Abstract Wilson Systems. Part I: Theory

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ABSTRACT. The Wilson orthonormal basis was constructed in 1991 by I. Daubechies, S. Jaffard and J.-L. Journé from elements of Gabor tight frame with redundancy 2. Having fixed a Hilbert space $\mathcal{H}$, discrete group $G$ and unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$, the vector $f \in \mathcal{H}$ is the tight frame atom if there exists $C > 0$ such that

$$\sum_{g \in G} |(h, \pi(g)f)|^2 = C\|h\|^2_{G}.$$ 

For the vector $f \in \mathcal{H}$ and an automorphism $\tau : G \to G$ of prime order $M$ we introduce a definition of the *algebraically summed system* being

$$\left( \sum_{i=0}^{M} \pi \circ \tau^i(g)f \right)_{g \in G}.$$

The frame operator of such a system is just a multiple by $M^2$ of the frame operator of suitably defined Wilson system. On the other hand, the frame operator of the *algebraically summed system* is, under some conditions, a multiple by $M$ of the frame operator of the system $(\pi(g)f)_{g \in G}$. The conditions differ and in case of $\pi, \tau$ equivalent, they provide a characterization for the atoms for which Wilson systems are tight frames, while in the case $\pi \circ \tau^i$ and $\pi \circ \tau$ mutually strongly disjoint they always hold.

The Wilson system definition we introduce coincides with the classical one for the group $G_2$ defined as

$$G_2 = \{(a, b, a \# b = ab, a^2 = 1, ab = ba = b\})$$

for the representation $\pi : G_2 \to L^2(\mathbb{R})$

$$\pi(a) = M_{a, 1}, \pi(b) = T_1, \pi(1) = Id_{\mathcal{H}},$$

where $M_{a, b}, T_q$ are unitary operators of modulation and translation defined as

$$M_{a, b}f(x) = e^{2\pi i a x}f(x), \quad T_qf(x) = f(x - q).$$

The corresponding automorphism is $\tau : G_2 \to G_2$ defined on the generators as

$$\tau(a) = a, \quad \tau(b) = e^{i\pi b}, \quad \tau(1) = 1.$$

The typical atom of the Wilson basis in the generalized definition is

$$\tau^{-1/2} \left( \sum_{i=0}^{M} \pi \circ \tau^i(g)f \right),$$

but one needs to make appropriate modifications for these $g \in G$, where $\pi \circ \tau(g) \in \mathbb{C} : \pi(g)$.

This generalization of Wilson systems 1) leads us to modified formulae for Wilson orthonormal basis in case of redundancy 2. 2) enables us to characterize in geometric language all tight frame atoms whose Wilson systems are tight frames (comp. P. Auscher’s analytical characterization). 3) gives both the characterizations and formulae for the systems that are analogues of Wilson systems for different automorphisms and/or different groups.

1. Introduction


**Theorem 1.1.** Let $f \in L^2(\mathbb{R}), \|f\| = 1$, and $f$ be real-valued. Let $Mh(x) = M_{1, 2}h(x) = e^{2\pi i x}h(x), T_{1, 2}h(x) = T_{1, 2}h(x) = h(x - 1)$ be the modulation and translation operators (frequency- and time-shifts, resp.).
If \((M^m T^n f)_{m,n \in \mathbb{Z}}\) is a tight frame in \(L^2(\mathbb{R})\), then the system composed of

- \((M^{2m} f)_{m \in \mathbb{Z}}\) and
- \(\left[2^{-1/2} (M^m T^n f + (-1)^{m+n} M^m T^{-n} f)\right]_{n \geq 1, m \in \mathbb{Z}}\)

is an orthonormal basis in \(L^2(\mathbb{R})\).

1.2. Characterization of Wilson basis atom. P. Auscher in 1994 gave characterization of the atoms for which the Wilson system is an orthonormal basis [1]. If we assume that a vector already is a tight frame atom, his characterization reads as follows:

**Theorem 1.2.** If \((M^m T^n f)_{m,n \in \mathbb{Z}}\) is a tight frame in \(L^2(\mathbb{R})\), where \(M h(x) = e^{i \pi x} h(x), T h(x) = h(x - 1)\) are modulation and translation operators (frequency- and time-shifts resp.) and \(\|f\| = 1\), then the both systems defined above are orthonormal bases if and only if

\[
E_k(x) = \sum_{n \in \mathbb{Z}} (-1)^n \overline{f(x - k - n/2 - 1/2)} f(-x - n/2) = 0
\]

for almost all \(x \in [0, 1/2]\).

P. Auscher also indicated the additional ‘complementary’ Wilson basis.

**Theorem 1.3.** Under the same assumptions as in Theorem 1.1 also the system composed of

- \((M^{2m+1} f)_{m \in \mathbb{Z}}\) and
- \(\left[2^{-1/2} (M^m T^n f - (-1)^{m+n} M^m T^{-n} f)\right]_{n \geq 1, m \in \mathbb{Z}}\)

is an orthonormal basis in \(L^2(\mathbb{R})\).

1.3. Unconditionality in Coorbit and Bargmann spaces. In 1992 H.G. Feichtinger, K. Gröchenig, and D. Walnut in joint works \([6], [8]\) showed, in particular, that Wilson basis is unconditional in the coorbit spaces \(Co(L^p)\) or, equivalently, its image under Bargmann transform is unconditional in Fock-Bargmann spaces.

1.4. Wilson Systems for Higher Redundancies. In 1996-1997 H. Bölcskei, K. Gröchenig, F. Hlawatsch, and H.G. Feichtinger proved that the following analogue of the Wilson system for even redundancy \(2N\) for \(N \in \mathbb{Z}\) is the tight frame with the frame bound reduced by \(2\) \([3], [4]\).

**Theorem 1.4.** If \(\|f\| = 1\) and \((M^m T^n f)_{m,n \in \mathbb{Z}}\) is a tight frame with the bound \(A\) in \(L^2(\mathbb{R})\), where

\[
M h(x) = e^{i \pi x/N} h(x), \quad T h(x) = h(x - 1),
\]

then the system composed of

- \((M^{2m} f)_{m \in \mathbb{Z}}\) and
- \(\left[2^{-1/2} (M^m T^n f + (-1)^{m+n} M^m T^{-n} f)\right]_{n \geq 1, m \in \mathbb{Z}}\)

is a tight frame with the bound \(A/2\) in \(L^2(\mathbb{R})\) if and only if

\[
\forall m \in \mathbb{Z} \quad D^{(g,g)}_m(t) := \sum_{l \in \mathbb{Z}} (-1)^l g(t - l) g(-t - l + \frac{m - 1/2}{2N}) = 0 \quad a.e.
\]
1.5. **K. Gröchenig’s Question.** The inspiring and challenging question was posed by K. Gröchenig whether there exists or does not exist a Wilson basis (or its analogue) for the case of redundancy 3 ([7], pp. 167-174).

Let $f \in L^2({\mathbb R})$, $\|f\| = 1$. Let

$$M' h(x) = M_{1/3} h(x) = e^{2\pi i x /3} h(x), \quad Th(x) = T_1 h(x) = h(x - 1).$$

This question can be actually seen twofolds:

1. Which formula should be used to derive an orthonormal basis from a tight frame $(M^{m,n} f)_{m,n \in \mathbb{Z}}$ in $L^2({\mathbb R})$?
2. What additional conditions the atom $f$ has to satisfy so that the system described in 1. can be an orthonormal basis?

1.6. **Recent results.** In [2] K. Bittner demonstrated that polynomials are reproduced by Wilson bases and described the rate of linear approximation in these bases. G. Kutyniok and Th. Strohmer generalized the notion of Wilson system to the case of symplectic lattices [13].

1.7. **P. Auscher’s approach.** Since the proof by P. Auscher seemed to the author as being only on the algebraic structure of the problem, the attempt was made to transfer it to this setting. In fact, the approach here mimics his and differs from his only by giving the abstract algebraical flavour at some points.

1.8. **Organization.** The paper is a first issue in the series intended to give:

- theoretical understanding of Wilson basis/frames phenomenon in sufficiently large generality,
- application of the described approach to the case of non-classical sign sequences for redundancy group $G_2$, groups corresponding to different overamplings and different groups including finite permutation groups and finite discrete Heisenberg groups,
- extensions of the approach to automorphisms of non-prime order, general definition of complementary Wilson systems etc.

The section 2 describes necessary assumptions, while in Section 3 there is a sketch motivating this paper’s approach to the problems related to Wilson bases. Then next sections give: the overview of the results (Sec. 4), the definition of Wilson system with some justifying comments (Sec. 5), the proofs of the paper’s main results (Sec. 6) and discussion of the assumptions used (Sec. 7).

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2. Assumptions

Having fixed a Hilbert space $\mathcal{H}$, discrete group $G$ and unitary representation $\pi : G \to U(\mathcal{H})$, the vector $f \in \mathcal{H}$ is the tight frame atom if and only if there exists $C > 0$ such that

$$\sum_{g \in G} |\langle h, \pi(g)f \rangle|^2 = C\|h\|^2_{\mathcal{H}}.$$ 

The frame operator of the system $(\pi(g)f)_{g \in G}$ is defined by

$$S_h = \sum_{g \in G} \langle h, \pi(g)f \rangle \pi(g)f.$$ 

Let $G$ be a discrete countable group and $Z(G)$ be the center of $G$. In the paper we shall consider only such groups $G$ that the quotient group $G/Z(G)$ is finite.

For the vector $f \in \mathcal{H}$ and an automorphism $\tau : G \to G$ of prime order $t$ we introduce a definition of algebraically summed system being

$$\left(\sum_{i=0}^{t-1} \pi \circ \tau^i(f)\right)_{g \in G}.$$ 

The frame operator of such a system is just a multiple by $t^2$ of the frame operator of suitably defined Wilson system. On the other hand, the frame operator of algebraically summed system is, under some conditions, a multiple by $t$ of the frame operator of the system $(\pi(g)f)_{g \in G}$. The conditions differ and in case of $\pi$ and $\pi \circ \tau$ equivalent, they provide a characterization for the atoms for which Wilson systems are tight frames, while in the case $\pi \circ \tau^i$ and $\pi \circ \tau^j$ mutually strongly disjoint they always hold.

Let $\mathcal{H} = L^2(X, dx, \mathbb{C})$, where $X \subset \hat{Z}(G)$ and $dx$ is a finite measure on $\hat{Z}(G)$. In particular, it can be singular or even atomic. Let $\pi$ be defined as a measurable function from $X \times G$ into $U(\mathcal{H})$ such that $\pi(x, \cdot) : G \to U(\mathcal{H})$ are representations of $G$. We need also the consistency assumption: For all $x \in X$ the representation $\pi(x, \cdot)$ restricted to $Z(G)$ coincides with the character $x \in X \subset \hat{Z}(G)$, i.e.

$$\forall_{x \in X \subset \hat{Z}(G)} \forall_{z \in Z(G)} \pi(x, z) = x(z)\text{Id}_{\mathbb{C}^r}.$$ 

For consistency of the automorphism $\tau$ with the representation, we assume

$$\forall_{x \in X} \forall_{g \in G} \pi(x, g) \in \mathbb{C}\text{Id}_{\mathbb{C}^r} \Rightarrow \pi(x, \tau g) \in \mathbb{C}\text{Id}_{\mathbb{C}^r}.$$ 

To reflect the situation we had with the integer oversampling for the classical Wilson system, we will assume that all $\pi_x$ are irreducible.

Let us recall that the representations $\pi$ and $\rho$ of the same group $G$ in the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, are equivalent if there exists such a unitary operator $J : \mathcal{H}_1 \to \mathcal{H}_2$ that $J \pi(g)J^{-1} = \rho(g)$ for all $g \in G$. For equivalent representations we write $\pi \approx \rho$. On the other hand, two representations $\pi$ and $\rho$ are strongly disjoint if for any operator $J : \mathcal{H}_1 \to \mathcal{H}_2$

$$[\forall_{g \in G} J \pi(g) = \rho(g)J ] \implies J = 0.$$ 

We denote the strong disjointness of the representations by $\pi \perp \rho$. 
3. Motivation

3.1. The role of automorphism. Let us take a look at the formula for 'typical' Wilson system element in the classical case ($n \geq 1, m \in \mathbb{Z}$):

\[ u_{mn} = 2^{-1/2} \left( M^m T^n f + (-1)^{m+n} M^m T^{-n} f \right). \]

If we will have a bijection $F$ mapping $M$ to $-M$ and $T$ to $-T^{-1}$, then (assuming preservation of operator composition) the operator $M^m T^n$ will be mapped to

\[ F(M^m T^n) = F(M)^m F(T^n) = F(M)^m F(T)^n = (-M)^m (-T^{-1})^n = (-1)^{m+n} M^m T^{-n}, \]

which is exactly the operator part of the second member of the above formula. Now a bijection preserving the operator composition can be modeled as a homomorphism or, better, isomorphism of the $C^*$-algebra generated by the operators $M$ and $T$. In fact, we may go a bit further, and introduce the abstract group with the multiplication law corresponding to the composition laws in the mentioned algebra of operators. Such a group in this case will be $G_2$ with the following generators/relations definition:

\[ G_2 = \langle a, b, \varepsilon | ab = \varepsilon ba, \varepsilon^2 = \varepsilon, \varepsilon a = \varepsilon a, b \varepsilon = \varepsilon b \rangle. \]

The group $G_2$ can be embedded as a subgroup into the Heisenberg group. One of the possibility of such embedding $i : G_2 \to H$ is

\[ i(a) = (1/2, 0, 0), \quad i(b) = (0, 1, 0), \quad i(\varepsilon) = (0, 0, 1/2). \]

However, this embedding is not unique and can be replaced by any of

\[ i(a) = (a/2, 0, 0), \quad i(b) = (0, a^{-1}, 0), \quad i(\varepsilon) = (0, 0, 1/2), \]

for $a \in \mathbb{R} \setminus \{0\}$.

The element $a$ is to be thought of as a counterpart of $M$, $b$ as a counterpart of $T$, and $\varepsilon$ of $-\mathrm{Id}_H$. We can formalize it, introducing the representation $\pi : G_2 \to L^2(\mathbb{R})$ defined as

\[ \pi(a) = M_{1/2}, \quad \pi(b) = T_1, \quad \pi(\varepsilon) = -\mathrm{Id}_H. \]

Note that we can define a mapping $\pi' : G_2 \to L^2(\mathbb{R})$ as

\[ \pi'(a) = M_{a/2}, \quad \pi'(b) = T_{a^{-1}}, \quad \pi'(\varepsilon) = -\mathrm{Id}_H, \]

and it will be again a representation of $G_2$. So $G_2$ is not just one subgroup of Heisenberg group, but it is isomorphic to a quite a bunch of them. It can be thought as an oversampling ratio, but this will be cleared up later.

Now the discussed algebra isomorphism can be modeled as the automorphism $\tau$ of the group $G_2$. Indeed, let us define $\tau$ to act as

\[ \tau(a) = \varepsilon a, \quad \tau(b) = \varepsilon b^{-1}, \quad \tau(\varepsilon) = \varepsilon. \]

Then it is easy to verify that under this automorphism a first member of Wilson system element definition is given by $\pi(g) f$, where $g = a^m b^n$, and the second one is given by $\pi(\tau g) f$.

Hence the idea comes of considering the systems with 'typical' element given by

\[ 2^{-1/2} \left( \pi(g) f + \pi(\tau g) f \right) \]

or, more generally,

\[ t^{-1/2} \left( \sum_{i=0}^{n-1} \pi(\tau^i g) f \right), \]
where \( t \) is the order of automorphism \( \tau \). (Note that \( \tau \) introduced above is of order 2.)

There are also some 'untypical' elements of the Wilson system, but these correspond to 'fixed points' of \( \tau \) in the sense that \( \pi(\tau g) \in \mathbb{C} \cdot \pi(g) \). The appropriate modifications can be made and we will explain them in the sequel.

3.2. How to attack classical case with this machinery? In the classical case with \( \mathcal{H} = L^2(\mathbb{R}) \), with the group \( G_2 \), and with the representation \( \pi \) defined above, the set \( X = [0, 1/2) \times [-1/2, 1/2) \), the dimension \( r = 2 \), the transformation from \( \mathcal{H} = L^2(\mathbb{R}) \rightarrow L^2(X, dx, \mathbb{C}^r) \) is furnished by the mapping \( \Omega : L^2(\mathbb{R}) \rightarrow L^2(X, dx, \mathbb{C}^r) \) defined for given \( f \in L^2(\mathbb{R}) \) as

\[
\Omega f(t, \omega) = \begin{bmatrix}
Zf(t, \omega) \\
Zf(t + \frac{1}{2}, \omega)
\end{bmatrix}.
\]

Note that the mapping \( \Omega \) is the Piecewise Zak Transform introduced by Z. Zibulski and Y. Zeevi in [17], and \( Z : L^2(\mathbb{R}) \rightarrow L^2([0, 1]^2) \) is defined by

\[
Zf(t, \omega) = 2^{-1/2} \sum_{n \in \mathbb{Z}} f(2(t - n)) e^{2\pi i n \omega}.
\]

For \( x = (t, \omega) \in X = [0, 1/2) \times [-1/2, 1/2) \) the representations \( \pi_x \) are given by

\[
\begin{align*}
\pi_x((t, \omega), a) &= \begin{bmatrix}
e^{2\pi i t} & 0 \\
0 & -e^{2\pi i t}
\end{bmatrix}, \\
\pi_x((t, \omega), b) &= \begin{bmatrix}
0 & e^{-2\pi i \omega} \\
1 & 0
\end{bmatrix}, \\
\pi_x((t, \omega), c) &= \begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix},
\end{align*}
\]

One can verify that the representations defined this way are irreducible for all \( x = (t, \omega) \in X \).

3.3. Decomposability of frame operator. In this context the decomposability of frame operator \( S \) of the system \((M^n T^n f)_{m,n \in \mathbb{Z}} \) means that

\[
S(t, \omega) = \mathcal{M}[Zf(t, \omega)]^2 + |Zf(t + \frac{1}{2}, \omega)|^2.
\]

In the case of rational oversampling, it means that \( S \) is equivalent to the operator of multiplication by the matrix function, where the matrix has the size \( p \times p \) (cf. [17]).

3.4. Applications. When set in work, such an approach brings the answers to questions like

- Is it possible to have a Wilson system with the 'typical' formula as
  \[
  u_{mn} = 2^{-1/2} (M^m T^n f + (-1)^m M^m T^{-n} f),
  \]
  \[
  u_{mn} = 2^{-1/2} (M^m T^n f + (-1)^n M^m T^{-n} f),
  \]
  \[
  u_{mn} = 2^{-1/2} (M^m T^n f + M^m T^{-n} f),
  \]

- How to build up the formula for case of redundancy 3 (K. Gröchenig’s Conjecture)?
- Is it possible to have some formula with 4 members and a tight frame with redundancy 4/4=1?
4. Outline of the results

Let \((\pi(g)f)_{g \in G}\) be a Bessel system, i.e. has an upper frame bound, in \(\mathcal{H}\). Let us introduce the notation \(S_{(\pi^*)_{x \in \mathbb{D}}}\) for the frame operator of the system \((x_n)_{n \in \mathbb{D}}\).

Claim 1. The frame operator of of the algebraically summed system

\[
\left( \sum_{i=0}^{t-1} \pi(t^i g)f \right)_{g \in G}
\]
equals to the frame operator of the “suitably defined” Wilson system multiplied by \(t^2\). In other words,

\[
S_{(\sum_{i=0}^{t-1} \pi(t^i g)f)_{x \in \mathbb{D}}} = t^2 \cdot S_{W(\pi, G, \tau, J)}.
\]

Claim 2. If \(\pi \circ \tau^i\) and \(\pi \circ \tau^j\) are mutually strongly disjoint for \(i \neq j\) and \(i, j = 0, \ldots, t - 1\), then the frame operator of the algebraically summed system

\[
\left( \sum_{i=0}^{t-1} \pi(t^i g)f \right)_{g \in G}
\]
is a \(t\)-multiplication of the frame operator of \((\pi(g)f)_{g \in G}\).

\[
t \cdot S_{\pi(g)f \mid x \in \mathbb{D}} = S_{(\sum_{i=0}^{t-1} \pi(t^i g)f)_{x \in \mathbb{D}}}.
\]

Claim 3. If \(\pi\) and \(\pi \circ \tau\) are equivalent, \(J : \mathcal{H} \rightarrow \mathcal{H}\) is the equivalence operator \(\forall g \in G : J \pi(g)J^{-1} = \tau(\pi(g))\), \(\theta\) is the automorphism of \(\widetilde{Z(G)}\) dual to \(\tau_{Z(G)}\), then the frame operator of the algebraically summed system

\[
\left( \sum_{i=0}^{t-1} \pi(t^i g)f \right)_{g \in G}
\]
is a \(t\)-multiplication of the frame operator of \((\pi(g)f)_{g \in G}\)

\[
t \cdot S_{\pi(g)f \mid x \in \mathbb{D}} = S_{(\sum_{i=0}^{t-1} \pi(t^i g)f)_{x \in \mathbb{D}}}.
\]

if and only if

\[
\langle f(x), [J^k f](x) \rangle_{cr} = 0
\]

for \(dx\)-almost all \(x \in X\).

If the measure \(dx\) of the set of fixed points \(6x = x\) is non-zero, the atom \(f \in \mathcal{H}\) should satisfy

\[
|\langle f(x), u_i(x) \rangle_{cr}|^2 = \lambda
\]

for \(dx\) almost all \(x \in X\) such that \(6x = x\), where the vectors \(u_i(x)\) are normed eigenvectors of operator \(J'(x)\) such that \([J f](x) = J'(x) f(x)\), if all eigenvalues of \(J'(x)\) are different. If they are not, no such atom exists.

Claim 4. The operator \(R : \mathcal{H} \rightarrow \mathcal{H}\) defined as

\[
\langle Rh_1, h_2 \rangle = \sum_{g \in G} \langle h_1, \pi(g)f \rangle \langle \rho(g)f, h_2 \rangle
\]
establishes the equivalence between the representations \(\pi\) and \(\rho\), i.e.

\[
\forall g \in G \quad R \pi(g) = \rho(g) R.
\]
In particular, if \( \rho(g) = \pi(\tau g) \),
\[
\forall g \in G \quad R \pi(g) = \pi(\tau g) R.
\]

**Example 4.1.** Let \( \pi : G \to \mathcal{H} \) be defined as
\[
\pi(a) = M_{1/2}, \quad \pi(b) = T, \quad \pi(\varepsilon) = -\text{Id}_{\mathcal{H}}.
\]
Since \( L^2(\mathbb{R}) \) is identified with \( L^2(X, dx, \mathbb{C}^2) \), it doesn’t matter whether we work in one space or another. Let us use the automorphism \( \tau \)
\[
\tau(a) = \varepsilon a, \quad \tau(b) = \varepsilon b^{-1}, \quad \tau(\varepsilon) = \varepsilon.
\]
Let now \( \pi' = \pi \) and \( \rho' = \pi \circ \tau \) and \( R \) be defined as in the above claim for \( \pi' \) and \( \rho' \). Then from the definition
\[
R h = \sum_{g \in G} \langle h, \pi'(g)f \rangle \rho'(g)f = \sum_{m, n \in \mathbb{Z}} \langle h, M^m T^n f \rangle (-1)^{m+n} M^m T^{-n} f.
\]
On the other hand, since \( R \) establishes the equivalence, one obtains
\[
\forall m, n \in \mathbb{Z} \quad R M^m T^n = (-1)^{m+n} M^m T^{-n} R.
\]

**Proof.**
\[
\langle R \pi(g') h_1, h_2 \rangle = \sum_{g \in G} \langle \pi(g') h_1, \pi(g)f \rangle \langle \rho(g)f, h_2 \rangle = \sum_{g \in G} \langle h_1, \pi(g^{-1}) f \rangle \langle \rho(g)f, h_2 \rangle = \sum_{g \in G} \langle h_1, \pi(g)f \rangle \langle \rho(g') f, \rho(g^{-1}) h_2 \rangle = \langle R h_1, \rho(g^{-1}) h_2 \rangle = \langle \rho(g') R h_1, h_2 \rangle.
\]
Hence,
\[
\forall g \in G \quad R \pi(g) = \rho(g) R.
\]

\( \square \)

5. Suitable definition of Wilson system

To give the “suitable” definition of the Wilson system, we start with the definition of index set.

**Definition 5.0.1.** Given \( G, \pi, \tau, f \) and \( \mathcal{H} = L^2(X, dx, \mathbb{C}^2) \), define the set \( K_1 \subset G \) of elements of the group \( G \) such that \( \pi(\tau g) \notin \mathbb{C} \pi(g) \) containing a single representative of the \( t \)-length orbit. Then define the set \( K_1 \subset G \) consisting of these elements with \( 1 \)-element orbit (for which \( \pi(\tau g) = \pi(g) \)). Then define \( K = K_1 \cup K_1 \).

Now we are ready to define the Wilson system.

**Definition 5.0.2.** Given \( G, \pi, \tau, f \) and \( \mathcal{H} = L^2(X, dx, \mathbb{C}^2) \), the Wilson system \( \mathcal{W}(\pi, G, \tau, f) \) is a mapping \( \mathcal{W} \) from the set \( K \subset G \) into \( \mathcal{H} \) such that
- if \( k \in K_1 \),
  \[
  W(k) = t^{-1/2} \left( \sum_{i=0}^{t-1} \pi(\tau^i g) f \right).
  \]
- if \( k \in K_1 \),
  \[
  W(k) = t^{-1} \left( \sum_{i=0}^{t-1} \pi(\tau^i g) f \right) = \pi(g) f.
  \]
Observe that in accordance with motivation the typical entry of the Wilson system defined this way is defined as mentioned earlier for a representative of each orbit under $\tau$-action. So if $\pi(\tau g) \notin S^1 \cdot \pi(g)$, the vector
\[
u(g) = t^{-1/2} \left( \sum_{i=0}^{t-1} \pi(\tau^i g)f \right),
\]
is the same for different elements of the same orbit $u(g) = u(\tau^i g)$ for $i = 0, \ldots, t-1$. We will pick only one of them to Wilson system.

For “untypical entries” there are two possibilities. Either $\pi(\tau g) = \pi(g)$, or $\pi(\tau g) \in (S^1 - 1) \cdot \pi(g)$. In case of $\pi(\tau g) = \pi(g)$, we just set $u(g) = t^{-1} \left( \sum_{i=0}^{t-1} \pi(\tau^i g)f \right) = \pi(g)f$, which is similar to $n = 0$ and $m$ even in the classical case.

Since the order $t$ of $\tau$ is prime, one is able to show that in case of $\pi(\tau g) \in (S^1 - 1) \cdot \pi(g)$, the corresponding $\left( \sum_{i=0}^{t-1} \pi(\tau^i g)f \right)$ will be a null vector. Indeed, if $\pi(\tau g) = \lambda \pi(g)$, $\pi(\tau^i g) = \lambda^i \pi(g)$, and
\[
\left( \sum_{i=0}^{t-1} \pi(\tau^i g)f \right) = \sum_{i=0}^{t-1} \lambda^i \pi(g) = 0.
\]

NOTE: For the above argument we need an assumption that $\tau$ preserves the group of these elements $g$ of $G$ for which $\pi(g) \in \mathbb{C} \cdot \text{Id}_H$. In the cases important for applications it holds.

6. Proofs


Proof of Claim 1. Note that we can define a frame operator even for one-element set. If $k \in K_1$, define $s(k) = \left( \sum_{i=0}^{t-1} \pi(\tau^i k)f \right)$. Also $\tau k, \tau^2 k, \ldots, \tau^{t-1} k \in K_t$. Observe that $s(k) = s(\tau k) = s(\tau^2 k) = s(\tau^3 k) = \cdots = s(\tau^{t-1} k)$. The Wilson system element $W(k) = t^{-1/2} \left( \sum_{i=0}^{t-1} \pi(\tau^i k)f \right) = t^{-1/2} s(k)$ has the frame operator equal to
\[
\langle h, W(k) \rangle W(k) = \left( \langle h, t^{-1/2} s(k) \rangle \right) t^{-1/2} s(k) = t^{-1} \langle h, s(k) \rangle s(k).
\]

Now let us sum up over $k \in \mathcal{O}_{K_t}(\tau)$
\[
\sum_{k \in \mathcal{O}_{K_t}(\tau)} \langle h, W(k) \rangle W(k) = \sum_{k \in \mathcal{O}_{K_t}(\tau)} t^{-1} \langle h, s(k) \rangle s(k) = \sum_{k \in K_t} t^{-2} \langle h, s(k) \rangle s(k).
\]

If we add to it the appropriate frame operators for 1-element orbits, we’ll be done. For the 1-element orbits things look easier. Namely, for $k \in K_1$ the Wilson system element $W(k) = t^{-1} \left( \sum_{i=0}^{t-1} \pi(\tau^i k)f \right) = t^{-1} s(k)$ has the frame operator equal to
\[
\langle f, W(k) \rangle W(k) = \left( \langle f, t^{-1} s(k) \rangle \right) t^{-1} s(k) = t^{-2} \langle f, s(k) \rangle s(k) = t^{-2} \langle f, s(k) \rangle s(k).
\]

\[\square\]
6.2. Representation theory’s proofs.

**Lemma 6.1.** If \( \pi \) is an irreducible representation of a finite group \( G \) in \( \mathcal{H} \), the frame operator of \( (\pi(g)f)_{g \in G} \) defined by

\[
\langle Sk_1, k_2 \rangle = \sum_{g \in G} \langle k_1, \pi(g)f \rangle \langle \pi(g)f, k_2 \rangle
\]

equals

\[
S = \frac{\#G}{\dim \mathcal{H}} \langle f, f \rangle \text{Id}_{\mathcal{H}}.
\]

**Proof.** The frame operator \( S \) commutes with \( \pi(g) \) for all \( g \in G \). From Schur’s lemma \( S = \lambda \text{Id}_{\mathcal{H}} \). Now, the trace of \( S \) being the nuclear operator is equal to

\[
\text{tr} S = \sum_{g \in G} \langle \pi(g)f, \pi(g)f \rangle = \#G \langle f, f \rangle.
\]

On the other hand, trace of the operator \( \lambda \text{Id}_{\mathcal{H}} \) equals to

\[
\text{tr}(\lambda \text{Id}_{\mathcal{H}}) = \lambda \dim \mathcal{H}.
\]

Comparing traces of both sides yields the desired formula. \( \square \)

**Lemma 6.2.** If \( \pi \) is an irreducible representation of \( G \) in \( \mathcal{H} \), where \( G/Z(G) \) is finite, let \( \{g_i\}_{i=1}^{\#G/Z(G)} \) be the representatives of the cosets of \( G \) with respect of \( Z(G) \). The frame operator of \( (\pi(g_i)f)_{i=1}^{\#G/Z(G)} \) defined by

\[
\langle Sk_1, k_2 \rangle = \sum_{i=1}^{\#G/Z(G)} \langle k_1, \pi(g_i)f \rangle \langle \pi(g_i)f, k_2 \rangle
\]

equals

\[
S = \frac{\#G/Z(G)}{\dim \mathcal{H}} \langle f, f \rangle \text{Id}_{\mathcal{H}}.
\]

**Proof.** If the frame operator \( S \) again commutes with \( \pi(g) \) for all \( g \in G \), from Schur’s lemma \( S = \lambda \text{Id}_{\mathcal{H}} \) and we conclude as in Lemma 6.1. To see that it actually commutes, let, without loss of generality, \( g \in G \) map \( g_i \) to \( h_i g_{\sigma(i)} \) for some \( h_i \in Z(G) \) and permutation \( \sigma \) of the set \( \{1, \ldots, \#G/Z(G)\} \). Note that both \( h_i \)'s and \( \sigma \) depend on \( g \), but with \( g \) fixed they are uniquely defined. We show that \( S \) commutes with \( \pi(g^{-1}) \). Since \( \pi \) is irreducible, all \( (\pi(h))_{h \in Z(G)} \) are scalar multiples of identity on \( \mathcal{H} \). Hence,

\[
\langle S \pi(g^{-1})k_1, k_2 \rangle = \sum_i \langle \pi(g^{-1})k_1, \pi(g_i)f \rangle \langle \pi(g_i)f, k_2 \rangle =
\]

\[
= \sum_i \langle k_1, \pi(g_i)f \rangle \langle \pi(g_i)f, k_2 \rangle = \sum_i \langle k_1, \pi(h_i g_{\sigma(i)}f) \rangle \langle \pi(g_i)f, k_2 \rangle =
\]

\[
= \sum_i \langle k_1, \pi(g_{\sigma(i)}f) \rangle \langle \pi(g_i^{-1}g_{\sigma(i)}f), k_2 \rangle = \sum_i \langle k_1, \pi(g_i)f \rangle \langle \pi(g_{\sigma^{-1}(i)} f), k_2 \rangle,
\]

where two last equalities follow from substitution \( i' = \sigma(i) \) and the fact that \( g^{-1} \) maps \( g_i \) to \( h_{\sigma^{-1}(i)}^{-1} g_{\sigma^{-1}(i)} \) as needed. So

\[
\langle S \pi(g^{-1})k_1, k_2 \rangle = \langle Sk_1, \pi(g)f k_2 \rangle = \langle \pi(g^{-1})Sk_1, k_2 \rangle.
\]
Hence, the commutativity of $S$ and $\pi(g)$ for all $g \in G$ is proved. \hfill \Box

**Lemma 6.3.** If $\pi$ satisfies assumptions, then the frame operator

\[
S h = \sum_{g \in G} \langle h, \pi(g)f \rangle \pi(g)f'
\]

can be represented as:

\[
S h(x) = \frac{\#G/Z(G)}{\dim \mathbb{C}} \langle f'(x), f(x) \rangle_{\mathbb{C}} h(x) 1_X(x).
\]

**Proof.** If $\pi$ satisfies assumptions for some $X \subset Z(G)$, $[\pi(z)f](\zeta) = \zeta(1)1_X(\zeta)f(\zeta)$ for $\zeta \in Z(G)$. From Plancherel Theorem

\[
\sum_{z \in \mathbb{C}} \langle h, [\pi(z)f] \rangle \langle \pi(z)f', H' \rangle =
\]

\[
= \sum_{z \in \mathbb{C}} \int_{\mathbb{C}} 1_X(\zeta) \langle h(\zeta), [\pi(z), \zeta]f(\zeta) \rangle d\zeta \times \int_{\mathbb{C}} 1_X(\eta) \langle [\pi(\eta, \zeta)f'(\eta), H'(\eta) \rangle d\eta =
\]

\[
= \sum_{z \in \mathbb{C}} \int_{\mathbb{C}} 1_X(\zeta) \langle h(\zeta), f(\zeta) \rangle 1_X(\zeta) \langle f'(\zeta), H'(\zeta) \rangle d\zeta = \int_{X} \langle h, f(x) \rangle \langle f'(x), H' \rangle.
\]

The conclusion follows from Lemma 6.2 above applied to each $\mathbb{C}$-co-py separately. \hfill \Box

**Lemma 6.4.** Let $\pi$ satisfy assumptions. If for certain $x, y \in X$ $\pi(x, g) \approx \pi(y, \tau g)$ for all $g \in G$, then $x = \theta y$.

**Proof.** Let $z \in Z(G)$. From assumption $\pi(x, z) = \pi(y, \tau z)$. Since $\pi(x, z) = x(z)1_{\mathbb{C}}$ for $x \in X$, $x(z) = y(\tau z)$. From definition of $\theta$, $x = \theta y$. \hfill \Box

The following is the application of the Theorem 2.7 ([12], p.105) to the case of $\pi$ and $\pi \circ \tau$:

**Lemma 6.5.** Let $\pi, \pi \circ \tau$ are equivalent and $J$ establishes this equivalence, then there exists such an operator function $J' \in L^2(X, U(\mathbb{C}))$ such that

- $[Jf](x) = J'(x) f(\theta(x))$,
- $J'(x) \pi(g)(\theta(x)) = \pi(\tau g | x) J'(x)$ for all $g \in G$ and almost all $x \in X$.

### 6.3. The frame operator of algebraically summed system.

**Proof.** of Claim 3. Let $S_{ij}$ be the operator defined as

\[
\langle S_{ij}h_1, h_2 \rangle = \sum_{g \in G} \langle h_1, \pi(\tau^g)i \rangle \langle \pi(\tau^g)i, h_2 \rangle
\]

First, applying Lemma 6.3 for $f' = J^{-1} f$ ($h_{1,2} \in \mathcal{H}$).

\[
\langle S_{0}h_1, h_2 \rangle = \sum_{g \in G} \langle h_1, \pi(g)f \rangle \langle \pi(\tau g)f, \pi(\tau g)^{-1} h_2 \rangle = \sum_{g \in G} \langle h_1, \pi(g)f \rangle \langle \pi(g)^{1}, J^{-1} f, h_2 \rangle = \int_X \frac{\#G/Z(G)}{\dim \mathbb{C}} \langle [J^{-1} f](x), f(x) \rangle_{\mathbb{C}} \langle h_1(x), [J^* h_2](x) \rangle_{\mathbb{C}} dx.
\]
Analogously we deal with all $S_{ij}$ obtaining that the frame operator $R$ of the system
\[
\left(\sum_{i=0}^{t-1} \pi(\tau^{-i} g)f\right)_{g \in G}
\] is equal to
\[
\langle Rh_1, h_2 \rangle =
\]
\[
(6.1) \quad \langle Rh_1, h_2 \rangle =
\]
\[
(6.2) \quad \sum_{g \in G} \left\langle h_1, \left(\sum_{i=0}^{t-1} \pi(\tau^{-i} g)f\right) \right\rangle \left\langle \left(\sum_{i=0}^{t-1} \pi(\tau^{-i} g)f\right), h_2 \right\rangle = \sum_{i,j=0}^{t-1} \langle S_{ij}h_1, h_2 \rangle =
\]
\[
(6.3) \quad \sum_{i,j=0}^{t-1} \int_X \frac{\#G/Z(\langle G \rangle)}{\dim \mathbb{C}^{r \oplus t}} \left\langle [J^{i+j} f](x), f(x) \right\rangle \left\langle h_1(x), [J^{i+j} h_2](x) \right\rangle \, dx =
\]
\[
(6.4) \quad \frac{\#G/Z(\langle G \rangle)}{\dim \mathbb{C}^{r \oplus t}} \int_X \left\langle h_1(x), \sum_{k=0}^{t-1} \langle f(x), [J^k f](x) \rangle [J^k h_2](x) \right\rangle \, dx.
\]

We justify below that from Lemma 6.5 and lemma below the expression (6.1) is equal to $\int_X \lambda \left\langle h_1(x), h_2(x) \right\rangle \, dx$ if and only if

- $\langle f(x), [J^k f](x) \rangle = 0$
  for $dx$-almost all $x \in X$ such that $(\theta^k x)_{k=0}^{t-1}$ consists of $t$ different elements.
  
- $| \langle f(x), u_t(x) \rangle |^2 = \lambda$
  for all $x \in X$ such that $\theta x = x$, unless the measure $dx$ of this set is $0$.

Indeed, the whole $X$ splits into orbits under action of $\theta$. Let $\Phi$ be the set of these orbits. Since $\theta^t = id_X$, $t$ being a prime number, the orbits are of twofold type. Either of 1 element (the point is fixed); denote the appropriate subset of $\Phi$ by $\Phi_1$, or of $t$ elements (all points of orbit are different); denote the appropriate subset by $\Phi_2$.

In the case of $t$-element orbit, $\theta$ acts on the sequence $(x, \theta x, \ldots, \theta^{t-1} x)$ as a shift (where $\varphi \in \Phi_2$ and $x \in \varphi$ is a representant of the orbit). For the illustration of the argument let us take $t = 3$ and $x_0$ be an element of $X$ with 3-element orbit. Then $x_1 = \theta x_0, x_2 = \theta x_1, \text{and } x_3 = \theta x_2$.

For given $x_0 \in X$ consider the mapping $P_{x_0} : L^2(\mathbb{C}^r) \to \mathbb{C}^r$ given by $P_{x_0}(f) = (f(x_0), f(x_1), f(x_2))$. By Lemma 6.5

\[
[Jh](x_i) = J'(x_i) h(x_{i+1} \mod 3) \quad \text{and} \quad [J^2 h](x_i) = J'(x_i) J'(x_{i+1} \mod 3) h(x_{i+2} \mod 3).
\]

So

\[
P_{x_0} [Jh] = \begin{bmatrix} J'(x_0) h(x_1) \\ J'(x_1) h(x_2) \\ J'(x_2) h(x_3) \end{bmatrix} \quad P_{x_0} [J] = \begin{bmatrix} 0 & J'(x_0) & 0 \\ 0 & 0 & J'(x_1) \\ J'(x_2) & 0 & 0 \end{bmatrix}.
\]

Thus, we obtain $P_{x_0} [J]$ and analogously $P_{x_0} [J^2]$.

\[
P_{x_0} [J^2] = \begin{bmatrix} 0 & 0 & J'(x_1) J'(x_2) \\ J'(x_2) J'(x_1) & 0 & 0 \\ 0 & J'(x_0) J'(x_2) & 0 \end{bmatrix}.
\]
If by $a_i^k$ we denote $\langle f(x_i), [J^k f](x_i) \rangle$, we obtain that

$$P_{x_i}[S] = \begin{bmatrix}
    a_0^i & \text{Id}_{C^r} & a_1^i & J'(x_0) & a_2^i J'(x_1) J'(x_2) \\
    a_0^i J'(x_2) & a_1^i & \text{Id}_{C^r} & a_2^i J'(x_1) \\
    a_0^i J'(x_2) & a_1^i J'(x_3) & a_2^i J'(x_4) & \text{Id}_{C^r}
\end{bmatrix}.$$ 

Such a matrix can be a scalar multiple of identity $\text{Id}_{C^r}$ if and only if $a_i^k = 0$ for $i = 0, 1, 2$ and $k = 1, 2$ and $a_0^i = a_1^i = a_2^i = \lambda$. In other words,

$$\langle f(x), [J^k f](x) \rangle_{C^r} = 0$$

for $dx$-almost all $x \in X$.

In the case of 1-element orbits, i.e., fixed points the situation is a little different. If $x = 0$, $[J f](x) = J'(x) f(x)$ and

$$t \frac{\# G/Z(G)}{\dim C^r} \langle h_1(x), \sum_{k=0}^{t-1} \langle f(x), [J^k f](x) \rangle \langle [J^k h_2](x) \rangle \rangle =$$

$$t \frac{\# G/Z(G)}{\dim C^r} \sum_{k=0}^{t-1} \langle f(x), [J^k f](x) \rangle \langle [J^k h_2](x), h_1(x) \rangle =$$

Assume that $J'(x)$ has eigenvalues $j_i$ and eigenvectors $u_i$. Assume further that $\langle f(x), u_i \rangle = f_i$ and $\langle h_1(x), u_i \rangle = h_1^i$, $\langle h_2(x), u_i \rangle = h_2^i$. Then using Plancherel Theorem

$$\sum_{k=0}^{t-1} \langle f(x), [J^k f](x) \rangle \langle [J^k h_2](x), h_1(x) \rangle = \sum_{i, i' = 1}^{r} \sum_{k=0}^{t-1} j_{i, i'}^k |f_i|^2 \overline{h_2^i} h_1^i.$$ 

If some $j_i$’s coincide the expression can never be equal to $\lambda \langle h_1(x), h_2(x) \rangle$. On the other hand, if all $j_i$’s are different the expression above reduces to

$$\sum_{i=0}^{r-1} |f_i|^2 \overline{h_2^i} h_1^i,$$

and can be equal to $\lambda \langle h_2(x), h_1(x) \rangle$ if and only if $|f_i|^2 = \lambda$ for all $i = 1, \ldots, r$. □

7. Discussion.

7.1. Discussion of the group center. Let $G$ be a discrete countable group and $Z(G)$ be the center of $G$. In the paper we shall consider only such groups $G$ that the quotient group $G/Z(G)$ is finite.

Certainly, the finite groups belong to this class. The class is a little narrower than the groups of type I, since discrete groups of this type are characterized by containing abelian normal subgroup $H$ with finite index. However, it may happen that the group of type I will contain the abelian normal subgroup of finite index, while the index of the center will be infinite. In particular, the discrete subgroups of Heisenberg group have this property, i.e. the quotient is $G/Z(G)$ finite, if they relate to the rational oversampling. On the other hand, if such a subgroup relates to irrational oversampling, one can verify that it has a trivial center, so the quotient group $G/Z(G)$ is infinite.
7.2. Discussion of the group and representation - integer vs. rational oversampling. The example of the group for which our theory will apply to is the group $G_2$ defined as a group with three generators $a, b, \varepsilon$ with the relations as below

$$G_2 = \langle a, b, \varepsilon \mid ab = \varepsilon ba, \varepsilon^2 = e, \varepsilon a = \varepsilon a, \varepsilon b = \varepsilon b \rangle.$$ 

The group $G_2$ can be embedded as a subgroup of a Heisenberg group, but in fact it is isomorphic to many subgroups generated by the sets \{$(a/2, 0, 0), (0, a^{-1}, 0), (0, 0, 1/2)$\}, where $a \in \mathbb{R}$, are all isomorphic and isomorphic to $G_2$. The same holds for $G_q$ defined by

$$G_q = \langle a, b, \varepsilon \mid ab = \varepsilon ba, \varepsilon^2 = e, \varepsilon a = \varepsilon a, \varepsilon b = \varepsilon b \rangle,$$

and the subgroups generated by the sets \{$(a/q, 0, 0), (0, a^{-1}, 0), (0, 0, 1/q)$\}.

It may seem that the group $G_q$ is the group for integer oversampling case only. However, the numerator of the redundancy can be changed by a suitable choice of the representation. The following example shows two representations $\pi_1$ and $\pi_2$ of the group $G_3$ in the Hilbert space $L^2(\mathbb{R})$ that correspond to the cases of redundancy $1/3$ and $2/3$, respectively. Let

$$\pi_1(a) = M_{1/3}, \quad \pi_1(b) = T_1, \quad \pi(\varepsilon) = e^{2\pi i/3}I,$$

and

$$\pi_2(a) = M_{2/3}, \quad \pi_2(b) = T_1, \quad \pi(\varepsilon) = e^{2\pi i/3}I.$$

The check that they are representations of $G_3$ or that $\pi_i(gg') = \pi_i(g)\pi_i(g')$ for all $g, g' \in G_3$ and for $i = 1, 2$ is immediate. Note that these representations are not equivalent. Specifically, $\pi_1$ is a subrepresentation of $\pi_2$ and $\pi_2 = \pi_1 \oplus \pi_1$. In this paper we limit our attention to the representations corresponding to the integer oversampling case. In the abstract approach it will be reflected by the assumption that the representation $\pi$ is, in some sense, multiplicity free.

References


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