A Categorification of the Skein Module of Tangles

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A CATEGORIFICATION OF THE SKEIN MODULE OF TANGLES

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Abstract. We define new homology groups of tangles which are invariant under isotopies and prove that they categorify the Kauffman bracket skein module of tangles or, equivalently, the Temperley-Lieb algebra. Our homology of 0-tangles coincides (up to normalization) with Khovanov link homology. Our homology of 1-tangles has an interpretation in terms of the reduced Khovanov homology. For any tangle \( T \) composed of tangles \( T_1 \) and \( T_2 \) we define a spectral sequence converging to the homology of \( T \) whose second term is composed of homologies of \( T_1 \) and \( T_2 \).

1. Introduction

Let \( I_0, \ldots, I_{2n-1} \) be distinct intervals placed consecutively in clockwise direction in the boundary of a disk \( D^2 \). A framed \( n \)-tangle in \( D^3 = D^2 \times (-1, 1) \) is a disjoint union of \( n \) bands, \( b : [0, 1] \times [0, 1] \hookrightarrow D^3 \), and a finite number of annuli, \( a : S^1 \times [0, 1] \hookrightarrow D^3 \), embedded into \( D^3 \) such that for any band \( b \), \( b([0, 1] \times \{0\}) = I_i \) and \( b([0, 1] \times \{1\}) = I_j \) for some \( i \neq j \). Additionally, we assume that all bands in a tangle have an “integral framing”. In other words, each tangle is represented by a tangle diagram in \( D^2 \) with blackboard framing. Since tangles are considered up to an ambient isotopy, two tangle diagrams are equivalent if they differ by second, third, and balanced first Reidemeister moves.

\[
\begin{array}{c}
\circ \circ \\
\text{Balanced first Reidemeister move}
\end{array}
\]

Let \( B_n \) be the set of all \( n \)-tangle diagrams in \( D^2 \) with no crossings and no closed components. Hence \( |B_n| \) is the \( n \)-th Catalan number, \( \frac{1}{n+1} \binom{2n}{n} \). Note that a 0-tangle is a framed link in \( D^3 \) and \( B_0 = \{ \emptyset \} \).

For any \( n \)-tangle \( T \) we define homology groups \( H_{i,j,s}(T) \), for \( i, j \in \mathbb{Z}, s \in B_n \), which are invariant under isotopies of \( T \) and whose properties are discussed in Sections 3 and 4.

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2. Definition

Consider a tangle diagram $D$ whose crossings are ordered by consecutive integers $1, 2, \ldots$. Following [Vi, APS1], a state of $D$ is an assignment of $+$ or $-$ sign to each of the crossings of $D$ and an additional assignment of $+$ or $-$ sign to each circle in the diagram obtained by smoothing the crossings of $D$ according to the following convention:

\[
\begin{array}{c|c}
+1 \text{ marker} & -1 \text{ marker} \\
\hline
\end{array}
\]

(Note that arcs obtained by smoothing of crossings of $D$ do not carry any labels.) Our construction of the chain complex associated with $D$ is analogous to that of [APS1]:

For any state $S$ of $D$ let [APS1]:

\[
\begin{align*}
I(S) &= \sharp\{\text{positive crossing markers}\} - \sharp\{\text{negative crossing markers}\}, \\
J(S) &= I(S) + 2 (\sharp\{\text{positive circles}\} - \sharp\{\text{negative circles}\}).
\end{align*}
\]

Furthermore, we define $\Psi(S)$ as an element of $B_n$ obtained from the state $S$ by removing all closed components. Let $S_{i,j,s}(D)$ be the set of all states $S$ of $D$ with $I(S) = i, J(S) = j$, and $\Psi(S) = s$. Let $G_{i,j,s}(D)$ be the free abelian group generated by states in $S_{i,j,s}(D).$ We define the coincidence number between states following [APS1, Definition 3.1]:

$[S : S']_\nu = 1$ if the following four conditions are satisfied:

1. the crossing $v$ is marked by $+$ in $S$ and by $-$ in $S'$,
2. $S$ and $S'$ assign the same markers to all the other crossings,
3. the labels of the common circles in $S$ and $S'$ are unchanged,
4. $J(S) = J(S'), \Psi(S) = \Psi(S').$

Otherwise $[S : S']_\nu$ is equal to 0.

All possible types of coincident states are shown in the table below, where $\varepsilon = +$ or $-$. 

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<td>$+ \varepsilon$</td>
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</tbody>
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Let \( t(S,v) \) denote the number of negative markers assigned to crossings bigger than \( v \). Then
\[
d_{i,j,s} : C_{i,j,s}(D) \to C_{i-2,j,s}(D)
\]
is defined by
\[
d_{i,j,s}(S) = \sum_v (-1)^t(S,v)d_{i,j,s,v}(S),
\]
where
\[
d_{i,j,s,v}(S) = \sum_{S' \in S_{i-2,j,s}} [S : S']_v S'.
\]
As in [APS1, APS2], \( d \) has degree \(-2\) with respect to the first index. Note also that \( C_{i,s,s}(D) = 0 \) for \( i \) not equal to the number of crossings of \( D \) mod 2.

In the next section we will prove

**Proposition 1.** \( d^2 = 0 \).

Therefore for any abelian group \( G \), \( (C_{*,j,s}(D) \otimes G, d_{*,j,s}) \) is a chain complex whose homology groups we denote by \( H_{*,j,s}(D; G) \), as usual abbreviating \( H_{i,j,s}(D; \mathbb{Z}) \) to \( H_{i,j,s}(D) \). By the argument of [APS1, Sec. 10.1] we see that \( H_{i,j,s}(D; G) \) does not depend (up to isomorphism) on the ordering of crossings of \( D \).

**Remark 2.** One can extend our definition of tangles to include framed tangles in \( F \times I \), where \( F \) is any orientable surface with a boundary. For such tangles there is an analogous construction of homology groups.

### 3. Embeddings into surfaces, Reidemeister moves

Consider an orientable surface \( F \) (possibly with boundary) and an open disk \( D_0^2 \subset F \). Let \( \gamma_1, \ldots, \gamma_m : I \to F \setminus D_0^2 \) be disjoint arcs properly embedded into \( F \setminus D_0^2 \). (Hence the endpoints of the arcs lie in \( \partial D_0^2 \).) Assume additionally that \( \gamma_1, \ldots, \gamma_m \) considered as elements of \( \pi_1(F/D_0^2) \) are such that
\[
\gamma_{i_1}^{\varepsilon_1} \cdots \gamma_{i_k}^{\varepsilon_k} \neq e,
\]
for any sequence of \( k \) distinct integers \( 1 \leq i_1, \ldots, i_k \leq n \) and \( \varepsilon_1, \ldots, \varepsilon_k \in \{\pm 1\} \).

For any tangle diagram \( D \) in \( D^2 \), let \( \tilde{D} \) denote a framed link diagram in \( F \) obtained by gluing \( D^2 \) into the place of \( D_0^2 \subset F \) and connecting the endpoints of \( D \) with the endpoints of \( \gamma_1, \ldots, \gamma_m \) so that the 0th endpoint of \( D \) (corresponding to \( I_0 \subset \partial D^2 \)) is identified with \( \gamma_1(0) \). (Here, and throughout the paper we assume that any tangle diagram \( D \) of a tangle \( T \) is composed of arcs and loops representing the cores of bands and annuli of \( T \). By definition, the endpoints of \( D \) are the centers of the intervals \( I_0, \ldots, I_{2n-1} \subset \partial D^2 \).)
If \( s \in B_n \) then \( \tilde{s} \) is a link diagram in \( F \) composed of disjoint noncontractible loops. Let \( \hat{s} \) be an enhanced state of \( \tilde{s} \). In other words \( \hat{s} \) denotes a labeling of components of \( \tilde{s} \) by \( \pm \) signs. For any \( S \in S_{i,j,s}(T) \) let \( \Lambda_i(S) \in S_{i,j,s}(\tilde{T}) \) be the state of \( \tilde{T} \) obtained by attaching to \( S \) the arcs \( \gamma_1, \ldots, \gamma_n \) labeled as in \( \hat{s} \). (The definition of \( S_{s,s}(L) \) for link diagrams, \( L \), in surfaces can be found in [APS2, Sec. 2].)

**Lemma 3.** For any arcs \( \gamma_1, \ldots, \gamma_n \) as above, any \( s \in B_n \), and any enhanced state \( \hat{s} \) of \( s \), \( \Lambda_i \) extends to an isomorphism

\[
\Lambda_i : C_{i,j,s}(T) \rightarrow C_{i,j,s}(\tilde{T})
\]

commuting with the differentials.

**Proof.** The statement follows directly from the definitions of the chain complexes associated with \( T \subset D^2 \) and \( \tilde{T} \subset F \). \( \square \)

The lemma implies Proposition 1. Furthermore, by [APS2, Theorem 2] we have:

**Theorem 4.** Let \( i, j \in \mathbb{Z} \), \( s \in B_n \).

1. If \( D' \) is related to \( D \) by the first Reidemeister move consisting of adding a negative kink to \( D \) then \( H_{i,j,s}(D') = H_{i-1,j-3,s}(D) \).

2. \( H_{i,j,s}(D) \) is invariant (up to an isomorphism) under the second and third Reidemeister moves.

For 0-tangles \( T \), the groups \( H_{i,j,0}(T; G) \) are Khovanov’s link homology groups – more precisely, they coincide with the groups \( H_{i,j,0}(T; G) \) of [APS1]. Furthermore, for any tangle \( T \), the groups \( H_{i,j,s}(T; G) \) coincide with stratified homology groups \( H_{i,j,s}(D; G) \) defined in [APS2] for the link diagram \( D \) constructed as above.

Further properties of our homology are described below.

4. Properties

4.1. **Rotation of a tangle.** For any \( n \)-tangle \( T \) let \( \lambda(T) \) denote \( T \) with its endpoint labels increased by 1 mod \( 2n \).
4.2. Short exact sequence. Any three skein related $n$-tangle diagrams

\[
\begin{array}{c}
\overline{D_n} \\
\overline{D_p} \\
\overline{D_0}
\end{array}
\]

define a short exact sequence

\[(1) \quad 0 \to C_{i,j,s}(D_\infty) \xrightarrow{\alpha} C_{i,j-1,s}(D_p) \xrightarrow{\beta} C_{i,j-2,s}(D_0) \to 0\]

for any $s \in B_n$, $j \in \mathbb{Z}$, where the maps $\alpha : C_{i,j,s}(D_\infty) \to C_{i,j-1,s}(D_p)$ and $\beta : C_{i,j-1,s}(D_p) \to C_{i,j-2,s}(D_0)$ are defined as in [APS1, Sec 7]. This sequence leads to the long exact sequence

\[(2) \quad \cdots \to H_{i,j,s}(D_\infty) \xrightarrow{\alpha} H_{i-1,j-1,s}(D_p) \xrightarrow{\beta} H_{i-2,j-2,s}(D_0) \xrightarrow{\gamma} H_{i-2,j,s}(D_\infty) \to \cdots\]

4.3. Categorification of the skein module of $n$-tangles in $D^2$. Formal $\mathbb{Z}[A^{\pm 1}]$-linear combinations $n$-tangles quotiented by the skein relations of the Kauffman bracket

\[
\otimes = A \otimes + A^{-1} \otimes, \quad L \cup \bigcirc = -(A^2 + A^{-2})L,
\]

form the Kauffman bracket skein module of $n$-tangles, $S(D^3, n)$, (called the relative Kauffman bracket skein module in [Pr]), which is isomorphic as a $\mathbb{Z}[A^{\pm 1}]$-module to the Temperley-Lieb algebra on $n$ strands, $S(D^3, n)$ has a basis composed of tangles represented by diagrams in $B_n$. Hence

\[
\langle T \rangle = \sum_{s \in B_n} \langle T \rangle_s s \in S(D^3, n),
\]

where $\langle T \rangle_s \in \mathbb{Z}[A^{\pm 1}]$ are uniquely defined for any $n$-tangle $T$ and $s \in B_n$. Let $\chi_A(H_{ss})$ denote the polynomial Euler characteristic of a bigraded group $H_{ss}$, cf. [APS2, Sec 1]:

\[(3) \quad \chi_A(H_{ss}) = \sum_{i,j} A^j (-1)^{i-j} r k H_{i,j}.
\]

Our homology theory categorifies the skein module of $n$-tangles in $D^2$:

**Proposition 5.** For any tangle $T$, $\chi_A(H_{ss}(T)) = \langle T \rangle_s$.  

Proof. By definition
\[ \langle \bigtimes \rangle_s = A \langle \bigotimes \rangle_s + A^{-1} \langle \bigcup \rangle_s, \quad \langle \bigcup \bigcap \rangle_s = -(A^2 + A^{-2}) \langle \bigcup \rangle_s, \]
and, by (2), analogous identities hold for \( \chi_A(H_{\text{ass}}(T)) \). Therefore, it is enough to assume that the tangle diagram \( T \) has no crossings and no trivial components. Under these assumptions \( T \in B_n \), and
\[ \chi_A(H_{\text{ass}}(T)) = \chi_A(C_{\text{ass}}(T)) = rk C_{0,0,s}(T) = \delta_{s,T} = \langle T \rangle_s, \]
where \( \delta_{s,T} = \begin{cases} 1 & \text{if } s = T \\ 0 & \text{otherwise.} \end{cases} \)

4.4. Reduced Link Homology. For any link diagram \( L \) with a specified one of its components, Khovanov defined reduced homology of \( L \), \( H_{i,j}(L) \), [K3, Sh]. His construction has the following interpretation in our setting: If \( \hat{L} \) is a 1-tangle obtained by cutting \( L \) at an arbitrary point on its specified component, then \( H_{i,j}(L) \) is isomorphic (up to normalization of indices) to \( H_{i,j,\alpha}(\hat{L}) \), where \( \alpha \) is the unique element of \( B_1 \). In particular, \( H_{i,j,\alpha}(\hat{L}) \) does not depend on the choice of the cutting point on the distinguished component of \( L \), despite the fact that different cutting points may give non-isomorphic 1-tangles.

4.5. Composition of tangles. Tangles \( T_1 \in B_{n_1}, T_2 \in B_{n_2} \) can be composed by connecting \( d \) consecutive endpoints of \( T_1 \) with \( d \) endpoints of \( T_2 \), to form a new \((n_1 + n_2 - d)\)-tangle. A tangle obtained by connecting endpoints \( 0, \ldots, d - 1 \) of \( T_1 \) with endpoints \( 0, \ldots, d - 1 \) of \( T_2 \) will be denoted by \( T_1 \#_d T_2 \). We assume that the endpoints of \( T_1 \#_d T_2 \) are clockwise ordered starting with the 0th endpoint which is the \( d \)th endpoint of \( T_1 \) with respect to the old ordering. After the last endpoint of \( T_1 \) follows \( d \)th endpoint of \( T_2 \) (with respect to the old ordering).

Note that any partial composition of tangles \( T_1 \) and \( T_2 \) can be realized as \( \lambda^a(\lambda^b(T_1)) \#_d \lambda^c(T_2)) \) for some \( a, b, c \in \mathbb{Z}, d \geq 0 \).

We study the algebraic relations between \( H_{\text{ass}}(T_1), H_{\text{ass}}(T_2), \) and \( H_{\text{ass}}(T_1 \#_d T_2) \) below.

Tensor product of tangles. The simplest type of composition occurs when \( d = 0 \). We denote \( T_1 \otimes T_2 \) by more standard notation: \( T_1 \bigotimes T_2 \). In particular, we have \( s_1 \otimes s_2 \in B_{n_1+n_2} \) for any \( s_1 \in B_{n_1}, s_1 \in B_{n_1} \). If \( T_1 \) is an \( n_1 \)-tangle and \( T_2 \) is an \( n_2 \)-tangle then \( H_{\text{ass}}(T_1 \otimes T_2) = 0 \) unless \( s = s_1 \otimes s_2 \) for some \( s_1 \in B_{n_1}, s_2 \in B_{n_2} \). If \( s = s_1 \otimes s_2 \) then \( s_1 \) and \( s_2 \) are
unique and $C_{\text{ass}}(T_1 \otimes T_2)$ is the tensor product of (filtered) chain complexes, $C_{\text{ass}_1}(T_1)$ and $C_{\text{ass}_2}(T_2)$. By taking a proper care of $j$-grading and applying the Künneth formula we get a short exact sequence

$$0 \to \bigoplus_{i_1+j_1=i, \quad j_1+j_2=j} H_{i_1,j_1,s}(T) \otimes H_{i_2,j_2,s}(T') \to H_{i,j,s}(T \otimes T') \to$$

$$\to \bigoplus_{i_1+j_1=i, \quad j_1+j_2=j} \text{Tor}(H_{i_1,j_1,s_1}(T_1), H_{i_2,j_2,s_2}(T_2)) \to 0$$

which is non-canonically split. In particular the homology groups of $T_1 \otimes T_2$ are determined by the homology groups of $T_1$ and $T_2$.

$d=1$. The homology groups of an $(n_1 + n_2 - 1)$-tangle $T_1 \#_1 T_2$ obtained by connecting the 0th endpoint of an $n_1$-tangle $T_1$ with the 0th endpoint of an $n_2$-tangle $T_2$, are determined by $H_{\text{ass}}(T_1)$, and $H_{\text{ass}}(T_2)$ as well. As before, any $s \in B_n$ such that $H_{\text{ass}}(T_1 \#_1 T_2) \neq 0$ comes from a composition of a unique $s_1 \in B_{n_1}$ and a unique $s_2 \in B_{n_2}$. Furthermore $C_{\text{ass}}(T_1 \#_1 T_2)$ is the tensor product of (filtered) chain complexes $C_{\text{ass}_1}(T_1)$ and $C_{\text{ass}_2}(T_2)$ and the short exact sequence (4) holds.

However, for $d > 1$ the relations between homology groups of $T_1, T_2$ and $T_1 \#_d T_2$ are more involved.

Capping a tangle. A capping an $n$-tangle $T$ means creating an $(n-1)$-tangle $c(T)$ by connecting the 0th endpoint and the 1st endpoint of $D$ together. In other words, $c(T) = T \#_2 a$, where $a$ is the 1-tangle diagram composed of a single arc with no self-intersections, $B_1 = \{a\}$.

Note that every partial composition of tangles $T_1$ and $T_2$ can be obtained from a tensor product of $\lambda^a(T_1)$ and $\lambda^b(T_2)$, for some $a, b \in \mathbb{Z}$, by successive capping operations and rotations by $\lambda^c$, $c \in \mathbb{Z}$. Therefore, describing algebraic relations between $H_{\text{ass}}(T)$ and $H_{\text{ass}}(c(T))$ is crucial for understanding relations between homology groups of tangles and their partial compositions.

Denote the 0th and the 1st endpoint of $T$ by $p_0$ and $p_1$ respectively. (In this way we avoid confusion caused by different labels of endpoints in $T$ and in $c(T)$.) For $s \in B_{n-1}$ let $\tilde{s}$ be the state in $B_n$ obtained by adding points $p_0$ and $p_1$ between the last, $(2n - 3)\gamma$, and the 0th endpoints of $s$, and by connecting $p_0$ and $p_1$ by an arc. In other words, $\tilde{s}$ is the unique state in $B_n$ such that $c(\tilde{s}) = s \cup \gamma$ where $\gamma$ is a closed loop passing through $p_0$ and $p_1$. Let $D$ be a diagram of $T$ and let $C_{i,j,s,+}(c(D)) \subset S_{i,j,s}(c(D))$ be the subgroup generated by states of $c(D)$ which contain a loop passing through $p_0$ and $p_1$ labeled by $+$. Similarly, let $C_{i,j,s,-}(c(D)) \subset C_{i,j,s}(c(D))$ be the subgroup generated by the states containing a negatively labeled loop passing through $p_0, p_1$. Finally, let $C_{i,j,s,0}(c(D))$ be the subgroup of $C_{i,j,s}(c(D))$ generated by states belonging neither to $C_{i,j,s,+}(c(D))$ nor to $C_{i,j,s,-}(c(D))$. In other words, $C_{i,j,s,0}(c(D))$ is generated by states in which
\(p_0\) and \(p_1\) belong to an arc rather than a loop. We have
\[
C_{i,j,s}^*(e(D)) = C_{i,j,s,0}^*(e(D)) \oplus C_{i,j,s,0}^*(e(D)) \oplus C_{i,j,s,0}^*(e(D)).
\]

Note that \(C_{i,j,s,0}^*(e(D))\) and \(C_{i,j,s,0}^*(e(D)) \oplus C_{i,j,s,0}^*(e(D))\) are subcomplexes of \(C_{i,j,s}^*(e(D))\) and, therefore, they define a filtration giving rise to a three-row spectral sequence \(E\). We have \(E_{i,0}^1 = C_{i,j,s,0}^*(e(D))\), \(E_{i,1}^1\) is the quotient of \(C_{i,j,s,0}^*(e(D)) \oplus C_{i,j,s,0}^*(e(D))\) by \(C_{i,j,s,0}^*(e(D))\), and \(E_{i,2}^1\) is the quotient of \(C_{i,j,s,0}^*(e(D))\) by \(C_{i,j,s,0}^*(e(D)) \oplus C_{i,j,s,0}^*(e(D))\). Note that \((E_{i,0}^1, d^1)\) and \((E_{i,2}^1, d^1)\) can be identified with \((C_{i,j,s}^*, d_{i,j,s})\). Furthermore, \((E_{i,1}^1, d^1)\) is isomorphic to the sum
\[
\bigoplus_{s' \in B_n} \langle C_{i,j,s'}^*(D), d_{i,j,s'} \rangle.
\]

Consequently, \(E_{i,0}^2 = E_{i,2}^2 = H_{i,j,s}^*(D)\) and
\[
E_{i,0}^2 = \bigoplus_{s' \in B_n} H_{i,j,s'}^*(D).
\]

The spectral sequence converges to \(H_{i,j,s}^*(e(D))\) and therefore provides an indirect algebraic relationship between \(H_{i,j,s}^*(e(D))\) and \(H_{i,j,s}^*(D)\). The groups \(H_{s,a}^*(e(D))\), however, are in general impossible to determine without the knowledge of differentials \(d^2\) and \(d^3\).

If we denote the homology of \(C_{i,j,s,0}^*(e(D)) \oplus C_{i,j,s,0}^*(e(D))\) by \(H_{i,j,s,0}^*(e(D))\), then \(H_{s,a}^*(e(D))\) and \(H_{s,a}^*(D)\) are related via \(H_{s,a}^*(e(D))\), by the following two long exact sequences:

\[
\rightarrow H_{i,j,s}^*(D) \rightarrow \bigoplus_{s' \in B_n} H_{i,j,s'}^*(D) \rightarrow H_{i,j,s,0}^*(e(D)) \rightarrow H_{i-2,j,s}^*(D) \rightarrow
\]

and

\[
\rightarrow H_{i,j,s,0}^*(e(D)) \rightarrow H_{i,j,s}^*(e(D)) \rightarrow H_{i,j,s}^*(D) \rightarrow H_{i-2,j,s,0}^*(e(D)) \rightarrow
\]

The arguments above can be extended to consider the influence of several cappings on the homology of a tangle. We summarize the results here without getting into the technical details. Consider \(k\) consecutive cappings \(T' = c...c(T)\) of an \(n\)-tangle \(T\). For any \(s \in B_{n-k}\) there is a stratification
\[
C_{i,j,s}(T') = \bigoplus_{-k \leq r \leq k} C_{i,j,s,r}(T')
\]
such that for any \(r_0\)
\[
D_{i,j,s,r_0}(T') = \bigoplus_{r \leq r_0} C_{i,j,s,r}(T')
\]
is a chain subcomplex of \(C_{i,j,s}(T')\) and for any \(-k < r \leq k\) the quotient chain complex \(D_{i,j,s,r}(T')/D_{i,j,s,r-1}(T')\) can be identified with a direct sum of complexes \(C_{i,j,s'}(T)\) for certain \(s' \in B_n\). As before, the above filtration of
$C_{s,j}(T')$ leads to a spectral sequence $E$ with $2k + 1$ rows, which converges to $H_{s,j}(T')$. Furthermore, the summands of $E^2$ are sums of homology groups $H_{i,j}(T)$ for certain $i' \in \mathbb{Z}_{\geq 0}$ and $s \in B_n$.

4.6. Khovanov's tangle homology. Following Khovanov's terminology, an $(m, n)$-tangle $T$ is a tangle in a square with distinguished $2n$ points on the top and $2m$ points on the bottom of it. Khovanov constructed homology groups of $T$, $H_{i,j}(T)$, which are invariant under isotopies of $T$. Additionally each such group is an $(H^n, H^m)$-bimodule for certain rings $H^n, H^m$ defined in [K2, Ja]. As groups, $H_{i,j}(T)$ are sums of Khovanov homology groups $H_{i,j}(L)$ over all links $L$ which are flat closures of $T$. A flat closure of $T$ is a link obtained by closing its $2n$ top points by $n$ non-intersecting arcs and, analogously, its $2m$ bottom points by $m$ non-intersecting arcs. By our earlier remarks, for any $(m, n)$-tangle $T'$ there is a spectral sequence $E$ converging to $H_{i,j}(T)$ whose summands of $E^2$ are sums of certain groups $H_{i,j}(T')$. We do not know if there is a more explicit relation between Khovanov's and our homology groups. Nor can we interpret the bimodule structure of Khovanov's groups in our setting.

References


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