by an $L^1$ radial decreasing function then $h_j$ converge pointwise almost everywhere to $h \in L^p$, $1 \leq p \leq \infty$ as $j \to \infty$, [15, Theorem 2.1]. Recall [15, Definition 1.2]:
Definition 2. Let there be given $h \in L^2$, and $r$-regular MRA. The orthogonal projection of $h$ onto $V_j$, $j \in \mathbb{Z}$, is given by

$$h_j(x) = E_j h(x) = \int h(y) E_j(x,y) dy, \quad x \in \mathbb{R}.$$ 

The sequence of projections $(h_j)_{j \in \mathbb{Z}}$ is called multiresolution expansion of $h$. The scaling expansion of $h \in L^2$ is given by

$$\sum_{k \in \mathbb{Z}} \int h(y) \phi(y - k) dy \phi(x - k) + \sum_{j \geq 0} \sum_{k \in \mathbb{Z}} \int h(y) \psi_{j,k}(y) dy \psi_{j,k}(x), \quad x \in \mathbb{R}.$$ 

whereas the wavelet expansion of $f \in L^2$ is given by

$$\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \int h(y) \psi_{j,k}(y) dy \psi_{j,k}(x), \quad x \in \mathbb{R}.$$ 

Obviously, for given $j_0 > 0$ and $x \in \mathbb{R}$ we have

$$h_{j_0}(x) = E_{j_0} h(x) = \sum_{k \in \mathbb{Z}} \int h(y) \phi(y - k) dy \phi(x - k)$$

$$+ \sum_{0 \leq j \leq j_0} \sum_{k \in \mathbb{Z}} \int h(y) \psi_{j,k}(y) dy \psi_{j,k}(x)$$

$$= \sum_{j \leq j_0} \sum_{k \in \mathbb{Z}} \int h(y) \psi_{j,k}(y) dy \psi_{j,k}(x).$$

Therefore, the scaling and wavelet expansion are the limits of $h_j$ as $j \to \infty$.

Our main result is the following.

Theorem 3. Let $\phi \in \tilde{S}_r$. Let $E_{j_0}(x,y)$, $j \in \mathbb{Z}$, denote corresponding kernels given by (9) and (11). Let $\sigma \in \tilde{S}_{r+1}$. Then

i) the sequence $\langle k E_{j_0}(kx,ky), \sigma(x) \rangle$ converges to $\sigma(y)$ in $S_r$ as $k \to \infty$, for any given $j \in \mathbb{Z}$.

ii) the sequence $\langle k E_{j}(kx,ky), \sigma(x) \rangle$ converges to $\sigma(y)$ in $S_r$ as $k \to \infty$.

iii) the sequence $\langle E_{j}(x,y), \sigma(x) \rangle$ converges to $\sigma(y)$ in $S_r$ as $j \to \infty$.

Remark 4. Although Theorem 3 can be related to Theorem 5, we give both proofs since the ideas behind them are based on different approaches, leading to different consequences.

Proof. As already mentioned, we may allow $j$ and $k$ to be reals. For that reason we do not consider relation between $h_j$ and coefficients in corresponding scaling and wavelet expansions in this paper. Furthermore, we assume that $j \in \mathbb{N}$, and $k > 0$ which is not a restriction.
Since \( E_j(x, y) = kE_\varphi(kx, ky), (x, y) \in \mathbb{R}^2 \), with \( k = 2^j \), (iii) follows from (i). Also, by the inspection of the proof of (i) we can see that if in the proof of (i) we put \( j = k \) in the index, nothing is changed and we obtain (ii). So we have to prove (i).

We give the proof in two steps. An alternative proof can be derived by the use of Theorem 1.

**Step 1.** Let there be given \( j \in \mathbb{N} \). We first show that the family

\[
\left\{ \langle kE_j(kx, k \cdot \cdot), \sigma(x) \rangle : k > 0 \right\}
\]

is bounded in \( S_{r+1} \). We will prove this by showing that \( |a(y) - b(y)| \leq C_1 \) and \( b(y) \leq C_2 \), \( y \in \mathbb{R} \), where

\[
a(y) := (1 + y^2)^{-p/2} \int_{-\infty}^{\infty} kE_j(kx, ky)\sigma(x)dx,
\]

\[
b(y) := (1 + y^2)^{-p/2} \int_{-\infty}^{\infty} kE_j(kx, ky)dx, \quad y \in \mathbb{R},
\]

\( 0 \leq p \leq r + 1 \). Both inequalities will be proved in the following Theorem. The proof is given in a separate section.

**Theorem 4.** Let there be given \( \sigma \in S_{r+1} \), and let \( E(x, y) \) be the kernel of the projection operator given by (9) with \( \varphi \in S_r \) and \( E_j(x, y) \) given by (11). Then for every \( p \), \( 0 \leq p \leq r + 1 \) there is a constant \( C > 0 \) such that

a) \( \sup_{y \in \mathbb{R}} |a(y) - b(y)| \leq C \).

b) \( \sup_{y \in \mathbb{R}} |b(y)| \leq C \).

**Step 2.** By Theorem 2, we have that \( \langle \partial_x^\alpha E_j(\cdot, y), \sigma(y) \rangle \) converges uniformly to \( \sigma^{(\alpha)} \), on every compact set and for all \( |\alpha| \leq r \).

Now we use the following necessary and sufficient condition for convergence in \( S_r \).

**Lemma 1.** Let there be given a sequence \( \{\sigma_n\}_{n \in \mathbb{N}}, \sigma_n \in S_{r+1}, n \in \mathbb{N}. \) The sequence \( \{\sigma_n\}_{n \in \mathbb{N}} \) converges to \( \sigma \in S_{r+1} \) in the norm of \( S_r \) if and only if

a) for every compact set \( K \subset \mathbb{R} \) and every \( a, a = 0, 1, \ldots, r \), \( \sigma_n^{(\alpha)} \)

converges uniformly to \( \sigma^{(\alpha)} \),

b) \( \sup_{x \in \mathbb{R}, n \in \mathbb{N}} \left( 1 + |x|^{2r/2} |\sigma_n^{(\alpha)}(x)| \right) \leq C_\alpha, a = 0, 1, \ldots, r + 1. \) That is, the sequence \( \{\sigma_n\}_{n \in \mathbb{N}} \) is bounded in \( S_{r+1} \).

**Proof.** Let there be given \( a, |\alpha| \leq r \). Then

\[
\sup_{x \in \mathbb{R}} \left( 1 + |x|^{2r/2} \left| (\sigma_n - \sigma)^{(\alpha)}(x) \right| \right)
\]
\[
\leq \sup_{|x| \leq K} (1 + |x|^2)^{r/2} \left| (\sigma_n(x) - \sigma^{(\alpha)}(x)) \right| + \sup_{|x| > K} (1 + |x|^2)^{r/2} \left| (\sigma_n(x) - \sigma^{(\alpha)}(x)) \right|
\]

The condition a) implies that for any \( \varepsilon > 0 \) and \( K > 0 \) there exists \( n_0 \in \mathbb{N} \) such that

\[
\sup_{|x| \leq K} (1 + |x|^2)^{r/2} \left| (\sigma_n(x) - \sigma^{(\alpha)}(x)) \right| < \frac{\varepsilon}{2}, \quad n \geq n_0.
\]

On the other hand,

\[
\sup_{|x| > K} (1 + |x|^2)^{r/2} \left| (\sigma_n(x) - \sigma^{(\alpha)}(x)) \right| \leq \frac{1}{1 + K^2} \sup_{|x| > K} (1 + |x|^2)^{(r+1)/2} \left( |(\sigma_n^{(\alpha)}(x))| + |\sigma^{(\alpha)}(x)| \right) < \frac{\varepsilon}{2}
\]

for \( K \) large enough. Note that a) and b) together imply

\[
(1 + |x|^2)^{(r+1)/2} \left| \sigma^{(\alpha)}(x) \right| \leq C \quad \text{for} \quad \alpha \leq r.
\]

Therefore

\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}, \frac{\alpha}{r} \leq \alpha \leq r} (1 + |x|^2)^{r/2} \left| (\sigma_n(x) - \sigma^{(\alpha)}(x)) \right| = 0.
\]

ThusLemma 1 is proved. \( \blacksquare \)

Theorem 4 together with Lemma 1 imply the convergence of the sequence \( \langle kE_j(kx, k\lambda) \sigma(x) \rangle \) to \( \sigma(y) \) in \( S_r \) as \( k \to \infty \). \( \blacksquare \)

Remark 5. Theorem 3 gives also an improvement of [28, Lemma 1], since apart from the boundedness of the corresponding family here we obtain the limit function as well.

As a consequence of Theorem 4 iii) we have the next corollary.

**Corollary 1.** Let there be given \( r \)-regular MRA. Let \( S_j \) and \( W_j \) be partial sums of the scaling and the wavelet expansion of \( \sigma \in S_r \) at the level \( j \geq 0 \), that is

\[
S_j \sigma(x) = \sum_{k \in \mathbb{Z}} \int \sigma(y) \phi(y-k) dy \phi(x-k) + \sum_{b \leq j \leq k} \sum_{k \in \mathbb{Z}} \int \sigma(y) \psi_{j,k}(y) dy \psi_{j,k}(x)
\]

\[
W_j \sigma(x) = \sum_{j \leq k} \sum_{k \in \mathbb{Z}} \int \sigma(y) \psi_{j,k}(y) dy \psi_{j,k}(x).
\]

Then both \( S_j \sigma \) and \( W_j \sigma \) converge to \( \sigma \) in \( S_r \) as \( j \) tends to infinity.
Definition 3. Let there be given a tempered distribution \( h \in \mathcal{S}' \) of order \( r \). Choose an \( r \)-regular MRA. The multiresolution expansion of \( h \) is given by the sequence \( \{ h_j \}_{j \in \mathbb{Z}} \) defined by

\[
\langle h_j, \sigma \rangle = \langle E_j h, \sigma \rangle = \langle h, E_j \sigma \rangle, \quad \sigma \in \mathcal{S}.
\]

Corollary 2. Let there be given a tempered distribution \( h \in \mathcal{S}' \). Then there exists \( r \in \mathbb{N} \), such that every \( r \)-regular MRA defines the multiresolution expansion of \( h \) which converges to \( h \) in \( \mathcal{S}' \) as \( j \to \infty \).

Proof. For a given \( h \in \mathcal{S}' \) there exists \( r_0 \in \mathbb{N} \) such that \( h \) belongs to \( \mathcal{S}^{r_0} \), the dual space of \( \mathcal{S}_{r_0} \), that is, \( h \) is of order \( r_0 \). We take \( r_0 - 1 \)-regular MRA. Definition 3 and Theorem 3 applied to the corresponding multiresolution expansion imply the result. ■

4. A REFORMULATION OF RESULTS

Let \( F \) be a normed space of functions or distributions. Recall, [2, 3, 4, 5, 9, 14, 11, 12, 13, 19, 29], a closed subspace \( S \) of \( F \) is shift-invariant if \( f \in S \) implies \( f(\cdot + k) \in S \), \( k \in \mathbb{Z} \). Let there be given a generator function \( \phi \in F \). The corresponding principal shift-invariant space (PSI) is denoted by \( S_\phi \),

\[
S_\phi = \text{span}\{\phi(\cdot - k), \ k \in \mathbb{Z}\}.
\]

The stationary ladder of spaces \( \{S^h_\phi\}_{h \geq 0} \) is given by

\[
S^h_\phi := \{ s(\cdot/h) : h > 0, s \in S_\phi \}.
\]

We say that \( S_\phi \) provides approximation order \( k \) in \( F \) if, for every sufficiently smooth \( f \),

\[
\inf_{g \in S^h_\phi} \| f - g \|_F \leq C h^k,
\]

where the positive constant \( C \) depends on \( f \). \( S_\phi \) provides density order \( k \) in \( F \) if for every sufficiently smooth \( f \),

\[
\lim_{h \to 0} \inf_{g \in S^h_\phi} \frac{\| f - g \|_F}{h^k} = 0.
\]

We recall the notion of the approximation order of an operator ([21]). Let there be given an integral operator \( E \) of the form

\[
(Ef)(x) = \int E(x, y) f(y) dy, \quad x \in \mathbb{R}.
\]

We assume that \( E(x - k, y) = E(x, y + k) \), \( k \in \mathbb{Z} \), \( x, y \in \mathbb{R} \). For \( h > 0 \) we define \( E_h := \sigma_h E \sigma_{1/h} \), where \( \sigma \) denotes the scaling operator \( \sigma_h f := f(\cdot/h) \).
We say that the integral operator $E$ given by (17) provides approximation order $k$ in $F$ if for every sufficiently smooth $f$,
\[\|E_h f - f\|_F \leq C h^k,\]
where the positive constant $C$ depends on $f$.

The approximation order of an operator in the space of tempered distributions is defined as follows.

**Definition 4.** Let there be given $f \in \mathcal{S}'$ of order $r$ ($f \in \mathcal{S}'_r$). Let $E(x, y)$, $x, y \in \mathbb{R}$, be the kernel of an integral operator $E : \mathcal{S}'_r \to \mathcal{S}'_r$. Distribution $Ef$ is given by
\[\langle Ef, \varphi \rangle := \langle f, E\varphi \rangle, \varphi \in \mathcal{S}_r.\]

The operator $E$ provides approximation order $k$ in $\mathcal{S}_r'$ if
\[|\langle Ef, \varphi \rangle - \langle f, \varphi \rangle| \leq C_{f, \varphi} h^k, \quad \varphi \in \mathcal{S}_r.\]

We say that an operator $E$ provides approximation order $k$ in $\mathcal{S}_r$ if
\[\|E_h \varphi - \varphi\|_{\mathcal{S}_r} \leq C_{\varphi} h^k,\]
for every sufficiently smooth $\varphi \in \mathcal{S}_r$.

**Remark 6.**
1. In the $L^p$ case, the notion "sufficiently smooth function $\varphi$" (see Definition 4) usually means $\varphi \in W^s_p$ where $s$ stands for the order of approximation. In our case, "sufficiently smooth" has a different meaning, see the proof of Theorem 5.

2. In [9], for $F = W^s_2$, the approximation order $k$ is defined by
\[\inf_{g \in \mathcal{S}_r^s} \|f - g\|_{W^s_2} \leq C h^{k-s} \|f\|_{W^s_2},\]
for $f \in W^s_2$, and $k > s$. Although this definition has some advantages, we decided to use (16) here.

Theorem 5 gives sharper results than Theorem 3 with additional assumptions and another approach in the proof.

**Theorem 5.** Let there be given compactly supported function $\phi \in \mathcal{S}_{r+k}$, such that the integer shifts of $\phi$ form an orthonormal basis of $S_{\phi}$ (with respect to the inner product in $L^2$). Let $E(x, y) = \sum_{k \in \mathbb{Z}} \phi(x - k) \phi(y - k)$, $x, y \in \mathbb{R}$, be the kernel of the integral operator $E$ given by (17). We then have

a) $S_{\phi}$ contains polynomials up to the order $r$.

b) $S_{\phi}$ provides approximation order $k$ and density order $k - 1$ in $\mathcal{S}_r$. 

c) \( E \) provides approximation order \( k \) in \( S_r \).

d) \( E \) provides approximation order \( k \) in \( S'_r \).

Remark 7. 1. Results related to Theorem 5 can be found in [17, 21] where the kernel of the integral operator \( E \) given by (17) satisfies some rather weak integrability conditions. This leads to the approximation results in \( L^p \) spaces, \( 1 \leq p \leq \infty \), and Sobolev and Triebel-Lizorkin type spaces.

2. The assumption (4), that the integer shifts of \( \phi \) form an orthonormal basis might be relaxed to the Riesz basis condition. In that case, with some additional hypotheses, Theorem 5 still holds true. The precise statement is out of scope of the present paper. We refer to [20] for a discussion in that direction.

Proof. The proof of a) is essentially the same as the proof of [25, Theorem 4, page 33].

b) Follows from a) and [21, Corollary 3.4] since \( E \) is the projection onto \( S_\delta \).

c) We assume that \( \sigma \in S_{r+k} \) for arbitrary \( k \in \mathbb{N} \). We present an explicit calculation of \( \|E_\delta \sigma - \sigma\|_{S_r} \) and prove that, for any \( p \leq r \) and \( h > 0 \)

\[
\sup_{x \in \mathbb{R}} (1 + |x|^2)^{\gamma/2} \left| \frac{d^p}{dx^p} \left(k \langle E_\delta (x, y), \sigma(y) \rangle - \sigma(x) \right) \right| < C \|\sigma\|_{S_{r+k}, h^k}.
\]

Note that the relation between the index \( j \) from the previous section, and \( h \) is \( h = 2^{-j} \). We use the notation and the formulation of Theorem 1. It implies, for \( 0 \leq p \leq r \),

\[
\sup_{x \in \mathbb{R}} (1 + |x|^2)^{\gamma/2} \left| \frac{d^p}{dx^p} \left( \langle E_\delta (x, y), \sigma(y) \rangle - \sigma(x) \right) \right|
= \sup_{x \in \mathbb{R}} (1 + |x|^2)^{\gamma/2} \frac{1}{h} \int g(\frac{x - y}{h}) \left( \sigma^{(j)}(y) - \sigma^{(j)}(x) \right) dy
+ \frac{1}{h} \int R_{\delta} \left( \frac{x - y}{h} \right) \left( \sigma^{(j)}(y) - \sigma^{(j)}(x) \right) dy \leq I + II,
\]

where

\[
I = \sup_{x \in \mathbb{R}} (1 + |x|^2)^{\gamma/2} \frac{1}{h} \int g(\frac{x - y}{h}) \left( \sigma^{(j)}(y) - \sigma^{(j)}(x) \right) dy
\]

and

\[
II = \sup_{x \in \mathbb{R}} (1 + |x|^2)^{\gamma/2} \frac{1}{h} \int R_{\delta} \left( \frac{x - y}{h} \right) \left( \sigma^{(j)}(y) - \sigma^{(j)}(x) \right) dy.
\]
We first estimate $I$. We fix $g \in \mathcal{D}$ with the properties $\int g(x)dx = 1$, and $\int g(x)x^\alpha dx = 0, \quad 0 < |\alpha| \leq \max\{r, k - 1\}$. Let $c$ be a constant such that $\text{supp } g \subseteq [-c, c]$. We have

$$I = \sup_{h \in \mathbb{R}} \left(1 + |hx|^{2r/2}\right) \left| \int_{|y - x| \leq c} g(x - y)(\sigma^{(r)}(h y) - \sigma^{(r)}(hx))dy\right|.$$

We assume $h \in (0, 1)$ without lost of generality. The smoothness of $\sigma \in \mathcal{S}_{r+k} \subseteq C^{r+k}$, implies

$$\sigma^{(r)}(h y) = \sigma^{(r)}(hx) + (y - x)h\sigma^{(r+1)}(hx) + \ldots + \frac{(y - x)^{r-1}}{(r-1)!} h^{r-1}\sigma^{(r+k-1)}(hx) + \frac{(y - x)^r}{k!} h^r\sigma^{(r+k)}(\xi(y))$$

where $\xi(y) = hx + \theta h(y - x) \in [hx - c, hx + c]$, for some $\theta \in (0, 1)$. The choice of $\theta$ implies that

$$\int_{|y - x| \leq c} g(x - y)((y - x)h\sigma^{(r+1)}(hx) + \ldots + \frac{(y - x)^{r-1}}{(r-1)!} h^{r-1}\sigma^{(r+k-1)}(hx))dy = 0.$$

Hence

$$I \leq \sup_{h \in \mathbb{R}} \left(1 + |hx|^{2r/2}\right) \left| \int_{|y - x| \leq c} g(x - y)\frac{(y - x)^r}{k!} h^r\sigma^{(r+k)}(\xi(y))dy\right|$$

$$\leq C \sup_{h \in \mathbb{R}} \left(1 + |hx|^{2r/2}\right) \sup_{|x - y| \leq c} |\sigma^{r+k}(\xi(y))| h^k \leq C ||\sigma||_{S_{r+k}} h^k,$$

where we used the following estimate

$$\sup_{h \in \mathbb{R}} \left(1 + |hx|^{2r/2}\right) \sup_{\xi(y) \in [hx - c, hx + c]} |\sigma^{r+k}(\xi(y))|$$

$$= \sup_{s \in \mathbb{R}} \left(1 + |s|^{2r/2}\right) \sup_{t \in [-c, c]} |\sigma^{r+k}(t)|$$

$$\leq \sup_{s \in \mathbb{R}} \left(1 + ||s|^{2r/2}|(|s| - c|^{2r/2} + (\sup_{t \in [-c, c]} |\sigma^{r+k}(t)|)$$

$$\leq C ||\sigma||_{S_{r+k}},$$

where $C = \sup_{s \in \mathbb{R}} \left(1 + ||s|^{2r/2}$$

It order to show $II \leq C ||\sigma||_{S_{r+k}} h^k$ we use additional assumptions about the scaling function $\phi$. Note that the fact that $\phi$ is compactly supported implies that there exists $M > 0$ such that $E(x, y) = 0$ for $|x - y| > M$. 

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Moreover, \( \int g(t)x' dt = 0, 0 < |a| \leq \max\{r, k-1\}, \) and \( E(x, y)y^a dy = x^a, \)

\[
\int R_\alpha(x, y)y^a dy = \partial_x^\alpha \int E(x, y)y^a dy - \partial_x^\alpha \int g(x - y)y^a dy
\]

\[= \partial_x^\alpha x^a - \partial_x^\alpha \int g(t)(x - t)^a dt = \partial_x^\alpha x^a - \partial_x^\alpha \int g(t)x^a dt = \partial_x^\alpha x^a - \partial_x^\alpha x^a = 0,
\]

and therefore

\[
II = \sup_{h \in \mathbb{R}} (1 + |hx|^2)^{\gamma/2} \left| \int R_\gamma(x, y)(\sigma^{(p)}(h y) - \sigma^{(p)}(hx)) dy \right|
\]

\[= \sup_{h \in \mathbb{R}} (1 + |hx|^2)^{\gamma/2} \left| \int R_\gamma(x, y)((y - x)h_\sigma^{(p+1)}(hx) + \ldots + \frac{(y - x)^k}{(k-1)!} - \frac{(y - x)^k}{k!} h^k \sigma^{(p+k)}(\xi(y))) dy \right|
\]

\[= \frac{h^k}{k!} \sup_{h \in \mathbb{R}} (1 + |hx|^2)^{\gamma/2} \left| \int R_\gamma(x, y)((y - x)^k \sigma^{(p+k)}(\xi(y))) dy \right|
\]

\[\leq \frac{h^k}{k!} \sup_{h \in \mathbb{R}} (1 + |hx|^2)^{\gamma/2} \left| \sigma^{(p+k)}(\xi(y)) \right| \int_{|y - x| \leq M} R_\gamma(x, y)^{(y - x)^k} dy.
\]

Now, \( \xi(y) = hx + \theta h(y - x) \in [hx - hM, hx + hM], \) for some \( \theta \in (0, 1) \)
implies as in (18)

\[
II \leq C \|\sigma\|_{S_{r+k}} h^k.
\]

Finally,

\[
\|E_\sigma \sigma - \sigma\|_{S_r} \leq 1 + I \leq C \|\sigma\|_{S_{r+k}} h^k.
\]

d) The argument is standard. Let \( f \in S_r' \subset S_{r+k}' \). From c) it follows that for any \( \varphi \in S_{r+k}, k \in \mathbb{N}, \) we have

\[
\|E_\sigma f - f\|_{S_{r+k}} = \sup_{\|\varphi\|_{S_{r+k}} = 1} |\langle E_\sigma f, \varphi \rangle - \langle f, \varphi \rangle| \leq \sup_{\|\varphi\|_{S_{r+k}} = 1} \|\varphi\|_{S_{r+k}} h^k \leq C \|\varphi\|_{S_{r+k}} h^k \leq C \|\varphi\|_{S_{r+k}} h^k.
\]
5. PROOF OF THEOREM 4

Proof. a) We rewrite $|a(y) - b(y)|, y \in \mathbb{R}$, as

$$
(1 + y^2)^{r+s} \left| \frac{d^p}{dy^p} \int_{-\infty}^{y+c} kE_j(kx, ky)\sigma(x)dx \right.
nul
+ \left. \frac{d^p}{dy^p} \int_{y-c}^{y+c} kE_j(kx, ky)(\sigma(x) - \sigma(y))dx \right.
nul
+ \left. \frac{d^p}{dy^p} \int_{y+c}^{\infty} kE_j(kx, ky)\sigma(x)dx \right.
nul
- \sigma(y) \frac{d^p}{dy^p} \left( \int_{-\infty}^{y-c} kE_j(kx, ky)dx + \int_{y+c}^{\infty} kE_j(kx, ky)dx \right)

\leq (1 + y^2)^{r+s} \left| \frac{d^p}{dy^p} \int_{y+c}^{\infty} kE_j(kx, ky)\sigma(x)dx \right|, \ y \in \mathbb{R},
$$

where $I_j, j = 1, 2, 3, 4, 5$ denote corresponding terms in the above expression. First, we show that

$$(1 + y^2)^{r+s} |I_3| = (1 + y^2)^{r+s} \left| \frac{d^p}{dy^p} \int_{y+c}^{\infty} kE_j(kx, ky)\sigma(x)dx \right|, \ y \in \mathbb{R},$$

is bounded. Assume that $p = 1$. We have ($y \in \mathbb{R}$)

$$
\frac{d}{dy} \int_{y+c}^{\infty} kE_j(kx, ky)\sigma(x)dx
$$

$$
= 2^j k \frac{d}{dy} \int_{y+c}^{\infty} \phi(2^j kx - l)\sigma(x)dx \cdot \phi(2^j ky - l)
$$

$$
= 2^j k \sum_{l \in \mathbb{Z}} \frac{d}{dy} \int_{y+c}^{\infty} \phi(2^j kx - l)\sigma(x)dx \cdot \phi(2^j ky - l)
$$

$$
+ 2^j k \sum_{l \in \mathbb{Z}} \int_{y+c}^{\infty} \phi(2^j kx - l)\sigma(x)dx \cdot \frac{d}{dy} \phi(2^j ky - l)
$$

$$
= -2^j k \sum_{l \in \mathbb{Z}} \phi(2^j k(y + c) - l)\sigma(y + c)\phi(2^j ky - l)
$$

$$
+ 2^j k \sum_{l \in \mathbb{Z}} \int_{y+c}^{\infty} \phi(2^j kx - l)\sigma(x)dx \cdot \frac{d}{dy} \phi(2^j ky - l) = A + B.
$$

Since $\phi \in \mathcal{S}_r$ and $\sigma \in \mathcal{S}_{r+1}$, for arbitrary $\delta \in \mathbb{R}$ and $0 \leq s \leq r + 1$ we have

$$
\sum_{l \in \mathbb{Z}} |\phi(2^j k(y + c) - l)||\sigma(y + c)||\phi(2^j ky - l)|
$$

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$$\leq \frac{C}{(1 + |y + c|)^{\epsilon}} \sum_{t \in \mathbb{Z}} \frac{1}{(1 + [2^j k(y + c) - l])^{\epsilon + 2}} \frac{1}{(1 + [2^j ky - l])^{\epsilon + 2}}$$

$$\leq \frac{C}{(1 + |y + c|)^{\epsilon}} \sum_{t \in \mathbb{Z}} \frac{1}{(1 + [2^j k(y + c) - l])^{\epsilon}} \frac{1}{(1 + [2^j ky - l])^{\epsilon}}$$

$$\leq \frac{C}{(1 + |y + c|)^{\epsilon}} \sum_{t \in \mathbb{Z}} \frac{1}{(1 + [2^j ky - l])^{\epsilon + 1}}$$

(Note: $\sum_{\mathbb{Z}} \frac{1}{(1 + [2^j ky - l])^{\epsilon + 1}} \leq 1 + \sum_{\mathbb{Z} \setminus \{0\}} \frac{1}{l^2}$.)

Thus $A \leq \frac{C}{(1 + |y + c|)^{\epsilon}} \frac{1}{(1 + [2^j ky - l])^{\epsilon + 1}}$.

Similar arguments imply

$$\sum_{t \in \mathbb{Z}} \left( \int_{y + c}^{\infty} |\phi(2^j kx - l)| |\sigma(x)| \, dx \right) \cdot \left| \frac{d}{dy} \phi(2^j ky - l) \right|$$

$$\leq C \sum_{t \in \mathbb{Z}} \frac{1}{(1 + [2^j ky - l])^{\epsilon + 1}} \int_{y + c}^{\infty} \phi(2^j kx - l) ||\sigma(x)|| \, dx$$

$$\leq \frac{C}{(1 + |y + c|)^{\epsilon}} \sum_{t \in \mathbb{Z}} \frac{1}{(1 + [2^j ky - l])^{\epsilon + 1}} \int_{y + c}^{\infty} \frac{1}{(1 + [2^j kx - l])^{\epsilon + 1}} \, dx$$

$$\leq \frac{C}{(1 + |y + c|)^{\epsilon}} \int_{y + c}^{\infty} \left( \sum_{t \in \mathbb{Z}} \frac{1}{(1 + [2^j ky - l])^{\epsilon + 1}} \frac{1}{(1 + [2^j kx - l])^{\epsilon + 1}} \right) \, dx$$

$$\leq \frac{C}{(1 + |y + c|)^{\epsilon}} \int_{y + c}^{\infty} \left( \sum_{t \in \mathbb{Z}} \frac{1}{(1 + [2^j ky - l])^{\epsilon + 1}} \frac{1}{(1 + [2^j kx - l])^{\epsilon + 2}} \right) \, dx$$

$$\int_{y + c}^{\infty} \left( \sum_{t \in \mathbb{Z}} \frac{1}{(1 + [2^j ky - l])^{\epsilon + 1}} \frac{1}{(1 + [2^j kx - l])^{\epsilon + 2}} \right) \, dx$$

$$\leq \frac{C}{(1 + |y + c|)^{\epsilon}} \int_{y + c}^{\infty} \frac{1}{(1 + [2^j ky - l])^{\epsilon + 1}} \, dx$$

$$\leq \frac{C}{(1 + |y + c|)^{\epsilon}} \int_{y + c}^{\infty} \frac{1}{(1 + [2^j kx - l])^{\epsilon + 2}} \, dx$$

$$\leq \frac{C}{(1 + |y + c|)^{\epsilon}} \int_{y + c}^{\infty} \frac{1}{(1 + [2^j k(x - y)])^{2}} \, dx$$

Thus $B \leq \frac{C}{(1 + |y + c|)^{\epsilon}} \frac{1}{(1 + [2^j ky])^{\epsilon + 1}}$.

We conclude that

$$\left(1 + y^2\right)^{\frac{\epsilon + 1}{2}} \left| \frac{d}{dy} \int_{y + c}^{\infty} \kappa E_j(kx, ky) \sigma(x) \, dx \right| \leq \frac{C}{(1 + |y + c|)^{\epsilon}} \int_{y + c}^{\infty} \frac{1}{(1 + [2^j ky])^{\epsilon + 1}} \, dx \leq C.$$
We omit the proof of
\[
(1 + y^2)^{\frac{\nu + 1}{2}} \left| \frac{d^p}{dy^p} \int_{y+c}^{\infty} k E_j(kx, ky)\sigma(x)dx \right| \leq C, \quad y \in \mathbb{R},
\]
for any \( p \leq r + 1 \), since it differs from the case \( p = 1 \) only in the number of terms to be estimated and the technique is the same. Therefore we conclude that \( (1 + y^2)^{\frac{\nu + 1}{2}} |I_2|, \quad y \in \mathbb{R} \), is bounded. The boundedness of the sum
\[
(1 + y^2)^{\frac{\nu + 1}{2}} \left( |I_1| + |I_4| + |I_5| \right), \quad y \in \mathbb{R},
\]
can be obtained in a similar way. It remains to show that \( (1 + y^2)^{\frac{\nu + 1}{2}} |I_2| \) is bounded as well. Like in the previous part, we show only the case \( p = 1 \).

\[
\frac{d}{dy} \int_{y-c}^{y+c} k E_j(kx, ky)(\sigma(x) - \sigma(y))dx
\]
\[
= k E_j(k(y + c), ky)(\sigma(y + c) - \sigma(y)) - k E_j(k(y - c), ky)(\sigma(y - c) - \sigma(y))
\]
\[+ \int_{y-c}^{y+c} \frac{d}{dy} k E_j(kx, ky)(\sigma(x) - \sigma(y))dx - \int_{y-c}^{y+c} k E_j(kx, ky)\sigma'(y)dx
\]
\[
= k E_j(k(y + c), ky)(\sigma(y + c) - 2\sigma(y)) - k E_j(k(y - c), ky)(\sigma(y - c) - 2\sigma(y))
\]
\[+ \int_{y-c}^{y+c} \frac{d}{dy} k E_j(kx, ky)\sigma(x)dx - \int_{y-c}^{y+c} k E_j(kx, ky)\sigma'(y)dx.
\]
The terms outside the integrals are bounded by \( C_s \sigma(y)|k E_j(k(y + c), ky)| \) for arbitrary \(|s| \leq r + 1 \). Indeed
\[
\sigma(y)|k E_j(k(y + c), ky)| \leq \frac{C_s}{(1 + |y|)^s} |k E_j(k(y + c), ky)|
\]
\[
\leq \frac{C_s}{(1 + |y|)^s} \frac{2^j k}{(1 + k^j(c))^j}, \quad s \in \mathbb{N},
\]
and similar estimates hold for the other summands. Also
\[
|\sigma'(y)| \int_{y-c}^{y+c} k E_j(kx, ky)dx \leq \frac{C_s}{(1 + |y|)^{s-2}} \frac{2^j k}{(1 + |k(x - y)|)^j} dx
\]
\[
\leq \frac{C_s}{(1 + |y|)^{s-2}} \int_{y-c}^{y+c} \frac{2^j k}{(1 + |k(t)|)^j} dt \leq \frac{C_s}{(1 + |y|)^{s-2}}.
\]
It remains to estimate \( \int_{y-c}^{y+c} \frac{d}{dy} k E_j(kx, ky)\sigma(x)dx \). Since
\[
\left| \int_{y-c}^{y+c} \frac{d}{dy} k E_j(kx, ky)\sigma(x)dx \right| \leq \frac{C_s}{(1 + |y - c|)^s} \int_{y-c}^{y+c} \frac{d}{dy} k E_j(kx, ky)dx,
\]
it remains to estimate the integral. We have
\[ \int_{y-c}^{y+c} \frac{d}{dy} k E_j(kx, ky) \, dx \leq \int_{y-c}^{y+c} 2^j k \sum_{l \in \mathbb{Z}} |\phi(2^j k x - l)| \frac{d}{dy} \phi(2^j k y - l) \, dx \]

\[ \leq \int_{y-c}^{y+c} 2^j k \sum_{l \in \mathbb{Z}} \frac{1}{(1 + |2^j k x - l|)^{\frac{j}{2} + \frac{1}{2}}} \frac{1}{(1 + |2^j k y - l|)^{\frac{j}{2} + \frac{1}{2}}} \, dx \]

\[ \leq \int_{y-c}^{y+c} 2^j k \frac{1}{(1 + |2^j k (x - y)|)^{\frac{j}{2}} \int_{-2^j k c}^{2^j k c} \frac{1}{(1 + |t|)^{\frac{j}{2}}} \, dt, \]

which is bounded. Therefore
\[ (1 + y^2) \frac{r+s}{2} \left| \frac{d}{dy} \int_{y-c}^{y+c} k E_j(kx, ky)(\sigma(x) - \sigma(y)) \, dx \right| \leq C, \]

and the proof of a) is completed.

\[ b) \] The statement is an easy consequence of the fact that \( \sigma \in S_{r+1}, \) and the estimate (13). Namely
\[ \sup_{y \in \mathbb{R}} (1 + y^2) \frac{r+s}{2} \sigma(y) \frac{d}{dy} \int_{-\infty}^{\infty} k E_j(kx, ky) \, dx \]

\[ \leq \sup_{y \in \mathbb{R}} (1 + y^2) \frac{r+s}{2} \sigma(y) \int_{-\infty}^{\infty} C(1 + |x-y|)^{-\frac{3}{2}} \, dx \]

\[ \leq \sup_{y \in \mathbb{R}} C (1 + y^2) \frac{r+s}{2} \sigma(y) \leq C_1, \]

and the proof is complete.

\section{6. Application: Quasiasymptotics at Infinity}

Quasiasymptotic behavior of a distribution turned out to be more appropriate for Abelian and Tauberian type theorems for several integral transforms such as Fourier, Laplace, Stieltjes and Mellin transform, than some other types of asymptotic behaviors. It is also used as an asymptotic analysis tool [7]. In particular, quasiasymptotic behavior of fundamental solutions of linear hyperbolic partial differential operators and of linear passive systems was obtained in [33]. We will not go into details concerning various definitions of asymptotic behaviors and their applications in PDEs and mathematical physics (cf. [7, 27, 30, 33] and the references given there).

Note that a function may have the quasiasymptotic behavior different from its classical asymptotic behavior, and it may have a quasiasymptotic behavior even if its classical asymptotic behavior does not exist at all (e.g., \( e^{ix} \) at infinity, delta distribution as well as its derivatives). This is another motivation for the study of quasiasymptotic behavior. In [28] we studied...
quasiassymptotic behavior of a tempered distribution at a finite point via its multiresolution expansion \( h_j, j \in \mathbb{N} \). We recall the result for the sake of completeness.

**Theorem 6.** [28] a) Let a distribution \( h \in \mathcal{S}' \) have quasiassymptotics at \( x_0 \), related to \( \rho \) equal to \( \gamma \neq 0 \) (\( h \sim \gamma \)). Then, for an \( r \)-regular MRA with sufficiently large \( r \), \( h_j \) has quasiassymptotics at \( x_0 \) related to \( \rho \) equals to \( \gamma \neq 0 \) (\( h_j \sim \gamma \)), \( j \in \mathbb{N} \), as well.

b) Let \( h_j(x), j \in \mathbb{N} \), have the quasiassymptotics at \( x_0 \) equal to \( \gamma_j \), and let \( \gamma_j \to \gamma \neq 0 \) as \( j \to \infty \). Moreover, assume that the family \( \{h(\varepsilon)/\rho(\varepsilon)\} \varepsilon \in (0,1) \) is bounded. Then \( h \) has quasiassymptotics at \( x_0 \) equal to \( \gamma \).

The definition of quasiassymptotics at \( x_0 \) can be found in [27, 28, 30]. We found that some parts of the proof of main results in [28] should be given with more details and in this sense this paper outpass the incompleteness.

In the rest of the paper we apply the result of the previous section to the study of quasiassymptotic behavior of a tempered distribution at infinity.

Definitions and properties given below can be found in [27, 30].

**Definition 5.** Let \( h \in \mathcal{S}' \) and let \( c(x), x \in (a, \infty), \ a > 0 \), be a continuous positive function. We say that \( h \) has the quasiassymptotics at infinity (in \( \mathcal{S}' \)) related to \( c(k) \), if there exists \( \sigma \in \mathcal{S} \), \( \sigma \neq 0 \), such that

\[
\lim_{k \to \infty} \left< \frac{h(kx)}{c(k)} \sigma(x) \right> \sigma(x), \quad \sigma \in \mathcal{S}.
\]

In this case we write: \( h \sim c(k) \) at \( \infty \) related to \( c(k) \) in \( \mathcal{S}' \).

This definition can be extended to the space of distributions \( \mathcal{D}' \). The relation between the quasiassymptotics at infinity in \( \mathcal{S}' \) and in \( \mathcal{D}' \) is given in [26]. The quasiassymptotics at infinity of a distribution \( f \) is related to the quasiassymptotics of its Fourier transform \( \hat{f} \) at zero [27, Proposition 8.4].

Recall, \( L : (a, \infty) \to \mathbb{R}^+, \ a > 0 \), is slowly varying at infinity if

\[
\lim_{k \to \infty} \frac{L(\lambda k)}{L(k)} = 1, \quad \lambda > 0.
\]

A measurable function \( \rho : (a, \infty) \to \mathbb{R}^+, \ a > 0 \), is regularly varying at infinity if there exists \( \alpha \in \mathbb{R} \) such that for all \( \lambda > 0 \)

\[
\lim_{k \to \infty} \frac{\rho(\lambda k)}{\rho(k)} = \lambda^\alpha.
\]

A function is regularly varying if and only if it can be written as \( \rho(x) = x^\alpha L(x), \ x > a \), for some \( \alpha \in \mathbb{R} \) and some slowly varying function \( L \) at infinity.
Let \( h \) and \( c \) satisfy the conditions of Definition 5. Then it is known (cf. [30]) that \( c \) is regularly varying at infinity. Moreover, \( g \) is homogenous with order of homogeneity \( \nu \), i.e.,

\[
g(mx) = m^\nu g(x), \quad x \in \mathbb{R}, \quad m > 0.
\]

Regularly varying functions play an important role in the qualitative analysis of the asymptotic behavior of solutions of certain differential equations. We refer to [24] for detailed exposition and examples.

In the following theorems we characterize the quasiasymptotic behavior of a distribution \( h \) at infinity throughout its projections \( h_j \), and vice versa (\( j \to \infty \)). Theorem 7 is an Abelian and Theorem 8 is a Tauberian type result.

**Theorem 7.** Let a distribution \( h \in \mathcal{S}' \) have quasiasymptotics at infinity related to \( \rho \) equal to \( \gamma \neq 0 \) (\( h_j \sim \gamma \)). Then there exists \( r > r_0 \), where \( r_0 \) is the order of \( h \), such that for a scaling function \( \phi \in \mathcal{S} \), and \( E_j \) given by (9) and (11), \( h_j(x) = \langle h(y), E_j(x, y) \rangle, \ j \in \mathbb{N}, \) has quasiasymptotics at infinity related to \( \rho \) equal to \( \gamma \neq 0 \) (\( h_j \sim \gamma \)).

**Proof.** Since \( \frac{h(kx)}{\rho(k)} \) converges in \( \mathcal{S} \) as \( k \to \infty \), there exists \( r \in \mathbb{N} \), \( r > r_0 \), such that

\[
\lim_{k \to \infty} \left\langle \frac{h(kx)}{\rho(k)}, \theta(x) \right\rangle = \langle \gamma(x), \theta(x) \rangle, \quad \theta \in \mathcal{S}_r.
\]

It follows that we may take \( \theta = \sigma \in \mathcal{S}_{r+1} \) and

\[
\lim_{k \to \infty} \left\langle \frac{h(kx)}{\rho(k)}, \sigma(x) \right\rangle = \langle \gamma(x), \sigma(x) \rangle, \quad \sigma \in \mathcal{S}_{r+1}.
\]

Then

\[
\left\langle \frac{h(kx)}{\rho(k)}, \sigma(x) \right\rangle = \left\langle \frac{h(ky)}{\rho(k)} \cdot E_j(kx, y), \sigma(x) \right\rangle = \left\langle \frac{h(ky)}{\rho(k)}, \langle E_j(kx, y), \sigma(x) \rangle \right\rangle
\]

\[
= \left\langle \frac{h(ky)}{\rho(k)}, \langle kE_j(kx, ky), \sigma(x) \rangle \right\rangle
\]

Theorem will be proved if the last expression tends to \( \langle \gamma(x), \sigma(x) \rangle \) as \( k \to \infty \). We have

\[
\lim_{k \to \infty} \left\langle \frac{h(ky)}{\rho(k)}, \langle kE_j(kx, ky), \sigma(x) \rangle \pm \sigma(y) \right\rangle =
\]

\[
\lim_{k \to \infty} \left\langle \frac{h(ky)}{\rho(k)}, \langle kE_j(kx, ky), \sigma(x) \rangle - \sigma(y) \right\rangle + \lim_{k \to \infty} \left\langle \frac{h(ky)}{\rho(k)}, \sigma(y) \right\rangle.
\]
Furthermore, \(|\frac{h(ky)}{\rho(k)} \cdot (kE_j(kx, ky), \sigma(y) - \sigma(y))|\) is less than or equal to

\[
\| \frac{h(ky)}{\rho(k)} \|_{\mathcal{L}(S_{r+1}, \mathbb{C})} \cdot \| \langle kE_j(kx, ky), \sigma(x) \rangle - \sigma(y) \|_{S_{r+1}}.
\]

where \(\| \cdot \|_{\mathcal{L}(S_{r+1}, \mathbb{C})}\) denotes the norm of a linear functional over \(S_{r+1}\).

Then 3 implies

\[
\lim_{k \to \infty} \left\langle \frac{h(ky)}{\rho(k)}, (kE_j(kx, ky), \sigma(x) - \sigma(y)) \right\rangle = 0.
\]

This finally gives

\[
\lim_{k \to \infty} \left\langle \frac{h_j(kx)}{\rho(k)}, \sigma(x) \right\rangle = \langle \gamma(x), \sigma(x) \rangle \quad \sigma \in S.
\]

**Theorem 8.** Let there be given \(h \in \mathcal{S}'\) of order \(r_0\). Let \(\phi \in \mathcal{S}_r\), for \(r > r_0\), \(h_j = E_jh, E_j\) being given by (9) and (11). Let \(h_j, j \in \mathbb{N}\), have the quasiasymptotics at infinity equal to \(\gamma_j\), and let \(\gamma_j \to \gamma \neq 0\) as \(j \to \infty\). Moreover, assume that the family \(\{h(ky)/\rho(k) | k \in \mathbb{N}\}\) is bounded in \(S_r\). Then \(h\) has the quasiasymptotics at infinity equal to \(\gamma\).

**Proof.** Put \(h_j(\cdot) = \langle h(y), E_j(\cdot, y) \rangle, j > 0\). Thus

\[
h_j(kx) = \langle h(y), E_j(kx, y) \rangle = \langle h(ky), kE_j(kx, ky) \rangle, \quad x \in \mathbb{R}, \quad k > 0.
\]

Let \(\sigma \in S\). By generalized Fubini theorem (cf. [26, 30])

\[
\left\langle \frac{h_j(kx)}{\rho(k)}, \sigma(x) \right\rangle = \left\langle \frac{1}{\rho(k)} \langle h(ky), kE_j(kx, ky) \rangle, \sigma(x) \right\rangle
\]

\[
= \left\langle \frac{h(ky)}{\rho(k)}, \langle kE_j(kx, ky), \sigma(x) \rangle \right\rangle.
\]

We have

\[
\left\langle \frac{h(ky)}{\rho(k)}, \sigma(y) \right\rangle = \left\langle \frac{h_j(ky)}{\rho(k)}, \sigma(y) \right\rangle + \left\langle \frac{h(ky) - h_j(ky)}{\rho(k)}, \sigma(y) \right\rangle.
\]

Let there be given \(\varepsilon > 0\). Take \(j \in \mathbb{N}\) such that \(|\langle \gamma_j - \gamma, \sigma \rangle| < \frac{\varepsilon}{2}|\) and let \(k_0\) be chosen so that

\[
\left| \left\langle \frac{h_j(ky)}{\rho(k)} - \gamma, \sigma(y) \right\rangle \right| < \frac{\varepsilon}{2} \quad \text{for} \quad k > k_0.
\]

Thus if we prove that for already fixed \(j\)

\[
\left| \left\langle \frac{h(ky) - h_j(ky)}{\rho(k)}, \sigma(y) \right\rangle \right| < \frac{\varepsilon}{2} \quad \text{for} \quad k > k_1 \geq k_0.
\]
then we will have
\[
\left| \left\langle \frac{h(ky)}{\rho(k)} - \gamma, \sigma(y) \right\rangle \right| < \varepsilon \quad \text{for} \quad k > k_1.
\]
So we have to estimate
\[
\left| \left\langle \frac{h(ky)}{\rho(k)} \sigma(y) - \langle kE_j(kx, ky), \sigma(x) \rangle \right\rangle \right|
\leq C \left\| \frac{h(ky)}{\rho(k)} \right\|_{L^\infty(s, \mathcal{C})} \left\| \sigma(y) - \langle kE_j(kx, ky), \sigma(x) \rangle \right\|_{S_r},
\]
Now we use the fact that \( \{ h(k\cdot)/\rho(k) \mid k > 0 \} \) is bounded in \( S'_r \) and the result of Lemma 1, that is
\[
\lim_{k \to \infty} \left\| \sigma(y) - \langle kE_j(kx, ky), \sigma(x) \rangle \right\|_{S_r} = 0.
\]
The theorem is proved.

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