Perturbations of Jacobi Polynomials and Piece-Wise Hypergeometric Orthogonal Systems

Yu.A. Neretin

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Neretin Yu. A.

We construct a family of noncomplete orthogonal systems of functions on the ray $[0, \infty]$; these systems depend on 3 real parameters $\alpha$, $\beta$, $\theta$. Elements of a system are piece-wise hypergeometric functions, having a singularity at $x = 1$. For $\theta = 0$ these functions vanish on $[1, \infty)$ and our system is reduced to the Jacobi polynomials $P_n^{\alpha, \beta}$ on the segment $[0, 1]$. In a general case, our functions can be considered as an interpretation of $P_n^{\alpha, \beta \pm \epsilon}$. Also, our functions are solutions of some exotic boundary problem for the hypergeometric differential operator. We find the spectral measure for this problem. This gives a new analog of the Weyl-Olevsky index hypegeometric transform.

1. Formulation of result

1.1. Jacobi polynomials. Preliminaries. Recall that the Jacobi polynomials $P_n^{\alpha, \beta}$ are the polynomials on the segment $[-1, 1]$ orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(y)g(y)(1 - y)^\alpha(1 + y)^\beta \, dy, \quad \alpha > -1, \beta > -1 \quad (1.1)$$

These polynomials are given by explicit formulae, (see [HTF], 10.8(16)),

$$P_n^{\alpha, \beta}(y) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)n!} F \left[ \frac{-n, n + \alpha + \beta + 1, 1 - y}{\alpha + 1 1/2} \right] \quad (1.2)$$

$$= \frac{(-1)^n \Gamma(n + \beta + 1)}{\Gamma(\beta + 1)n!} F \left[ \frac{-n, n + \alpha + \beta + 1, 1 + y}{\beta + 1 1/2} \right] \quad (1.3)$$

$$= \frac{(-1)^n \Gamma(n + \beta + 1)}{\Gamma(\beta + 1)n!} \left( \frac{1 - y}{2} \right)^{-\alpha} F \left[ \frac{n + \beta + 1, -\alpha - n, 1 - y}{\beta + 1 1/2} \right] \quad (1.4)$$

Here $F = \frac{\Gamma}{\Gamma}$ is the Gauss hypergeometric function,

$$F[a, b; c; x] = F \left[ \frac{a, b}{c}; x \right] := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} x^k$$

and $(a)_k := a(a + 1) \ldots (a + k - 1)$ is the Pochhammer symbol.

The expressions (1.2), (1.3) are polynomials since $(-n)_k = 0$ for $k > n$. The last expression (1.4) is a series, it can be obtained from (1.3) by the transformation (see [HTF]),

$$F[a, b; c; x] = (1 - x)^{c-a-b} F [c - a, c - b; c; x] \quad (1.5)$$
Norms of the Jacobi polynomials with respect to the inner product (1.1) are given by

\[ \|P_n^{\alpha,\beta}\|^2 = (P_n^{\alpha,\beta}, P_n^{\alpha,\beta}) = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)} \quad (1.6) \]

The Jacobi polynomials are the eigen-functions of the differential operator

\[ D := (1 - y^2) \frac{d^2}{dy^2} + [(\beta - \alpha - (\alpha + \beta + 2)y] \frac{d}{dy} \quad (1.7) \]

Precisely,

\[ DP_n^{\alpha,\beta} = -n(n+\alpha+\beta+1)P_n^{\alpha,\beta} \]

1.2. Orthogonal systems. Now, fix \( \theta \in \mathbb{C} \) such that

\[ 0 < \text{Re} \, \theta < 1 \quad (1.8) \]

Also, fix \( \alpha, \beta \in \mathbb{C} \) such that

\[ -1 < \text{Re} \, \alpha < 1, \, \alpha \neq 0, \quad \text{Re} \, \beta > -1 \quad (1.9) \]

Consider the space of functions on the half-line \( x > 0 \) equipped with the bilinear scalar product

\[ \{f, g\} = \int_0^1 f(x)g(x)(1-x)^{\alpha}x^{\beta} \, dx + \frac{\sin(\alpha + \theta)\pi}{\sin \theta \pi} \int_1^\infty f(x)g(x)(x-1)^{\alpha}x^{\beta} \, dx \quad (1.10) \]

Denote by \( H(x) \) the Heaviside function

\[ H(x) = \begin{cases} 
1, & x > 0; \\
0, & x < 0 
\end{cases} \]

Let \( p \in \mathbb{C} \) ranges in the set

\[ p - \theta \in \mathbb{Z}, \quad \text{Re}(2p + \alpha + \beta + 1) > 0, \quad 1 + p + \alpha \neq 0 \quad (1.11) \]

Define the piece-wise hypergeometric functions \( \Phi_p(x) \) on the half-line \([0, \infty)\) by

\[ \Phi_p(x) = \frac{\Gamma(2p + \alpha + \beta + 2)}{\Gamma(\beta + 1)} F \left[ \begin{array}{c} -p, p + \alpha + \beta + 1 \\ \beta + 1 \end{array} ; x \right] H(1-x) + \frac{\Gamma(1 + p + \alpha)}{\Gamma(-p)} F \left[ \begin{array}{c} p + \alpha + 1, p + \alpha + \beta + 1 \\ 2p + \alpha + \beta + 2 \end{array} ; x \right] x^{-\alpha-\beta-\frac{p}{2}-1} H(x-1) \quad (1.12) \]

Theorem. The functions \( \Phi_p \) are orthogonal with respect to the symmetric bilinear form (1.10),

\[ \{\Phi_p, \Phi_q\} = 0 \quad \text{for} \quad p \neq q \quad (1.13) \]
and

\[ \{\Phi_p, \Phi_p\} = \frac{\Gamma^2(2p + \alpha + \beta + 2)\Gamma(1 + p + \alpha)\Gamma(p + 1)}{(2p + \alpha + \beta + 1)\Gamma(p + \beta + 1)\Gamma(p + \alpha + \beta + 1)} \quad (1.14) \]

We can consider also the Hermitian inner product

\[ \langle f, g \rangle = \int_0^1 f(x)g(x)(1-x)^{\alpha}x^{\beta}dx + \frac{\sin(\alpha + \theta)\pi}{\sin\theta\pi} \int_1^\infty f(x)g(x)(x-1)^{\alpha}x^{\beta}dx \quad (1.15) \]

here we must assume \( \theta, \alpha, \beta \in \mathbb{R} \), and

\[ 0 \leq \theta < 1, \quad -1 < \alpha < 1, \quad \beta > -1, \quad 2p + \alpha + \beta + 1 > 0 \]

By our theorem, the functions \( \Phi_p \) are orthogonal with respect to the inner product (1.15). If the factor \( \sin(\alpha + \theta)\pi/\sin(\alpha\pi) \) is positive, then our inner product also is positive definite.

**Remark.** The system \( \Phi_p \) is not a basis in our Hilbert space.

**1.3. Comparison with the Jacobi polynomials.** Let us show, that our construction is reduced to the Jacobi polynomials in the case \( \theta = 0 \).

First, assume \( x = (1 + y)/2 \) in the formulae (1.1), (1.3). We observe that the first summand in (1.12) is a Jacobi polynomial. The second summand in (1.12) is 0 since it contains the factor \( \Gamma(-p)^{-1} \). This factor is 0 if \( p = 0, 1, 2, \ldots \).

Thus, for an integer \( p \),

\[ \Phi_p(x) = \frac{\Gamma(2p + \alpha + \beta)\Gamma(p + 1)}{\Gamma(p + \beta + 1)} P^{\alpha,\beta}_p(2x - 1)H(1 - x) \]

Hence for \( \theta = 0 \) our orthogonality relations are the orthogonality relations for the Jacobi polynomials.

**1.4. Singular boundary problem.** Consider the differential operator

\[ D := x(x-1)\frac{d^2}{dx^2} + (\beta + 1 - (\alpha + \beta + 2)x)\frac{d}{dx} \quad (1.16) \]

The functions \( \Phi_p \) satisfy the equation

\[ D\Phi_p = -p(p + \alpha + \beta + 1)\Phi_p \quad (1.17) \]

on the both intervals \((0, 1), (1, \infty)\) (see formulae [HTF1], 2.9(1), (13) for the Kummer solutions of the hypergeometric equation).

Let \( \alpha \neq 0 \). Now we define a space \( E \) of functions on \([0, \infty)\). Its elements are functions \( f(x) \) that are smooth outside the singular points \( x = 0, x = 1, x = \infty \); at the singular points they satisfy the following boundary conditions (the conditions a) and b) are really important).

a) **The condition at \( \theta \).** A function \( f \) is smooth at 0.

\(^1\)Otherwise, below we have logarithmic asymptotics at \( x = 1 \).
b) The condition at 1. There are functions \( u(x), v(x) \) smooth at 1 such that

\[
f(x) = \begin{cases} 
    u(x) + v(x)(1-x)^{\alpha}, & x < 1; \\
    \frac{\sin \theta \pi}{\sin (\alpha + \theta) \pi} u(x) + v(x)(x-1)^{\alpha} & x > 1
\end{cases}
\]

c) The condition at \( \infty \). There is a function \( u(y) \) smooth at zero, such that

\[
f(x) = u(1/x)x^{n-\alpha-\beta-1} \quad \text{for large } x
\]

where \( r \) is the minimal possible value of \( p \).

**Lemma.** a) Let \( p \in \mathcal{E} \).

b) For \( f, g \in \mathcal{E} \),

\[
\{Df, g\} = \{f, Dg\}
\]

Obviously, this implies the orthogonality relations for \( p \neq q \). Indeed,

\[
\{D\Phi_p, \Phi_q\} = \{-p(p+\alpha+\beta+1)\{\Phi_p, \Phi_q\} = -q(q+\alpha+\beta+1)\{\Phi_p, \Phi_q\}
\]

and hence \( \{\Phi_p, \Phi_q\} = 0 \).

**Remark.** The functions \( \Phi_p \) satisfy the equation (1.17) for all \( p \in \mathbb{C} \). Also, for \( \Re p > -(\alpha + \beta + 1)/2, \Re q > -(\alpha + \beta + 1)/2 \), the integral \( \{\Phi_p, \Phi_q\} \) is convergent. Nevertheless, generally \( \{\Phi_p, \Phi_q\} \neq 0 \).

Denote by \( \mathcal{H} \) the Hilbert space with the inner product (1.15). Obviously, \( \mathcal{E} \subset \mathcal{H} \).

**Theorem.** The operator \( D \) is essentially self-adjoint on \( \mathcal{E} \).

**Remark.** a) We can replace the boundary condition at \( \infty \) by the following: \( f(x) = 0 \) for large \( x \).

b) If \( \beta > 1 \), then we can replace the condition at 0 by the following: \( f(x) = 0 \) at some neighborhood of 0.

c) For \( \beta < 1 \) the latter variant gives a non-self-adjoint operator. Possible self-adjoint conditions are enumerated by points \( \lambda : \mu \) of the real projective line; they can be given in the form

\[
f(x) = A(\lambda + \mu x^{-\beta}) + x\varphi(x) + x^{-\beta+1}\psi(x)
\]

where \( \varphi, \psi \) are functions smooth near 0, \( A \) ranges in \( \mathbb{C} \). The condition given above corresponds to \( \mu = 0 \).

1.5. Expansion in eigenfunctions. Hence the usual expansion of a function in the series of the Jacobi polynomials must be replaced by the eigenfunction expansion of \( D \) in spirit of Weyl and Titmarsh (see [DS]).
For $s \in \mathbb{R}$, we define the function $\Psi_s(x)$ on $[0, \infty)$ given by

$$\Psi_s(x) = F \left[ \frac{a+\beta+1}{2} + is, \frac{a+\beta+1}{2} - is; x \right] H(1 - x) +$$

$$+ \frac{2\Gamma(\beta + 1)}{\sin(\theta + \alpha)\pi} \cdot \text{Re} \left\{ \frac{\Gamma(-2is)\cos \left( \frac{a+\beta}{2} + \theta - is \right)\pi}{\Gamma \left( \frac{a+\beta+1}{2} - is \right)\Gamma \left( -\frac{a+\beta+1}{2} - is \right)} \right\} \times$$

$$\times F \left[ \frac{a+\beta+1}{2} + is, \frac{a+\beta+1}{2} + is; 1 \right] x^{-(a+\beta+1)/2 - is} H(x - 1) \quad (1.18)$$

Obviously,

$$\Psi_s(x) = \Psi_{-s}(x)$$

**Remark 1.** Let us show that the both summands of $\Psi_s$, are solutions of the equation

$$Df = -\left( \frac{1}{4}(a + \beta + 1)^2 + s^2 \right)f \quad (1.19)$$

Indeed, the first summand is the same as above, we substitute $x = -\frac{a+\beta+1}{2} + is$ to (1.12). The hypergeometric function in the second summand is a Kummer solution of the equation (1.19). Since the coefficients of $D$ are real, the complex conjugate function also is a solution of the same equation.

**Remark 2.** The functions $\Psi_s(x)$ satisfy the boundary condition at $x = 1$.

**Remark 3.** The functions $\Phi_p$ with $p - \theta \in \mathbb{Z}$ are all the $L^2$-eigenfunctions for the boundary problem formulated above. Also, the functions $\Psi_s(x)$ are all the remaining generalized eigenfunctions of the same boundary problem. See below Section 4.

This 3 remarks easily imply the explicit Plancherel measure for the operator $D$.

Consider the Hilbert space $V$, whose elements are pairs $(a(p), F(s))$, where $a(p)$ is a sequence ($p$ ranges in the same set as above), and $F(s)$ is a function on the half-line $s > 0$; the inner product is given by

$$[(a, F); (b, G)] = \sum_p a(p)\overline{b(p)} +$$

$$+ \frac{\sin \theta \pi \sin(\theta + \alpha)\pi}{4\pi \Gamma(\beta + 1)^2} \int_0^{\infty} \left| \frac{\Gamma \left( \frac{a+\beta+1}{2} - is \right)\Gamma \left( -\frac{a+\beta+1}{2} - is \right)}{\Gamma(2is)\cos \left( \frac{a+\beta+1}{2} + \theta - is \right)\pi} \right|^2 F(s)\overline{G(s)} ds$$

$$\quad (1.20)$$

We define the operator $U f \mapsto (a_p, F(s))$ from $\mathcal{H}$ to $V$ by

$$a(p) = \langle f, \Phi_p \rangle_{\mathcal{H}},$$

$$F(s) = \langle f, \Psi_s \rangle_{\mathcal{H}}$$

**Theorem.** The operator $U : \mathcal{H} \rightarrow V$ is a unitary invertible operator.
In particular, this theorem implies the **inversion formula**,
\[ U^{-1} = U^* \]

1.6. **Discussion: shift of the index** \( n \) **for classical orthogonal bases.**

Consider the group \( SL_2(\mathbb{R}) \) of real \( 2 \times 2 \) matrices whose determinant is 1. Let \( SL_2^0(\mathbb{R}) \) be its universal covering group. Recall that the **principal nonunitary series of representations** of \( SL_2^0(\mathbb{R}) \) is realized in the space of functions on \( \mathbb{R} \); the operators of a representation \( T_{\sigma, \tau} \), are given by the formula

\[
T_{\sigma, \tau}(a \begin{pmatrix} c & b \\ e & d \end{pmatrix} f(x) = f \left( \frac{a(x+c) + b}{c(x+d)} \right) (c(x+d) \tau)^{(c(x+d) \sigma)} \]

(1.21)

where \( \sigma, \tau \in \mathbb{C} \), and \((c(x+d) \tau)^{(c(x+d) \sigma)} \) is a branch holomorphic in the upper half-plane \( \text{Im} x \geq 0 \), and \((c(x+d) \tau)^{(c(x+d) \sigma)} \) is a branch holomorphic in the lower half-plane \( \text{Im} x \leq 0 \).

A representation \( T_{\sigma, \tau} \) is unitary in \( L^2(\mathbb{R}) \) if \( \text{Im} \sigma = \text{Im} \tau \), \( \text{Re} \sigma + \text{Re} \tau = -1 \). Such representations form the **principal unitary series**.

If \( \tau = 0 \), then the space of functions holomorphic in the upper half-plane is invariant with respect to the operators (1.21). Such representations in the space of holomorphic functions are named **highest weight representations**. If also \( \sigma < 0 \), then this representation is unitary.

If \( \sigma = l \) is a positive integer, and \( \tau = 0 \), then the space of polynomials of degree \( \leq l \) is invariant with respect to the operators (1.21), and thus we obtain a finite-dimensional representation of \( SL_2(\mathbb{R}) \).

We also define the following one-parameter subgroups in \( SL_2(\mathbb{R}) \).

- the group \( K \) consisting of orthogonal matrices \( \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} \);
- a conjugate subgroup \( L = L_g \) having the form \( gKg^{-1} \), where \( g \in SL_2(\mathbb{R}) \);
- the subgroup \( H \) consisting of diagonal matrices \( \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \);
- and the subgroup \( N \) consisting of upper triangular matrices \( \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \).

Consider an \((l+1)\)-dimensional representation of \( SL_2(\mathbb{R}) \), and the eigenbases \( e_k \) (resp. \( f_k \)) for the subgroup \( L \) (resp. \( K \)). Consider the transition matrix \( S_{k,l} := (e_k, f_l) \). We can consider the columns of the matrix as an orthogonal system of functions on the set \( 0, 1, \ldots, l \). This system of functions are the **Kravchuk polynomials** (for the Kravchuk polynomials see [HTF2], [KS], [NUS]).

The same construction applied to a unitary highest weight representation of \( SL_2(\mathbb{R}) \) gives the **Meixner polynomials**, they are orthogonal with respect to some weight supported by the set \( 0, 1, 2, \ldots \) (for the Meixner polynomials, see [HTF2], [KS], [NUS]).

Vilenkin and Klimyk ([VK], 1983) applied this construction to a general infinite dimensional unitary representation of \( SL_2(\mathbb{R}) \). This gives a (nonpolynomial) orthogonal system of functions on the lattice \( \mathbb{Z} \). Nonformally, the explicit formula for these functions has the form \( m_n + \phi \), where \( m_n \) are the Meixner polynomials; the parameter \( n \) in \( \mathbb{Z} \). For details, see [VK1], [GK]. In fact, this family
of systems contains the Meixner systems as degenerations and the Kravchuk systems as degenerations of analytic continuations.

The similar shift of an index is possible for some other classical orthogonal systems.

— **Deformed Laguerre systems** (see Groenevelt, Koelink, [GK]). We consider a representation of \( SL_2(\mathbb{R}) \) of the principal series and construct its model, where the subgroup \( N \) acts as multiplications by functions. Then we write the \( K \)-eigenbasis in this model. In the standard model (1.21) the group \( N \) acts by shifts \( f(x) \mapsto f(x+b) \). Hence we simply consider the Fourier transform of \( K \)-eigenbasis \((1+ix)^{-n} (1-ix)^{\sigma+n}\). Thus we obtain an orthogonal Laguerre-like system.

— **Deformed Meixner-Pollachek systems** (see [Ner2]). We repeat the same construction with the diagonal subgroup \( H \). In the standard realization (1.21) of representations of \( SL_2(\mathbb{R}) \) the subgroup \( H \) acts by dilatations \( f(x) \mapsto f(\lambda x) \lambda^{-\sigma-\tau} \), hence we simply consider the Mellin transform of \( L \)-eigenbasis. \(^2\) This construction gives hypergeometric orthogonal bases in the space of \( \mathbb{C}^2 \)-valued functions.

— **Deformed continuous dual Hahn systems** (see [Ner1]). We consider the Plancherel decomposition for the tensor product of two representations of the principal series of \( SL_2(\mathbb{R}) \) and write the images of \( K \)-eigenbases in this decomposition.

It is important, that the tables of basic formulae [HTF2], [KS] for classical orthogonal polynomials survive also for deformed bases (and this confirms the nontriviality of the phenomenon).

It seems that the for the Jacobi system there is an obstacle for such perturbation, since the numbers (1.6) lose their positivity if we replace \( n \in \mathbb{N} \) by \( n + \theta \) with \( \theta \in \mathbb{Z} \). As we have seen above, the Jacobi system survives, but lose its completeness.

1.7. Discussion: boundary problems with several singular points. Nonstandard boundary problems of the type discussed 1.4 are usual in the noncommutative harmonic analysis (see Molchanov, [Mol1]–[Mol3]), they arise in approximately following situations.

Consider the one-sheet hyperboloid \( SL_2(\mathbb{R})/H \). Evaluation of the Plancherel measure for \( SL_2(\mathbb{R})/H \) can be reduced to spectral decomposition of the Laplace operator restricted to the set \( H \)-invariant functions. This restriction is a Legendre differential operator on some contour passing through a singular point of the Legendre equation\(^3\).

There are many other situations of such kind (see also [Ner3]). Quite often such operator has a large discrete specter. I can not find a material problem of harmonic analysis, where our operator \( D \) appears.

1.8. Discussion. Degree of rigidity of the problem. There are two classical variants of expansion of the hypergeometric differential operator in eigenfunctions. One case gives the expansion in the Jacobi polynomials. An-

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\(^2\) The action of \( SL_2(\mathbb{R}) \) after the Mellin transform was written in [Vil], see also [VK1].

\(^3\) This interior singular point corresponds to light-like points on the hyperboloid.
other one gives the Weyl–Olevsky index hypergeometric transform (see detailed exposition in [DS]). An additional variant was obtained in [Ner1].

It seems that there are several other possibilities to write boundary conditions for the hypergeometric operator. But not all possible variants are reasonable.

For instance, consider the same operator $D$ defined in $L^2(0, 1)$ with respect to the same weight $x^\beta (1 - x)\alpha$. If $\alpha > 1$, $\beta > 1$, then the space $\mathcal{D}(0, 1)$ of compactly supported smooth functions on $(0, 1)$ is a domain of self-adjointness. If

$$-1 < \beta < 1, \quad -1 < \alpha < 1$$

then the deficiency indices of $D$ on $\mathcal{D}(0, 1)$ are $(2, 2)$. In fact, the both solutions of the equation $Df = \lambda f$ are in $L^2$ for all $\lambda$

Fix $\mu, \nu \in \mathbb{R}$. Let us write the boundary conditions

$$f(x) = A \left[ 1 - \mu \frac{\Gamma(-\beta)}{\Gamma(\beta)} x^{-\beta} \right] + x \varphi_0(x) + x^{-\beta+1} \psi_0(x) \quad \text{near } x = 0$$

$$f(x) = B \left[ 1 + \nu \frac{\Gamma(\alpha)}{\Gamma(-\alpha)} (1 - x)^{-\alpha} \right] + x \varphi_1(x) + (1 - x)^{-\alpha+1} \psi_1(x) \quad \text{near } x = 1$$

where $\varphi_0, \psi_0$ are smooth near $x = 0$ and $\varphi_1, \psi_1$ are smooth near $x = 1$.

Then the specter is discrete and $\lambda = -p(\alpha + \beta + 1 + p)$ is a point of the specter iff

$$\frac{1}{\Gamma(\beta + p + 1) \Gamma(-p - \alpha)} + \frac{\lambda}{\Gamma(p + 1) \Gamma(-\alpha - \beta - p)} = \frac{\mu}{\Gamma(-p) \Gamma(p + \alpha + \beta + 1)} + \frac{\nu \mu}{\Gamma(-\beta - p) \Gamma(\alpha + p + 1)}$$

The equation seems nice, but apparently it is nonsolvable.

1.9. Discussion. An attempt of an application. Denote

$$\xi_\mu = (1 - x)^\mu H(1 - x), \quad \nu \in \mathbb{R}$$

Equating

$$\langle \xi_\mu, \xi_\nu \rangle = \langle U \xi_\mu, U \xi_\nu \rangle$$
we obtain the following identity

\[ \pi^{-3} \sin \theta \sin (\theta + \alpha) \pi \times \]

\[ \times \int_{0}^{\infty} \left| \frac{\Gamma \left( \frac{\alpha + \beta + 1}{2} + is \right) \Gamma \left( \frac{\alpha + \beta + 1}{2} + \theta + is \right) \Gamma \left( \frac{\alpha - \beta + 1}{2} - \theta + is \right)}{\Gamma(2is) \Gamma(\mu + \frac{\alpha + \beta + 3}{2} + is) \Gamma(\nu + \frac{\alpha + \beta + 3}{2} + is)} \right|^2 ds \]

\[ + \frac{1}{\pi^2} \sin(\theta - \mu) \sin(\theta - \nu) \pi \sum_{p} (2p + \alpha + \beta + 1) \times \]

\[ \times \Gamma \left[ \begin{array}{c} p + \alpha + \beta + 1, p + \beta + 1, -\nu + p, -\mu + p \\ p + \alpha + \beta + \mu + 2, p + \alpha + \beta + \nu + 2, p + \alpha + 1, p + 1 \end{array} \right] = \]

\[ = \frac{\Gamma(\beta + 1) \Gamma(\alpha + \mu + \nu + 1)}{\Gamma(\alpha + \beta + \mu + \nu + 2) \Gamma(\mu + 1) \Gamma(\nu + 1) \Gamma(\alpha + \mu + 1) \Gamma(\alpha + \nu + 1)} \]

This identity is a kind of a beta-integral, continuous and discrete beta-integrals are well-known, see [Ask]. Our integral has a mixed continuous-discrete form, and apparently the summands in the left-hand side do not admit simple expressions (but apparently they can be written as $F_2\alpha$-functions at 1).

1.10. **Structure of the paper.** In Sections 2 and 3, we give two proofs of the orthogonality relations. In Section 3 we also discuss our boundary problem. In Section 4 we obtain the spectral decomposition of $D$.

**Acknowledgments.** I am grateful to V.F. Molchanov for a discussion of this subject.

2. Calculation

We use the notation

\[ \Gamma \left[ \begin{array}{c} a_1, \ldots, a_k \\ b_1, \ldots, b_l \end{array} \right] := \frac{\Gamma(a_1) \ldots \Gamma(a_k)}{\Gamma(b_1) \ldots \Gamma(b_l)} \]

2.1. **The Mellin transform.** For a function $f$ defined on the semi-line $x > 0$, its *Mellin transform* is defined by the formula

\[ \mathcal{M}f(s) = \int_{0}^{\infty} f(x) x^s dx / x \]  \hspace{1cm} (2.1)

In the cases that are considered below, this integral converges in some strip $\sigma < \text{Re} s < \tau$. The inversion formula is

\[ f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathcal{M}f(s) x^{-s} ds \]

there the integration is given over an arbitrary contour lying in the strip $\sigma < \text{Re} s < \tau$.

The multiplicative convolution $f * g$ is defined by

\[ f * g(x) = \int_{0}^{\infty} f(x) g(y/x) dx / x \]  \hspace{1cm} (2.2)
The Mellin transform maps the convolution to the product of functions,

$$\mathcal{M}[f * g](s) = \mathcal{M}(f) \cdot \mathcal{M}(g)$$

(2.3)

(if \(\mathcal{M}f(s), \mathcal{M}g(s)\) are defined in the common strip).

2.2. A way of proof of orthogonality. We write two explicit functions \(\mathcal{K}_1(s), \mathcal{K}_2(s)\) and evaluate their inverse Mellin transforms \(K_1, K_2\). Next, we write the identity

$$K_1 * K_2(1) = \mathcal{M}^{-1}[(\mathcal{K}_1 \mathcal{K}_2)'](1)$$

and observe that it coincides with the orthogonality relations for \(\Phi_p\) and \(\Phi_q\).

2.3. Evaluation of the convolution. We use the following Barnes-type integral [PBM3, 8.4.49.1]

$$\frac{1}{2\pi i} \int_{i\infty}^{+i\infty} \Gamma \left[ \begin{array}{c} c, 1 - b \\ a \\ \end{array} \right] x^{-s} ds =$$

$$= F \left[ \begin{array}{c} a, \frac{1}{x}; c \\ \frac{1}{a} \\ \end{array} \right] H(1 - x) +$$

$$+ x^{-a} \Gamma \left[ \begin{array}{c} c, 1 - b \\ c - a, 1 + a - b \\ \end{array} \right] F \left[ \begin{array}{c} a, 1 + a - c ; 1 \\ 1 + a - b \\ \end{array} \right] H(x - 1) \quad (2.4)$$

The integrand has two series of poles

$$s = 0, -1, -2, \ldots, \quad s = a, a + 1, \ldots$$

The integration is given over an arbitrary contour lying in the strip \(0 < \text{Re } s < \text{Re } a\) (such contour separates two series of poles). The condition of the convergence is \(\text{Re}(c - a - b) > -1\).

This identity can be easily proved using the Barnes residue method, see, for instance, ([Sla], [Mar]).

We consider two functions \(\mathcal{K}_1(s), \mathcal{K}_2(s)\) given by

$$\mathcal{K}_1(s) = \Gamma \left[ \begin{array}{c} \beta + 1, \frac{p + a + 1}{\beta + p + 1} \\ \beta + p + 1 \\ \end{array} \right] \cdot \Gamma \left[ \begin{array}{c} s, \beta + p + 1 - s \\ s + p + a + 1, \beta + 1 - s \\ \end{array} \right]$$

$$\mathcal{K}_2(s) := \Gamma \left[ \begin{array}{c} 2q + a + \beta + 2, -a - q \\ q + a + \beta + 1 \\ \end{array} \right] \Gamma \left[ \begin{array}{c} s + a + q, \beta + 1 - s \\ s, q + \beta + 2 - s \\ \end{array} \right]$$

we assume that \(p - \theta, q - \theta \in \mathbb{Z}\). Using formula (2.4), we evaluate their inverse Mellin transforms,

$$K_1(x) := F \left[ \begin{array}{c} p + \beta + 1, -p - a \\ \beta + 1 \\ \end{array} \right] H(1 - x) +$$

$$+ x^{-\beta - x - 1} F \left[ \begin{array}{c} \beta + 1, p + a + 1 \\ -p, 2p + a + \beta + 2 \\ \end{array} \right] F \left[ \begin{array}{c} \beta + p + 1, p + 1, 1 \\ 2p + a + \beta + 2, x \\ \end{array} \right] H(x - 1) \quad (2.5)$$
\[ K_2(x) = x^{2+\alpha} F \left[ \begin{array}{c} q + \alpha + \beta + 1, q + \alpha + 1 \\ 2q + \alpha + \beta + 2 \end{array} ; x \right] H(1-x) + \\
+ x^{\beta-1} \Gamma \left[ \begin{array}{c} 2q + \alpha + \beta + 2, -\alpha - q \\ q + 1, \beta + 1 \end{array} \right] F \left[ \begin{array}{c} q + \alpha + \beta + 1, -q, \frac{1}{x} \\ \beta + 1 \end{array} ; x \right] H(x - 1) \]

Now we write the identity
\[ K_1 \ast K_2(1) = \mathcal{M}^{-1} [\mathcal{K}_1 \mathcal{K}_2](1) \]
and multiply its both sides by
\[ \Gamma \left[ \begin{array}{c} 2p + \alpha + \beta + 2, q + 1 \\ \beta + 1, -\alpha - q \end{array} \right] \]

We obtain the following identity
\[
\Gamma \left[ \begin{array}{c} 2p + \alpha + \beta + 2, q + 1 \\ \beta + 1, -\alpha - q \end{array} \right] \Gamma \left[ \begin{array}{c} 2q + \alpha + \beta + 2, -\alpha - q \\ q + 1, \beta + 1 \end{array} \right] \times \\
\times \int_0^1 F \left[ \begin{array}{c} p + \beta + 1, -p - \alpha \\ \beta + 1 \end{array} ; x \right] x^{\beta + 1} F \left[ \begin{array}{c} q + \alpha + \beta + 1, -q \\ \beta + 1 \end{array} ; x \right] dx/ + (2.6) \\
+ \Gamma \left[ \begin{array}{c} 2p + \alpha + \beta + 2, q + 1 \\ \beta + 1, -\alpha - q \end{array} \right] \Gamma \left[ \begin{array}{c} \beta + 1, p + a + 1 \\ -p, 2p + \alpha + \beta + 2 \end{array} \right] \times \\
\times \int x^{\beta - p - 1} F \left[ \begin{array}{c} \beta + p + 1, p + 1, 1 \\ 2p + \alpha + \beta + 2 \end{array} ; x \right] x^{-\alpha} F \left[ \begin{array}{c} q + \alpha + \beta + 1, q + a + 1, 1 \\ 2q + \alpha + \beta + 2 \end{array} ; x \right] dx/ \times (2.7) \\
=
\frac{1}{2\pi i} \Gamma \left[ \begin{array}{c} 2p + \alpha + \beta + 2, q + 1 \\ \beta + 1, -\alpha - q \end{array} \right] \Gamma \left[ \begin{array}{c} \beta + 1, p + a + 1 \\ \beta + p + 1 \end{array} \right] \times \\
\times \int_{-\infty}^{+\infty} F \left[ \begin{array}{c} s, \beta + p + 1 - s \\ s + p + a + 1, \beta + 1 - s \end{array} ; 0 \right] \Gamma \left[ \begin{array}{c} s + a + q, \beta + 1 - s \\ s, \beta + 2 - s \end{array} \right] (2.8) \]

We must identify this identity with the following orthogonality identity for \( \Phi_p, \Phi_q \).
\[
\Gamma \left[ \begin{array}{c} 2p + \alpha + \beta + 2 \\ \beta + 1 \end{array} \right] \Gamma \left[ \begin{array}{c} 2q + \alpha + \beta + 2 \\ \beta + 1 \end{array} \right] \times \\
\times \int_0^1 F \left[ \begin{array}{c} p + \alpha + \beta + 1, -p \\ \beta + 1 \end{array} ; x \right] F \left[ \begin{array}{c} q + \alpha + \beta + 1, -q \\ \beta + 1 \end{array} ; x \right] x^{\beta - x} (1 - x)^\alpha dx/ \times (2.9) 
\]
\[ + \frac{\sin(\alpha + \theta)\pi}{\sin \theta \pi} \cdot \Gamma \left[ \frac{1 + p + \alpha, 1 + q + \alpha}{-p, -q} \right] \times \]

\[ \times \int_1^\infty F \left[ \frac{p + \alpha + \beta + 1}{2p + \alpha + \beta + 2} \right] F \left[ \frac{q + \alpha + \beta + 1}{2q + \alpha + \beta + 2} \right] \times \]

\[ \times x^{-2\alpha - \beta - \gamma - 3} (x - 1)^\alpha dx = \quad (2.10) \]

\[ = \frac{\delta_{\gamma - \eta, 0}}{2p + \alpha + \beta + 1} \Gamma \left[ \frac{2p + \alpha + \beta + 2, 2p + \alpha + \beta + 2, 1 + p + \alpha, p + 1}{p + \beta + 1, p + \alpha + \beta + 1} \right] \quad (2.11) \]

where \( \delta_{\gamma - \eta, 0} \) is the Kronecker symbol.

We identify (2.6)-(2.8) with (2.9)-(2.11) line-by-line.

1. The summand (2.6) equals the summand (2.9). We transform the first \( F \)-factor of the integrand by (1.5).

2. The summand (2.7) equals the summand (2.10). First, we transform the first \( F \)-factor of the integrand by (1.5). Secondly, we apply the reflection formula \( \Gamma(z)\Gamma(1 - z) = \pi / \sin(\pi z) \) to the gamma-product in (2.7).

\[ \frac{\Gamma(z + \alpha + q)\Gamma(1 + \alpha + p)\sin(1 + \alpha + q)}{\Gamma(-p)\Gamma(-q)\sin(1 + q)} \]

Next, we use \( q - \theta \in \mathbb{Z} \),

\[ \frac{\sin(1 + \alpha + q)\pi}{\sin(1 + q)\pi} = \frac{\sin(1 + \alpha + n + \theta)\pi}{\sin(1 + n + \theta)\pi} = \frac{(-1)^{n+1} \sin(\alpha + \theta)\pi}{(-1)^{n+1} \sin(\theta)\pi} \]

3. Right-hand sides, i.e., (2.8) and (2.11). We apply the Barnes-type integral

\[ \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma \left[ \frac{a + s, b - s}{c + s, d - s} \right] ds = \Gamma \left[ \frac{a + b, c + d - a - b - 1}{c + d - 1, c - a, d - b} \right] \]

see [PBM3], 2.2.1.3\(^4\) and obtain

\[ \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma \left[ \frac{\beta + 1 + p - s, a + q + s}{1 + p + \alpha + s, q + \beta + 2 - s} \right] ds = \]

\[ = \frac{1}{(\alpha + \beta + p + q + 1)\Gamma(p + q + 1)} \quad (2.12) \]

Since \( p \neq q \in \mathbb{Z} \), the latter expression is zero if \( p \neq q \).

2.4. Convergence of the integrals.

Lemma. Under our conditions (1.9), (1.11) the integral

\[ \int_0^1 \Phi_p(x)^2 x^\beta (1 - x)^\alpha dx + \frac{\sin(\alpha + \theta)\pi}{\sin(\theta)\pi} \int_1^\infty \Phi_p(x)^2 x^\beta (x - 1)^\alpha dx \quad (2.13) \]

\(^4\)This identity is a partial case of (2.4). We substitute \( x = 1 \) to (2.4) and apply the Gauss summation formula for \( F[a, b; c] \).
is absolutely convergent.

Since

$$|\Phi_{\beta}(x)\Phi_{\gamma}(x)| \leq \frac{1}{2}(|\Phi_{\beta}(x)|^2 + |\Phi_{\gamma}(x)|^2)$$

this lemma implies also the absolute convergence of

$$\int_0^1 \Phi_{\beta}(x)\Phi_{\gamma}(x)x^\beta(1-x)^\alpha \, dx + \frac{\sin(\alpha + \beta)\pi}{\sin(\beta\pi)} \int_0^\infty \Phi_{\beta}(x)\Phi_{\gamma}(x)x^\beta x(1-x)^\alpha \, dx \quad (2.14)$$

Proof. To follow the asymptotics, we use one of the Kummer relations (see [HTF1], 2.10(1)),

$$F\left[A, B; C; z\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F\left[A, B; C; 1 - z\right] + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{a-b} F\left[A, B; C; 1 - z\right] \quad (2.15)$$

The function $\Phi_{\beta}$ is continuous at $x = 0$, and hence the condition of the convergence of the integral is $\Re \beta > -1$.

The formula (2.15) gives the following asymptotics of $\Phi_{\beta}$ as $x \to 1 - 0$

$$C_1 + C_2 (1-x)^{-\alpha} \quad (2.16)$$

For $\Re \alpha > 0$ we have $\Phi_{\beta} \sim (1-x)^{-\alpha}$, and the condition of convergence of (2.13) is $\Re \alpha < 1$. For $\Re \alpha < 0$, the function $\Phi_{\beta}$ has a finite limit at $1 + 0$, and the condition of convergence (2.13) is $\Re \alpha > -1$.

Considering the right limit at $1$, we obtain the same restrictions for $\alpha$.

Obviously,

$$\Phi_{\beta}(x) \sim x^{-\alpha-\beta-p-1}, \quad x \to \infty$$

Thus the condition of the convergence is $\Re(2p + \alpha + \beta + 1) > 0$.

We also must avoid a pole in (1.12), and this gives $\alpha + p + 1 \neq 0$.

Denote $m = p - \alpha$, $n = q - \beta$.

Lemma. For fixed $m, n$, the integral (2.14) depends holomorphically on $\alpha, \beta, \theta$ in the allowed domain of parameters.

Proof. For each given point $(\alpha_0, \beta_0, \theta_0)$, the convergence of our integral is uniform in a small neighborhood of $(\alpha_0, \beta_0, \theta_0)$ (since our asymptotics are uniform). It remains to refer to the Morera Theorem (if each integral over closed contour is $0$, then the function is holomorphic).

2.5. Restrictions necessary for our calculation. First, we used the Mellin transform, and hence our functions $K_1, K_2$ must be locally integrable. The unique point of discontinuity is $x = 1$. We have

$$K_1(x) \sim A_1 + A_1^\pm (1-x)^{-\alpha}, \quad x \to 1 \pm 0;$$

$$K_2(x) \sim B_1 + B_1^\pm (x-1)^\alpha, \quad x \to 1 \pm 0;$$
This implies $\Re \alpha < 1$.

Second, we use the convolution theorem for the Mellin transform.

The Mellin transform (2.1) of $K_1$ absolutely converges in the strip

$$0 < \Re s < \beta + p + 1$$

The Mellin transform of $K_2$ absolutely converges in the strip

$$-\alpha - q < \Re \beta < 1$$

We can apply the convolution theorem (2.3) if the following conditions are satisfied

$$\begin{align*}
0 < \beta + p + 1 \\
0 < \beta + q + 1
\end{align*}$$

--- emptyness of strips

$$\begin{align*}
0 < \beta + 1 \\
0 < p + q + \alpha + \beta + 1
\end{align*}$$

--- emptyness of intersection of strips

This domain is nonempty, but it is smaller than the domain of convergence of (2.13). But the orthogonality identities (1.13), (1.14) have holomorphic left-hand sides and right-hand sides. Hence they are valid in the whole domain of the convergence.

2.6. Restrictions for \( \theta \). These restrictions (1.8) were not used in proof. In fact, \( \theta \) is defined up to a shift \( \theta \mapsto \theta + 1 \). This shift preserves the orthogonal system \( \Phi_2 \), but changes enumeration of the basic elements.

By this reason I'll explain how the functions \( X_1, X_2 \) were written.

2.7. Comments. The orthogonality relations for the Jacobi polynomials \( P_n^{\alpha, \beta} \) are well known but not self-evident. Let us try to prove them using the technique of Barnes integrals, see [Mar].

Our problem is reduced to an evaluation of the integral

$$\int_0^1 \binom{-n}{\beta + 1} x^\beta (1 - x)^\alpha dx \quad (2.17)$$

Denote

$$L_1(x) = (1 - x)^\alpha \binom{-m, m + \alpha + \beta}{\beta + 1} H(1 - x)$$

$$L_2(x) = x^{-\beta - 1} \binom{n, n + \alpha + \beta}{\beta + 1} \frac{1}{x} H(x - 1) + r(x)H(x - 1) \quad (2.18)$$

where \( r(x) \) is an arbitrary function.

Our integral (2.17) is the convolution \( L_1 \ast L_2(x) \) at the point \( x = 1 \). We intend to evaluate it using the Mellin transform.

The Mellin transform of \( L_1 \) is (see [PBMS], 8.4.49.1)

$$\mathcal{M} L_1(s) = \Gamma \left[ \frac{\beta + 1, \alpha + m + 1}{\beta + m + 1} \right] \Gamma \left[ \frac{s, \beta + m + 1 - s}{\alpha + m + 1 + s, \beta + 1 - s} \right]$$
Then we find a function of the form (2.18) in the table of inverse Mellin transforms (see [PBM3], 8.4.49.1). In fact, tables of integrals are not necessary here, since we must write a Barnes integral defining a given hypergeometric function on $[0,1]$, and it is more-or-less obvious how to do this.

We can assume

$\mathcal{M} L_2(s) = \Gamma \left[ 2n + \alpha + \beta + 2, -\alpha - n \right] \cdot \Gamma \left[ \frac{\alpha + n + s, \beta + 1 - s}{s, n + \beta + 2 - s} \right]$ 

and after this the desired calculation can be performed.

After this we change $m \rightarrow m + \theta$, $n \rightarrow n + \theta$.

2.8. Evaluation of summands in (2.9), (2.10). Let $p, q \in \mathbb{C}$. Let us evaluate

$$X := \int_0^1 \Phi_p(x) \Phi_q(x)x^\beta (1 - x)^\alpha dx, \quad Y := \int_1^\infty \Phi_p(x) \Phi_q(x)x^\beta (x - 1)^\alpha dx,$$

Denote

$$a(p, q) := \frac{1}{p + q + \alpha + \beta + 1} \Gamma \left[ 2p + \alpha + \beta + 2, 2q + \alpha + \beta + 2 \right]$$

$$b(p, q) := \Gamma \left[ \frac{q + 1, p + \alpha + 1}{p + \beta + 1, q + \alpha + \beta + 1} \right]$$

We write the equation (2.6)-(2.8) and the same equation with transposed $p$, $q$

$$X + \frac{\sin(\alpha + q)\pi}{\sin q\pi} Y = a(p, q)b(p, q)\frac{\sin(q - p)\pi}{(q - p)\pi}$$

$$X + \frac{\sin(\alpha + p)\pi}{\sin p\pi} Y = a(p, q)b(q, p)\frac{\sin(q - p)\pi}{(q - p)\pi}$$

It is a linear system of equations for $X$ and $Y$. Its determinant is

$$\frac{\sin(\alpha + p)\pi}{\sin p\pi} - \frac{\sin(\alpha + q)\pi}{\sin q\pi} = \frac{\sin\alpha\pi\sin(q - p)\pi}{\sin p\pi\sin q\pi}$$

Hence

$$Y = a(p, q) \frac{\sin p\pi\sin q\pi}{\pi(q - p)\sin\alpha\pi} \left[ b(q, p) - b(p, q) \right]$$

$$X = a(p, q) \frac{\sin p\pi\sin q\pi}{\pi(q - p)\sin\alpha\pi} \left[ \frac{\sin(\alpha + p)\pi}{\sin p\pi} b(p, q) - \frac{\sin(\alpha + q)\pi}{\sin q\pi} b(q, p) \right] =$$

$$= \frac{\pi a(p, q)}{(q - p)\sin\alpha\pi} \left[ \frac{1}{\Gamma\left[ p + \beta + 1, q + \alpha + \beta + 1, -q, -p - \alpha \right]} - \frac{1}{\Gamma\left[ q + \beta + 1, p + \alpha + \beta + 1, -p, -q - \alpha \right]} \right]$$

(2.19)
3. The boundary problem

In this section $\alpha \neq 0$. But complex values of $\alpha, \beta, \theta$ are admissible.

3.1. Symmetry of the boundary problem. We consider the hypergeometric differential operator $D$ given by (1.16) and the boundary problem for $D$ defined in Subsection 1.4. We intend to prove the identity

$$\{Df, g\} = \{f, Dg\}, \quad f, g \in \mathcal{E} \quad (3.1)$$

Let

$$H = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx}$$

be a differential operator on $[a, b]$ formally symmetric with respect to a weight $\mu(x)$, i.e., for smooth $f, g$ that vanish near the ends of the interval,

$$\int_a^b H f(x) \cdot g(x) \, dx = \int_a^b f(x) \cdot H g(x) \, dx$$

Equivalently, $(\mu f)' = \mu g$. Then for general $f, g$, we have

$$\int_a^b H f(x) \cdot g(x) \, dx - \int_a^b f(x) \cdot H g(x) \, dx =$$

$$= \left\{ \{f'(x)g(x) - g'(x)f(x)\} a(x)\mu(x) \right\} \bigg|_a^b \quad (3.2)$$

We apply this identity to the operator $D$ and to the segment $[a, b] = [0, 1 - \varepsilon]$.

Let on some segment $[1 - h, 1]$ we have

$$f(x) = u(x) + (1 - x)^{-\alpha} v(x), \quad g(x) = \bar{u}(x) + (1 - x)^{-\alpha} \bar{v}(x)$$

Then the correcting term (3.2) is

$$\text{det} \begin{pmatrix} u'(x) + (1 - x)^{-\alpha} v'(x) - \alpha (1 - x)^{-\alpha-1} v(x) & u(x) + (1 - x)^{-\alpha} v(x) \\ \bar{u}(x) + (1 - x)^{-\alpha} \bar{v}(x) - \alpha (1 - x)^{-\alpha-1} \bar{v}(x) & \bar{u}(x) + (1 - x)^{-\alpha} \bar{v}(x) \end{pmatrix} \times$$

$$\times x^{\beta+1} (1 - x)^{\alpha+1} \bigg|_{x=1-\varepsilon} \quad (3.3)$$

The last factor gives the power $\varepsilon^{\alpha+1}$; recall that $-1 < \alpha < 1$. The summands of the determinant have powers

$$1, \quad \varepsilon^{-\alpha}, \quad \varepsilon^{-2\alpha}, \quad \varepsilon^{-\alpha-1}, \quad \varepsilon^{-2\alpha-1}$$

But the term with $\varepsilon^{-2\alpha-1}$ in the determinant vanishes. Hence only the term with $\varepsilon^{-\alpha-1}$ gives a contribution to the limit as $\varepsilon \to +0$. Finally,

$$\lim_{\varepsilon \to +0} \int_0^{1-\varepsilon} \left( D f(x) g(x) - f(x) D g(x) \right) x^{\beta} (x-1)^{\alpha} \, dx = u(1) \bar{v}(1) - v(1) \bar{u}(1) \quad (3.4)$$
For $x > 1$, we have
\[
f(x) = \frac{\sin \theta \pi}{\sin(\alpha + \theta) \pi} u(x) + (1-x)^{-\alpha} v(x), \quad g(x) = \frac{\sin \theta \pi}{\sin(\alpha + \theta) \pi} \bar{u}(x) + (1-x)^{-\alpha} \bar{v}(x)
\]
In a similar way, we obtain
\[
\lim_{\varepsilon \to +0} \frac{\sin(\alpha + \theta) \pi}{\sin \theta \pi} \int_{1+x}^{\infty} (Df(x)g(x) - f(x)Dg(x)) x^\beta (x-1)^{\alpha} \, dx = -u(1)\bar{v}(1) + v(1)\bar{u}(1)
\]
This finishes the proof of the identity (3.1)

3.2. Boundary conditions for $\Phi_p$. Let us show that $\Phi_p(x)$ satisfy the boundary conditions at $x = 1$. It is given by a direct calculation, below we present its details.

We need in expressions for $\Phi_p$ having the form
\[
\Phi_p(x) = \begin{cases} A_1(p; x) + B_1(p; x)(1-x)^{-\alpha}, & x < 1 \\ A_2(p; x) + B_2(p; x)(1-x)^{-\alpha}, & x > 1 \end{cases}
\]
(3.5)

We intend to expand $\Phi_p$ in power series at $x = 1$, on the semi-segments $(1 - \varepsilon, 1], [1, 1 + \varepsilon]$.

We use the formula (2.15) for the left semi-segment and obtain
\[
\Phi_p(x) = \Gamma \left[ 2p + a + \beta + 2 \right] \left\{ \Gamma \left[ \beta + 1, -\alpha \right] F \left[ p, p + a + \beta + 1; 1 - x \right] + \right. \\
+ \Gamma \left[ -p, p + a + \beta + 1 \right] F \left[ p + \beta + 1, -p - \alpha; 1 - x \right] (1 - x)^{-\alpha} \right\}
\]
(3.6)

for $x < 1$

Next we use the identity
\[
F \left[ a, b, \frac{1}{c} ; x \right] = \Gamma \left[ c, c - a - b \right] \left[ \Gamma \left[ a + 1 - c \right] F \left[ a, a + 1 - c \right. \right. \\
+ \Gamma \left[ b, 1 - b \right] F \left[ c - b, 1 - b \right] (1 - x)^c (x - 1)^{a-b} \right]
\]
(3.7)

(this formula is a modified variant of [HTF1], 1.10(4). We obtain
\[
\Phi_p(x) = \Gamma \left[ p + a + 1, -p \right] \times \\
\left. \left\{ \Gamma \left[ 2p + a + \beta + 2, -\alpha \right] F \left[ p + a + \beta + 1, -p; 1 - x \right] + \right. \right.
\\
+ \Gamma \left[ 2p + a + \beta + 2, \alpha \right] F \left[ p + \beta + 1, -p - \alpha; 1 - x \right] (x - 1)^{-\alpha} \right\}
\]
(3.8)
for $x > 1$.

The expressions (3.6), (3.8) are the desired expansions (3.5). We observe, that

$$B_1(p, x) = B_2(p, x); \quad A_1(p, x)/A_2(p, x) = \frac{\sin(p + a)\pi}{\sin p\pi}$$

We have

$$\frac{\sin(p + a)\pi}{\sin p\pi} = \frac{\sin(\theta + a)\pi}{\sin \theta\pi} \quad (3.9)$$

and this implies our boundary condition.

Remark (it will be important below in Subsection 4.2). The property (3.9) is valid if $p - \theta \in \mathbb{Z}$. Indeed, the difference between the left-hand side and the right-hand side is

$$\frac{\sin \alpha \pi \sin(\theta - p)\pi}{\sin p\pi \sin \theta\pi}$$

3.3. Another proof of the orthogonality relations. For $p \neq q$, the orthogonality follows from the symmetry condition (3.1).

Let us evaluate

$$X := \int_0^1 \Phi_p(x)\Phi_q(x)x^\beta(x - 1)^\alpha$$

We preserve the notation (3.5). By formula (3.4),

$$\lim_{\varepsilon \to 0^+} \left\{ \int_0^{1-\varepsilon} D\Phi_p(x)\Phi_q(x)x^\beta(1-x)^\alpha dx - \int_0^{1-\varepsilon} \Phi_p(x)D\Phi_q(x)x^\beta(1-x)^\alpha dx \right\} =$$

$$= A(p, 1)B(q, 1) - A(q, 1)B(p, 1)$$

The constants $A(p, 1)$ etc. are the $\Gamma$-coefficients in (3.6) and (3.8); thus the right-hand side is known.

Since $\Phi_p$ are the eigenfunctions (see (1.17)), the left-hand side is

$$\left[ -p(p + \alpha + \beta + 1) + q(q + \alpha + \beta + 1) \right] \cdot X = (q - p)(q + p + \alpha + \beta + 1)X$$

After simple cancellations we obtain the expression (2.20).

In the same way, we obtain the expression (2.19) for $\int_1^\infty$.

4. The spectral measure

Now we intend to evaluate the spectral measure for the operator $D$ in the Hilbert space $\mathcal{H}$ using Weyl–Titchmarsh machinery, see [DS].

To avoid logarithmic asymptotics, we assume $\alpha \neq 0$, $\beta \neq 0$.

4.1. Eigenfunctions of the adjoint operator. Now we intend to discuss the adjoint operator $D^*$ for $D$.

Denote by $\text{Dom}(A)$ the domain of definiteness of a linear operator $A$. Recall that $f \in \mathcal{H}$ is contained in $\text{Dom}(D^*)$ if there exists a function $h \in \mathcal{H}$ such that for each $g \in \text{Dom}(D)$ we have

$$\langle f, Dg \rangle = \langle h, g \rangle$$
In this case, we claim \( h = D^* f \).

Since \( D \) is symmetric, we have

\[
\text{Dom}(D^*) \supset \text{Dom}(D) = \mathcal{E}
\]

Description of \( \text{Dom}(D^*) \) is not an important question, really it is necessary only description of eigenfunctions of \( D^* \).

**Lemma.** Let \( \Xi \) be an eigenfunction of \( D^* \). Then \( \Xi \) satisfies to the boundary conditions a), b) at \( x = 0 \) and \( x = 1 \) from 1.4.

**Proof.** The condition at 0. Let \( D^* \Xi = \lambda \Xi \), represent \( \lambda \) as

\[
\lambda = -p(p + a + \beta + 1)
\]

(4.1)

There are two solutions of the hypergeometric equation \( Df = \lambda f \) near \( x = 0 \); if \( \beta \) is not a non-negative integer, then they are given by

\[
S_1 = \Phi_p(x) = F \left[ -p, p + a + \beta + 1; \beta + 1; x \right]
\]

(4.2)

\[
S_2 = x^{-\beta}F \left[ -\beta, p, a + p + 1; 1 - \beta; x \right]
\]

(4.3)

If \( \beta \geq 1 \), the second solution is not in \( \mathcal{H} \), and the statement is obvious. \(^5\)

Let \( -1 < \beta < 1 \). Let \( f \in \mathcal{E} \), i.e. \( f \) is smooth near 0. Expand our eigenfunction \( \Xi \) as

\[
\Xi = u(x) + x^{-\beta}v(x), \quad u(x), v(x) \in C^\infty
\]

(4.4)

in fact, \( u(x) \) and \( v(x) \) are the hypergeometric functions defined from (4.2), (4.3) up to scalar factors. If \( \Xi \) is in \( \text{Dom}(D^*) \), then

\[
\langle Df, \Xi \rangle = \langle f, D^* \Xi \rangle = 0
\]

Repeating the calculation of Subsection 3.1, we obtain that this difference is

\[
f(0)v(0)
\]

Since \( f \) is arbitrary, then \( v(0) = 0 \). But \( v(x) = \text{const} \cdot F [ -\beta, p, a + p + 1; 1 - \beta; x ] \), we have const = 0.

The condition at \( x = 1 \). A proof is similar. A priori, we know that

\[
\Xi(x) = \begin{cases} 
  u_-(x) + v_-(x)(1-x)^{-a}, & x < 1 \\
  u_+(x) + v_+(x)(1-x)^{-a}, & x < 1 
\end{cases}
\]

In fact \( u_\pm \) and \( v_\pm \) are the hypergeometric functions in the right-hand sides of (3.6), (3.8) up to constant factors.

\(^5\)For integer \( \beta > 0 \), this also is valid.
Let \( f \in \mathcal{E} \), i.e.,
\[
f(x) = \begin{cases} 
a(x) + b(x)(1 - x)^{-\alpha}, & x < 1 \\
\frac{\sin \theta \pi}{\sin(\alpha + \theta)\pi}a(x) + b(x)(x - 1)^{\alpha}, & x > 1
\end{cases}
\]
here \( a(x), b(x) \) are smooth near \( x = 1 \).

If \( \Xi \in \text{Dom}(D^*) \), then the condition (4.4) is satisfied. Repeating the considerations of Subsection 3.1, we obtain that (4.4) is equal to
\[
(a(1)v_-(1) - b(1)u_-(1)) - \frac{\sin(\alpha + \theta)\pi}{\sin \theta \pi} \left( \frac{\sin \theta \pi}{\sin(\alpha + \theta)\pi}a(1)v_+(1) - b(1)u_+(1) \right)
\]
It is zero for all \( a(1), b(1) \) and hence
\[
v_-(1) = v_+(1), \quad u_+(1) = \frac{\sin \theta \pi}{\sin(\alpha + \theta)\pi}u_-(1)
\]
But a priori we know \( v_{\pm} \) and \( u_{\pm} \) up to constant factors, and hence, and this implies our statement.

4.2. \( L^2 \)-eigenfunctions of \( D^* \).

**Lemma.** If \( \Xi \in \mathcal{H} \) is an eigenfunction of \( D^* \), then \( \Xi = \Phi_{q} \) with \( q \in \theta + \mathbb{Z} \).

**Proof.** Let \( \lambda \) be an eigenvalue, let \( p \) is given by (4.1) with \( \text{Re} \, p > -(\alpha + \beta + 1)/2 \).

Due the boundary condition at 0, we have
\[
\Xi = \text{const} \, F[-p, p + \alpha + \beta + 1; \beta + 1; x]
\]
for \( x < 1 \).

Only one solution of the equation \( Df = \lambda f \) is contained in \( L^2 \) at infinity, it has the form
\[
\text{const} \cdot F[p + \alpha + 1, p + \alpha + \beta + 1; 2p + \alpha + \beta + 2; 1/x] x^{-\alpha - \beta - p - 1}
\]
on \([1, \infty)\). Thus, on the both segments the eigenfunction \( \Xi \) coincides with \( \Phi_{q} \) up to scalar factors. The gluing condition is (3.9). By the last remark of Subsection 3.2, \( p - \theta \in \mathbb{Z} \).

If
\[
\text{Re} \, p = -(\alpha + \beta + 1)/2
\]
then there is no \( L^2 \)-eigenfunctions at infinity. \( \square \)

4.3. Self-adjointness. By the previous lemma, the equations \( D^* f = \pm if \) have no solution in \( \mathcal{H} \). This implies the essential self-adjointness of \( D \).

4.4. Specter. The eigenvalues \( \lambda = -p(p + \alpha + \beta + 1) \) corresponding to the functions \( \Phi_{q} \) form a discrete specter. The remaining specter corresponds to the semi-line (4.6), i.e., \( \lambda \geq (\alpha + \beta + 1)^2/4 \).
Indeed, in all the other cases, we have precisely one $L^2$ solution $S_0(x)$ of the
differential equation $Df = \lambda f$ near 0, and precisely one $L^2$-solution $S_\infty(x)$
ear infinity. Hence we can write the Green kernel as it is explained in [DS].

4.5. **Almost $L^2$-eigenfunctions.** Let

$$p = -(\alpha + \beta + 1)/2 + is, \quad s \in \mathbb{R}$$

and $\lambda$ is given by (4.1).

**Lemma.** The function $\Psi_s$, given by (1.18) is a unique almost $L^2$-solution of
the equation $D\Xi = \lambda \Xi$.

**Proof.** Near $x = 0$ such solution must have the form (4.5).

We write the following basis $\Lambda(s, x)$, $\Lambda(\pm s, x)$ in the space of solutions
of the equation $Df = \lambda f$,

$$\Lambda(s, x) = F \left[ \frac{\alpha + \beta + 1}{2} + is, \frac{\alpha - \beta + 1}{2} + is, \frac{1}{x} \right] x^{-(\alpha + \beta + 1)/2} = is$$

(4.7)

The both solutions are almost $L^2$. Now we must satisfy the boundary conditions at $x = 1$. For this, we expand the 3 solutions (4.5) and $\Lambda(\pm s; x)$ near the
point $x = 1$. It remains to write the gluing conditions at $x = 1$. The calculation
is long, its reduced to usage of the complement formula for $\Gamma$ and elementary
trigonometry. We omit this.

The formula for the spectral measure follows from the explicit asymptotics
of almost $L^2$-solutions at $\infty$; this is explained in [DS].

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Math.Physics group, Institute of Theoretical and Experimental Physics, B.Cheremushkinskaya, 25, Moscow 117 259, Russia
neretin@mccme.ru
& Math.Dept, University of Vienna, Nordbergstrasse, 15, Vienna