Cohomology of Graded Lie Algebras of Maximal Class

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COHOMOLOGY OF GRADED LIE ALGEBRAS OF MAXIMAL CLASS

ALICE FIALOWSKI AND DMITRI MILLIONSCHIKOV

ABSTRACT. It was shown by A. Fialowski that an arbitrary infinite-dimensional \( \mathbb{N} \)-graded "filiform type" Lie algebra \( \mathfrak{g} = \bigoplus_{i} \mathfrak{g}_i \) with one-dimensional homogeneous components \( \mathfrak{g}_i \) such that \( [\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j}, \forall i \geq 2 \) over a field of zero characteristic is isomorphic to one (and only one) Lie algebra from three given ones: \( \mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{L}_1 \), where the Lie algebras \( \mathfrak{m}_0 \) and \( \mathfrak{m}_1 \) are defined by their structure relations: \( \mathfrak{m}_0: [\epsilon_i, \epsilon_i] = \epsilon_{i+1}, i = 2, \ldots, n-1 \). Evidently, \( \mathfrak{m}_0(n) \) is generated by \( \epsilon_1 \) and \( \epsilon_2 \). Another example of \( \mathbb{N} \)-graded two-generated filiform Lie algebra is \( \mathfrak{m}_2(n): [\epsilon_i, \epsilon_i] = \epsilon_{i+1}, i = 2, \ldots, n-1, [\epsilon_2, \epsilon_i] = \epsilon_{i+2}, i = 3, \ldots, n-2 \).

A. Fialowski classified in [4] all infinite-dimensional \( \mathbb{N} \)-graded two-generated Lie algebras \( \mathfrak{g} = \bigoplus_{i} \mathfrak{g}_i \) with one-dimensional homogeneous components \( \mathfrak{g}_i \). In particular, there are only three algebras in her list satisfying the "filiform property": \( \{\mathfrak{g}_1, \mathfrak{g}_2\} = \mathfrak{g}_{i+1}, \forall i \). They are \( \mathfrak{m}_0, \mathfrak{m}_2, \mathfrak{L}_1 \), where \( \mathfrak{m}_0, \mathfrak{m}_2 \) denote infinite-dimensional analogues of \( \mathfrak{m}_0(n), \mathfrak{m}_2(n) \), respectively and \( \mathfrak{L}_1 \) is the "positive" part of the Witt or Virasoro algebra. The classification of finite-dimensional \( \mathbb{N} \)-graded filiform Lie algebras over a field of zero characteristic was done in [9].

A. Shalev and E. Zelmanov defined in [11] the coklass (which might be infinity) of a finitely generated and residually nilpotent Lie algebra \( \mathfrak{g} \), in analogy with the case

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of (pro-)\(p\)-groups, as \(\text{cc}(\mathfrak{g}) = \sum_{i \geq 1} (\dim(C^i \mathfrak{g}/C^{i+1} \mathfrak{g}) - 1)\). Obviously the coclass of a filiform algebra is equal to one and the same is true for the infinite-dimensional algebras \(\mathfrak{m}_0, \mathfrak{m}_2, L_1\). Algebras of coclass 1 are also called algebras of \textit{maximal class}. They are also \textit{narrow} or \textit{thin} Lie algebras (A. Shalev, A. Caranti, M. Newman, et al.). Part of Fialowski’s classification in [4] can be reformulated in the following way: \textit{Up to an isomorphism there are only three \(\mathbb{N}\)-graded Lie algebras of maximal class with one-dimensional homogeneous components:} \(\mathfrak{m}_0, \mathfrak{m}_2, L_1\). The last statement was rediscovered in [11].

Algebras of maximal class are in the center of attention these days both in zero and positive characteristic. There are many open questions related to them. One natural question is their cohomology which is the subject of the present paper.

I. Goncharova calculated in 1973 [7] the Betti numbers \(b^q(L_1) = \dim H^q(L_1)\). Her result implies, as a corollary, the celebrated Euler identity in combinatorics:

\[
\prod_{j=1}^{\infty} (1-t^j) = \sum_{k=0}^{\infty} (-1)^k (t^{\frac{2\pi^2 k^2}{6}} + t^{\frac{-2\pi^2 k^2}{6}})
\]

(see [5]). The Betti numbers \(\dim H^q(\mathfrak{m}_0(n))\) (finite-dimensional case) were calculated in [1] (see also [2]), however, there are no explicit formulas for basic cocycles and no description of the multiplicative structure of \(H^*(\mathfrak{m}_0(n))\) was obtained.

We give a complete description of the cohomology algebras \(H^*(\mathfrak{m}_0)\) and \(H^*(\mathfrak{m}_2)\). The method we use is based on Dixmier’s exact sequence in Lie algebra cohomology [3]. In our considerations we use combinatorics: partitions and generating functions.

The paper is organized as follows. In Sections 1–2 we review all necessary definitions and facts concerning Lie algebra cohomology and \(\mathbb{N}\)-graded Lie algebras, in particular we recall Dixmier’s exact sequence in the cohomology [3]. We start our computations in Section 3 with the algebra \(H^*(\mathfrak{m}_0)\) (Theorem 3.4). In Section 4 we discuss the relations of our results with representations theory. It turns out that the basic cocycles representing \(H^q(\mathfrak{m}_0)\) are at the same time the highest weight vectors (primitive vectors) of the \(q\)-th exterior power \(\Lambda^q(V(\lambda))\) of the irreducible one-dimensional \(\mathfrak{sl}(2, \mathbb{K})\)-module \(V(\lambda)\) for some \(\lambda\) (Theorem 4.1). In Section 5 we apply Dixmier’s exact sequence and the results of Section 3 to compute \(H^*(\mathfrak{m}_2)\) (Theorem 5.5). Section 6 is devoted to finite-dimensional analogs of the algebras considered above. Recall that all of them are filiform Lie algebras. The Betti numbers \(\dim H^q(\mathfrak{m}_0(n))\) are known [1], [2]. Some of \(\dim H^q(L_1/L_{n+1})\) was found in [8]. The questions on the explicit formulas for representing cocycles and the multiplicative structure are still open for these algebras. At the end of the paper we consider the characteristic \(p\) analog of the algebras \(\mathfrak{m}_0\) and \(\mathfrak{m}_2\). We briefly remark that other computational tools such as spectral sequences or the Hodge Laplacian of the differential \(d\) lead to the same algebraic and combinatorial problems that are solved in the present paper.

1. Lie algebra cohomology and Dixmier’s exact sequence

Let us consider the cochain complex of a Lie algebra \(\mathfrak{g}\) over a field \(\mathbb{K}\) of zero characteristic:

\[
\mathbb{K} \xrightarrow{d_n = 0} \mathfrak{g}^* = C^1(\mathfrak{g}) \xrightarrow{d_1} C^2(\mathfrak{g}) \xrightarrow{d_2} \cdots \xrightarrow{d_{p-1}} C^p(\mathfrak{g}) \xrightarrow{d_p} \cdots
\]
where $C^p(g)$ denotes the vector space of continuous skew-symmetric $p$-linear forms on $g$, and the differential $d_p$ is defined by:

$$d_p c(X_1, \ldots, X_{p+1}) = \sum_{1 \leq i < j \leq p+1} (-1)^{i+j-1} c([X_i, X_j], X_1, \ldots, \hat{X}_j, \ldots, X_{p+1}).$$

The differential $d_1 : g^* \to \Lambda^2(g^*)$ coincides with the dual mapping of the Lie bracket $[,] : \Lambda^2 g \to g$ and

$$d(\rho \wedge \eta) = d\rho \wedge \eta + (-1)^{d_\rho \rho} \rho \wedge d\eta, \forall \rho, \eta \in \Lambda^*(g^*).$$

**Definition 1.1.** The cohomology of $(C^*(g), d)$ is called the cohomology (with trivial coefficients) of the Lie algebra $g$ and is denoted by $H^*(g)$.

Let $b$ be an ideal of codimension 1 in $g$. One can choose an element $X \in g$ such that $X \notin b$. This element determines a 1-form $\omega$ by the following condition:

$$\omega(X) = 1, \quad \omega(Y) = 0 \quad \text{for} \quad Y \in b.$$

As $b$ is an ideal in $g$, the 1-form $\omega \in g^*$ is closed and $b$ is an invariant subspace of $adX$.

**Theorem 1.2** ( Dixmier [3]). There exists a long exact sequence of Lie algebra cohomology:

$$\cdots \to (adX^*)^{-1} \xrightarrow{d} H^{i-1}(b) \xrightarrow{\omega \wedge} H^i(g) \xrightarrow{r_i} H^i(b) \xrightarrow{(adX^*)_i} H^i(b) \to \cdots$$

where

1) the homomorphism $r_i : H^i(g) \to H^i(b)$ is the restriction homomorphism;

2) $\omega \wedge : H^{i-1}(b) \to H^i(g)$ is induced by the multiplication $\omega \wedge : \Lambda^{i-1}(b^*) \to \Lambda^i(g^*)$;

3) the homomorphisms $(adX^*)_i : H^i(b) \to H^i(b)$ are induced by the derivation $adX^*_i$ of degree zero of $\Lambda^i(b^*)$ $(adX^*_i(a \wedge b) = adX^*a \wedge b + a \wedge adX^*b$ for all $a, b \in \Lambda^i(b^*)$) that extends the dual mapping $adX^* : b^* \to b^*$. The 0-derivation $adX^*$ commutes with $d$ and we will denote the corresponding mapping in the cohomology by the same symbol, more precisely, $(adX^*)_i$ denotes the mapping $H^i(b) \to H^i(b)$.

Dixmier’s theorem [3] is very important for our computations, so it appears to be useful to recall his proof. In fact, Dixmier considered the cohomology of a nilpotent Lie algebra $g$ with coefficients in arbitrary $g$-module $V$, but we restrict ourselves to the case of scalar coefficients.

**Proof.** Each form $f \in \Lambda^*(g)$ can be decomposed as $f = \omega \wedge f' + f''$, where $f' \in \Lambda^{*-1}(b^*)$ and $f'' \in \Lambda^*(b^*)$. One can write a short exact sequence of algebraic complexes

$$0 \to \Lambda^{*-1}(b^*) \xrightarrow{\omega \wedge} \Lambda^*(g^*) \xrightarrow{r_{-1}} \Lambda^*(b^*) \to 0$$

where $\Lambda^{*-1}(b^*)$ is taken with the differential $-d$ as $d(\omega \wedge c) = d_{\Lambda^*}(\omega \wedge c) = -\omega \wedge dc$.

This short exact sequence of algebraic complexes gives us a long exact sequence in the cohomology. To see this, we have to define the homomorphism $H^2(b) \to H^2(b)$ in the long exact sequence.

First of all let us introduce a new mapping

$$\Lambda^*(g^*) \to \Lambda^{*-1}(b^*), \quad f \in \Lambda^*(g^*) \to f_X \in \Lambda^{*-1}(b^*),$$

where $f_X(X_1, \ldots, X_q) = f(X, X_1, \ldots, X_q)$. That means $f_X = f'$ if $f = \omega \wedge f' + f''$. 


Then an obvious formula holds:

\[(2) \quad (df)_X (X_1, \ldots, X_{q+1}) = \sum_{1 \leq j \leq q+1} (-1)^j f(adX_j (X_1, \ldots, \hat{X}_j, \ldots, X_{q+1}))+\]

\[+ \sum_{1 \leq i < j \leq q+1} (-1)^{i+j} f([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{q+1}) =\]

\[= (a(X^*)_2(f) + d(f_X))(X_1, \ldots, X_{q+1}).\]

Hence the homomorphism \(H^q(b) \rightarrow H^q(b), [f] \rightarrow ([df])\) of the long exact sequence coincides with the homomorphism induced by \((adX^*)_q\).

\[\square\]

2. \(\mathbb{N}\)-graded Lie algebras

**Definition 2.1.** A Lie algebra \(\mathfrak{g}\) is called \(\mathbb{N}\)-graded, if it is decomposed to the direct sum of subspaces such that

\[\mathfrak{g} = \bigoplus_{i \in \mathbb{N}} \mathfrak{g}_i, \quad \forall i, j \in \mathbb{N}, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \quad i, j \in \mathbb{N}.

**Example 2.2.** The Lie algebra \(m_0\) is defined by its infinite basis \(e_1, e_2, \ldots, e_n, \ldots\) with commutator relations:

\[\{e_i, e_i\} = e_{i+1}, \quad \forall \ i \geq 2.

**Remark.** We always omit the trivial commutator relations \([e_i, e_j] = 0\) in the definitions of Lie algebras.

**Example 2.3.** The Lie algebra \(m_0\) is defined by its infinite basis \(e_1, e_2, \ldots, e_n, \ldots\) and commutator relations:

\[\{e_1, e_i\} = e_{i+1}, \quad \forall \ i \geq 2; \quad \{e_2, e_j\} = e_{j+2}, \quad \forall \ j \geq 3.

Consider now the algebra of polynomial vector fields on the real line \(\mathbb{R}^1\).

**Example 2.4.** Let us define the algebra \(L_k\) as the Lie algebra of polynomial vector fields on the real line \(\mathbb{R}^1\) with zero in \(x = 0\) of order not less then \(k + 1\).

The algebra \(L_k\) can be defined by its basis and the commutator relations are given by the natural commutator of vector fields

\[e_i = x^{i+1} \frac{d}{dx}, \quad i \in \mathbb{N}, \quad i \geq k, \quad \{e_i, e_j\} = (j - i)e_{i+j}, \quad \forall \ i, j \in \mathbb{N}.

\(L_k\) is a subalgebra of the Witt algebra \(W\), where \(W\) is spanned by all \(e_i = x^{i+1} \frac{d}{dx}, \quad i \in \mathbb{Z}\). Hence \(L_k\) can be regarded as a "positive" part of the Witt algebra.

The algebras \(m_0, m_2, L_1\) are \(\mathbb{N}\)-graded Lie algebras generated by two elements \(e_1, e_2\).

**Theorem 2.5** (follows from A. Fialowski [4]). Let \(\mathfrak{g} = \bigoplus_{i=1}^\infty \mathfrak{g}_i\) be a \(\mathbb{N}\)-graded Lie algebra such that:

\[(3) \quad \dim \mathfrak{g}_i = 1, \quad i \geq 1; \quad [\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j}, \quad \forall i \geq 2.

Then \(\mathfrak{g}\) is isomorphic to one (and only one) Lie algebra from the three given ones:

\[m_0, m_2, L_1.

**Remark.** All these Lie algebras have exactly two defining relations in degree 5 and 7.
The ideals $C^k\mathfrak{g}$ of the descending central series of a Lie algebra $\mathfrak{g}$ determine a decreasing filtration $C$ of $\mathfrak{g}$, i.e. $[C^k\mathfrak{g}, C^l\mathfrak{g}] \subset C^{k+l}\mathfrak{g}$, $k, l \geq 1$. One can consider the associated $\mathbb{N}$-graded Lie algebra $gr_C\mathfrak{g}$:

$$gr_C\mathfrak{g} = \oplus_{i \geq 1}(gr_C\mathfrak{g})_i = \oplus_{i \geq 1}C^i\mathfrak{g}/C^{i+1}\mathfrak{g}.$$  

We have the following obvious isomorphisms of $\mathbb{N}$-graded Lie algebras:

$$gr_C m_0 \cong gr_C L_1 \cong gr_C m_0 \cong m_0.$$

Remark. The Lie algebra $m_0$ has at least two different $\mathbb{N}$-gradings: one of them we have already discussed, and the second one (sometimes called natural) is defined by

$$m_0 = \oplus_{i \geq 1}(m_0)_i, \quad (m_0)_i = \langle e_1, e_2, \ldots, e_i \rangle, \quad (m_0)_i = \langle e_{i+1} \rangle, \quad i \geq 2.$$  

The first grading is more convenient in cohomological computations and we will use only this one.

**Theorem 2.6** (Vergne [12]). Let $\mathfrak{g} = \oplus_{i \geq 1}\mathfrak{g}_i$ be a $\mathbb{N}$-graded Lie algebra such that

$$\dim \mathfrak{g}_1 = 2, \dim \mathfrak{g}_i = 1, \ i \geq 2; \ [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, \ i \geq 1.$$  

Then $\mathfrak{g}$ is isomorphic to $m_0$.

A Lie algebra $\mathfrak{g}$ in Definition 1.1 is assumed to be topological and the spaces $C^\ast(\mathfrak{g})$ can have rather complicated nature, but here we will consider only $\mathbb{N}$-graded Lie algebras $\mathfrak{g} = \oplus_{i \geq 1}\mathfrak{g}_i = \oplus_{i \geq 1}\langle e_i \rangle$. Let us denote by $e_i^\ast(e_j) = \delta_{ij}$ the corresponding dual 1-forms. In this situation we assume $C^\ast(\mathfrak{g})$ to be a vector space of formal series of elements from $\Lambda^\ast(e^1, e^2, \ldots)$ (we will also use the notation $\Lambda^\ast(\mathfrak{g}^\ast)$ for this space).

One can define a second grading in the cochain complex $(C^\ast(\mathfrak{g}), d) = (\Lambda^\ast(\mathfrak{g}^\ast), d)$:

$$\Lambda^\ast(\mathfrak{g}^\ast) = \bigoplus_k \Lambda^k(\mathfrak{g}),$$

where a finite-dimensional subspace $\Lambda^k(\mathfrak{g})$ is spanned by $q$-forms $e^{i_1}\wedge \ldots \wedge e^{i_q}, i_1 < \ldots < i_q$ such that $i_1 + \ldots + i_q = k$. The symbol $\bigoplus_k$ means the completed direct sum.

The second grading is compatible with the differential $d$ and with the exterior product:

$$d \Lambda^k(\mathfrak{g}) \subset \Lambda^{k+1}(\mathfrak{g}), \quad \Lambda^k(\mathfrak{g}) \wedge \Lambda^l(\mathfrak{g}) \subset \Lambda^{k+l}(\mathfrak{g}).$$

The exterior product in $\Lambda^\ast(\mathfrak{g})$ induces a structure of a bigraded algebra in the cohomology $H^\ast(\mathfrak{g})$:

$$H^k_\ast(\mathfrak{g}) \wedge H^l_\ast(\mathfrak{g}) \to H^{k+l}_\ast(\mathfrak{g}).$$

**Theorem 2.7** (Goncharova [7]). The Betti numbers $b^k(L_1) = \dim H^k(L_1) = 2$ for every $q \geq 1$, more precisely

$$b^k_\ast(L_1) = \dim H^k_\ast(L_1) = \begin{cases} 1, & k = \frac{3q^2 + q}{2}, \\ 0, & \text{otherwise}. \end{cases}$$

The numbers $\frac{3q^2 + q}{2}$ are so called Euler pentagonal numbers. A sum of two arbitrary pentagonal numbers is not a pentagonal number, hence the algebra $H^\ast(L_1)$ has a trivial multiplication. One can consider the Euler characteristic

$$\chi_k(L_1) = \sum_q (-1)^q \dim C^q_k(L_1) = \sum_q (-1)^q b^q_k(L_1).$$
of each subcomplex $C^*_k(L_1), k \geq 0$, and associate the corresponding generating function $\chi_k(t) = \sum_{k=0}^{\infty} \chi_k (L_1)t^k$:
\[
\prod_{j=1}^{\infty} (1-t^j) = \sum_{k,q \geq 0} (-1)^q \dim C^*_k(L_1)t^k = \sum_{k,q \geq 0} (-1)^q b_q^k(L_1)t^k = \sum_{k=0}^{\infty} (-1)^k \left( t^{\frac{3q^2-k}{2}} + t^{\frac{3q^2+k}{2}} \right).
\]
This equality of two expressions for the Euler characteristic proves the celebrated Euler identity in combinatorics (see [5], [6] for details).

**Remark.** The $\mathbb{N}$-graded Lie algebras $L_1, m_0, m_2$ have the same cochain spaces (but different differentials), hence the cohomology $H^*(m_0)$ and $H^*(m_2)$ (that we are going to compute in the present article) also satisfy the "Euler property":
\[
\sum_{k=0}^{\infty} \sum_{q \geq 0} (-1)^q b_q^k(m_0)t^k = \sum_{k=0}^{\infty} \sum_{q \geq 0} (-1)^q b_q^k(m_2)t^k = \sum_{k=0}^{\infty} (-1)^k \left( t^{\frac{3q^2-k}{2}} + t^{\frac{3q^2+k}{2}} \right).
\]

### 3. The cohomology $H^*(m_0)$

Now we are going to apply Dixmier’s exact sequence for the situation
\[ g = m_0, \; \mathfrak{b} = \text{Span}(e_2, e_3, \ldots, e_k, \ldots), \; X = e_1.\]

Obviously the ideal $\mathfrak{b} = \text{Span}(e_2, e_3, \ldots)$ is an abelian Lie algebra and its cohomology is the exterior algebra $\Lambda^*(e_2, e_3, \ldots)$.

Let us denote by $D_1$ the operator $ade^*_1 : \Lambda^*(e_2, e_3, \ldots) \to \Lambda^*(e_2, e_3, \ldots)$. It can be defined explicitly by
\[
D_1(e^2) = 0, \; D_1(e^i) = e^{i-1}, \; \forall i \geq 3,
\]
\[
D_1(\xi \wedge \eta) = D_1(\xi) \wedge \eta + \xi \wedge D_1(\eta), \; \forall \xi, \eta \in \Lambda^*(e_2, e_3, \ldots).
\]

**Lemma 3.1.** The operator $D_1$ is surjective.

**Proof.** Let us define the operator $D_{-1} : \Lambda^*(e^2, e^3, \ldots) \to \Lambda^*(e^3, e^4, \ldots)$,
\[
D_{-1}e^i = e^{i+1}, \; D_{-1}(\xi \wedge e^i) = \sum_{I \geq 2} (-1)^I D_1^I(\xi) \wedge e^{i+1+I},
\]
where $i \geq 2$ and $\xi$ is an arbitrary form in $\Lambda^*(e^2, \ldots, e^{i-1})$. The sum in the definition (5) of $D_{-1}$ is always finite because $D_1^I$ decreases the second grading by $I$. For instance, $D_{-1}(e^i \wedge e^k) = \sum_{i=0}^{\infty} (-1)^i e^{i+1} \wedge e^{k+i+1}$. The operator $D_{-1}$ is right inverse to $D_1$, as $D_1D_{-1} = Id$ on $\Lambda^*(e^2, e^3, \ldots)$.

In fact, $D_{-1}D_1(e^i) = e^{i+1}, \; i \geq 2$ and for arbitrary $\xi \in \Lambda^*(e^2, \ldots, e^{i-1})$ we have
\[
D_1D_{-1}(\xi \wedge e^i) = \sum_{I \geq 0} (-1)^I D_1^I(\xi) \wedge e^{i+I+1+I} + \sum_{I \geq 0} (-1)^I D_1^I(\xi) \wedge e^{i+I} = \xi \wedge e^{i+1}.
\]

One can write the formula for $D_{-1}(e^i \wedge \ldots \wedge e^k \wedge e^l) = D_{-1}(0)$:
\[
\omega(e^i \wedge \ldots \wedge e^k \wedge e^l) = \sum_{I \geq 0} (-1)^I D_1^I(e^i \wedge \ldots \wedge e^l) = \omega(e^i \wedge \ldots \wedge e^l) \wedge e^{i+1+l}.
\]

This sum is also always finite and determines a homogeneous closed $(q+1)$-form of the second grading $i_1 + \ldots + i_{q+1} + 2k + 1$.

Let us consider the restriction $(D_1^2)^{q+1}_k$ of $D_1$ on $(q+1)$-forms of the second grading $k$. 
Lemma 3.2. Let $q \geq 1$. Then $\ker (D_1)^{q+1}_k$ is spanned by

$$\omega(e^{i_1} \wedge \ldots \wedge e^{i_q} \wedge e^{i_{q+1}}), \quad 2 \leq i_1 < \ldots < i_q, \quad i_1 + \ldots + i_{q+1} + 2i_q + 1 = k.$$ 

Proof. Obviously, if these forms exist, they are linearly independent. How many
they are? Let us consider the inclusion $A^{q+1}_{k-1} : A^{q+1}_{k-1}(e^2, e^3, \ldots) \rightarrow A^{q+1}_k(e^2, e^3, \ldots)$
defined on basic monomials by the shift of the last superscript:

$$A^{q+1}_{k-1}(\xi \wedge e^i \wedge e^j) = \xi \wedge e^i \wedge e^{i+1}.$$ 

It follows immediately that $\text{Im} A^{q+1}_{k-1}$ is spanned by monomials $e^{i_1} \wedge \ldots \wedge e^{i_q} \wedge e^{i_{q+1}}$, such that $j_2 + 1 - j_1 \geq 2$. Hence $\dim \text{Im} A^{q+1}_{k-1} = \dim A^{q+1}_k(e^2, e^3, \ldots)$ and

$$\text{Im} A^{q+1}_{k-1} \oplus \text{Span}(\{\omega(e^{i_1} \wedge \ldots \wedge e^{i_q} \wedge e^{i+1}), i_1 + \ldots + 2i_q + 1 = k\}) = A^{q+1}_k(e^2, e^3, \ldots).$$ 

On the other hand, $(D_1)^{q+1}_k$ is surjective and hence

$$\dim \ker (D_1)^{q+1}_k = \dim A^{q+1}_k(e^2, e^3, \ldots) - \dim A^{q+1}_{k-1}(e^2, e^3, \ldots)$$

which completes the proof. 

We will think of $\omega$ as a linear map defined on the subspace of $A^{q+1}_k(e^2, e^3, \ldots)$, spanned by the monomials $e^{i_1} \wedge \ldots \wedge e^{i_q} \wedge e^{i_{q+1}}$:

$$\omega(\sum_{i_1, \ldots, i_q} a_{i_1, \ldots, i_q} e^{i_1} \wedge \ldots \wedge e^{i_q} \wedge e^{i+1}) := \sum_{i_1, \ldots, i_q} a_{i_1, \ldots, i_q} \omega(e^{i_1} \wedge \ldots \wedge e^{i_q} \wedge e^{i+1}).$$

Let us denote by $P_q(k)$ the number of (unordered) partitions of a positive integer $k$ into $q$ parts, i.e. (see [6]) $P_q(k)$ is the number of solutions in positive integers $x_i$ of

$$k = x_1 + \cdots + x_q, \quad 1 \leq x_1 \leq x_2 \leq \cdots \leq x_q.$$ 

Denote by $V_q(k)$ the number of partitions of a positive integer $k$ into $q$ distinct
parts, i.e. $V_q(k)$ is the number of solutions in positive integers $y_i$ of

$$k = y_1 + \cdots + y_q, \quad 1 < y_1 < y_2 < \cdots < y_q.$$ 

Lemma 3.3. $\dim A^{q}_k(e^2, e^3, \ldots) = V_q(k-\tilde{q}) = P_q\left(k-\frac{q(q+1)}{2}\right).$

Proof. It consists of two standard tricks in combinatorics. Let us consider

1) a one-to-one correspondence between the set of partitions of a positive integer $N$ exactly into $q$ parts and the standard basis of $A^{q+1}_N(e^2, e^3, \ldots)$:

$$(x_1, x_2, x_3, \ldots, x_q) \rightarrow e^{x_1 + 1} \wedge e^{x_2 + 3} \wedge e^{x_3 + 3} \ldots \wedge e^{x_q + q}, \quad N = x_1 + \cdots + x_q, \quad 1 \leq x_1 \leq x_2 \leq \cdots \leq x_q;$$

2) an analogous bijection between the partitions of $N$ into $q$ distinct parts and the standard basis of $A^{q+1}_N(e^2, e^3, \ldots)$:

$$(y_1, y_2, y_3, \ldots, y_q) \rightarrow e^{y_1 + 1} \wedge e^{y_2 + 1} \wedge e^{y_3 + 1} \wedge \ldots \wedge e^{y_q + 1}, \quad N = y_1 + \cdots + y_q, \quad 1 \leq y_1 < y_2 < \cdots < y_q.$$ 

$\square$
Theorem 3.4. The bigraded cohomology algebra \( H^*(m_0) = \bigoplus_{k,q} H^q_k(m_0) \) is spanned by the cohomology classes of the following homogeneous cocycles:

\[
\begin{align*}
\xi^1, \xi^2, \xi^3, \ldots \xi^q, \omega(\xi^1 \wedge \ldots \wedge \xi^q \wedge \xi^{q+1}) &= \sum_{l \geq 0} (-1)^l (a_d e_l^1)^l (\xi^1 \wedge \ldots \wedge \xi^q \wedge \xi^q \wedge \xi^{q+1})^l, \\
\end{align*}
\]

where \( q \geq 1 \), \( 2 \leq i_1 < i_2 < \ldots < i_q \), in particular

\[
\dim H^q_k(m_0) = P_q(k) - P_q(k-1).
\]

The multiplicative structure is defined by

\[
\begin{align*}
\omega^1 \wedge \omega(\xi^1 \wedge \xi^{q+1}) &= 0, \quad \omega^2 \wedge \omega(\xi^1 \wedge \xi^{q+1}) = \omega(\xi^1 \wedge \xi^{q+1}), \\
\omega(\xi^1 \wedge \xi^{q+1}) \wedge \omega(\eta^1 \wedge \eta^{q+1}) &= \sum_{l=0}^{j-i+1} (-1)^l \omega((a_d e_l^1)^l (\xi^1 \wedge \eta^1) \wedge \xi^{q+1+l} \wedge \alpha \eta^{q+1+l}) + \\
&+ (-1)^{j-i+deg} \sum_{l=0}^{l-1} \omega((a_d e_l^1)^l (\xi^1 \wedge \eta^1) \wedge \xi^{q+1+l} \wedge \alpha \eta^{q+1+l}) + \\
&+ (-1)^{j-i+deg+1} \sum_{l=0}^{l-1} \omega((a_d e_l^1)^l (\xi^1 \wedge \eta^1) \wedge \xi^{q+1+l} \wedge \alpha \eta^{q+1+l}),
\end{align*}
\]

where \( i \leq j \), \( \xi \) and \( \eta \) are arbitrary homogeneous forms in \( \Lambda^*(\xi^1, \ldots, \xi^{q+1}) \) and \( \Lambda^*(\eta^1, \ldots, \eta^{q+1}) \), respectively.

Proof. It follows from Lemma 3.1 that in our case Dixmier’s sequence is equivalent to the following exact sequences:

\[
0 \to H^q_k(m_0) \to H^q_k(m_0) \to 0;
\]

Here \( \text{Ker}(a_d e_l^1) = \text{Ker}(D_1)^l \) and the dimensions \( \dim \text{Ker}(D_1)^l \) were found in Lemma 3.2 and the final formula for \( \dim H^q_k(m_0) \) follows from Lemma 3.3.

Let us remark that the sum (7) is always finite. Namely, the maximal value of the superscript \( l \) is equal to \( l_{\text{max}} = i_1 + \ldots + i_q - \frac{2(q+3)}{2} \) and the corresponding summand in (7) is \( (-1)^{j-i+deg} a(i_1, \ldots, i_q) \xi^1 \wedge \eta^1 \wedge \ldots \wedge \xi^{q+1} \wedge \eta^{q+1} \) (see Example 3.7 below).

We will obtain the formulas for the multiplicative structure of \( H^*(m_0) \) by using the explicit expressions (6) for basic cocycles. An arbitrary homogeneous \( q \)-cocycle \( \omega(\xi^1 \wedge \ldots \wedge \xi^q \wedge \xi^{q+1}) \) is completely determined by its leading term \( \xi^1 \wedge \ldots \wedge \xi^q \wedge \xi^{q+1} \) - the unique monomial \( \alpha \xi^1 \wedge \ldots \wedge \xi^q \wedge \xi^{q+1} \), \( i_1 < \ldots < i_q \) in its decomposition such that \( i_q - i_{q-1} = 1 \). Hence we only have to consider the summands with this property in the right part of the formula

\[
\omega(\xi^1 \wedge \eta \wedge \xi^{q+1}) = \sum_{l,i,k \geq 0} (-1)^{l+i+k} D_i^k (\xi \wedge \eta) \wedge \xi^{q+1} + D^k (\eta \wedge \xi) \wedge \xi^{q+1+1}.
\]

They are of the following three kinds:

1) \( (-1)^l D_i^k (\xi \wedge \eta) \wedge \xi^{q+1} \wedge \xi^{q+1} \) for \( i + 1 \leq j + 2 \);
2) \( (-1)^{i-j-k} D_i^k (\xi \wedge \eta) \wedge \xi^{q+1} \wedge \eta^{q+1+k} \) for \( k \geq 1 \);
3) \( (-1)^{i-j-k} D_i^k (\xi \wedge \eta) \wedge \xi^{q+1+k} \wedge \eta^{q+1+k} \) for \( k \geq 1 \).
Taking \( \omega \) of their sum we get formula (8) for the multiplication in \( H^*(m_0) \) (recall that \( D_1 \) denotes the operator \( ade^*_1 \)).

**Example 3.5.** We choose the following basis of \( H^2(m_0) \):

\[
e^2 \wedge e^3, e^3 \wedge e^4 - e^2 \wedge e^5, \ldots, \omega(e^j \wedge e^{j+1}) = \sum_{l=0}^{j-2} (-1)^l e^{l-j} \wedge e^{j+l+1}, \ldots
\]

From this it is clear that

\[
\dim H^2_k(m_0) = \begin{cases} 
1, & k = 2j + 1 \geq 5, \\
0, & \text{otherwise}
\end{cases}
\]

and for \( 2 \leq i < j \) we have

\[
\omega(e^2 \wedge e^3) \wedge \omega(e^i \wedge e^{i+1}) = \omega(e^2 \wedge e^3 \wedge e^j \wedge e^{j+1}),
\]

\[
\omega(e^i \wedge e^{i+1}) \wedge \omega(e^j \wedge e^{j+1}) = \sum_{l=0}^{j-i+1} (-1)^l \omega(e^{j-l} \wedge e^{i+1+l} \wedge e^j \wedge e^{j+1}) +
\]

\[
+ (-1)^{j-i} \sum_{s=1}^{2i-j-1} \omega(e^{2i-j-1-s} \wedge e^j \wedge e^{j+s} \wedge e^{j+s+1}) +
\]

\[
+ (-1)^{j-i+1} \sum_{s=1}^{2i-j-3} \omega(e^{2i-j-1-s} \wedge e^j \wedge e^{j+s} \wedge e^{j+s+2}).
\]

**Corollary 3.6.** \( H^2_k(m_0) = 0 \) if \( k < 0 \),

and for \( k \geq 0 \) we have

\[
\dim H^2_k(m_0) = \begin{cases} 
l - 1, & k = 6l + r, r = 0, 1, 2, 4, \\
l, & k = 6l + r, r = 3, 5.
\end{cases}
\]

This follows from the formulas for \( P_3(k) \) given in [6]. Using other remarks on \( P_3(k) \) in [6], one can show that \( \dim H^2_k(m_0) \) is a polynomial of degree \( q - 2 \) in \( k \) with leading term \( \frac{k^{q-2}}{(q-2)!} \), and the other coefficients depend on residue \( k \) modulo \( q! \).

**Example 3.7.**

\[
\omega(e^5 \wedge e^6 \wedge e^7) = e^5 \wedge e^6 \wedge e^7 - e^4 \wedge e^6 \wedge e^8 + (e^3 \wedge e^6 + e^4 \wedge e^5) \wedge e^9 -
\]

\[
- (e^2 \wedge e^6 + 2e^3 \wedge e^5) \wedge e^{10} + (3e^2 \wedge e^5 + 2e^3 \wedge e^4) \wedge e^{11} - 5e^2 \wedge e^4 \wedge e^{12} + 5e^2 \wedge e^3 \wedge e^{13}.
\]

Now we can finish this section with the formula for the generating function for Betti numbers \( b^2_k(m_0) \):

\[
\sum_{k=0}^{\infty} \sum_{q=0}^{k} b^2_k(m_0)t^k x^q = t(1 + x) + (1 - t) \prod_{j=2}^{\infty} (1 + xt^j),
\]

and verify the Euler property:

\[
\sum_{k=0}^{\infty} \sum_{q \geq 0} (-1)b^2_k(m_0)t^k = 1 - t - t^2 + \sum_{q \geq 2} t^q \sum_{q \geq 2} (-1)^q (V_q(k-q) - V_q(k-q-1)) =
\]

\[
= 1 - t - t^2 + \prod_{j=2}^{\infty} (1 - t^j - 1 + t^2) - t \prod_{j=2}^{\infty} (1 - t^j) - 1 = (1 - t) \prod_{j=2}^{\infty} (1 - t^j) = \prod_{j=1}^{\infty} (1 - t^j).
\]
4. The cohomology $H(m_\mathcal{K})$ and highest weight representations of $\mathfrak{sl}(2, \mathbb{K})$

This section was influenced by Appendix B in Borelmann's article [2].

Let $X, Y, H$ denote the standard basis of $\mathfrak{sl}(2, \mathbb{K})$:

$$[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y.$$ 

One can define an infinite-dimensional $\mathfrak{sl}(2, \mathbb{K})$-module $V(\lambda)$ by its basis $\{f_i, i \geq 0\}$ and the well-known classical formulas ([10]):

$$H f_i = (\lambda - 2i) f_i,$$

$$Y f_i = (i + 1) f_{i+1},$$

$$X f_i = (\lambda - i + 1) f_{i-1},$$

where we set $f_{-1} = 0$ and $\lambda \in \mathbb{K}$. The $\mathfrak{sl}(2, \mathbb{K})$-module $V(\lambda)$ is generated by its highest weight vector (primitive vector [10]) $f_0$: $H f_0 = \lambda f_0, X f_0 = 0, Y f_0 = Y f_0 / i!$.

The module $V(\lambda)$ is irreducible if and only if $\lambda \notin \mathbb{N}, \lambda \neq 0$. Sometimes the module $V(\lambda)$ is called the standard cyclic module or the Verma module.

Consider now the $q$-th exterior power $\Lambda^q(V(\lambda))$. It is decomposed to the direct sum of its weight subspaces $(\Lambda^q(V(\lambda)))^\alpha = \{v \in \Lambda^q(V(\lambda)), H(v) = \alpha v\}$:

$$\Lambda^q(V(\lambda)) = \bigoplus_k (\Lambda^q(V(\lambda)))^{\lambda q - 2k} = \bigoplus_k \Lambda^q_k(V(\lambda)),$$

where $\Lambda^q_k(V(\lambda))$ is spanned by monomials $f_{i_1} \wedge \ldots \wedge f_{i_k}$ such that $i_1 + \ldots + i_k = k$.

Observe $H(f_{i_1} \wedge \ldots \wedge f_{i_k}) = (\lambda q - 2 \sum_{t=1}^q i_t) f_{i_1} \wedge \ldots \wedge f_{i_k}$.

Now take $\lambda \notin \mathbb{N}, \lambda \neq 0$ and consider a new basis of $V(\lambda)$:

$$\tilde{f}_0 = f_0, \quad \tilde{f}_i = f_i / \prod_{t=1}^i (\lambda - t + 1), \quad i \geq 1, \quad X \tilde{f}_i = \tilde{f}_{i-1}, \quad i \geq 0.$$

In fact, we can reformulate Theorem 3.4 as follows.

**Theorem 4.1.** Let $q \geq 2$ and $V(\lambda)$ be an infinite-dimensional irreducible $\mathfrak{sl}(2, \mathbb{K})$-module, then its $q$-th exterior power $\Lambda^q(V(\lambda))$ is reducible. The subspace of its primitive vectors of weight $\lambda q - 2k$ has dimension $P_q(k - (q-3)/2) - P_q(k - (q-3)/2) - 1$ and we can take the following basis:

$$(\lambda q - 2k = i) \quad \omega(\tilde{f}_{i_1} \wedge \ldots \wedge \tilde{f}_{i_q} \wedge \tilde{f}_{i_q} \wedge \tilde{f}_{i+1}) = \sum_{t=1}^q (-1)^{t} X^{t} (\tilde{f}_{i_1} \wedge \ldots \wedge \tilde{f}_{i_q} \wedge \tilde{f}_{i_q} \wedge \tilde{f}_{i+1}),$$

where $0 \leq i_1 < i_2 < \ldots < i_{q-2} < i_q$, $\sum_{t=1}^q i_t = 2q - 2 < k$.

That means the module $\Lambda^q(V(\lambda))$ is decomposed to the direct sum of its highest weight submodules $\Lambda^q(V(\lambda)) = \bigoplus V_{i_1 \ldots i_q \circ 2 \circ 2},$ where the highest weight vector of $V_{i_1 \ldots i_q \circ 2 \circ 2}$ is $\omega(\tilde{f}_{i_1} \wedge \ldots \wedge \tilde{f}_{i_q} \wedge \tilde{f}_{i+1}).$

**Corollary 4.2.** Let $\lambda \notin \mathbb{N}$ and $q \geq 2$, then each submodule $V_{i_1 \ldots i_q \circ 2 \circ 2}$ in the decomposition of $\Lambda^q(V)$ considered above is irreducible (and hence isomorphic to $V(\lambda)$).

**Proof.** The operator $D_1 : \text{Span}(e^2, e^3, \ldots) \rightarrow \text{Span}(e^2, e^3, \ldots)$ considered above plays the role of the operator $X$ in an infinite-dimensional irreducible $\mathfrak{sl}(2, \mathbb{K})$-module $V(\lambda) = \text{Span}(e^2, e^3, \ldots)$. We just rescale $\tilde{f}_i = e^{i+2}, \ i \geq 0$, where $\{\tilde{f}_i, \ i \geq 0\}$
is the basis (4) of \( V(\lambda) \). The subspace of primitive elements of weight \( \lambda_q - k \) coincides by definition with \((\text{Ker} D_1)^2_{k+2q}\) and the theorem follows from Lemma 3.2.
\(\square\)

5. The cohomology \( H^*(m_2) \)

Let us set in Dixmier’s exact sequence
\[
\mathfrak{g} = m_2, \quad \mathfrak{b} = \text{Span}(\epsilon_1, \epsilon_3, \ldots), \quad X = \epsilon_2.
\]
The ideal \( \mathfrak{b} \) is isomorphic to \( m_2 \) by shifting \( \epsilon_1 = \epsilon'_1, \epsilon_i = \epsilon'_i, i \geq 3 \). Hence it follows from the results of the previous section that the cohomology \( H^*(\mathfrak{b}) \) is spanned by the cohomology classes of
\[
e^{\epsilon_1}, e^{\epsilon_3}, \omega_k(e^{\epsilon_1} \wedge \cdots \wedge e^{\epsilon_p} \wedge e^{\epsilon_p+1}), \quad p \geq 1, \quad 3 \leq i_1 < \cdots < i_p,
\]
where in the formula \( \omega_k(\xi \wedge e^{\epsilon_1} \wedge e^{\epsilon_2+1}) = \sum_{\ell \geq 0} D_\ell^2 (\xi \wedge e^{\epsilon_1} \wedge e^{\epsilon_2+1}), \xi \in \Lambda^*(e^{\epsilon_2}, \ldots, e^{\epsilon_1}) \) and we also assume that \( D_1(e^{\epsilon_3}) = 0 \), for instance \( \omega_k(e^{\epsilon_3} \wedge e^{\epsilon_4}) = e^{\epsilon_3} \wedge e^{\epsilon_4} \).

The operator \( D_2 = a \delta e^*_2 : \Lambda^*(\mathfrak{b}) \to \Lambda^*(\mathfrak{b}) \) inducing \( D_3 : H^*(\mathfrak{b}) \to H^*(\mathfrak{b}) \) can be defined by
\[
D_2(e^{\epsilon_1}) = D_2(e^{\epsilon_3}) = 0, \quad D_2(e^{\epsilon_i}) = e^{\epsilon_{i-2}}, \quad i \geq 5,
\]
(11)
\[
D_2(\xi \wedge \eta) = D_2(\xi) \wedge \eta + \xi \wedge D_2(\eta), \quad \forall \xi, \eta \in \Lambda^*(\mathfrak{b}).
\]

It is immediate that
\[
D_2(e^{\epsilon_3}) = e^{\epsilon_1}, \quad D_2(e^{\epsilon_1}) = 0,
\]
(12)
\[
D_2(\omega_k(e^{\epsilon_3} \wedge e^{\epsilon_4})) = 0, D_2(\omega_k(e^{k} \wedge e^{k+1})) = -2 \omega_k(e^{k-1} \wedge e^k), \quad k \geq 4.
\]

**Proposition 5.1.** Let \( 3 \leq i_1 < \cdots < i_{p-1} < i \) and \( \xi = e^{\epsilon_{i_1}} \wedge \cdots \wedge e^{\epsilon_{i_{p-1}}} \), then
\[
D_2(\omega_k(\xi \wedge e^{\epsilon_1} \wedge e^{\epsilon_2+1})) = \omega_k((D_2 + D_1^2)(\xi) \wedge e^{\epsilon_1} \wedge e^{\epsilon_2+1}) - 2 \omega_k(\xi \wedge e^{\epsilon_{i-1}} \wedge e^{\epsilon_i}).
\]

Proof. First of all let us remark that in general \( D_2 \) and \( D_1^2 \) coincide only on \( \Lambda^1(\mathfrak{b}) \):
\[
D_1^2(e^{\epsilon_1}) = D_2(e^{\epsilon_1}) = e^{\epsilon_{i-2}}, \quad i \geq 5, \quad D_1^2(\xi \wedge \eta) = D_1^2(\xi) \wedge \eta + 2 D_1(\xi) \wedge D_1(\eta) + \xi \wedge D_1^2(\eta).
\]

An arbitrary cocycle is completely determined by its terms \( a e^{\epsilon_{i_1}} \wedge \cdots \wedge e^{\epsilon_{i_{p-1}}} \wedge e^{\epsilon_i} \).

On the other hand, the operator \( D_2 \) decreases the difference between the two last superscripts of some monomial \( e^{\epsilon_{i_1}} \wedge \cdots \wedge e^{\epsilon_i} \) by two. Denoting all "non-interesting" terms by dots, we obtain an expression for \( D_2(\omega_k(\xi \wedge e^{\epsilon_1} \wedge e^{\epsilon_2+1})) \):
\[
D_2(\xi \wedge e^{\epsilon_1} \wedge e^{\epsilon_2+1}) - D_1(\xi \wedge e^{\epsilon_1} \wedge e^{\epsilon_2+2} + D_1^2(\xi \wedge e^{\epsilon_1} \wedge e^{\epsilon_2+3} + \cdots) =
\]
\[
= D_2(\xi) \wedge e^{\epsilon_1} \wedge e^{\epsilon_2+1} + \xi \wedge e^{\epsilon_1} \wedge D_2(e^{\epsilon_2+1}) -
\]
\[
- \xi \wedge D_1(e^{\epsilon_1} \wedge D_1(e^{\epsilon_2+2}) + D_1^2(\xi) \wedge e^{\epsilon_1} \wedge D_2(e^{\epsilon_2+3}) + \cdots =
\]
\[
= D_2(\xi) \wedge e^{\epsilon_1} \wedge e^{\epsilon_2+1} + \xi \wedge e^{\epsilon_1} \wedge e^{\epsilon_2+1} - \xi \wedge e^{\epsilon_2+1} \wedge e^{\epsilon_1} + D_1^2(\xi) \wedge e^{\epsilon_1} \wedge e^{\epsilon_2+1} + \cdots.
\]
\(\square\)

**Lemma 5.2.** \( H^*(\mathfrak{b}) = \text{Im} D_2 \oplus \langle e^3 \rangle \).

Proof. We have already seen (12) that \( D_2 \) is surjective on \( H^2(\mathfrak{b}) \). Now for an arbitrary \( \xi = e^{\epsilon_1} \wedge \cdots \wedge e^{\epsilon_{i-1}} \) with \( q \geq 2 \) and \( 3 \leq i_1 < \cdots < i_{p-1} < i \), define an operator \( D_{-2} : H^2(\mathfrak{b}) \to H^2(\mathfrak{b}) \) by the formula:
\[
D_{-2}(\omega_k(\xi \wedge e^{\epsilon_1} \wedge e^{\epsilon_2+1})) = \sum_{\ell \geq 0} \frac{1}{2^{\ell+1}} \omega_k \left( (D_2 + D_1^2)(\xi) \wedge e^{\epsilon_1} \wedge e^{\epsilon_2+1} \right).
\]
Like in the definition of $D_{-1}$, remark that the sum (5) is always finite: the operator $(D_3 + D_1)^l$ decreases the second grading of an arbitrary homogeneous $\xi$ by $2l$. Now using formula (5.1) we obtain $D_3 D_{-3} = -Id$.

Let us consider the restriction $(D_3)^{l+1}_k = D_3 : H^{l+1}_k(b) \rightarrow H^{l+1}_{k-1}(b)$.

**Corollary 5.3.** Let $q \geq 1$. Then $\dim \ker (D_3)^{l+1}_k = \dim H^{l+1}_k(b) - \dim H^{l+1}_{k-1}(b)$.

**Lemma 5.4.** The space $\ker D_3$ is spanned by

$$e^1, \omega_k(e^3 \wedge e^4) = e^3 \wedge e^4, \sum_{l \geq 1} \frac{1}{l!} \omega_k((D_3 + D_1)^l (e^1 \wedge \cdots \wedge e^l) \wedge e^{l+1} \wedge e^{l+2} \wedge \cdots)$$

where $1 \leq q$, $3 \leq i_1 < i_2 < \cdots < i_q$.

**Proof.** A mimic of the proof of Lemma 3.2. \hfill \Box

**Theorem 5.5.** The bigraded cohomology algebra $H^*(m_2) = \oplus q,k H^q_k(m_2)$ is spanned by cohomology classes of the following homogeneous cocycles:

$$w_{i_1, \ldots, i_q}, i_1, i_2, \ldots, i_q + 2$$

$$u_{i_1, \ldots, i_q}, i_1, i_2, \ldots, i_q + 2$$

where $1 \leq q$, $3 \leq i_1 < i_2 < \cdots < i_q$, in particular for $q \geq 3$,

$$\dim H^q_k(m_2) = H^q_k(m_2)^{r_1} - H^q_k(m_2)^{r_2}$$

where $r_1 = q - 1$, $r_2 = 0$.

**Proof.** Dixmier’s sequence is equivalent to the following exact sequences:

$$0 \rightarrow H^0(b) \rightarrow H^1(m_2) \rightarrow \langle e^1 \rangle \rightarrow 0$$

$$0 \rightarrow \langle e^3 \rangle \rightarrow H^2(m_2) \rightarrow \langle \omega_k(e^3 \wedge e^4) \rangle \rightarrow 0$$

The equality $\ker (D_3)^{l+1}_q = \ker (D_3)^{l+1}_q$ was found in Lemma 5.4. Formulas for $\dim H^q_k(m_2)$ follows from Corollary 5.3 and Theorem 3.4. The last remark is that $\omega_k(\xi)$ represents the inverse image of $\omega_k(\xi) \in \ker (D_3)^{l+1}_q$ with respect to $r_3 : H^q_k(m_2) \rightarrow H^q_k(b)$, for instance $r_3^{-1}(\omega_k(e^3 \wedge e^4)) = r_3^{-1}(e^3 \wedge e^4) = e^3 \wedge e^4 - e^2 \wedge e^5$.

**Example 5.6.**

$$w_{3, 4, 7} = \omega(e^5 \wedge e^6 \wedge e^7) + \omega(e^3 \wedge e^7 \wedge e^8) = e^3 \wedge e^6 \wedge e^7 + (e^3 \wedge e^7 \wedge e^8) - e^3 \wedge e^4 \wedge e^5 \wedge e^8 +$$

$$+ (e^4 \wedge e^5 \wedge e^7 \wedge e^8) - e^4 \wedge e^7 \wedge e^8 + (e^4 \wedge e^6 \wedge e^8) \wedge e^8 + e^4 \wedge e^6 \wedge e^8 \wedge e^{10} + e^4 \wedge e^6 \wedge e^8 \wedge e^{11} + e^4 \wedge e^6 \wedge e^8 \wedge e^{12} + e^2 \wedge e^3 \wedge e^{13}.$$

**Corollary 5.7.** 1) The space $H^2(m_2)$ is two-dimensional and it is spanned by the cohomology classes represented by cocycles $e^2 \wedge e^3$ and $e^3 \wedge e^4 - e^2 \wedge e^5$ with second grading 5 and 7 respectively;

2) The space $H^5(m_2)$ is infinite-dimensional and it is spanned by

$$w_{k, k+1, k+2} = \sum_{l \geq 0} \omega (e^{k-2l} \wedge e^{k+1} \wedge e^{k+2} \wedge \cdots)$$

where $k \geq 3$.

Hence $\dim H^q_k(m_2) = \begin{cases} 1, & q = 3k + 3 \geq 12, \\ 0, & \text{otherwise}. \end{cases}$
Remark. Again (see [6]), one can show that $b_k^2(m_2)$ for $q \geq 3$ is a polynomial of degree $q - 3$ with leading term $\frac{2k^{q-3}}{q-3}$ and other coefficients depending on residue $k$ mod $q$. We have the following identity for the corresponding generating function:

$$\sum_{k=0}^{\infty} \sum_{q=0}^{\infty} b_k^2(m_2)t^qx^2 = (1 + x)(t + t^2 - t^3 + xt^5) + (1 - t - t^3 + t^5) \prod_{j=3}^{\infty}(1 + xt^j).$$

It is easy to verify that the Euler property of $b_k^2(m_2)$ is equivalent to the following obvious equality: $(1-t-t^2+t^3)\prod_{j=3}^{\infty}(1-t^j) = \prod_{j=1}^{\infty}(1-t^j)$.

6. Finite-dimensional analogs and their cohomology

Let $\mathfrak{g}$ be a $\mathbb{N}$-graded Lie algebra, then $\mathfrak{g}_{i=0}^{n+1}\mathfrak{g}_{i}$ is an ideal and we can consider the corresponding quotient Lie algebra $\mathfrak{g}/\mathfrak{g}_{i=0}^{n+1}\mathfrak{g}_{i}$ and denote it by $\mathfrak{g}(n)$. The quotient Lie algebras $m_0(n), m_2(n)$ and $V_n = L_{n}/L_{n+1}$ are $n$-dimensional nilpotent Lie algebras with the same length $s(\mathfrak{g}) = n - 1$ of the descending central series $\{C^n\mathfrak{g}\}$. In fact, $n - 1$ is the maximum of $s(\mathfrak{g})$ in the set of $n$-dimensional nilpotent Lie algebras. Studying this class of nilpotent Lie algebras was initiated by Vergne in [12].

Recall that a nilpotent $n$-dimensional Lie algebra $\mathfrak{g}$ is called filiform Lie algebra, if its descending central series $\{C^n\mathfrak{g}\}$ has the length (nil-index) $s(\mathfrak{g}) = n - 1$.

The filiform Lie algebra $m_0(n)$ plays a special role in the theory of filiform Lie algebras: an arbitrary $n$-dimensional filiform Lie algebra can be obtained as a nilpotent deformation of $m_0(n)$ ([12]). Vergne’s explicit formulas for basic cocycles $\Psi_{k,r}$ of the second cohomology $H^2(m_0, m_0)$ with coefficients in the adjoint representation are one of the main tools in the study of filiform Lie algebras via the deformation theory. We can mention that the cohomology groups $H^2(m_0)$ and $H^3(m_2)$ were also found by M. Vergne ([12]).

Later all the Betti numbers of $m_0(n)$ were calculated by M. Bordemann in [2] (Appendix B). We would like to discuss his elegant approach. We have also to remark that similar results are well known in combinatorics (R. Stanley and his school), see [1] for references. By arguments equivalent to Dixmier’s exact sequence, Bordemann reduced the computation of $\dim H^3(m_0(n))$ to the problem of finding $\ker(D_1_{i})$:

$$\dim H^3(m_0(n)) = \dim \ker(D_1_{i-1})_{2} + \dim \ker(D_1_{i})_{2-1}.$$ 

Let us take $\lambda = n - 2$ and consider the irreducible $(n - 1)$-dimensional $\mathfrak{sl}(2, \mathbb{K})$-module $V(n-2) = \text{Span}(e_1, \ldots, e_n)$, where $X = D_1$. The dimension of $\ker(D_1_{i-1})_{2}$ is equal to the number of irreducible $\mathfrak{sl}(2, \mathbb{K})$-modules in the decomposition of $\Lambda^2(V(n - 2))$ and the last number is equal in its turn to the dimension of the zero-eigenspace plus the dimension of one-eigenspace of $H : \Lambda^2(V(n - 2)) \rightarrow \Lambda^2(V(n - 2))$.

Now we rescale as before $\tilde{f}_i = e^{i+2}$, $i = 0, \ldots, n - 2$.

$$H(\tilde{f}_{i_1} \wedge \ldots \wedge \tilde{f}_{i_q}) = (\lambda - 2) \sum_{i=1}^{\lambda} \tilde{f}_{i_1} \wedge \ldots \wedge \tilde{f}_{i_q}, \lambda = n - 2.$$ 

Hence $\dim \ker(D_1_{i-1})_{2}$ is equal to the number of solutions of the equation

$$\sum_{i=1}^{\lambda} i = \left\lfloor \frac{(n - 2)}{2} \right\rfloor, 0 \leq i_1 < \cdots < i_q \leq n - 2.$$
In other words, \( \dim \ker(D_1(n))_q = V_{q,n-1}(\lfloor \frac{nq}{2} \rfloor) \), where we denoted by \( V_{q,n-1}(N) \) the number of partitions of a positive integer \( N \) into \( q \) distinct summands such that \( 1 \leq i_1 < \cdots < i_q \leq n-1 \) and \( \lfloor x \rfloor \) stands for the integer part of \( x \in \mathbb{Q} \). We conclude that

\[
\dim H^q(m_0(n)) = V_{q,n-1}(\lfloor qn/2 \rfloor) + V_{q-1,n-1}(\lfloor (q-1)n/2 \rfloor).
\]

It is possible to write some of the first formulas in terms of more convenient combinatorial functions:

\[
\dim H^2(m_0(n)) = \left\lceil \frac{n+1}{2} \right\rceil, \quad \dim H^3(m_0(n)) = \left\lceil \frac{n+1}{3} \right\rceil + \frac{1}{8},
\]

\[
\dim H^4(m_0(n)) = \left\lceil \frac{4(n+1)}{5} \right\rceil + \frac{4n+13}{36}.
\]

**Remark.** The last formulas give no information about the bigraded structure of \( H^*(m_0(n)) \). Also we have no explicit formulas for basic cocycles. Later Borlemann’s results were generalized in [1] to the case of an arbitrary finite-dimensional nilpotent Lie algebra with an abelian ideal of codimension one (the preprint version of [2] appeared earlier than [1]).

It was shown in [8] that the case \( g(n) \) can be more complicated then its infinite-dimensional analog \( g \). The Betti numbers of \( L_1/L_{n+1} \) stabilize as \( n \to \infty \) and \( \dim H^3(L_1/L_{n+1}) \) are equal to the Fibonacci numbers \( F_{n+2} \) for sufficiently large \( n \gg q \). For instance,

\[
\dim H^3(L_1/L_{n+1}) = 3, \quad \dim H^3(L_1/L_{n+1}) = 5, \quad \dim H^4(L_1/L_{n+1}) = 8, \ldots.
\]

However, the full description of \( H^*(L_1/L_{n+1}) \) is still an open question. The same situation is with the cohomology \( H^*(m_2) \).

The classification of finite-dimensional \( \mathbb{N} \)-graded filiform Lie algebras \( g = \bigoplus \langle e_i \rangle \) with one-dimensional homogeneous components (there is the one-parametric family \( g_{\alpha} \) of non-isomorphic algebras in each dimension \( 7 \leq \dim g \leq 11 \), see [9]) also shows that one can expect the difficulties in the finite-dimensional case.

### 7. Final remarks

1) We can consider the Lie algebras \( m_0 \) and \( m_2 \) not only over a field of zero characteristic but over an arbitrary field of positive characteristic. From the proof of Theorem 3.4 it follows that the statement is also valid over an arbitrary field and Theorem 5.5 is valid over any field of non-even characteristic. This remark appears important because of applications of Lie algebras of maximal class in the theory of (pro-)\( p \)-groups (see [11]).

2) Let \( g = \bigoplus_{i \geq 1} \langle e_i \rangle \) be an \( \mathbb{N} \)-graded Lie algebra with one-dimensional homogeneous components. We can equip the (finite-dimensional) space \( \Lambda^k_{\mathbb{R}}(g^*) \) with an euclidean scalar product \( \langle \cdot, \cdot \rangle \), such that the basic monomials \( e^{i_1} \wedge \ldots \wedge e^{i_k} \) form anortonormal basis of \( \Lambda^k_{\mathbb{R}}(g^*) \):

\[
\langle e^{i_1} \wedge \ldots \wedge e^{i_k}, e^{j_1} \wedge \ldots \wedge e^{j_k} \rangle = \delta^{i_1}_{j_1} \ldots \delta^{i_k}_{j_k}.
\]

One of the main methods to compute the cohomology \( H^*(g) \) of the cochain complex \( (\Lambda^*(g), d) \) is to find the zero-eigenspace of the Hodge Laplacian \( dd^* + d^*d \) (see [5]).
Proposition 7.1. The Hodge Laplacian $dd^* + d^*d$ of the Lie algebra $\mathfrak{m}_0$ satisfies the following properties:

\begin{equation}
(dd^* + d^*d)(e^1) = 0, \quad (dd^* + d^*d)(\alpha \wedge \eta) = \alpha \wedge D_1 \eta, \quad \eta \notin \Lambda^{r+1}(\mathfrak{g}^*)
\end{equation}

where $D_1 = ade_1$ is the 0-derivation of the exterior algebra $\Lambda^*(\mathfrak{g})$ that extends the operator $ade_1 : \mathfrak{m}_0 \to \mathfrak{m}_0$.

Let us study the kernel of the Hodge Laplacian. We have proved in Lemma 3.1 that $D_1 : \Lambda^*(e^2, e^3, \ldots) \to \Lambda^*(e^2, e^3, \ldots)$ is surjective, hence $\text{Ker} D_1 = 0$, moreover it is easy to see that $e^1 \notin \text{Im} D_1$ and on the another hand, $\text{Ker} D_1 D_1 = 0$. Hence $\text{Ker} (dd^* + d^*d) \subseteq e^1 \oplus \Lambda^*(e^2, e^3, \ldots)$ and the problem of determining the zero-eigenspace of $dd^* + d^*d$ reduces to the computation of $\text{Ker} D_1 D_1 = \text{Ker} D_1$. The basic cocycles $\omega(e^1 \wedge \ldots \wedge e^{n+1})$ from Theorem 3.4 together with $e^1, e^2$ form the basis of the harmonic forms.

3) One can compute the cohomology of $\mathfrak{m}_0$ directly without Dixmier’s exact sequence, but for the Lie algebra $\mathfrak{m}_2$ such a computation is not so easy. We chose Dixmier’s method (see also [1]) to demonstrate a general principle which is applicable for other finite or infinite dimensional Lie algebras as well. Dixmier remarked in [3] that his exact sequence in the cohomology is equivalent to some cohomological spectral sequence.

Let $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ be an arbitrary $\mathbb{N}$-graded Lie algebra of the filiform type, i.e. $\dim \mathfrak{g}_i = 1$ for all $i$ and $[\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j}$, $i \geq 2$. We showed earlier that it is filtered by the ideals $\{ C^i \mathfrak{g} \}$ of the descending central series and the corresponding associated graded Lie algebra $gr_C \mathfrak{g}$ is isomorphic to $\mathfrak{m}_0$. It is easy to see that this filtration induces a filtration in the cochain complex $\Lambda^*(\mathfrak{g})$ and the first term $E_1$ of the corresponding spectral sequence $(E_r, d_r)$ is isomorphic to the cohomology $H^*(gr_C \mathfrak{g}) = H^*(\mathfrak{m}_0)$. Moreover, the spectral sequence $(E_r, d_r)$ degenerates at the first term $E_1$ in the case $\mathfrak{g} = m_0$, and at the second term $E_2$ in the case $\mathfrak{g} = m_1$, respectively. In both cases the differential $d_1$ is equal to $e^1 \wedge D_1$, and $d_1 = 0$ for $\mathfrak{g} = m_0$, while $d_1 = e^2 \wedge D_2$ for $\mathfrak{g} = m_1$. The case of $\mathfrak{g} = L_1$ is much more complicated. We have infinite number of non-trivial differentials $d_r$ of the spectral sequence $E_r$ and computing all of them would give a new proof of Goncharova’s theorem. This approach might be helpful to obtain explicit formulas for Goncharova’s cocycles which have not yet been found so far.

References


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