Decay of Correlations for the Rauzy–Veech–Zorich Induction Map on the Space of Interval Exchange Transformations

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1 Introduction

The aim of this paper is to prove a stretched-exponential bound for the decay of correlations for the Rauzy-Veech-Zorich induction map on the space of interval exchange transformations (Theorem 4). A Corollary is the Central Limit Theorem for the Teichmüller flow (Theorem 10).

The proof of Theorem 4 proceeds by approximating the induction map by a Markov chain satisfying the Doeblin condition, the method of Sinai [13] and Bunimovich-Sinai [14]. The main “loss of memory” estimate is Lemma 4.

1.1 Interval exchange transformations.

Let $m$ be a positive integer. Let $\pi$ be a permutation on $m$ symbols. The permutation $\pi$ will always be assumed irreducible, which means that $\pi \{1, \ldots, k\} = \{1, \ldots, k\}$ only if $k = m$.

Let $\lambda$ be a vector in $\mathbb{R}_+^m$, $\lambda = (\lambda_1, \ldots, \lambda_m)$, $\lambda_i > 0$ for all $i$. Denote

$$|\lambda| = \sum_{i=1}^{m} \lambda_i.$$

Consider the half-open interval $[0, |\lambda|)$. Consider the points $\beta_i = \sum_{j<i} \lambda_j$, $\beta_i^+ = \sum_{j<i} \lambda_j$.

Denote $I_i = [\beta_i, \beta_{i+1})$, $I_i^+ = [\beta_i^+, \beta_{i+1}^+]$. The length of $I_i$ is $\lambda_i$, whereas the length of $I_i^+$ is $\lambda_{i+1}$.

Set

$$T_{(\lambda, \pi)}(x) = x + \beta_{i+1}^+ - \beta_i \text{ for } x \in I_i.$$

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The map $T_{(\lambda, \pi)}$ is called an interval exchange transformation corresponding to $(\lambda, \pi).

The map $T_{(\lambda, \pi)}$ is an order-preserving isometry from $I_i$ onto $I_i^\pi$.

We say that $\lambda$ is irrational if there are no rational relations between $|\lambda|$, $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$.

**Theorem 1 (Oseledets([5]), Keane([9]))** Let $\pi$ be irreducible and $\lambda$ irrational. Then for any $x \in [0, \sum_{i=1}^m \lambda_i]$, the set $\{T_{(\lambda, \pi)}^n x, n \geq 0\}$ is dense in $[0, \sum_{i=1}^m \lambda_i]$.

### 1.2 Rauzy operations $a$ and $b$.

Let $(\lambda, \pi)$ be an interval exchange. Assume that $\pi$ is irreducible and $\lambda$ is irrational.

Following Rauzy [6], consider the induced map of $(\lambda, \pi)$ on the interval $[0, \lambda - \min(\lambda_m, \lambda_{m-1}(\pi)))$. The induced map is again an interval exchange of $m$ intervals. For $i, j = 1, \ldots, m$, denote by $E_{ij}$ an $m \times m$ matrix of which the $i, j$-th element is equal to 1, all others to 0. Let $E$ be the $m \times m$-identity matrix.

#### 1.2.1 Case $a$: $\lambda_{m-1} > \lambda_m$.

Define

$$A(a, \pi) = \sum_{i=1}^{\pi^{-1}(m)} E_{ii} + E_{m, \pi^{-1}m + 1} + \sum_{i=\pi^{-1}m+1}^{m} E_{i, i+1}$$

$$a \pi(j) = \begin{cases} \pi j, & \text{if } j \leq \pi^{-1}m; \\ \pi m, & \text{if } j = \pi^{-1}m + 1; \\ \pi (j-1), & \text{other } j. \end{cases}$$

If $\lambda_{m-1} > \lambda_m$, then the induced interval exchange of $T_{(\lambda, \pi)}$ on the interval $[0, \sum_{i \neq m} \lambda_i]$ is $T_{(\lambda', \pi')}$, where $\lambda' = A(a, \pi)^{-1} \lambda$ and $\pi' = a \pi$.

#### 1.2.2 Case $b$: $\lambda_m > \lambda_{m-1}$.

Define

$$A(b, \pi) = E + E_{m, \pi^{-1}m}$$

$$b \pi(j) = \begin{cases} \pi j, & \text{if } \pi j \leq \pi m; \\ \pi j + 1, & \text{if } \pi m < \pi j < m; \\ \pi m + 1, & \text{if } \pi j = m. \end{cases}$$

If $\lambda_{m-1} < \lambda_m$, then the induced interval exchange of $T_{(\lambda, \pi)}$ on the interval $[0, \sum_{i \neq m-1} \lambda_i]$ is $T_{(\lambda', \pi')}$, where $\lambda' = A(b, \pi)^{-1} \lambda$ and $\pi' = b \pi$.

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Note that operations $a$ and $b$ are invertible on the space of permutations, namely, we have:

\[
a^{-1} \pi(j) = \begin{cases} 
\pi(j), & \text{if } j \leq \pi^{-1}(m); \\
\pi(j + 1), & \text{if } \pi^{-1}(m) + 1 < j < m; \\
\pi(\pi^{-1}(\pi(m) + 1)), & \text{if } j = m.
\end{cases}
\]

\[
b^{-1} \pi(j) = \begin{cases} 
\pi(j), & \text{if } \pi(j) \leq \pi(m) \\
m, & \text{if } j = \pi^{-1}(\pi(m) + 1); \\
\pi(j) - 1, & \text{if } \pi(j) > \pi(m) + 1.
\end{cases}
\]

For $(\lambda, \pi) \in \Delta(\mathcal{R})$, denote

\[
T_{a^{-1}}(\lambda, \pi) = (A(a^{-1}, a)\lambda, a^{-1} \pi), \quad T_{b^{-1}}(\lambda, \pi) = (A(b^{-1}, b)\lambda, b^{-1} \pi). \quad (1)
\]

The interval exchange $T_{a^{-1}}(\lambda, \pi)$ is the preimage of $(\lambda, \pi)$ under the operation $a$, and the interval exchange $T_{b^{-1}}(\lambda, \pi)$ is the preimage of $(\lambda, \pi)$ under the operation $b$.

Normalize (dividing by $|\lambda| = \lambda_1 + \cdots + \lambda_m$) and set:

\[
t_{a^{-1}}(\lambda, \pi) = \frac{A(a^{-1}, a)\lambda}{|A(a^{-1}, a)\lambda|} a^{-1} \pi, \quad t_{b^{-1}}(\lambda, \pi) = \frac{A(b^{-1}, b)\lambda}{|A(b^{-1}, b)\lambda|} b^{-1} \pi. \quad (2)
\]

### 1.3 Rauzy class and Rauzy graph.

If $\pi$ is an irreducible permutation, then its **Rauzy class** is the set of all permutations that can be obtained from $\pi$ by applying repeatedly the operations $a$ and $b$; the Rauzy class of the permutation $\pi$ is denoted $\mathcal{R}(\pi)$. Rauzy class has a natural structure of an oriented labelled graph: namely, the permutations of the Rauzy class are the vertices of the graph, and if $\pi = a\pi'$ then we draw an edge from $\pi$ to $\pi'$ and label it by $a$, and if $\pi = b\pi'$ then we draw an edge from $\pi$ to $\pi'$ and label it by $b$. This labelled graph will be called the **Rauzy graph** of the permutation $\pi$.

For example, the Rauzy graph of the permutation $(4321)$ is

\[
\begin{array}{c}
(3142) \xrightarrow{a} (4132) \xrightarrow{b} (4321) \xrightarrow{b} (2431) \xrightarrow{a} (2413) \xrightarrow{a} (2143) \xrightarrow{b} (4213) \xrightarrow{a} (3241)
\end{array}
\]

For a permutation $\pi$, consider the set $\{a^n \pi, n \geq 0\}$. This set forms a cycle in the Rauzy graph which will be called the *a-cycle* of $\pi$. Similarly, the set $\{b^n \pi, n \geq 0\}$ will be called the *b-cycle* of $\pi$. 

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1.4 The Rauzy-Veech-Zorich induction.

Denote
\[ \Delta_{m-1} = \{ \lambda \in \mathbb{R}^+_0 : |\lambda| = 1 \}, \]
\[ \Delta^+ = \{ \lambda \in \Delta_{m-1}, \lambda_{r-1} > \lambda_m \}, \Delta^- = \{ \lambda \in \Delta_{m-1}, \lambda_m > \lambda_{r-1} \}, \]
\[ \Delta(\mathcal{R}) = \Delta_{m-1} \times \mathcal{R}(\pi). \]

Define a map
\[ T : \Delta(\mathcal{R}) \to \Delta(\mathcal{R}) \]
by
\[ T(\lambda, \pi) = \begin{cases} (\frac{\lambda}{\lambda_{r-1}}, a\pi), & \text{if } \lambda \in \Delta^+; \\ (\frac{\lambda}{\lambda_m}, b\pi), & \text{if } \lambda \in \Delta^-; \end{cases} \]

Each \((\lambda, \pi) \in \Delta(\mathcal{R})\) has exactly two preimages under the map \(T\), namely, \(t_{a-1}(\lambda, \pi)\) and \(t_{b-1}(\lambda, \pi)\) (2).

The set \(\Delta(\mathcal{R})\) is a finite union of simplices. Let \(\mathbf{m}\) be the Lebesgue measure on \(\Delta(\mathcal{R})\) normalized in such a way that \(\mathbf{m}(\Delta(\mathcal{R})) = 1\).

**Theorem 2 (Veech[1])** The map \(T\) has an infinite conservative ergodic invariant measure, absolutely continuous with respect to Lebesgue measure on \(\Delta(\mathcal{R})\).

From this result Veech [1] derives that almost all (with respect to \(\mathbf{m}\)) interval exchange transformations are uniquely ergodic.

Denote
\[ \Delta^+ = \cup_{r' \in \mathcal{R}(\pi)} \Delta^+_{r'}, \Delta^- = \cup_{r' \in \mathcal{R}(\pi)} \Delta^-_{r'}. \]

Following Zorich [4], we define the function \(n(\lambda, \pi)\) in the following way.

\[ n(\lambda, \pi) = \begin{cases} \inf \{k > 0 : T^k(\lambda, \pi) \in \Delta^-\}, & \text{if } \lambda \in \Delta^+; \\ \inf \{k > 0 : T^k(\lambda, \pi) \in \Delta^+\}, & \text{if } \lambda \in \Delta^-; \end{cases} \]

Define
\[ \mathcal{G}(\lambda, \pi) = T^n(\lambda, \pi)(\lambda, \pi). \]

The map \(\mathcal{G}\) will be referred to as the Rauzy-Veech-Zorich induction map [6, 1, 4].

For \((\lambda, \pi) \in \Delta(\mathcal{R})\), denote
\[ t_{a-1}(\lambda, \pi) = t_{a-1}(\lambda, \pi), t_{b-1}(\lambda, \pi) = t_{b-1}(\lambda, \pi), T_{a-1}(\lambda, \pi) = T_{a-1}(\lambda, \pi), T_{b-1}(\lambda, \pi) = T_{b-1}(\lambda, \pi). \]

Under the map \(\mathcal{G}\), each interval exchange \((\lambda, \pi)\) has countably many preimages:
\[ \mathcal{G}^{-1}(\lambda, \pi) = \begin{cases} \{t_{a-1}(\lambda, \pi), n \in \mathbb{N}\}, & \text{if } (\lambda, \pi) \in \Delta^+; \\ \{t_{b-1}(\lambda, \pi), n \in \mathbb{N}\}, & \text{if } (\lambda, \pi) \in \Delta^-. \end{cases} \]
Theorem 3 (Zorich[4]) The map $\mathcal{G}$ has an ergodic invariant probability measure, absolutely continuous with respect to Lebesgue on $\Delta(\mathcal{R})$.

Denote this invariant measure by $\nu$; the probability with respect to $\nu$ will be denoted by $\mathbb{P}$.

Let $\rho(\lambda, \pi)$ be the density of $\nu$ with respect to the Lebesgue measure $\mu$. Zorich [4] showed that for any $\pi \in \mathcal{R}$ there exist two positive rational homogeneous of degree $-m$ functions $\rho^+_{\pi}, \rho^-_{\pi}$ such that

$$
\rho(\lambda, \pi) = \begin{cases} 
\rho^+_{\pi}(\lambda), & \text{if } \lambda \in \Delta^+; \\
\rho^-_{\pi}(\lambda), & \text{if } \lambda \in \Delta^-.
\end{cases}
$$

Remark. In particular, the invariant density is bounded from below: there exists a positive constant $C(\mathcal{R})$, depending on the Rauzy class only and such that $\rho(\lambda, \pi) \geq C(\mathcal{R})$ for any $(\lambda, \pi) \in \Delta(\mathcal{R})$.

The map $\mathcal{G}$ is not mixing: indeed, from the definition of $\mathcal{G}$, we have

$$
\mathcal{G}(\Delta^+) = \Delta^-; \mathcal{G}(\Delta^-) = \Delta^+.
$$

Let $\mathcal{B}$ be the Borel $\sigma$-algebra on $\Delta(\mathcal{R})$, and let $\mathcal{B}_n = \mathcal{G}^{-n}\mathcal{B}$. We have $\mathcal{B}_{n+2} \subseteq \mathcal{B}_n$. Recall [23] that exactness of the map $\mathcal{G}^2$ means, by definition, that the $\sigma$-algebra $\cap_{n=1}^{\infty} \mathcal{B}_n$ is trivial [23] (in other words, that Kolmogorov’s 0 − 1 law holds for the map $\mathcal{G}^2$.)

Proposition 1 The map $\mathcal{G}^2 : \Delta^+ \to \Delta^+$ is exact with respect to $\nu|_{\Delta^+}$.

This Proposition is proven in Section 4; it implies strong mixing for the map $\mathcal{G}^2$.

1.5 The main result

Introduce a metric on $\Delta_{m-1}$ by setting

$$
d(\lambda, \lambda') = \log \frac{\max\{\frac{M_{\lambda\lambda'}}{M_{\lambda'\lambda'}}, \frac{M_{\lambda'\lambda}}{M_{\lambda\lambda'}}\}}{\min\{\frac{M_{\lambda\lambda'}}{M_{\lambda'\lambda'}}, \frac{M_{\lambda'\lambda}}{M_{\lambda\lambda'}}\}}.
$$

Now introduce a metric on $\Delta(\mathcal{R})$ by setting

$$
d((\lambda, \pi), (\lambda', \pi')) = \begin{cases} 
2 + d(\lambda, \lambda'), & \text{if } \pi \neq \pi'; \\
d(\lambda, \lambda'), & \text{if } \pi = \pi'.
\end{cases}
$$

For $\alpha > 0$, let $H_\alpha$ be the space of functions $\phi : \Delta(\mathcal{R}) \to \mathbb{R}$ such that if $d((\lambda, \pi), (\lambda', \pi')) \leq 1$, then $|\phi(\lambda, \pi) - \phi(\lambda', \pi')| \leq C d((\lambda, \pi), (\lambda', \pi'))^\alpha$ for some constant $C$.

Define

$$
C_{H_\alpha}(\phi) = \max_{d((\lambda, \pi),(\lambda', \pi')) \leq 1} \frac{|\phi(\lambda, \pi) - \phi(\lambda', \pi')|}{d((\lambda, \pi), (\lambda', \pi'))^\alpha},
$$

The main result of this paper is
Theorem 4 Let $G : \Delta(\mathcal{R}) \to \Delta(\mathcal{R})$ be the Rauzy-Veech-Zorich induction map and let $\nu$ be the absolutely continuous invariant measure.

Let $p > 2$. Then, for any $\alpha > 0$, there exist positive constants $C, \delta$ such that for any $\phi \in H_\alpha \cap L_p(\Delta^+(\mathcal{R}), \nu)$ and $\psi \in L_2(\Delta^+(\mathcal{R}), \nu)$ we have

$$|\int \phi \times \psi \circ G^{2n} d\nu - \int \phi d\nu \int \psi d\nu| \leq C \exp(-\delta n^{1/4})(C_{H_\alpha}(\phi) + \|\phi\|_{L_p}(\|\psi\|_{L_2})).$$

Denote by $N(0, \sigma)$ the Gaussian distribution with mean 0 and variance $\sigma$. By [7, 8, 17], we have

Corollary 1 Let $\phi \in H_\alpha \cap L_p(\Delta(\mathcal{R})^+, \nu)$, $\int \phi d\nu = 0$. Assume that there does not exist $\psi \in L_2(\Delta(\mathcal{R})^+, \nu)$ such that $\phi = \psi \circ G^2 - \psi$. Then there exists $\sigma > 0$ such that

$$\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \phi \circ G^{2n} \xrightarrow{d} N(0, \sigma) \text{ as } N \to \infty.$$ 

1.6 Veech’s space of zippered rectangles

A zippered rectangle associated to the Rauzy class $\mathcal{R}$ is a quadruple $(\lambda, h, a, \pi)$, where $\lambda \in \mathbb{R}^\ast, h \in \mathbb{R}^m, a \in \mathbb{R}^m, \pi \in \mathcal{R}$, and the vectors $h$ and $a$ satisfy the following equations and inequalities (one introduces auxiliary components $a_0 = h_0 = a_{m+1} = h_{m+1} = 0$, and sets $\pi(0) = 0, \pi^{-1}(m + 1) = m + 1$):

$$h_i - a_i = h_{z-1(i+1)} - a_{z-1(i+1)} - 1, \quad i = 0, \ldots, m$$
$$h_i \geq 0, \quad i = 1, \ldots, m, \quad a_i \geq 0, \quad i = 1, \ldots, m - 1,$$
$$a_i \leq \min(h_i, h_{i+1}) \text{ for } i \neq m, i \neq \pi^{-1}m,$$
$$a_m \leq h_m, \quad a_m \geq -h_{z-1m}, \quad a_{z-1m} \leq h_{z-1m+1}.$$

The area of a zippered rectangle is given by the expression $\lambda_1 h_1 + \cdots + \lambda_m h_m$. Following Veech, we denote by $\Omega(\mathcal{R})$ the space of all zippered rectangles, corresponding to a given Rauzy class $\mathcal{R}$ and satisfying the condition

$$\lambda_1 h_1 + \cdots + \lambda_m h_m = 1.$$ 

We shall denote by $x$ an individual zippered rectangle.

Veech further defines a map $U$ and a flow $P^t$ on the space of zippered rectangles in the following way:

$$P^t(\lambda, h, a, \pi) = (e^t \lambda, e^{-t} h, e^{-t} a, \pi).$$

$$U(\lambda, h, a, \pi) = \begin{cases} (A^{-1}(a, \pi) \lambda, A^t(a, \pi) h, a', a \pi), & \text{if } (\lambda, \pi) \in \Delta^- \\ (A^{-1}(b, \pi) \lambda, A^t(b, \pi) h, a'', b \pi), & \text{if } (\lambda, \pi) \in \Delta^+. \end{cases}$$
where

\[
  a'_j = \begin{cases} 
    a_i, & \text{if } j < \pi^{-1}m, \\
    h_{x^{-1}m} + a_{m-1}, & \text{if } i = \pi^{-1}m, \\
    a_{i-1}, & \text{other } i.
  \end{cases}
\]

and

\[
  d'_i = \begin{cases} 
    a_i, & \text{if } j < m, \\
    -h_{x^{-1}m} + a_{x^{-1}m-1}, & \text{if } i = m.
  \end{cases}
\]

The map \( \mathcal{U} \) is invertible; \( \mathcal{U} \) and \( P^t \) commute ([1]). Denote

\[
  \tau(\lambda, \pi) = (\log(|\lambda| - \min(\lambda_m, \lambda_{x^{-1}m})),
\]

and for \( x \in \Omega(\mathcal{R}) \), \( x = (\lambda, h, a, \pi) \), write

\[
  \tau(x) = \tau(\lambda, \pi).
\]

Now define

\[
  \mathcal{Y}(\mathcal{R}) = \{ x \in \Omega(\mathcal{R}) : |\lambda| = 1 \}.
\]

and

\[
  \Omega_\theta(\mathcal{R}) = \bigcup_{x \in \mathcal{Y}(\mathcal{R}), 0 \leq t \leq \tau(x)} P^t x.
\]

\( \Omega_\theta(\mathcal{R}) \) is a fundamental domain for \( \mathcal{U} \) and, identifying the points \( x \) and \( \mathcal{U}x \) in \( \Omega_1(\mathcal{R}) \), we obtain a natural flow, also denoted by \( P^t \), on \( \Omega_1(\mathcal{R}) \).

The space \( \Omega(\mathcal{R}) \) has a natural Lebesgue measure class and so does the transversal \( \mathcal{Y}(\mathcal{R}) \). Veech [1] has proved the following Theorem.

\textbf{Theorem 5} There exists a measure \( \mu_\mathcal{R} \) on \( \Omega(\mathcal{R}) \), absolutely continuous with respect to Lebesgue, preserved by both the map \( \mathcal{U} \) and the flow \( P^t \) and such that \( \mu_\mathcal{R}(\Omega_\theta(\mathcal{R})) < \infty \).

For \( x \in \mathcal{Y}(\mathcal{R}) \), define

\[
  S(x) = \mathcal{U} P^\tau(x)(x).
\]

The map \( S \) is a lift of \( \mathcal{T} \) to the space of zippered rectangles; indeed, if

\[
  S(\lambda, h, a, \pi) = (\lambda', h', a', \pi'),
\]

then \( (\lambda', \pi') = \mathcal{T}(\lambda', \pi') \).

Since \( \mathcal{Y}(\mathcal{R}) \) is a transversal to the flow, the measure \( \mu_\mathcal{R} \) induces an absolutely continuous measure \( \mu_\mathcal{R}^{(1)} \) on \( \mathcal{Y}(\mathcal{R}) \); since \( \mu_\mathcal{R} \) is both \( \mathcal{U} \) and \( P^t \)-invariant, the measure \( \mu_\mathcal{R}^{(1)} \) is \( S \)-invariant. Since \( \mu_\mathcal{R}(\Omega_\theta(\mathcal{R})) < \infty \), the measure \( \mu_\mathcal{R}^{(1)} \) is conservative; it is, however, infinite (Veech [1]).
Zorich [4] constructed a different section for the flow $P^t$, for which the restricted measure has finite total mass.

Following Zorich [4], define

$$\Omega^+(\mathcal{R}) = \{x = (\lambda, h, a, \pi) : (\lambda, \pi) \in \Delta^+, a_m \geq 0\}.$$

$$\Omega^-(\mathcal{R}) = \{x = (\lambda, h, a, \pi) : (\lambda, \pi) \in \Delta^-, a_m \leq 0\},$$

$$\mathcal{V}^+(\mathcal{R}) = \mathcal{V}(\mathcal{R}) \cap \Omega^+(\mathcal{R}), \mathcal{V}^-(\mathcal{R}) = \mathcal{V}(\mathcal{R}) \cap \Omega^-(\mathcal{R}), \mathcal{V}^\pm(\mathcal{R}) = \mathcal{V}^+(\mathcal{R}) \cup \mathcal{V}^-(\mathcal{R}).$$

Take $x \in \mathcal{V}^\pm(\mathcal{R}), x = (\lambda, h, a, \pi)$, and define

$$\mathcal{F}(x) = \mathcal{S}^n(\lambda, \pi) x.$$

The map $\mathcal{F}$ is a lift of the map $\mathcal{G}$ to the space of zippered rectangles: if

$$\mathcal{F}(\lambda, h, a, \pi) = (\lambda', h', a', \pi'),$$

then $(\lambda', \pi') = \mathcal{G}(\lambda, \pi)$.

We shall see, moreover, that the map $\mathcal{F}$ can be almost surely (with respect to Lebesgue) identified with the natural extension of the map $\mathcal{G}$ (Section 3).

If $x \in \mathcal{V}^+$, then $\mathcal{F}(x) \in \mathcal{V}^-$, and if $x \in \mathcal{V}^-$, then $\mathcal{F}(x) \in \mathcal{V}^+$. The map $\mathcal{F}$ is the induced map of $\mathcal{S}$ to the subset $\mathcal{V}^\pm(\mathcal{R})$.

Since $\mathcal{V}^\pm(\mathcal{R})$ is a transversal to the flow $P^t$, the measure $\mu_{\mathcal{R}}$ naturally induces an absolutely continuous measure $\nu$ on $\mathcal{V}^\pm(\mathcal{R})$; since $\mu_{\mathcal{R}}$ is both $\mathcal{U}$ and $P^t$-invariant, the measure $\nu$ is $\mathcal{F}$-invariant.

Zorich [4] proved

**Theorem 6** The measure $\nu$ is finite and ergodic for $\mathcal{F}$.

Since the map $\mathcal{G}$ is exact (as is shown in Section 4), the map $\mathcal{F}$ satisfies the $K$-property of Kolmogorov, and, in particular, is strongly mixing. Decay of correlations is proven for the map $\mathcal{F}$ as well. Define a metric on the space of zippered rectangles in the following way. Take two zipperized rectangles $x = (\lambda, h, a, \pi)$ and $x' = (\lambda', h', a', \pi')$. Write

$$d((\lambda, h, a), (\lambda', h', a')) = \log \frac{\max |\lambda_i - \lambda_i'| |h_i - h_i'| |a_i - a_i'|}{\min |\lambda_i - \lambda_i'| |h_i - h_i'| |a_i - a_i'|}.$$

Define the metric on $\Omega(\mathcal{R})$ by

$$d(x, x') = \begin{cases} d((\lambda, h, a), (\lambda', h', a')) & \text{if } \pi = \pi' \text{ and } \frac{a_m}{a_m'} > 0; \\ 2 + d((\lambda, h, a), (\lambda', h', a')), & \text{otherwise.} \end{cases}$$

As above, for $\alpha > 0$, let $H_\alpha$ be the space of functions $\phi : \mathcal{V}^\pm(\mathcal{R}) \to \mathbb{R}$ such that $d(x, x') \leq 1$, then $|\phi(x) - \phi(x')| \leq C d(x, x')^\alpha$ for some constant $C$.

Note that the distance $d(x, x')$ is not defined if $a_i = 0$ or $a_i' = 0$ for some $i = 1, \ldots, m$; nothing, therefore, is said about the values of a function from $H_\alpha$. 

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at such points. This does not represent a problem, however, since we only need
the space \( H_n \) for the Central Limit Theorem, and for such a result we may
deal with functions defined almost everywhere.

Define

\[
C_{H_n}(\phi) = \max_{d(x, x') \leq 1} \frac{|\phi(x) - \phi(x')|}{d(x, x')^\alpha}.
\]

**Theorem 7** Let \( F : \mathcal{Y}^+ (\mathcal{R}) \to \mathcal{Y}^+ (\mathcal{R}) \) be the Rauzy-Veech-Zorich induction
map on the space of zippered rectangles and let \( \mathcal{F} \) be the absolutely continuous
invariant probability measure. Let \( \rho > 2 \). Then, for any \( a > 0 \), there exist
positive constants \( C, \delta \) such that for any \( \phi, \psi \in H_n \cap L_p(\mathcal{Y}(\mathcal{R}), \mathcal{F}) \) we have

\[
|\int \phi \times \psi d\mathcal{F}^2 \, d\mathcal{F} - \int \phi \, d\mathcal{F} \int \psi \, d\mathcal{F}| \leq C \exp(-\delta n^{1/\delta}) (C_{H_n}(\phi) + |\phi|_{L_p}) (C_{H_n}(\psi) + |\psi|_{L_p})
\]

Theorem 7 will be established simultaneously with the Theorem 4. Indeed, the
map \( F \) can be almost surely identified with the natural extension of the map
\( G \), and the method of Markov approximations of of Sinai [13] and Bunimovich-
Sinai [14] allows to obtain the decay of correlations for the invertible case sim-
ultaneously with that for the noninvertible one.

Since the flow \( P^t \) is a special flow over the map \( F \), by the Theorem of
Melbourne and Török [15], the decay of correlations for the map \( F \) allows to
obtain the Central Limit Theorem for the flow \( P^t \).

Denote by \( X_t \) the derivative with respect to the flow \( P^t \).

**Theorem 8** Let \( \rho > 2 \) and let \( \phi \in H_n(\Omega_1(\mathcal{R})), L_p(\Omega_1(\mathcal{R}), \mu_{\mathcal{F}}) \) satisfy \( \int \phi \, d\nu \equiv 0 \). Assume that there does not exist \( \psi \in L_2(\Omega_2(\mathcal{R}), \mu_{\mathcal{F}}) \) such that \( \phi = X_t \psi \).

Then there exists \( \sigma > 0 \) such that

\[
\frac{1}{\sqrt{T}} \int_0^T \phi \circ P^t \, d\mathcal{T} \to N(0, \sigma) \quad \text{as} \quad T \to \infty.
\]

This Theorem will be proved in Section 16.

1.7 Zippered rectangles and the moduli space of holomorphic differentials.

Let \( g \geq 2 \) be an integer. Take an arbitrary integer vector \( \kappa = (k_1, \ldots, k_n) \) such
that \( k_1 > 0, k_1 + \cdots + k_n = 2g - 2 \).

Denote by \( \mathcal{M}_\kappa \) the moduli space of Riemann surfaces of genus \( g \) endowed
with a holomorphic differential of area 1 with singularities of orders \( k_1, \ldots, k_n \).
(the *stratum* in the moduli space of holomorphic differentials). Denote by \( g_t \)
the Teichmüller flow on \( \mathcal{M}_\kappa \) (see [10], [21], [28], [29]). The flow \( g_t \)
preserves a natural absolutely continuous probability measure on \( \mathcal{M}_\kappa \) ([21], [1], [29]). We
denote that measure by \( \mu_\kappa \).
A zippered rectangle naturally defines a Riemann surface endowed with a holomorphic differential of area 1. The orders of the singularities of \( \omega \) are uniquely defined by the Rauzy class of the permutation \( \pi ([1]) \).

For any \( \mathcal{R} \) we thus have a map

\[
\pi_{\mathcal{R}} : \Omega_{\mathcal{R}} \rightarrow \mathcal{M}_\kappa,
\]

where \( \kappa \) is uniquely defined by \( \mathcal{R} \).

Veech [1] proved

**Theorem 9 (Veech)**

1. The set \( \pi_{\Omega_{\mathcal{R}}} (\mathcal{R}) \) is a connected component of \( \mathcal{M}_\kappa \). Any connected component of any \( \mathcal{M}_\kappa \) has the form \( \pi_{\mathcal{R}} (\Omega_{\mathcal{R}} (\mathcal{R})) \) for some \( \mathcal{R} \).

2. The map \( \pi_{\mathcal{R}} \) is finite-to-one and almost everywhere locally bijective.

3. \( \pi_{\Omega_{\mathcal{R}}}(x) = \pi_{\mathcal{R}}(x) \).

4. The flow \( P^t \) on \( \Omega_{\mathcal{R}}(\mathcal{R}) \) projects under \( \pi_{\mathcal{R}} \) to the Teichmüller flow \( g_t \) on the corresponding connected component of \( \mathcal{M}_\kappa \).

5. \( (\pi_{\mathcal{R}})_* \mu_\kappa = \mu_\mathcal{R} \).

A detailed treatment of the relationship between Rauzy classes, zippered rectangles and connected components is given by M. Kontsevich and A. Zorich in [26].

Say that a function \( \psi : \mathcal{M}_\kappa \rightarrow \mathbb{R} \) is Hölder in the sense of Veech if there exists a Hölder function \( \phi : \Omega_{\mathcal{R}}(\mathcal{R}) \rightarrow \mathbb{R} \) such that \( \psi \circ \pi_{\mathcal{R}} = \phi \).

**Remark.** This definition has a natural interpretation in terms of cohomological coordinates of Hubbard and Masur [28]. Indeed, under the map \( \pi_{\mathcal{R}} \) the Veech coordinates on the space of zippered rectangles correspond, up to a linear change of variables, to the cohomological coordinates of Hubbard and Masur. Locally, one can associate a Hilbert metric to those coordinates. A function Hölder in the sense of Veech if and only if it is Hölder with respect to that metric. Note that the thus defined local Hilbert distance between two elements in \( \mathcal{M}_\kappa \) majorates the Teichmüller distance between their underlying surfaces.

Therefore, if a function \( \phi : \mathcal{M}_\kappa \rightarrow \mathbb{R} \) is a lift of a smooth function from the underlying moduli space \( \mathcal{M}_g \) of compact surfaces of genus \( g \), then \( \phi \) is Hölder in the sense of Veech.

Denote by \( X_t \) the derivative in the direction of the flow \( g_t \).

Theorem 8 and Theorem 9 imply the following

**Theorem 10** Let \( \mathcal{H} \) be a connected component of \( \mathcal{M}_\kappa \). Let \( p > 2 \), and let \( \psi \in L_p(\mathcal{H}, \mu_\kappa) \) be Hölder in the sense of Veech and satisfy \( \int \phi d\mu_\kappa = 0 \). Assume that there does not exist \( \psi \in L_2(\mathcal{H}, \mu_\kappa) \) such that \( \phi = X_t \psi \). Then there exists \( \sigma > 0 \) such that

\[
\frac{1}{\sqrt{T}} \int_0^T \phi \circ g_t dt \overset{d}{\to} \mathcal{N}(0, \sigma) \quad \text{as} \quad T \to \infty.
\]
1.8 Outline of the Proof of Theorem 4.

First, one takes a subset of the space $\Delta(R)$ such that the induced map of $G$ is uniformly expanding (namely, the set of all interval exchanges such that the renormalization matrix for them is a fixed matrix all whose elements are positive, see Proposition 4; note that the return map on such a subset is an essential element in Veech’s proof of unique ergodicity [1]). Then one estimates the statistics of return times in this subset, in the spirit of Lai-Sang Young [11]. After that, the method of Markov approximations, due to Sinai [13], Bunimovich and Sinai [14], is used to complete the proof.

The paper is organized as follows. In Section 2, we state auxiliary propositions about unimodular matrices. In Section 3, following Veech [1] and Zorich [4], we construct symbolic dynamics for the Rauzy-Veech-Zorich induction map $G$, compute its transition probabilities in the sense of Sinai [13], and identify the natural extension of $G$ with $F$. In Section 4, we establish the exactness of $G^2$. In Section 6, we state the main Lemma 4, whose proof takes Sections 6 – 10. In the remainder of the paper we apply the Markov approximation method of Sinai [13], Bunimovich and Sinai [14], in order to obtain the decay of correlations for $G$ and $F$. In the final Section, we apply the Theorem of Melbourne and Török to obtain the Central Limit Theorem for the Teichmüller flow.

2 Matrices

Let $A$ be an $m \times m$-matrix with positive entries.

Denote

$$|A| = \sum_{i,j=1}^{m} A_{ij}$$

$$col(A) = \max_{i,j,k} \frac{A_{ij}}{A_{kj}}$$

$$row(A) = \max_{i,j,k} \frac{A_{ij}}{A_{ik}}$$

**Proposition 2** Let $Q$ be a matrix with positive entries, $A$ a matrix with non-negative entries without zero columns or rows.

Then all entries of the matrices $AQ$ and $QA$ are positive, and, moreover, we have

$$row(AQ) \leq row(Q), col(QA) \leq col(Q)$$

**Corollary 2** Let $Q$ be a matrix with positive entries, $A$ a matrix with non-negative entries without zero columns or rows.

$$row(QAQ) \leq row(Q), col(QAQ) \leq col(Q)$$

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Let $A$ be an $m \times m$ matrix with nonnegative entries and determinant 1. Consider the map $J_A : \Delta_{m-1} \to \Delta_{m-1}$ given by

$$J_A(\lambda) = \frac{A\lambda}{|A\lambda|}.$$  

Then

$$\det DJ_A(\lambda) = \frac{1}{|A\lambda|^m}. \quad (5)$$

Suppose all entries of $A$ are positive; then, for any $\lambda, \lambda' \in \Delta_{m-1}$, we have

$$\row(A)^{-m} \leq \frac{\det DJ_A(\lambda)}{\det DJ_A(\lambda')} \leq \row(A)^m, \quad (6)$$

whence we have the following

**Proposition 3** Let $C \subset \Delta_{m-1}$ and let $A$ be a matrix with positive entries and determinant 1. Then

$$\row(A)^{-m} \frac{\mathbf{m}(C_1)}{\mathbf{m}(C_2)} \leq \frac{\mathbf{m}(J_A(C_1))}{\mathbf{m}(J_A(C_2))} \leq \row(A)^m \frac{\mathbf{m}(C_1)}{\mathbf{m}(C_2)}.$$  

We also note the following well-known Lemma (see, for example, [17]):

**Lemma 1** Suppose all entries of the matrix $A$ are positive. Then the map $J_A$ is uniformly contracting with respect to the Hilbert metric.

3 Symbolic dynamics for $G$.

First, following Veech [1] and Zorich [4], we describe a Markov partition and a symbolic dynamics for the map $G^\sigma$, then we identify almost surely the induction map $F$ on the space of zippered rectangles with the natural extension of $G$, and, finally, we compute for $G$ its transition rectangles in the sense of Sinai [25].

3.1 The alphabet

Let $\pi \in R$, and let $n$ be a positive integer.

Set

$$A(a, n, \pi) = \{ \lambda : \text{there exists } (\lambda', \pi') \text{ such that } \lambda' \in \Delta_1^+, \text{ and } (\lambda, \pi) = t_{a-z}(\lambda', \pi') \}$$

$$\Delta(a, n, \pi) = \{ (\lambda, \pi), \lambda \in A(a, n, \pi) \}$$

In other words, $\Delta(a, n, \pi)$ is the set of interval exchange transformations such that the application of the Zorich induction results in the application of the $a$-operation $n$ times.

The sets $\Delta(a, n, \pi)$ and $\Delta(a, n', \pi')$ are disjoint unless $n = n'$, $\pi = \pi'$, and
\[ \Delta^* = \bigcup_{n=1}^{\infty} \Delta(a, n, \pi) \]
up to a set of measure zero (namely, a union of countably many hyperplanes on which Zorich induction is not defined).

If \( \pi' = a^n \pi \), then we have

\[ G\Delta(a, n, \pi) = \Delta^*_n. \]

Similarly, for \( \pi \in \mathcal{R} \), and \( n \) a positive integer, set

\[ \Lambda(b, n, \pi) = \{ \lambda : \text{there exists } (\lambda', \pi') \text{ such that } \lambda' \in \Delta^*_n, \text{ and } (\lambda, \pi) = t_{b^{-n}}(\lambda', \pi') \}. \]

\[ \Delta(b, n, \pi) = \{ (\lambda, \pi), \lambda \in \Lambda(b, n, \pi) \}. \]

In other words, \( \Delta(b, n, \pi) \) is the set of interval exchange transformations such that the application of the Zorich induction results in the application of the \( b \)-operation \( n \) times.

The sets \( \Delta(b, n, \pi) \) and \( \Delta(b, n', \pi') \) are disjoint unless \( n = n' \), \( \pi = \pi' \), and

\[ \Delta^*_n = \bigcup_{n=1}^{\infty} \Delta(b, n, \pi) \]

up to a set of measure zero (namely, a union of countably many hyperplanes on which the Zorich induction is not defined).

If \( \pi' = b^n \pi \), then, clearly,

\[ G(\Delta(b, n, \pi)) = \Delta^*_n. \]

Note that the sets \( \Delta(a, n, \pi) \) and \( \Delta(b, n', \pi') \) are always disjoint, since we have \( \Delta(a, n, \pi) \subset \Delta^*_n \), \( \Delta(b, n', \pi') \subset \Delta^*_n \).

The sets \( \Delta(a, n, \pi) \), \( \Lambda(b, n, \pi) \), for all \( n > 0 \) and all \( \pi \in \mathcal{R} \), form a Markov partition for \( G \).

### 3.2 Words

Consider the alphabet

\[ A = \{ (c, n, \pi) : c = a \text{ or } b \} \]

For \( w_1 \in A \), \( w_1 = (c_1, n_1, \pi_1) \), we write \( c_1 = c(w_1) \), \( n_1 = n(w_1) \).

For \( w_1, w_2 \in A \), \( w_1 = (c_1, n_1, \pi_1) \), \( w_2 = (c_2, n_2, \pi_2) \), define the function \( B(w_1, w_2) \) in the following way: \( B(w_1, w_2) = 1 \) if \( c_1^{n_1} \pi_1 = \pi_2 \) and \( c_1 \neq c_2 \) and \( B(w_1, w_2) = 0 \) otherwise.

Let

\[ W_{A,B} = \{ w = w_1 \ldots w_n, w_i \in A, B(w_i, w_{i+1}) = 1 \text{ for all } i = 1, \ldots, n \}. \]
For $w_1 \in \mathcal{A}$, $w_1 = (c_1, n_1, \pi_1)$, set
\[
A(w) = A(c_1, c_1^{-n_1} \pi_1) \ldots A(c_1, c_1^{-1} \pi_1)A(c_1, \pi_1),
\]
and for $w \in W_{A,B}$, $w = w_1 \ldots w_n$, set
\[
A(w) = A(w_1) \ldots A(w_n).
\]

Also, for $w_1 \in \mathcal{A}$, $\pi \in \mathcal{R}$, set $w_1^{-1} \pi = c_1^{-n_1} \pi$, and for $w \in W_{A,B}$, $w = w_1 \ldots w_n$, set
\[
w^{-1} \pi = w_1^{-1} \ldots w_n^{-1} \pi.
\]

For $w \in W_{A,B}$, define a map $t_w : \Delta(\mathcal{R}) \to \Delta(\mathcal{R})$ by
\[
t_w(\lambda, \pi) = \left( \frac{A(w|\lambda)}{|A(w)|^{w^{-1} \pi}} \right)
\]

Consider also the map
\[
T_w(\lambda, \pi) = (A(w)\lambda, w^{-1} \pi)
\]

For $w_1 \in \mathcal{A}$, $w_1 = (c_1, n_1, \pi_1)$, we write $\Delta(w_1) = \Delta(c_1, n_1 \pi_1)$.

For $w \in W_{A,B}$, $w = w_1 \ldots w_n$, denote
\[
\Delta(w) = t_w(\Delta(\mathcal{R})).
\]

Then, by definition,
\[
\Delta(w) = \{ (\lambda, \pi) : (\lambda, \pi) \in \Delta(w_1), \mathcal{G}(\lambda, \pi) \in \Delta(w_2), \ldots, \mathcal{G}^{n-1}(\lambda, \pi) \in \Delta(w_n) \}.
\]

Say that $w_1 \in \mathcal{A}$ is compatible with $(\lambda, \pi) \in \Delta(\mathcal{R})$ if
1. either $\lambda \in \Delta^\pm$, $c_1 = a$, and $a^{n_1} \pi_1 = \pi$
2. or $\lambda \in \Delta^\pm$, $c_1 = b$, and $b^{n_1} \pi_1 = \pi$.

Say that a word $w \in W_{A,B}$, $w = w_1 \ldots w_n$ is compatible with $(\lambda, \pi)$ if $w_n$ is compatible with $(\lambda, \pi)$.

We can write
\[
\mathcal{G}^{-n}(\lambda, \pi) = \{ t_w(\lambda, \pi) : |w| = n \text{ and } w \text{ is compatible with } (\lambda, \pi) \}.
\]

Suppose that a word $w \in W_{A,B}$ is compatible with both $(\lambda, \pi)$ and $(\lambda', \pi)$. Then
\[
d(t_w(\lambda, \pi), t_w(\lambda', \pi)) \leq d((\lambda, \pi), (\lambda', \pi')).
\]

If, moreover, all entries of the the matrix $A(w)$ are positive, then, by Lemma 1, there exists $\alpha(w)$, $0 < \alpha(w) < 1$, such that
\[
d(t_w(\lambda, \pi), t_w(\lambda', \pi)) \leq \alpha(w)d((\lambda, \pi), (\lambda', \pi')).
\]

We therefore have

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Proposition 4 Let $w \in W_{A,B}$ be such that all entries of the matrix $A(w)$ are positive. Then the return map of $\mathcal{G}$ on $\Delta(w)$ is uniformly expanding with respect to the Hilbert metric.

3.3 Sequences

Now let

$$\Omega_{A,B} = \{ \omega = \omega_1 \ldots \omega_n, \ldots, \omega_n \in A, B(\omega_n, \omega_{n+1}) = 1 \text{ for all } n \in \mathbb{N} \}$$

and

$$\Omega_{A,B}^\mathbb{Z} = \{ \omega = \ldots \omega_n \ldots \omega_n, \ldots, \omega_n \in A, B(\omega_n, \omega_{n+1}) = 1 \text{ for all } n \in \mathbb{Z} \}$$

Denote by $\sigma$ the shift on both these spaces.

There is a natural map $\Phi : \Delta \to \Omega_{A,B}$ given by the formula

$$\Phi(\lambda, \pi) = \omega_1 \ldots \omega_n \ldots$$

if

$$\mathcal{G}^n(\lambda, \pi) \in \Delta(\omega_n)$$

The measure $\nu$ projects under $\Phi$ to a $\sigma$-invariant measure on $\Omega_{A,B}$; probability with respect to that measure will be denoted by $\mathbb{P}$.

For $w \in W_{A,B}$, $w = w_1 \ldots w_n$, let

$$C(w) = \{ \omega \in \Omega_{A,B} : \omega_1 = w_1, \ldots, \omega_n = w_n \}.$$  

We have then

$$\Delta(w) = \Phi^{-1}(C(w)).$$

W. Veech [1] has proved the following

Proposition 5 The map $\Phi$ is $\nu$-almost surely bijective.

We thus obtain a symbolic dynamics for the map $\mathcal{G}$.

3.4 The natural extension.

Consider the natural extension for the map $\mathcal{G}$.

The phase space is the space of sequences of interval exchanges; it will be convenient to number them by negative integers. We set:

$$\overline{\Sigma}(\mathcal{R}) =$$

\{ x = \ldots (\lambda(-n), \pi(-n)), \ldots, (\lambda(0), \pi(0)) | \mathcal{G}(\lambda(-n), \pi(-n)) = (\lambda(1-n), \pi(1-n)), n = 1, \ldots \}$$

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The map $\mathcal{G}$ and the invariant measure $\nu$ are extended to $\overline{\Sigma}$ in the natural way. We shall still denote the probability with respect to the extended measure by $\P$.

We extend the map $\Phi$ to a map

$$\overline{\Phi} : \overline{\Sigma} \rightarrow \Omega^\infty_{A, B},$$

$$\overline{\Phi}(\lambda) = \ldots \omega_{-n}, \ldots \omega_0 \ldots \omega_n \ldots ,$$

if $(\lambda(-n), \pi(-n)) \in \Delta(\omega_{-n})$, and $\mathcal{G}^0(\lambda(0), \pi(0)) \in \Delta(\omega_0)$.

Now take a zippered rectangle $x \in \Omega(\mathcal{R})$, $x = (\lambda, h, a, \pi)$. Set $\mathcal{F}^n(x) = (\lambda(n), h(n), a(n), \pi(n))$.

Consider a map

$$\hat{\Phi} : \mathcal{Y}(\mathcal{R}) \rightarrow \Omega^\infty_{A, B},$$

given by

$$(\lambda, h, a, \pi) \rightarrow \ldots \omega_{-n} \ldots \omega_0 \ldots \omega_n \ldots ,$$

where

$$(\lambda(n), \pi(n)) \in \Delta(\omega_n)$$

for all $n \in \mathbb{Z}$.

Under the natural projection $(\lambda, h, a, \pi) \rightarrow (\lambda, \pi)$, the $\mathcal{F}$-invariant measure $\mathcal{P}$ on $\mathcal{Y}^\infty(\mathcal{R})$ is mapped to the $\mathcal{G}$-invariant measure $\nu$ on $\Delta(\mathcal{R})$, whence the measure $\hat{\Phi} \cdot \mathcal{P}$ is exactly the probability measure $\P$ on the space of bi-infinite sequences. To complete the identification of the spaces $(\mathcal{Y}^\infty(\mathcal{R}), \mathcal{P})$ and $(\Omega^\infty_{A, B}, \P)$, it remains to show that almost surely there is at most one zippered rectangle corresponding to a given symbolic sequence.

**Proposition 6** Let $q \in W_{A, B}$ be such that all entries of the matrix $A(q)$ are positive. Let $\omega \in \Omega^\infty_{A, B}$ be such that the word $q$ occurs infinitely many times in $\omega$. Then there exists at most one zippered rectangle corresponding to $\omega$.

**Proof.** Write

$$\omega = \ldots \omega_{-n} \ldots \omega_0 \ldots \omega_n \ldots ,$$

and let $(\lambda, h, a, \pi)$ be a zippered rectangle corresponding to $\omega$; we want to show that $(\lambda, h, a, \pi)$ is uniquely defined by $\omega$.

First, $(\lambda, \pi)$ is uniquely defined by the "future" $\omega_0 \ldots \omega_n \ldots$ of $\omega$.

Denote $w(n) = \omega_{-n} \ldots \omega_0$, $(\lambda(-n), h(-n), a(-n), \pi(-n)) = \mathcal{F}^{-n}(\lambda, h, a, \pi)$.

For any $n$, the interval exchange $(\lambda(-n), \pi(-n))$ corresponds to the symbolic sequence $\omega_{-n} \ldots \omega_0 \ldots$, and, again, is uniquely defined by that sequence.

By definition of the map $\mathcal{F}$, we have

$$\lambda(-n) = \frac{A(w(n))\lambda}{|A(w(n))\lambda|}, \quad h(-n) = (A(w(n))^t)^{-1}h \cdot |A(w(n))\lambda|.$$
Projectively, therefore, we have

\[ \mathbb{R}_+ h \subset A(w(n))\mathbb{R}_+^m. \]

Since the subword \( q \) occurs infinitely many times, the intersection

\[ \bigcap_{n=1}^{\infty} A(w(n))\mathbb{R}_+^m \]

consists of a single line and the vector \( h \) is therefore uniquely determined by the condition \( \langle \lambda, h \rangle = 1 \).

It remains to determine the vector \( a \).

By definition of the map \( F \), for any \( n \) there exists an orthogonal matrix \( U(-n) \), uniquely determined by \( \omega \), and a vector \( v(-n) \), uniquely determined by the the vectors \( h(-n), \ldots, h(0) \) and \( \omega \), such that

\[ \frac{U(-n)a(-n) + v(-n)}{|A(w(n))\lambda|} = a. \tag{8} \]

Now let \( n \) be a moment such that all \( \lambda(-n)_i > \frac{1}{100m} \) (there are infinitely many such moments). Then \( |a(-n)_i| < 100m \) for all \( i = 1, \ldots, m \) and, \( (8) \) since \( |A(w(n))\lambda| \to \infty \) as \( n \to \infty \), \( (8) \) implies that \( a \) is also uniquely determined by \( \omega \).

The proof is complete.

### 3.5 Transition probabilities.

Take a sequence \( c_1 \ldots c_n \ldots \in \Omega_{A,B} \). Following Sinai [25], consider the transition probability

\[ \mathbb{P}(\omega_1 = c_1 | \omega_2 = c_2, \ldots, \omega_n = c_n, \ldots) = \lim_{n \to \infty} \frac{\mathbb{P}(c_1 \ldots c_n)}{\mathbb{P}(c_2 \ldots c_n)}. \]

In this subsection, we give a formula for this probability in terms of \( (\lambda, \pi) = \Phi^{-1}(c_1 \ldots c_n \ldots) \).

Assume \( w_1 \in A \) is compatible with \( (\lambda, \pi) \).

Denote

\[ \mathbb{P}(w_1 | (\lambda, \pi)) = \mathbb{P}(((\lambda(-1), \pi(-1)) = t_{w_1}(\lambda(0), \pi(0)) | (\lambda(0), \pi(0)) = (\lambda, \pi)). \]

If \( w_1 \in A \) is compatible with \( (\lambda, \pi) \), from the definition of \( G \) and from (5) we have

\[ \mathbb{P}(w_1 | (\lambda, \pi)) = \frac{\rho(t_{w_1}(\lambda, \pi))}{\rho(\lambda, \pi)|A(w_1)\lambda|^m} \tag{9} \]

Since the invariant density is a homogeneous function of degree \( -m \), we have

\[ \rho(T_{w_1}(\lambda, \pi)) = \frac{\rho(t_{w_1}(\lambda, \pi))}{|A(w_1)\lambda|^m}, \]

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and we can rewrite (9) as follows:

$$\mathbb{P}(w_1 | (\lambda, \pi)) = \frac{\rho(T_{w_1} (\lambda, \pi))}{\rho(\lambda, \pi)}$$  \hspace{1cm} (10)$$

Let \( w = w_1 \ldots w_n \) be compatible with \((\lambda, \pi)\).

Denote

$$\mathbb{P}(w | (\lambda, \pi)) = \mathbb{P}((\lambda(-k), \pi(-k)) = t_{w_{n-k+1}} (\lambda(1-k), \pi(1-k)), k = 1, \ldots, n | (\lambda(0), \pi(0)) = (\lambda, \pi)).$$

From (9), by induction, we have

$$\mathbb{P}(w | (\lambda, \pi)) = \frac{\rho(t_w (\lambda, \pi))}{\rho(\lambda, \pi) |A(w)|^m}$$  \hspace{1cm} (11)$$

Since the invariant density is a homogeneous function of degree \(-m\), we have

$$\rho(T_w (\lambda, \pi)) = \frac{\rho(t_w (\lambda, \pi))}{|A(w)|^m},$$

and we can rewrite (11) as follows:

$$\mathbb{P}(w | (\lambda, \pi)) = \frac{\rho(T_w (\lambda, \pi))}{\rho(\lambda, \pi)}$$  \hspace{1cm} (12)$$

**Corollary 3** There exists \( C > 0 \) such that the following is true. Suppose \( w \in W_{\lambda,B} \) is compatible with \((\lambda, \pi)\). Then

$$\mathbb{P}(w | (\lambda, \pi)) \geq \frac{C}{\rho(\lambda, \pi) |A(w)|^m}$$

Proof: recall that the invariant density is a positive homogeneous function of degree \(-m\) and therefore is bounded from below: there exists \( C > 0 \) such that \( \rho(\lambda, \pi) > C \) for all \((\lambda, \pi) \in \Delta(\mathcal{R})\). In particular, \( \rho(t_w (\lambda, \pi)) > C \). Substituting into (11), we obtain the result.

For \( \epsilon : 0 < \epsilon < 1 \), let

$$\Delta_\epsilon = \{(\lambda, \pi) \in \Delta(\mathcal{R}), \min |\lambda_i| \geq \epsilon \}.$$

For any \( \epsilon > 0 \) there exists a constant \( C(\epsilon) \) such that for any \((\lambda, \pi) \in \Delta_\epsilon \), we have \( \rho(\lambda, \pi) < C(\epsilon) \).

**Corollary 4** For any \( \epsilon > 0 \) there exists \( C(\epsilon) > 0 \) such that if \((\lambda, \pi) \in \Delta_\epsilon \), then

$$\mathbb{P}(w | (\lambda, \pi)) \geq \frac{C(\epsilon)}{|A(w)|^m}.$$
4 Proof of the Exactness

First, one notes that the discrete parameter $\pi$ does not give rise to any period, and then the proof follows the standard pattern [27, 17]: since almost every point of any measurable subset is a density point, bounded distortion estimates of Proposition 3 imply that if the measure of a tail event is positive, then it must be arbitrarily close to 1.

In more detail, observe that there exists an integer $M$ such that for any $n > M$ and for any $\pi, \pi' \in \mathcal{R}$ there exist $k_1, \ldots, k_{2n}$ such that $a^{k_1} b^{k_2} \ldots a^{k_{2n-1}} b^{k_{2n}} \pi = \pi'$. This follows from connectedness of the Rauzy graph and the fact that for any $\pi \in \mathcal{R}$ there exist $n_1, n_2$ such that $a^{n_1} \pi = b^{n_2} \pi = \pi$.

Let $\alpha_0$ be the partition of $\Delta^+$ into $\Delta^+_\pi$, $\pi \in \mathcal{R}$, and let $\alpha_n$ be the partition into the cylinders $\Delta(w)$, where $w \in \mathcal{W}_{\mathcal{A}, \mathcal{B}}$, $|w| = 2n$.

**Lemma 2** There exists $k > 0$ such that the following is true. Suppose $C \subset \Delta^+$, and there exists $\pi \in \mathcal{R}$ such that $\Delta^+_\pi \subset C$. Then $\mathcal{G}^{2k} C = \Delta^+(\mathcal{R})$.

This implies

**Lemma 3** There exists $k > 0$ such that the following holds. For any $\varepsilon > 0$ there is $\delta > 0$ such that for any $C \subset \Delta^+(\mathcal{R})$ satisfying $\mathcal{m}(C \triangle \Delta^+_\pi) < \delta$, we have $\mathcal{m}(\mathcal{G}^{2k} C \triangle \Delta^+) < \varepsilon$.

Now suppose $C \subset \Delta^+$ is a $\mathcal{G}^2$-tail event, i.e., for any $n > 0$ there exists $B_n$ such that $C = \mathcal{G}^{2n} B_n$ and $0 < \nu(C) < 1$. Then $\nu(B_n) \sim \nu(C)$ and, by Lemma 3, we can assume that there exists $\varepsilon > 0$ such that for any $\pi \in \mathcal{R}$, we have

$$\mathcal{m}((\Delta^+ \setminus C) \cap \Delta^+_\pi) \geq \varepsilon$$ (13)

Let $q = q_1 \ldots q_k$ be a word such that the matrix $A(q)$ is positive.

For almost any $(\lambda, \pi) \in C$ we have

$$\lim_{n \to \infty} \frac{\mathcal{m}(\alpha_n(\lambda, \pi) \cap C)}{\mathcal{m}(\alpha_n(\lambda, \pi))} = 1$$ (14)

Now let $n$ be such that $\mathcal{G}^{2n}(\lambda, \pi) \in \Delta(q)$. Denote $(\lambda', \pi') = \mathcal{G}^{2n}(\lambda, \pi)$. Let $A$ be the corresponding renormalization matrix, that is, $\lambda = J_A \lambda'$. Then $A = A_1 A(q)$ for some (unimodular nonnegative integer) matrix $A_1$. We have $\alpha_n(\lambda, \pi) = J_A(\Delta^+_\pi)$. By Proposition 3, from (13), we deduce that there exists $\varepsilon'$, not depending on $n$ such that

$$\frac{\mathcal{m}(\alpha_n(\lambda, \pi) \cap (\Delta^+ \setminus C))}{\mathcal{m}(\alpha_n(\lambda, \pi))} \geq \varepsilon'.$$

Since, by ergodicity, for almost any $(\lambda, \pi)$ we can find infinitely many $n$ such that $\mathcal{G}^{2n}(\lambda, \pi) \in \Delta(q)$, we arrive at a contradiction with (14), which gives the exactness of $\mathcal{G}^2$. 
5 The Main Lemma

We shall suppose from now on that the Rauzy class $\mathcal{R}$ is fixed and will often suppress it from notation.

For $\epsilon : 0 < \epsilon < 1$, define, in the same way as above,

$$\Delta_\epsilon = \{ (\lambda, \pi) \in \Delta(\mathcal{R}), \min |\lambda_i| \geq \epsilon\}.$$ 

Lemma 4 There exist positive constants $\gamma, K, p$ such that the following is true for any $\epsilon > 0$. Suppose $(\lambda, \pi) \in \Delta_\epsilon$. Then

$$\mathbb{P}\{ \exists n \leq K | \log \epsilon \| (\lambda(-n), \pi(-n)) \in \Delta_\gamma | (\lambda(1), \pi(1)) = (\lambda, \pi) \} \geq p.$$ 

From Corollary 4, we obtain

Corollary 5 Let $q \in W_{\Delta, R}$, $q = q_1 \ldots q_l$ be such that all entries of the matrix $A(q)$ are positive. Then there exist positive constants $K(q), p(q)$ such that the following is true for any $\epsilon > 0$. Suppose $(\lambda, \pi) \in \Delta_\epsilon$. Then

$$\mathbb{P}\{ \exists n \leq K(q) | \log \epsilon \| (\lambda(-n), \pi(-n)) \in \Delta_\gamma | (\lambda(1), \pi(1)) = (\lambda, \pi) \} \geq p(q).$$ 

Informally, the proof of Lemma 4 proceeds by getting rid of small intervals. For $\gamma > 0$, $k \leq m$, denote

$$\Delta_{\gamma, k} = \{ (\lambda, \pi) : \exists i_1, \ldots, i_k : \lambda_{i_1}, \ldots, \lambda_{i_k} \geq \gamma \}.$$ 

and

$$\Delta_{\gamma, k, \epsilon} = \{ (\lambda, \pi) : \lambda_i \geq \epsilon \text{ for all } i = 1, \ldots, m \text{ and } \exists i_1, \ldots, i_k : \lambda_{i_1}, \ldots, \lambda_{i_k} \geq \gamma \}.$$ 

Lemma 4 follows from

Lemma 5 There exist constants $L, K, p$, depending only on the Rauzy class, such that the following is true for any $\gamma, k, \epsilon$.

Assume $(\lambda, \pi) \in \Delta_{\gamma, k, \epsilon}$.

Then

$$\mathbb{P}\{ \exists n \leq K | \log \epsilon \| (\lambda(-n), \pi(-n)) \in \Delta_{\gamma/L, k+1, \epsilon/L} | (\lambda(1), \pi(1)) = (\lambda, \pi) \} \geq p.$$ 

Lemma 5 is proved in the next four sections.
6 An estimate on the number of Rauzy operations.

Recall that, if $\lambda, \pi \in \Delta^\pm$, then the $G$-preimages of $(\lambda, \pi)$ are the exchanges $t_{a^{-n}}(\lambda, \pi), n = 1, \ldots$ whereas if $\lambda, \pi \in \Delta^-$, then the $G$-preimages of $(\lambda, \pi)$ are the exchanges $t_{b^{-n}}(\lambda, \pi), n = 1, \ldots$

Denote

$$p_n(\lambda, \pi) = \begin{cases} \mathbb{P}((\lambda(-1), \pi(-1)) = t_{a^{-n}}(\lambda, \pi)((\lambda(0), \pi(0)) = (\lambda, \pi)), & \text{if } (\lambda, \pi) \in \Delta^+; \\
\mathbb{P}((\lambda(-1), \pi(-1)) = t_{b^{-n}}(\lambda, \pi)((\lambda(0), \pi(0)) = (\lambda, \pi)), & \text{if } (\lambda, \pi) \in \Delta^-.
\end{cases}$$

For $\lambda \in \mathbb{R}_{\frac{1}{\Delta^+}}$, set

$$T_{a^{-}\lambda}(\lambda) = A(a^{-1}, a)\lambda, \quad t_{a^{-}\lambda}(\lambda) = \frac{A(a^{-1}, a)\lambda}{A(a^{-1}, a)\lambda},$$

and

$$T_{b^{-}\lambda}(\lambda) = T_{a^{-}\lambda}(\lambda) \cdots T_{a^{-}\lambda}(\lambda), \quad t_{b^{-}\lambda}(\lambda) = t_{a^{-}\lambda}(\lambda) \cdots t_{a^{-}\lambda}(\lambda),$$

so that we have

$$t_{a^{-n}}(\lambda, \pi) = (t_{a^{-}\lambda}(\lambda, a^{-n} \pi), T_{a^{-n}}(\lambda, \pi) = (T_{a^{-}\lambda}(\lambda, a^{-n} \pi),$$

$$t_{b^{-n}}(\lambda, \pi) = (t_{b^{-}\lambda}(\lambda, b^{-n} \pi), T_{b^{-n}}(\lambda, \pi) = (T_{b^{-}\lambda}(\lambda, b^{-n} \pi).$$

Lemma 6 If $(\lambda, \pi) \in \Delta^+$, then, for any $N \geq 1$, we have

$$\sum_{n=N+1}^{\infty} p_n(\lambda, \pi) = \frac{\rho_{a^{-}\lambda}^{-1} T_{a^{-}\lambda}(\lambda)}{\rho_{a^{-}\lambda}^{-1}}(\lambda)$$

If $(\lambda, \pi) \in \Delta^-$, then, for any $N \geq 1$, we have

$$\sum_{n=N+1}^{\infty} p_n(\lambda, \pi) = \frac{\rho_{b^{-}\lambda}^{-1} T_{b^{-}\lambda}(\lambda)}{\rho_{b^{-}\lambda}^{-1}}(\lambda)$$

Proof: We only consider the case $(\lambda, \pi) \in \Delta^+$. In this case, the formula (10) can be written as

$$p_n(\lambda, \pi) = \frac{\rho_{a^{-}\lambda}^{-1} (T_{a^{-}\lambda}(\lambda))}{\rho_{a^{-}\lambda}^{-1}}(\lambda),$$

whence we can write
\[
\rho^+_\pi (\lambda) = \sum_{n=1}^{\infty} \rho_{a-n \pi}^-(T^{(\pi)}_{a-n} \lambda). \tag{15}
\]

Note that this formula is true for any permutation \(\pi\) and any \(\lambda\) (i.e., even if \(\lambda \notin \Delta^+_a\), the formula, being an identity between rational functions, still holds).

Since, for any \(\lambda\), we have

\[
T^{(\pi)}_{a-n N} \lambda = T^{(a^{-n} \pi)}_{a-n} (T^{(\pi)}_{a-n} \lambda),
\]

from (15) we obtain

\[
\rho^+_{a-N \pi} (T^{(\pi)}_{a-N} \lambda) = \sum_{n=1}^{\infty} \rho_{a-n \pi}^- T^{(\pi)}_{a-n-N} \lambda = \rho^+_\pi (\lambda)( \sum_{n=N+1}^{\infty} \mathbf{P}_n(\lambda, \pi)),
\]

and the Lemma is proved.

### 6.1 Bounded growth

Let \((\lambda, \pi) \in \Delta(\mathcal{R})\).

Define

\[
(\lambda^{(n)}, \pi^{(n)}) = \begin{cases} 
    t_{a-s}(\lambda, \pi), & \text{if } (\lambda, \pi) \in \Delta^+; \\
    t_{b-s}(\lambda, \pi), & \text{if } (\lambda, \pi) \in \Delta^-.
\end{cases}
\]

\[
(\Lambda^{(n)}, \pi^{(n)}) = \begin{cases} 
    T_{a-s}(\lambda, \pi), & \text{if } (\lambda, \pi) \in \Delta^+; \\
    T_{b-s}(\lambda, \pi), & \text{if } (\lambda, \pi) \in \Delta^-.
\end{cases}
\]

We have

\[
\mathcal{G}^{-1}(\lambda, \pi) = \{(\lambda^{(n)}, \pi^{(n)}), n = 1, \ldots\}.
\]

and

\[
\mathbf{P}_n = \mathbb{P}( (\Lambda(-1), \pi(-1)) = (\lambda^{(n)}, \pi^{(n)})) | (\lambda(0), \pi(0)) = (\lambda, \pi)).
\]

For any \(n \in \mathbb{N}\), there exists \(i(n) \in \{1, \ldots, m\}\) such that

\[
|\Lambda^{(n)}| - |\Lambda^{(n-1)}| = \lambda_{i(n)}.
\]

If \((\lambda(-1), \pi(-1))\) is a \(\mathcal{G}\)-preimage of \((\lambda, \pi)\) and \((\lambda(-1), \pi(-1)) = t_{c-s}(\lambda, \pi),\ c = a\ or\ b,\) then we define a vector \(\Lambda(-1)\) by the relation \((\Lambda(-1), \pi(-1)) = T_{c-s}(\lambda, \pi)\) (in other words, \((\Lambda(-1), \pi(-1))\) is the Zorich preimage without normalization).

**Lemma 7** There exists a constant \(C(\mathcal{R})\), depending on the Rauzy class only, such that for any \((\lambda, \pi) \in \Delta(\mathcal{R})\) we have

\[
\mathbb{P}( |\Lambda(-1)| > K |(\lambda(0), \pi(0)) = (\lambda, \pi)) < \frac{C(\mathcal{R})}{K - 2}
\]

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For definiteness, assume \( \lambda \in \Delta^r \) (the proof is completely identical in the other case). Then \( \mathcal{G} \)-preimages of \((\lambda, \pi)\) are \((\lambda^{(r)}, \pi^{(r)}) = t_{b-n}(\lambda, \pi), n = 1, 2, \ldots\).

By construction [4], the invariant density \( \rho^r_\pi \) has the form

\[
\rho^r_\pi(\lambda) = \sum_{i=1}^{N} \frac{1}{l_i(\lambda)^{l_i(\lambda)} \cdots l_m(\lambda)}
\]

where the functions \( l_{ij} \) are linear:

\[
l_{ij}(\lambda) = a^{(1)}_{ij} \lambda_1 + \cdots + a^{(m)}_{ij} \lambda_m,
\]

and all \( a^{(r)}_{ij} \) are nonnegative (in fact, \( a^{(r)}_{ij} = 0 \) or 1, but we do not need this fact here).

Let \( l \) be the length of the \( a \)-cycle of \( \pi \), that is, the smallest such number that \( d^l \pi = \pi \).

Since for any \( k > 0 \) we have \( a^{-kl} \pi = \pi \), from Lemma 6 we obtain

\[
\sum_{n=kl+1}^{\infty} p_{n}(\lambda, \pi) = \frac{\rho^r_\pi(\Lambda^{(kl)})}{\rho^r_\pi(\lambda)}.
\]

As noted above, for any \( n > 0 \) there exists \( \lambda_i(n) \) such that

\[
|\Lambda^{(n)}| - |\Lambda^{(n-1)}| = \lambda_i(n),
\]

and, in fact,

\[
\Lambda^{(n)} = (\lambda_1, \ldots, \lambda_{m-1}, \lambda_m + \lambda_i(1) + \cdots + \lambda_i(n)).
\]

Since

\[
\sum_{n=kl+1}^{\infty} p_{n}(\lambda, \pi) \to 0 \text{ as } k \to \infty,
\]

for any \( i = 1, \ldots, N \) there exists \( j \) such that \( a^{(m)}_{ij} > 0 \). Renumbering, if necessary, the linear forms \( l_{ij} \), we may assume that \( a^{(m)}_{ij} > 0 \) for any \( i \). Denote \( \epsilon = \min\{a^{(m)}_{ij}\} \) and \( L = \max\{a^{(r)}_{ij}\} \). For any \( \lambda \in \mathbb{R}^m \) we have then

\[
\epsilon \lambda_m \leq l_i(\lambda) \leq L |\lambda|,
\]

whence

\[
\frac{\rho^r_\pi(\Lambda^{(kl)})}{\rho^r_\pi(\lambda)} \leq \frac{L}{\epsilon (\lambda_m + \lambda_i(1) + \cdots + \lambda_i(N))}. \tag{16}
\]

Let \( N \) be the smallest number such that \( |\Lambda(-N)| > K \) and let \( s \) be the largest such integer that \( sl < N \). Then \( |\Lambda(-sl)| > K - 1 \) (because all \( \lambda_i(s+1), \ldots, \lambda_i(N) \) are all distinct) and \( \lambda_m + \lambda_i(1) + \cdots + \lambda_i(sl) > K - 2 \) (because \( |\Lambda(-sl)| = 1 + \lambda_i(1) + \cdots + \lambda_i(sl) \)).
Therefore, by (16), we obtain
\[
\frac{\rho_{\pi}^{*}(\Lambda^{(k|)})}{\rho_{\pi}^{*}(\lambda)} \leq \frac{L}{\epsilon K - 2},
\]
and the Lemma is proved.

**Lemma 8** Suppose \((\lambda, \pi) \in \Delta^+\), and let \(l\) be the length of the a-cycle of \(\pi\).

Then, for any \(k \geq 1\), we have
\[
\sum_{n=k+1}^{\infty} p_n(\lambda, \pi) \geq \left( \frac{\lambda_{\pi^{-1}m}}{\lambda_{\pi^{-1}m} + k} \right)^m.
\]

Suppose \((\lambda, \pi) \in \Delta^-\), and let \(l\) be the length of the b-cycle of \(\pi\).

Then, for any \(k \geq 1\), we have
\[
\sum_{n=k+1}^{\infty} p_n(\lambda, \pi) \geq \left( \frac{\lambda_m}{\lambda_m + k} \right)^m.
\]

**Proof.** Again, we only consider the case \((\lambda, \pi) \in \Delta^-\), as the proof of the other case is identical.

\[
\sum_{n=k+1}^{\infty} p_n(\lambda, \pi) = \frac{\rho_{\pi}^{*}(\Lambda^{(k|)})}{\rho_{\pi}^{*}(\lambda)}.
\]

Set \(\Lambda^{(k|)} = (\Lambda_1^{(k|)}, \ldots, \Lambda_m^{(k|)})\).

For \(k = 1\) we have \(\Lambda_i^{(1|)} = \lambda_i\) for \(i < m\) and \(\Lambda_i^{(l|)} = \lambda_i + \lambda_{i(1)} + \cdots + \lambda_{i(l)}\),

and for arbitrary \(k\) by induction we obtain \(\Lambda_i^{(k|)} = \lambda_i\) for \(i < m\) and \(\Lambda_i^{(k|)} = \lambda_i + k(\lambda_{i(1)} + \cdots + \lambda_{i(l)})\).

Note that \(\lambda_{i(1)} + \cdots + \lambda_{i(l)} \leq 1\) (since \(i(1), \ldots, i(l)\) are all distinct).

As in the proof of the previous Lemma, write
\[
\rho_{\pi}^{*}(\lambda) = \sum_{i=1}^{N} \frac{1}{l_i(\lambda) l_{i2}(\lambda) \cdots l_{im}(\lambda)},
\]
whence
\[
\rho_{\pi}^{*}(\Lambda^{(k|)}) \geq \min_i \frac{l_i(\lambda) l_{i2}(\lambda) \cdots l_{im}(\lambda)}{l_i(\Lambda^{(k|)}) l_{i2}(\Lambda^{(k|)}) \cdots l_{im}(\Lambda^{(k|)})}.
\]

For any linear form \(l(\lambda) = a_1 \lambda_1 + \cdots + a_m \lambda_m, a_i \geq 0\), we have
\[
\frac{l(\Lambda^{(k|)})}{l(\lambda)} \geq \frac{\lambda_m}{\lambda_m + k(\lambda_{i(1)} + \cdots + \lambda_{i(l)})} \geq \frac{\lambda_m}{\lambda_m + k},
\]
and the Lemma follows.
7  An estimate on the probability of stopping.

Lemma 9  For any \( \gamma > 0 \), there exists \( c(\gamma) > 0 \) such that if \( \lambda_{i(N)} > \gamma \), then

\[
\frac{P_N(\lambda, \pi)}{\sum_{n=N+1}^{\infty} P_n(\lambda, \pi)} \geq c(\gamma)
\]

From Lemma 8 we immediately have the following Corollary.

Corollary 6  For any \( \gamma > 0 \), there exists \( c(\gamma) > 0 \) such that the following is true.

Assume \((\lambda, \pi) \in \Delta^+, \lambda_{i(N)} > \gamma, \lambda_{\tau-1} > \gamma\). Then

\[
P_N \geq \frac{c(\gamma)}{N^m}.
\]

Similarly, if \((\lambda, \pi) \in \Delta^-, \lambda_{i(N)} > \gamma, \lambda_m > \gamma\), then

\[
P_N \geq \frac{c(\gamma)}{N^m}.
\]

If \((\lambda, \pi) \in \Delta^+, \) then, by the definition of \( P_n(\lambda, \pi) \) and by Lemma 6, we have

\[
P_N(\lambda, \pi) = \frac{\rho_{\pi - N}^N(T^{[\tau]}_{\pi - N})}{\rho_\pi(\lambda)},
\]

\[
\sum_{n=N+1}^{\infty} P_n(\lambda, \pi) = \frac{\rho_{\pi - N}^N(T^{[\tau]}_{\pi - N})}{\rho_\pi(\lambda)},
\]

and, therefore,

\[
\frac{P_N(\lambda, \pi)}{\sum_{n=N+1}^{\infty} P_n(\lambda, \pi)} = \frac{\rho_{\pi - N}^N(T^{[\tau]}_{\pi - N})}{\rho_\pi(\lambda)}.
\]

Lemma 9 follows now from the following

Lemma 10  For any \( \gamma > 0 \) there exists a constant \( c(\gamma) > 0 \) such that the following is true. Let \((\lambda, \pi) \in \Delta(R)\). If \( \lambda_{\pi-1} > \gamma \), then

\[
\frac{\rho_{\pi}^N(\lambda)}{\rho_\pi^{N+1}(\lambda)} \geq c(\gamma).
\]

If \( \lambda_{\pi-1}(\pi_{m+1}) > \gamma \), then

\[
\frac{\rho_{\pi}^N(\lambda)}{\rho_\pi^{N+1}(\lambda)} \geq c(\gamma).
\]

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The proof of Lemma 10 will take the remainder of this section.

First, we modify Veech’s coordinates on the space of zippered rectangles. Take a zippered rectangle \((\lambda, h, a, \pi) \in \Delta(R)\), and introduce the vector \(\delta = (\delta_1, \ldots, \delta_m) \in \mathbb{R}^m\) by the formula

\[
\delta_i = a_{i-1} - a_i, \quad i = 1, \ldots, m
\]

(here we assume, as always, \(a_0 = a_{m+1} = 0\)).

**Proposition 7** The data \((\lambda, \pi, \delta)\) determine the zippered rectangle \((\lambda, h, a, \pi)\) uniquely.

**Remark.** The coordinates \((\lambda, \pi, \delta)\) on the space of zippered rectangles have a natural interpretation in terms of the cohomological coordinates of Hubbard and Masur [28]: namely, the \(\lambda_i\) are the real parts of the corresponding cycles, and the \(\delta_i\) are (minus) the imaginary parts.

Proof of Proposition 7. For any \(i = 1, \ldots, m\), we have

\[
a_i = -\delta_i - \cdots - \delta_i,
\]

so the vector \(a\) is uniquely defined by \(\delta\). It remains to show that the vector \(h\) is uniquely defined by \(\delta\), and, to do this, we shall express the \(h\) through the \(a\). First note that

\[
h_{\pi^{-1}m} = a_{\pi^{-1}m} - a_m.
\]

Now, if \(i \neq \pi^{-1}m\), then \(i = \pi^{-1}(k - 1)\) for some \(k \in \{1, \ldots, m\}\). The equation

\[
h_i - a_i = h_{\pi^{-1}(\pi(i)+1)} - a_{\pi^{-1}(\pi(i)+1)-1},
\]

then takes the form

\[
h_{\pi^{-1}(k-1)} - a_{\pi^{-1}(k-1)} = h_{\pi^{-1}(k)} - a_{\pi^{-1}(k)-1},
\]

or, equivalently,

\[
h_{\pi^{-1}(k)} = a_{\pi^{-1}(k)-1} + h_{\pi^{-1}(k-1)} - a_{\pi^{-1}(k-1)}.
\]

Since

\[
h_{\pi^{-1}1} = a_{\pi^{-1}1-1},
\]

by induction, we obtain

\[
h_{\pi^{-1}k} = a_{\pi^{-1}k-1} + \sum_{l=1}^{k-1} (a_{\pi^{-1}l-1} - a_{\pi^{-1}l})
\]

for any \(k = 1, \ldots, m\), and the Lemma is proved.

The above computations give us the following expression for \(h\) in terms of \(\delta\):

\[
h_{\pi^{-1}k} = -\sum_{i=1}^{\pi^{-1}k-1} \delta_i + \sum_{l=1}^{k-1} \delta_{\pi^{-1}(l)}
\]

\[26\]
or, equivalently,

\[ h_r = - \sum_{i=1}^{r-1} \delta_i + \sum_{i=1}^{\tau(r)-1} \delta_{\tau^{-1}i}. \]  

(21)

Rewriting the inequalities defining the zippered rectangle in terms of \( \delta \), we obtain by a straightforward computation the following system:

\[
\begin{align*}
\delta_1 + \cdots + \delta_i &\leq 0, \quad i = 1, \ldots, m-1, \\
\delta_{\tau^{-1}i} + \cdots + \delta_{\tau^{-1}i} &\geq 0, \quad i = 1, \ldots, m-1.
\end{align*}
\]

The parameter \( a_m = - (\delta_1 + \cdots + \delta_m) \) can be both positive and negative. Introduce the following cones in \( \mathbb{R}^m \):

\[
K_\tau = \{ \delta = (\delta_1, \ldots, \delta_m) : \delta_1 + \cdots + \delta_i \leq 0, \delta_{\tau^{-1}i} + \cdots + \delta_{\tau^{-1}i} \geq 0, i = 1, \ldots, m-1 \},
\]

\[
K_\tau^+ = K_\tau \cap \{ \delta : \sum_{i=1}^m \delta_i \leq 0 \}, \quad K_\tau^- = K_\tau \cap \{ \delta : \sum_{i=1}^m \delta_i \geq 0 \}.
\]

We have established the following

**Proposition 8.** For \((\lambda, \pi) \in \Delta(\mathbb{R})\) and an arbitrary \( \delta \in K_\tau \) there exists a unique zippered rectangle \((\lambda, h, a, \pi)\) corresponding to the parameters \((\lambda, \pi, \delta)\).

In what follows, we shall simply refer to the zippered rectangle \((\lambda, \pi, \delta)\).

**Remark.** It would be interesting to write down explicitly the generating vectors for the cones \( K_\tau, K_\tau^+, K_\tau^- \); in particular, that would allow to give an explicit expression for the invariant densities of Veech [1] and Zorich [4].

Denote by \( \text{Area}(\lambda, \pi, \delta) \) the area of the zippered rectangle \((\lambda, \pi, \delta)\). We have:

\[
\text{Area}(\lambda, \pi, \delta) = \sum_{r=1}^m \lambda_r h_r = \sum_{r=1}^m \lambda_r (- \sum_{i=1}^{r-1} \delta_i + \sum_{i=1}^{\tau(r)-1} \delta_{\tau^{-1}i}) = \\
\sum_{i=1}^m \delta_i (- \sum_{r=1+i}^m \lambda_r + \sum_{r=\tau(i)+1}^m \lambda_{\tau^{-1}r}) = 1.
\]  

(22)

A straightforward computation shows that in the coordinates \((\lambda, \pi, \delta)\) the Rauzy induction map is written as follows:

\[
\mathcal{T}(\lambda, \pi, \delta) = \begin{cases} 
\left( \frac{A(\pi, b)^{-1}}{|A(\pi, b)|}, b\pi, A(\pi, b)^{-1}\delta \cdot |A(\pi, b)^{-1}\lambda| \right), & \text{if } \lambda \in \Delta_\tau^+; \\
\left( \frac{A(\pi, a)^{-1}}{|A(\pi, a)|}, a\pi, A(\pi, a)^{-1}\delta \cdot |A(\pi, a)^{-1}\lambda| \right), & \text{if } \lambda \in \Delta_\tau^-.
\end{cases}
\]

For \( \lambda \in \mathbb{R}_+^m \), denote

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\[ K(\lambda, \pi) = \{ \delta : \text{Area}(\lambda, \pi, \delta) \leq 1 \}, \]
\[ K^+(\lambda, \pi) = \{ \delta : \text{Area}(\lambda, \pi, \delta) \leq 1 \}, \]
\[ K(\lambda, \pi) = \{ \delta : \text{Area}(\lambda, \pi, \delta) \leq 1 \}. \]

Denote by \( \text{vol}_m \) the Lebesgue measure in \( \mathbb{R}^m \).

Set
\[ r(\lambda, \pi) = \text{vol}_m (K(\lambda, \pi)), r^+(\lambda, \pi) = \text{vol}_m (K^+(\lambda, \pi)), r^-(\lambda, \pi) = \text{vol}_m (K^-(\lambda, \pi)). \]

By definition, the functions \( r, r^+, r^- \) are positive rational functions, homogeneous of degree \( -m \).

**Lemma 11**

1. \( r^-(\lambda, \pi) = r(T_{\delta^1}, (\lambda, \pi)) \).
2. \( r^+(\lambda, \pi) = r(T_{\delta^m}, (\lambda, \pi)) \).
3. \( r(\lambda, \pi) = r(T_{\delta^1}, (\lambda, \pi)) + r(T_{\delta^m}, (\lambda, \pi)) \).

**Proof.** If
\[ \delta = (\delta_1, \ldots, \delta_m) \in K^-(\lambda, \pi), \]
then
\[ \hat{\delta} = (\delta_1, \ldots, \delta_{m-1}, \delta_m + \delta_{m-1}) \in K(T_{\delta^1}, (\lambda, \pi)), \]
and vice versa. This gives a volume-preserving bijection between \( K^-(\lambda, \pi) \) and \( K(T_{\delta^1}, (\lambda, \pi)) \), whence \( r^-(\lambda, \pi) = r(T_{\delta^1}, (\lambda, \pi)) \). The second assertion is proved in the same way, and the third follows from the first two.

**Corollary 7**

\[ r^+(\lambda, \pi) = \sum_{n=1}^{\infty} r^-(T_{\delta^1}, (\lambda, \pi)). \]
\[ r^-(\lambda, \pi) = \sum_{n=1}^{\infty} r^+(T_{\delta^m}, (\lambda, \pi)). \]

We only prove the first assertion. We have
\[ r^+(\lambda, \pi) = r(T_{\delta^1}, (\lambda, \pi)) = r^+(T_{\delta^1}, (\lambda, \pi)) + r^-(T_{\delta^1}, (\lambda, \pi)) = r(T_{\delta^1}, (\lambda, \pi)) + r^-(T_{\delta^1}, (\lambda, \pi)). \]

Proceeding by induction,
\[ r^+(\lambda, \pi) = \sum_{n=1}^{N} r^-(T_{\delta^1}, (\lambda, \pi)) + r(T_{\delta^1}, (\lambda, \pi)). \]

Since
\[ T_{\delta_{m-1}}(\lambda, \pi) = (T_{\delta_{m-1}}(\lambda, \pi), a^{-N-1} \pi), \]

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and $|T_{\lambda}^{(\pi)}(\lambda)| \to \infty$ as $N \to \infty$, we obtain $r(T_{\lambda}^{(\pi)}(\lambda)) \to 0$ as $N \to \infty$, and the Corollary is proved.

Since the functions $r, r^+, r^-$ are positive, rational and homogeneous of degree $-m$, Corollary 7 implies that, for some positive constant $C(\mathcal{R})$, depending only on the Rauzy class $\mathcal{R}$, we have

$$\rho^+(\lambda, \pi) = C(\pi)r^+(\lambda, \pi), \rho^-(\lambda, \pi) = C(\pi)r^-(\lambda, \pi).$$

By construction, for any $\lambda \in \mathbb{R}_+^m$ we have

$$r^+_m(\lambda_1, \ldots, \lambda_m) = \frac{1}{\lambda_m} \frac{d}{d\lambda_m} J^m_{\theta}(\lambda_1, \ldots, \lambda_m).$$

In view of this observation, it suffices to prove only the first assertion of the Lemma 10, as the second one follows automatically.

Take $\delta = (\delta_1, \ldots, \delta_m) \in \mathbb{R}^m$, and, for $\theta > 0$, define

$$J^m_{\theta}(\delta) = (\delta_1, \ldots, \delta_m + \theta), \quad J^{(\pi^{-1}m)}_{\theta}(\delta) = (\delta_1, \ldots, \delta_{\pi^{-1}m} - \theta, \ldots, \delta_m).$$

**Proposition 9** Let $\theta > 0$. If $\delta \in K_{\pi}$, then $J^m_{\theta}(\delta) \in K_{\pi}$, $J^{(\pi^{-1}m)}_{\theta}(\delta) \in K_{\pi}$. If $\delta \in K_{\pi}^-$, then $J^m_{\theta}(\delta) \in K_{\pi}^-$. If $\delta \in K_{\pi}^+$, then $J^{(\pi^{-1}m)}_{\theta}(\delta) \in K_{\pi}^+$. This follows directly from the definition of the cones $K_{\pi}, K_{\pi}^-, K_{\pi}^+$. From (22) we obtain

$$\text{Area}(\lambda, \pi, J^m_{\theta}(\delta)) = \text{Area}(\lambda, \pi, \delta) + \theta \sum_{r=\pi(m)+1}^m \lambda_{\pi^{-1}r},$$

$$\text{Area}(\lambda, \pi, J^{(\pi^{-1}m)}_{\theta}(\delta)) = \text{Area}(\lambda, \pi, \delta) + \theta \sum_{r=\pi^{-1}(m)+1}^m \lambda_r,$$

which implies

**Proposition 10**

$$\text{Area}(\lambda, \pi, \delta) \leq \text{Area}(\lambda, \pi, J^m_{\theta}(\delta)) \leq \text{Area}(\lambda, \pi, \delta) + \theta |\lambda|,$$

$$\text{Area}(\lambda, \pi, \delta) \leq \text{Area}(\lambda, \pi, J^{(\pi^{-1}m)}_{\theta}(\delta)) \leq \text{Area}(\lambda, \pi, \delta) + \theta |\lambda|.$$  

For $s \in \mathbb{R}$ and a hyperplane of the form $\delta + \cdots + \delta_m = s$, let $\text{vol}_{m-1}$ stand for the induced $(m-1)$-dimensional volume form on the hyperplane.

Denote

$$K_{s,\pi} = K_{\pi} \cap \{\delta : \sum_{i=1}^m \delta_i = s\},$$

$$K_s(\lambda, \pi) = K(\lambda, \pi) \cap K_{s,\pi},$$

$$V_s(\lambda, \pi) = \text{vol}_{m-1}(K_s(\lambda, \pi)).$$

Denote by $a_{\text{max}}$ the maximal possible value of $\delta_1 + \cdots + \delta_m = -a_m$ in $K(\lambda, \pi)$.
**Proposition 11** Assume \(0 \leq s \leq a_{\text{max}}^{-}\). Then

\[
V_s(\lambda, \pi) \leq (1 + s)^{m-1}V_0(\lambda, \pi).
\]

Proof: Indeed, if \((\lambda, \pi, \delta) \in V_s(\lambda, \pi)\), then Proposition 10 implies

\[
(\lambda, \pi, \frac{j(\tau^{-1}m)\delta}{1 + s}) \in V_0(\lambda, \pi),
\]

and the assertion follows.

**Proposition 12** Assume \(s, 0 \leq s \leq 1\) is such that \(\frac{\lambda}{\rho} \leq a_{\text{max}}^{-}\). Then

\[
(1 - \frac{s}{\lambda})^{m-1}V_0(\lambda, \pi) \geq V_0(\lambda, \pi).
\]

Denote \(\theta = \frac{s}{\lambda}\), then \(s = \frac{\lambda}{1 + \theta}\). If \((\lambda, \pi, \delta) \in V_0(\lambda, \pi)\), then

\[
(\lambda, \pi, \frac{j(\tau^{-1}m)\delta}{1 + \theta}) \in V_0(\lambda, \pi),
\]

and, again, the assertion follows.

Propositions 11, 12 imply

**Lemma 12** For any \(C_1 > 0\) there exists \(C_2 > 0\) such that the following is true.

Let \(a_{\text{max}}^{-}(\lambda, \pi) < C_1\). Then

\[
r^{-}(\lambda, \pi) < C_2V_{m-1}^{1}(\lambda, \pi).
\]

Note that there exists \(\epsilon > 0\), depending only on \(\mathcal{R}\) and such that for any \((\lambda, \pi) \in \Delta(\mathcal{R})\), we have \(a_{\text{max}}^{-} > \epsilon\). In conjunction with Propositions 11, 12, this implies

**Lemma 13** There exists a constant \(C_3\) such that for any \((\lambda, \pi) \in \Delta(\mathcal{R})\), we have

\[
\rho^{-}(\lambda, \pi) \geq C_3V_{m-1}^{1}(\lambda, \pi).
\]

Since \(a_m \leq h_{\pi^{-1}, m+1}\), we have

\[
a_{\text{max}}^{-}(\lambda, \pi) \leq \frac{1}{h_{\pi^{-1}, m+1}},
\]

which implies the following

**Corollary 8** For any \(C_4 > 0\) there exists \(C_5 > 0\) such that the following is true.

Assume \(\lambda_{\pi^{-1}, m+1} > C_4\). Then

\[
\frac{r^{-}(\lambda, \pi)}{r^{+}(\lambda, \pi)} < C_5,
\]

which implies Lemma 10.
8 Kerckhoff names

In the following two sections, we shall use Kerckhoff’s convention of numbering the sub-intervals of an interval exchange [20]; to avoid confusion, we shall speak of Kerckhoff names of subintervals.

Take an interval exchange \((\lambda, \pi)\). A Kerckhoff naming on the subintervals \((\lambda, \pi)\) is defined by an arbitrary permutation \(i_1, \ldots, i_m\) of the symbols \(\{1, \ldots, m\}\). Once such a permutation is given, we assign names \(I_{i_1}, \ldots, I_{i_m}\) to the subintervals of \((\lambda, \pi)\), from the left to the right (i.e., the subinterval \([0, \lambda_1)\) is named \(I_{i_1}\), the subinterval \([\lambda_1, \lambda_1 + \lambda_2)\) is named \(I_{i_2}\) and so forth).

A Kerckhoff naming of the subintervals of \((\lambda, \pi)\) induces a naming on the subintervals of \(\mathcal{T}(\lambda, \pi)\) in the following way. Assume \(\lambda_m < \lambda_{s-i_m}\) and the Rauzy operation \(a\) was applied to \((\lambda, \pi)\) in order to obtain \(\mathcal{T}(\lambda, \pi)\). Then the subintervals of \(\mathcal{T}(\lambda, \pi)\) are named, from the left to the right, by \(I_{i_1}, \ldots, I_{i_{s-i_m}}, I_{i_m}, I_{i_{s-i_m+1}}, \ldots, I_{i_m-1}\). If \(\lambda_m > \lambda_{s-i_m}\) and the Rauzy operation \(b\) was applied, then the subintervals of \(\mathcal{T}(\lambda, \pi)\) are just named, as before, by \(I_{i_1}, \ldots, I_{i_m}\), from the left to the right. Proceeding inductively, we obtain a naming for any \(\mathcal{G}^*(\lambda, \pi)\). Conversely, if we have a Kerckhoff naming of subintervals of \((\lambda, \pi)\), then, for any word \(w \in \mathcal{W}_{A,B}\) compatible with \((\lambda, \pi)\), we automatically obtain a Kerckhoff naming on the subintervals of \(t_w(\lambda, \pi)\) and \(T_w(\lambda, \pi)\).

Let \((\lambda, \pi)\) be an interval exchange with a Kerckhoff naming \(I_{i_1}, \ldots, I_{i_m}\). If \((\lambda, \pi) \in \Delta^+\), then we say that \(I_{i_{s-i_m}}\) is the subinterval in the critical position (we shall also sometimes say “in the \(a\)-critical position”). If \((\lambda, \pi) \in \Delta^-\), then we say that \(I_{i_m}\) is the subinterval in the critical position (we shall also sometimes say “in the \(b\)-critical position”).

9 Exponential growth.

Let \(x \in \overline{\Delta}\), that is, \(x = (\ldots, (\lambda(-n), \pi(-n)), \ldots, (\lambda, \pi))\), where, as usual, \(\mathcal{G}(\lambda(-n), \pi(-n)) = (\lambda(1-n), \pi(1-n))\). Define the words \(u(n)\) by the relation \((\lambda(-n), \pi(-n)) = t_{u(n)}(\lambda, \pi)\). Set \((\lambda(-n), \pi(-n)) = T_{u(n)}(\lambda, \pi)\).

**Lemma 14** There exists \(N\) such that the following is true. For any \(x \in \overline{\Delta}(R)\), there exist \(i_1, i_2 \in \{1, \ldots, m\}\) such that

\[
\Lambda(-N)_{i_1} + \Lambda(-N)_{i_2} \geq 2(\lambda(0)_{i_1} + \lambda(0)_{i_2})
\]

Proof:

Take a point \(x \in \overline{\Delta}\),

\[x = (\ldots, (\lambda(-n), \pi(-n)), \ldots, (\lambda, \pi)).\]

Give Kerckhoff names \(I_1, \ldots, I_m\) to the subintervals of the exchange \((\lambda, \pi)\) from the left to the right, so that the length of \(I_i\) is \(\lambda_i\). We thus automatically obtain a Kerckhoff naming for the subintervals of \((\lambda(-n), \pi(-n))\) for any \(n\).

Let \(I_{j_n}\) be the critical subinterval for \((\lambda(-n), \pi(-n))\).
Consider the infinite sequence

$$I_{j_1} \ldots I_{j_k} \ldots$$

(23)

Note that $j_n \neq j_{n+1}$. A subword $I_{j_k} \ldots I_{j_{k+1}}$ will be called a simple cycle if $I_{j_k} = I_{j_{k+1}}$, whereas $I_{j_{k+1}} \ldots I_{j_{k+1}+1}$ are all distinct. Naturally, $1 \leq t \leq m$. There are finitely many possible simple cycles, therefore there exists $N$, depending only on $m$, such that for any word of length $N$ in the alphabet $\{I_1, \ldots, I_m\}$, some simple cycle occurs at least $m$ times. Now take the word

$$I_{j_1} \ldots I_{j_N},$$

(24)

the beginning of the sequence (23), and take a simple cycle which occurs $m$ times, say

$$I_{t_1} \ldots I_{t_r},$$

(25)

Here, of course, $r \leq m$. Now estimate the non-renormalized length of the subintervals $I_{t_1}, \ldots, I_{t_r}$ ($r \leq m$). In the beginning, these are $\lambda_{t_1}, \ldots, \lambda_{t_r}$. The key observation is, as usual, that the interval in critical position at a given inverse Zorich step was, at the previous step, added to the previous critical interval. After the first occurrence of the cycle (25), therefore, the (non-normalized) length of $I_{t_1}$ is at least $\lambda_{t_1} + \lambda_{t_2}$, that of $I_{t_2}$ is at least $\lambda_{t_2} + \lambda_{t_3}$ and so forth. After the second occurrence of (25), the length of $I_{t_1}$ is at least $\lambda_{t_1} + \lambda_{t_2} + \lambda_{t_3}$, that of $I_{t_2}$ is at least $\lambda_{t_2} + \lambda_{t_3} + \lambda_{t_4}$, and so forth. Finally, after the $r$-th occurrence of (25), the length of $I_{t_1}$ is not less than $\lambda_{t_1} + \lambda_{t_2} + \cdots + \lambda_{t_r}$, that is, not less than $2\lambda_{t_1}$, since $\lambda_{t_1} = \lambda_{t_r}$. The Lemma is proven.

10 Proof of the Lemma 5

An informal sketch of the proof of Lemma 5. One divides the subintervals into “big” ones and “small” ones: the aim is to obtain one more “big” interval. For this, one must first put a small subinterval into critical position. This is achieved by Lemma 15. In the previous section, we have seen that the total length of the (non-renormalized) interval grows exponentially with the number of Zorich steps (with an exponent depending on $\epsilon$). When the total length of the interval doubles, we obtain a new “big” subinterval.

10.1 Putting a small interval into critical position

Take an interval exchange $(\lambda, \pi)$ and name the subintervals $I_1, \ldots, I_m$, from the right to the left.

**Proposition 13** Any interval can be put both in the $a$-critical and in the $b$-critical position.
Proof: First note that if an interval can be put in the \( a \)-critical position, then it can also be put into the \( b \)-critical position just by performing the entire \( a \)-cycle of the corresponding permutation. Since the permutation is irreducible, it suffices to prove that, if \( I_i \) can be put into critical position, then also all \( I_j \) for \( j > i \). To prove this, take the shortest word \( w \) that puts \( I_i \) into the \( a \)-critical position. Then, in the preimage, all \( I_j, j > i \), still stand to the right of \( I_i \), though perhaps in a different order (because an inversion of order between \( I_i \) and \( I_j \) can only happen once \( I_i \) reaches the critical position). Therefore, we can immediately place any of the \( I_j, j > i \), into the \( b \)-critical position, but then also into the \( a \)-critical position.

More precisely, pick a positive integer \( k \leq m \) and a real \( \gamma > 0 \). We say that we have a \((k, \gamma)\)-big-small decomposition if the intervals of the exchange are divided into two groups: \( I_{i_1}, \ldots, I_{i_k} \), each of length at least \( \gamma \), and the remaining ones (nothing is said about the length of the remaining ones).

Under the Kerckhoff convention, a big-small decomposition of \((\lambda, \pi)\) is inherited by all \( t_w(\lambda, \pi) \) (one just takes the intervals with the same names).

**Lemma 15** For any \( \gamma > 0 \), there exist constants \( p(\gamma), L(\gamma) \) such that the following is true. Let \((\lambda, \pi) \in \Delta_{k, \gamma} \) with a fixed big-small decomposition. Then there exists \( w \in \mathcal{W}_{A, B} \) such that

1. \( \mathbb{P}(w|\lambda, \pi) \geq p(\gamma) \).
2. \( |T_w(\lambda, \pi)| < L(\gamma) \).
3. the exchange \( t_w(\lambda, \pi) \) has a small interval in critical position.

Proof: Take the shortest word (in terms of the number of Zorich operations) that puts a small interval into critical position. Among all such words, pick the one that involves the smallest number of Rauzy operations. The length of this word, as well as the number of Rauzy operations involved, only depends on the Rauzy class. At each intermediate Rauzy step, all subintervals following the critical one either in the preimage or in the image must be big, otherwise there would exist a shorter word placing a small interval into critical position. Therefore, by Lemma 9 and the Corollary 6, the probability of each Zorich operation involved is bounded from below by a constant that only depends on \( \gamma \). The Lemma is proved.

### 10.2 Completion of the proof.

Proof: Take any \( x \in \mathbb{X} \). Take the first \( n \) such that \( |A(-n)| > 2 \). By Lemma 14, \( n < K[\log \epsilon] \). By Lemma 7, with positive probability depending only on \( M \), we can assume \( |A(-n)| < 2M \). Consider two cases:

1. at all steps from 1 to \( n \), only small intervals were added between themselves.
2. at some step a large interval was added to a small one.
Note, that since we start with a small interval in critical position, either one of the other case holds (for, in order that a small interval be added to a big interval, a big interval must first be placed into critical position, and for that it must first be added to a small one).

In the first case, the lengths of all large intervals remain the same, and after renormalization at step $n$, each large interval has length at least $\gamma/2M$. However, since $|\lambda(-n)| > 2$, there must be another interval of length at least $1/2M$, and the Lemma is proved.

In the second case, let $n_1$ be the first moment, at which a big interval is added to a small one. Then $|\lambda(-n_1)| < 2$ and, since at previous moments only small intervals were added between themselves, we have $k + 1$ intervals of length at least $\gamma/2$, and the Lemma is proved completely.

11 Estimate of the measure.

**Lemma 16** There exists a constant $C(R)$ depending only on the Rauzy class $R$ such that

$$\nu(\Delta(R) \setminus \Delta_r(R)) < C \epsilon$$

The proof repeats that of Proposition 13.2 in Veech [1].

Lemma 4 and Corollary 5 therefore imply the following

**Corollary 9** Let $q \in W_{A,B}$, $q = q_1 \ldots q_r$ be such that all entries of the matrix $A(q)$ are positive.

There exist $C > 0, \alpha > 0$ such that the following is true for any $n$.

$$\mathbb{P}(\{(\lambda, \pi) : G^{2k}(\lambda, \pi) \not\in \Delta(q) \text{ for all } k, 1 \leq k \leq n\} \leq C \exp(-\alpha \sqrt{n})$$

Proof: Let $n = r^2$ and denote

$$X(n, q) = \{(\lambda, \pi) : G^{2k}(\lambda, \pi) \not\in \Delta(q) \text{ for all } k, 1 \leq k \leq n\}.$$ 

Take

$$B(n) = \{(\lambda, \pi) : G^{2k}(\lambda, \pi) \not\in \Delta_{\exp(-r)} \text{ for some } k, 1 \leq k \leq n\}$$

Then, by the previous Lemma, $\nu(B(n)) \leq C r^2 \exp(-r)$, whereas, by Corollary 5,

$$\nu(X(n, q) \setminus B(n)) \leq (1 - p(q))^r,$$

and Corollary 9 is proven.

**Remark.** This result allows to use the tower method of L.-S. Young [11] and to obtain the decay rate $\exp(-\alpha \sqrt{n})$ for correlations of bounded Hölder functions. For bounded Lipschitz functions, one can also use the method of V. Maume-Deschamps [12] and obtain the uniform rate of decay at the rate $\exp(-\alpha n^{1/2-\epsilon})$. It is not clear to me, however, how to use either of these methods in the invertible case.
12 Inequalities

Let

\[ W_{A,B}^+ = \{ w \in W_{A,B} : |w| \text{ is even}, \; \Delta(w) \subset \Delta^+ \}. \]

**Lemma 17** For any \( C_1, C_2 > 0 \) there exists \( C_3 > 0 \) such that the following is true.

Suppose \( \text{row}(A) < C_1 \) and \( \lambda \in \Delta_{c_2} \).

Then

\[ \frac{1}{C_3} \leq \frac{1}{\prod_{j=1}^{m} \sum_{i=1}^{m} A_{ij}} \leq C_3 \]

Proof:

Denote \( A_j = \sum_{i=1}^{m} A_{ij} \), so that \( |A| = \sum_{j=1}^{m} A_j \).

Then

\[ \frac{A_j}{A_k} \leq \text{row}(A), \]

whence

\[ \frac{A_j}{|A|} \geq \frac{1}{m \text{row}(A)}. \]

Finally, if \( \lambda \in \Delta_{c_2} \), then

\[ |A\lambda| \geq C_2 |A|, \]

which completes the proof.

**Corollary 10** For any \( C_4 > 0, C_5 > 0 \) there exists \( C_6 > 0 \) such that the following is true. Suppose \( (\lambda, \pi) \in \Delta_{c_4} \). Suppose \( w \in W_{A,B} \) is compatible with \( (\lambda, \pi) \) and such that \( \text{row}(A(w)) < C_5 \). Then

\[ \frac{1}{C_6} \leq \frac{m(C(w))}{P(w(\lambda, \pi))) \leq C_6 \]

**Corollary 11** For any \( C_7 > 0, C_8 > 0 C_9 > 0 \), there exists \( C_{10} > 0 \) such that the following is true.

Suppose \( (\lambda, \pi) \in \Delta_{c_7} \).

Suppose \( w \in W_{A,B} \) is compatible with \( (\lambda, \pi) \) and furthermore satisfies

\[ \text{row}(A(w)) < C_8, \; \Delta(w) \subset \Delta_{c_9} \]

Then

\[ \frac{1}{C_{10}} \leq \frac{P(C(w))}{P(w(\lambda, \pi))) \leq C_{10} \]

**Corollary 12** Let \( M \) be such that for any \( n > M \) any two vertices in the Rauzy graph can be joined in \( n \) steps.

Then for any \( C_{17} > 0, C_{18} > 0 C_{19} > 0 \), there exists \( C_{20} > 0 \) such that the following is true.

Suppose \( (\lambda, \pi) \in \Delta^+ \cap \Delta_{c_{20}} \).
Suppose \( w \in W_{\Delta, B}^+ \) satisfies

\[
\text{row}(A(w)) < C_{18}, \quad \Delta(w) \subset \Delta^+ \cap \Delta_{C_{19}}
\]

Then for any \( n \geq M \), we have

\[
\frac{1}{C_{20}} \leq \frac{\mathbb{P}(C(w))}{\mathbb{P}(\mathbb{K}(\lambda, \pi))} \leq C_{20}
\]

From the definition (4) of the Hilbert metric it easily follows that for any \( \lambda, \lambda' \in \Delta_{m-1} \) we have

\[
e^{-d(\lambda, \lambda')} \lambda'_i \leq \lambda_i \leq e^{d(\lambda, \lambda')} \lambda'_i.
\]

**(Proposition 14)** Assume \( \lambda, \lambda' \in \Delta^+ \). Then

\[
\exp(-md(\lambda, \lambda')) \leq \frac{\rho(\lambda, \pi)}{\rho(\lambda', \pi)} \leq \exp(md(\lambda, \lambda'))
\]

Proof. Indeed, there exist linear forms

\[
t^{(j)}_i(\lambda) = \sum_{k=1}^{m} t^{(j)}_{ik} \lambda_k,
\]

where \( t^{(j)}_{ik} \) are nonnegative integers (in fact, either 0 or 1, but we do not need this here), such that

\[
\rho(\lambda, \pi) = \sum_{j=1}^{s} t^{(j)}_1(\lambda) t^{(j)}_2(\lambda) \ldots t^{(j)}_m(\lambda).
\]

Clearly, if for all \( i = 1, \ldots, m \) and some \( \alpha > 0 \), we have \( \alpha^{-1} \lambda_i \leq \lambda'_i \leq \alpha \lambda_i \), then

\[
\alpha^{-m} \leq \frac{\rho(\lambda, \pi)}{\rho(\lambda', \pi)} \leq \alpha^m,
\]

and the Proposition is proved.

For similar reasons we have

**(Proposition 15)** Assume \( \lambda, \lambda' \in \Delta^+ \) and let \( A \) be an arbitrary matrix with nonnegative integer entries. Then

\[
\exp(-md(\lambda, \lambda')) \leq \frac{\rho(A\lambda, \pi)}{\rho(A\lambda', \pi)} \leq \exp(md(\lambda, \lambda'))
\]

From these propositions and the formula 11 we obtain

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Corollary 13 Let $c \in \mathcal{A}$ be compatible with $\pi$. Then for any $\lambda, \lambda' \in \Delta_{\pi}^+$ we have

$$\exp(-2md(\lambda, \lambda')) \leq \frac{\mathbb{P}(c(\lambda, \pi))}{\mathbb{P}(c(\lambda', \pi))} \leq \exp(2md(\lambda, \lambda'))$$

This Corollary implies the following

Lemma 18 Let $w \in W_{\mathcal{A}, \mathcal{B}}^+$ be such that the cylinder $C(w)$ has finite Hilbert diameter.

Then for any $c$ compatible with $w$ and any $(\lambda_0, \pi) \in C(w)$ we have

$$\exp(-2m \text{diam}C(w)) \leq \frac{\mathbb{P}(c(\lambda_0, \pi))}{\mathbb{P}(\omega_0 = c \omega_{[1,|w|]} = w)} \leq \exp(2m \text{diam}C(w))$$

Proof: We have

$$\nu(C(cw)) = \int_{C(cw)} \mathbb{P}(c(\lambda, \pi))d\nu(\lambda, \pi)$$

Let $d = \text{diam}C(w)$. For any $(\lambda, \pi), (\lambda', \pi) \in C(w)$, we have, by Corollary 13,

$$\exp(-2md) \leq \frac{\mathbb{P}(c(\lambda, \pi))}{\mathbb{P}(c(\lambda', \pi))} \leq \exp(2md).$$

Fix an arbitrary $(\lambda_0, \pi) \in \Delta_w$. Then, from the above,

$$\nu(C(cw))P(c(\lambda_0, \pi))\exp(-2md) \leq \int_{C(cw)} P(c(\lambda, \pi))d\nu(\lambda, \pi) \leq \nu(C(cw))P(c(\lambda_0, \pi))\exp(2md),$$

and, since, by definition, we have

$$\mathbb{P}(\omega_0 = c \omega_{[1,|w|]} = w) = \frac{\mathbb{P}(cw)}{\mathbb{P}(w)},$$

the Lemma is proved.

For $N \in \mathbb{N}$ and $A \subset \Delta(\mathcal{R})$, we denote $\mathbb{P}^{(N)}(A(\lambda, \pi)) = \mathbb{P}(\lambda(-N), \pi(-N)) \in A(\lambda(0), \pi(0)) = (\lambda, \pi)$; for $w \in W_{\mathcal{A}, \mathcal{B}}^+$, we write $\mathbb{P}^{(N)}(w(\lambda, \pi)) = \mathbb{P}^{(N)}(\Delta(w))(\lambda, \pi)$.

Lemma 19 Let $M$ be a number such that for any $N \geq M$ any two vertices of the Rauzy graph can be connected in $N$ steps. For any $\gamma > 0$, $N \geq M$ there exists a constant $C_0$ depending only on $\gamma$ and $N$ such that for any word $w \in W_{\mathcal{A}, \mathcal{B}}^+$ and any $(\lambda, \pi) \in \Delta_{\gamma}$

$$\mathbb{P}^{(2N)}(w(\lambda, \pi)) \geq \frac{C_0}{|A(w)\lambda|^m}$$

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Proof:
Let \( w = w_1 \ldots w_{2n} \), and let \( w_{2n} = (a, m_1, \pi_1) \).
Let \( \pi'_1, \pi'_2 \ldots \pi'_{2N} \) a path of length \( 2N \) between \( \pi \) and \( \pi_1 \) (here \( \pi'_1 = \pi \), \( \pi'_2 = \pi_{2k+1} = a \pi_{2k} \), \( \pi_{2k+2} = b \pi_{2k+1} \)).
Denote \( w_{2n+2i+1} = (a, 1, \pi_{2i+1}) \), \( w_{2n+2i} = (b, 1, \pi_{2i}) \). In other words, the word \( w = w_{2n+1} \ldots w_{2n+2N} \in \mathcal{W}_{A,B} \) is the word corresponding to the path \( \pi'_1 \pi'_2 \ldots \pi'_{2N} \) in the Rauzy graph. Then \( w' = w_1 \ldots w_{2n+2N} \) is a word compatible with \( (\lambda, \pi) \).
Besides,
\[
|A(w_{2n+1}c_{n+2} \ldots w_{2n+2N})| < (2N)^{(2N)}.
\]
We have
\[
P^{(2n)}(w | (\lambda, \pi)) \geq P(w' | (\lambda, \pi)) = \frac{\rho(T_{w'}(\lambda), w' \pi)}{|A(w')\lambda\pi|^m \rho(\lambda, \pi)^m}.
\]
There exists a universal constant \( C_1 \) such that \( \rho(\lambda', \pi') \geq C_1 \) for any \( (\lambda', \pi') \in \Delta^+ \) (the density of the invariant measure is bounded from below).
Then, \( |A(w')\lambda\pi|^m \leq |A(w')|^m \leq (2N)^{2mN} |A(w)|^m \).
Finally, there exists a \( C_2 \) depending on \( c \) only such that if \( \lambda_i > c \) for all \( i \) then \( \rho(\lambda, \pi) > C_2 \).
Combining all of the above, we obtain the result of the Lemma.

13 Markov approximation and the Doeblin condition

13.1 Good cylinders

Let \( q = q_1 \ldots q_l \) be a word such that all entries of the matrix \( A(q) \) are positive. Fix \( \epsilon > 0 \) and let \( k_0 \) be such that
\[
\mathbb{P}(\Delta(q) \cap G^{-2n} \Delta(q)) \geq \epsilon \text{ for } n > k_0. \tag{27}
\]
Note that, due to mixing, Corollary 5 implies the following

**Proposition 16** Let \( q \in W_{A,B} \), \( q = q_1 \ldots q_l \) be such that all entries of the matrix \( A(q) \) are positive and that \( \Delta(q) \subset \Delta^+ \). Then there exist positive constants \( K(q) \), \( p(q) \) such that the following is true for any \( \epsilon > 0 \). Suppose \( (\lambda, \pi) \in \Delta_1 \cap \Delta^+ \) and set \( n \) to be the integer part of \( K(q) \log \epsilon \). Then
\[
\mathbb{P}(\{(\lambda(-2n), \pi(-2n)) \in \Delta(q) | (\lambda(0), \pi(0)) = (\lambda, \pi)\}) \geq p(q).
\]

Take \( k > k_0 \). Let \( r = 2(K+1)k + 2M \), where \( K \) is the constant from the Lemma 4 and \( M \) is the connecting constant of the Rauzy graph from Lemma 19. Let \( \theta, 0 < \theta < 1 \) be arbitrary. A word \( w = w_1 \ldots w_k \) is called good if
1. \( \Delta(w) \subset \Delta_{\exp(-k)} \).
2. the word $q$ appears at least $\frac{N^a}{r}$ times in $w$ (we only count disjoint appearances).

A word $w_1 \ldots w_r$ is called good if $w_1 \ldots w_k$ is good, a word $w_1 \ldots w_{N_r}$ is called good if all words $w_1 \ldots w_r, w_{r+1} \ldots w_{2r}, \ldots w_{(N-1)r+1} \ldots w_{N_r}$ are good, and a word $w_1 \ldots w_{N_r+k}$. $L < r$, is good if $w_1 \ldots w_{N_r}$ is good and either $L < k$ or $w_{N_r+1} \ldots w_{N_r+k}$ is good.

We denote by $G(N)$ the set of all good words of length $N$.

Let
$$\Delta(G(N)) = \cup_{w \in G(N)} \Delta(w),$$
and
$$\Delta(B(N)) = \Delta^+ \setminus \Delta(G(N))$$

By Corollary 9, there exist constants $C_{31}, C_{32}$ such that for all $r$ we have
$$\mathbb{P}(\Delta(B(N)) \leq C_{31} N \exp(-C_{32} r^{(1-\theta)/2}). \quad (28)$$
and, for any $(\lambda, \pi) \in \Delta(q)$, also
$$\mathbb{P}(\langle \lambda(-1), \pi(-1) \rangle \in \Delta(B(N)) | \langle \lambda(0), \pi(0) = (\lambda, \pi) \rangle \leq C_{31} N \exp(-C_{32} r^{(1-\theta)/2}). \quad (29)$$

### 13.2 Preliminary estimates for the Doeblin condition.

From Corollary 13 we deduce that there exists a constant $C_{33}$ such that for any $(\lambda, \pi), (\lambda', \pi) \in \Delta(q)$, and any word $w$ compatible with $q$, we have
$$\frac{1}{C_{33}} \leq \frac{\mathbb{P}(w | (\lambda, \pi))}{\mathbb{P}(w | (\lambda', \pi))} \leq C_{33}.$$

Finally, by Lemma 19, there exists a constant $C_{34}$ such that for any $w \in W_{A,B}$ and for any $N > M$ we have
$$\frac{1}{C_{34}} \leq \frac{\mathbb{P}(2N | (\lambda, \pi))}{\mathbb{P}(2N | (\lambda', \pi))} \leq C_{34}.$$

Take an arbitrary point $(\lambda, \pi) \in \Delta_q$. Define a new measure $\varphi$ on $\Delta^+$. Namely, for a set $A \subset \Delta^+$ put
$$\varphi(A) = \mathbb{P}(\lambda(-2M), \pi(-2M) \in A | \lambda(0), \pi(0) = (\lambda, \pi)).$$

**Lemma 20** There exists a constant $a > 0$ such that the following is true for any $r$. Let $C_1, C_2 \in G(r)$.

Then
$$\mathbb{P}(w | [1, r] = C_1, w | [r+1, 2r] \in G(r), w | [2r+1, 3r] = C_2) \geq a \varphi(C_1)$$

Indeed, we have the following propositions:
Proposition 17 There exist a constant \( p_1 \) such that the following is true for all \( r \) and all \( n \geq r \).

Let \( C_2 \in G(r) \), \((\lambda, \pi) \in C_2 \). Then
\[
\mathbb{P}\left( (\lambda(-2n), \pi(-2n)) \in \Delta(q) \mid (\lambda(0), \pi(0)) = (\lambda, \pi) \right) \geq p_1.
\]
This follows from the definition of a good cylinder and Corollary 5.

Proposition 18 There exists a constant \( p_2 \) such that the following is true for all \( k \).
\[
\mathbb{P}(\omega |_{[1,r]} \in G(r), \omega |_{[2M+1, r+M+1]} = q, \omega |_{[r+1, r+1+k]} = q) \geq p_2
\]
This follows from the estimates (28), (29) on the measure of bad cylinders and from Proposition 16.

Proposition 19 There exists a constant \( p_3 \) such that the following is true for all \( r \). Let \( c_1 \ldots c_n \cdots \in \Delta(q) \).
\[
\mathbb{P}(\omega |_{[1,r]} \in C_1, \omega_{r+1} = c_1, \omega_{r+2} = c_2, \ldots) \geq p_3 \varphi(C_1)
\]
This follows directly from Lemma 19.

The three Propositions imply Lemma 20.

13.3 Approximation by a Markov measure

We define a new measure \( p_{r,\theta} \) on the set \( G(r^2) \) of good cylinders of length \( r^2 \).

Let \( C = c_1 \ldots c_r \) be a \((r, \theta)\)-good cylinder. Set \( C_t = c_{(i+1)r} \).

Define
\[
p_{r,\theta}(C_t) = \mathbb{P}(\omega |_{[1,r]} \in C_t, \omega |_{[r+1, 2r]} = C_2, \omega |_{[r+1, 2r]} = C_3, \ldots, \omega |_{[r^2-1, r^2]} = C_r).
\]

If \( D \) is not a good cylinder, then \( p_{r,\theta}(D) = 0 \).

Normalize to get a probability measure:
\[
P_{r,\theta}(C) = \frac{p_{r,\theta}(C)}{\sum_{D \in G(r^2)} p_{r,\theta}(D)}.
\]

\( p_{r,\theta} \) is a Markov measure of memory \( r \) (in general, non-homogeneous), as is shown by the following well-known Lemma [14].

Lemma 21 For any \( k, 0 < k < r \), we have
\[
P_{r,\theta}(\omega |_{[k+r+1, (k+1)r]} = C_k, \omega |_{[(k+1)r+1, r^2]} = C_{k+1} \cdots C_r) = P_r,\theta(\omega |_{[k+r+1, (k+1)r]} = C_k, \omega |_{[(k+1)r+1, (k+2)r]} = C_{k+1}).
\]

From the Hölder property for the transition probability, we have
Proposition 20 There exist constants $C_{41}, C_{42}$ such that the following is true for any $r$.
Let $c_1, \ldots, c_r \in \Omega_{A,B}$ and assume $c_{n+1}, \ldots, c_{n+r} \in \mathbf{G}(r)$. Then
\[
\exp(-C_{41}\exp(-C_{43}k^g)) \leq \frac{P(\omega_1 = c_1, \ldots, \omega_n = c_n, \omega_{n+1} = c_{n+1}, \ldots, \omega_{n+r} = c_{n+r})}{P(\omega_1 = c_1, \ldots, \omega_n = c_n, \omega_{n+1} = c_{n+1}, \ldots, \omega_{n+i} = c_{n+i}, \ldots)} \leq \exp(C_{41}\exp(-C_{43}k^g))
\]

Corollary 14 There exist constants $C_{43}, C_{44}$ such that the following is true for any $r$. Let $A \in \mathcal{F}_n$, let $c_{n+1} \ldots, c_{n+r} \in \Omega_A$, and assume $c_{n+1}, \ldots, c_{n+r} \in \mathbf{G}(r)$. Then
\[
\exp(-C_{43}\exp(-C_{44}k^g)) \leq \frac{P(A|\omega_n = c_n, \omega_{n+1} = c_{n+1}, \ldots, \omega_n = c_n, \ldots)}{P(A|\omega_n = c_n, \omega_{n+i} = c_{n+i}, \ldots)} \leq \exp(C_{43}\exp(-C_{44}k^g))
\]

Applying $l$ times, we obtain

Lemma 22 There exist constants $C_{45}, C_{46}, C_{47}, C_{48}$ such that the following is true for any $r$. Let $c_1, \ldots, c_r \in \mathbf{G}(r^2)$. Then for any $l$, $1 \leq l \leq r$, we have
\[
\exp(-C_{43}l\exp(-C_{44}k^g)) \leq \frac{P(\omega_1 = c_1, \ldots, \omega_l = c_l)}{P(\omega_1 = c_1, \ldots, \omega_l = c_l)} \leq \exp(C_{45}l\exp(-C_{46}k^g))
\]
and
\[
\exp(-C_{47}l\exp(-C_{48}k^g)) \leq \frac{P(\omega_1 = c_1, \ldots, \omega_r = c_r)}{P(\omega_1 = c_1, \ldots, \omega_r = c_r)} \leq \exp(C_{47}l\exp(-C_{48}k^g))
\]

Summing over cylinders of length $lr$, we obtain

Corollary 15 There exist constants $C_{49}, C_{50}$ such that the following is true for any $r$. Let $c_1, \ldots, c_r \in \mathbf{G}(r^2)$. Then for any $l$, $1 \leq l \leq r$, and any $A \in \mathcal{F}_{lr}$, we have
\[
\exp(-C_{49}l\exp(-C_{50}k^g)) \leq \frac{P(A \cap \mathbf{G}(lr)|\omega_{lr+1} = c_{lr+1}, \ldots, \omega_r = c_r)}{P(\omega_1 = c_1, \ldots, \omega_r = c_r)} \leq \exp(C_{49}l\exp(-C_{50}k^g))
\]
and
\[
\exp(-C_{49}l\exp(-C_{50}k^g)) \leq \frac{P(A \cap \mathbf{G}(lr))}{P(\omega_1 = c_1, \ldots, \omega_r = c_r)} \leq \exp(C_{49}l\exp(-C_{50}k^g))
\]

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Using (28), we can estimate the total mass of the measure $\mu_{r,\vartheta}$.

**Corollary 16** There exist constants $C_{51}, C_{52}$ such that for any $r$ we have

$$\mu_{r,\vartheta}(G(r^2)) \geq \exp(-C_{51}r\exp(-C_{52}k^{(1-\vartheta)/2}))$$

We now have normalized versions of previous statements.

**Corollary 17** There exist constants $C_{53}, C_{54}, C_{55}, C_{56}$ such that the following is true for any $r$. Let $c_1 \ldots c_{r^2} \in G(r^2)$. Then for any $l, 1 \leq l \leq r$, and any $A \in \mathcal{F}_{lr}$, we have

$$\exp(-C_{53}l\exp(-C_{54}k^{\vartheta}) - C_{55}r\exp(-C_{56}k^{(1-\vartheta)/2})) \leq$$

$$\leq \frac{\mathbb{P}(A \cap G(lr)|\omega_{lr+1} = c_{lr+1}, \ldots, \omega_{r^2} = c_{r^2})}{\mu_{r,\vartheta}(A)} \leq$$

$$\leq \exp(C_{53}l\exp(-C_{54}k^{\vartheta}) + C_{55}r\exp(-C_{56}k^{(1-\vartheta)/2}))$$

and

$$\exp(-C_{53}l\exp(-C_{54}k^{\vartheta}) - C_{55}r\exp(-C_{56}k^{(1-\vartheta)/2})) \leq$$

$$\leq \frac{\mathbb{P}(A \cap G(lr))}{\mu_{r,\vartheta}(A)} \leq$$

$$\leq \exp(C_{53}l\exp(-C_{54}k^{\vartheta}) + C_{55}r\exp(-C_{56}k^{(1-\vartheta)/2}))$$

Using the Markov approximation, we can estimate conditional measures of good cylinders for the measure $\mathbb{P}$.

**Corollary 18** There exist constants $C_{57}, C_{58}, C_{59}, C_{60}$ such that the following is true for any $r$. Let $c_1 \ldots c_{r^2} \in G(r^2)$. Then for any $l, 1 \leq l \leq r$, we have

$$\mathbb{P}((\omega_1 \ldots \omega_{lr}) \in G(lr)|\omega_{lr+1} = c_{lr+1}, \ldots, \omega_{r^2} = c_{r^2}) \geq \exp(-C_{57}l\exp(-C_{58}k^{\vartheta}) - C_{59}r\exp(-C_{60}k^{(1-\vartheta)/2}))$$

**Proof:** Indeed,

$$\mu_{r,\vartheta}(\omega_1 \ldots \omega_{r^2} \in G(lr)|\omega_{lr+1} = c_{lr+1}, \ldots, \omega_{r^2} = c_{r^2}) = 1.$$

### 13.4 Doeblin Condition

**Proposition 21** There exists $C_{61}$ such that the following holds for any $r$. For any $C_1 \subset \Delta(q), C_2 \subset \Delta_q$, and any $C_3 \in G(r)$, we have either

$$\frac{1}{C_{61}} \leq \frac{\mu_{r,\vartheta}(C_3 | C_2)}{\mu_{r,\vartheta}(C_3 | C_1)} \leq C_{61},$$

or $\mu_{r,\vartheta}(C_3 | C_2) = \mu_{r,\vartheta}(C_3 | C_1) = 0$.

Considering $n$-step transition probabilities, we obtain
Proposition 22 There exists a constant $C_{62}$ such that the following holds for any $r$. For any $\mathfrak{C}_1 \subset \Delta(q)$, $\mathfrak{C}_2 \subset \Delta(q)$ any $\mathfrak{C}_3 \in \mathbf{G}(r)$, and any $n \geq M$, we have

$$\frac{1}{C_{62}} \leq \frac{p_{r, \theta}(\omega[|1, r|] = C_1 \omega[2n+2r^{2}+2r = C_3]}{p_{r, \theta}(\omega[|1, r|] = C_1 \omega[2n+2r^{2}+2r = C_3]} \leq C_{62}.$$ 

Now, mixing, Proposition 16 and Proposition 17, and the definition of a good cylinder imply that

Proposition 23 There exists a constant $C_{63}$ such that the following holds for any $r$. For any $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3 \in \mathbf{G}(r)$ we have

$$\frac{1}{C_{63}} \leq \frac{p_{r, \theta}(\omega[|1, r|] = C_1 \omega[2r^{2}+2r = C_3]}{p_{r, \theta}(\omega[|1, r|] = C_1 \omega[2r^{2}+2r = C_3]} \leq C_{63}.$$ 

Now let $c_1 \ldots c_{r+2} \in \mathbf{G}(r^2)$. Denote $\mathfrak{C}_i = c_{r+1+i(r+1)}$. Lemma 20, together with the above estimates, implies the following

Corollary 19 There exist constants $C_{71}, C_{72}$ such that the following is true. For any $l$, $1 \leq l \leq r$, we have

$$\mathbb{P}(\omega[|1, r|] \in \mathbf{G}(r), \omega[2l+1, l+1; r] = \mathfrak{C}_l, \omega[2l+2, l+2; r] \in \mathbf{G}(r)] = \mathfrak{C}_3 \geq C_{71} \times \varphi(\mathfrak{C}_1)$$

and

$$\mathbb{P}_{r, \theta}(\omega[|1, r|] \in \mathbf{G}(r), \omega[2l+1, l+1; r] = \mathfrak{C}_l, \omega[2l+2, l+2; r] \in \mathbf{G}(r)] = \mathfrak{C}_3 \geq C_{71} \times \varphi(\mathfrak{C}_1).$$

This is the Doeblin Condition for the measure $P_{r, \theta}$ (see [13], [14], [22]). The Doeblin Condition implies that there exist constants $C_{73}, C_{74}$ such that for any $\mathfrak{C}_1, \mathfrak{C}_2 \in \mathbf{G}(r)$, we have

$$\exp(-C_{73} \exp(-C_{74} r)) \leq \frac{p_{r, \theta}(\omega[|1, r|] = \mathfrak{C}_1 \omega[2r^{2}+2r = \mathfrak{C}_3]}{p_{r, \theta}(\mathfrak{C}_1)} \leq \exp(C_{73} \exp(-C_{74} r)).$$

whence we obtain

Proposition 24 There exist constants $C_{81}, C_{82}, C_{83}, C_{84}$ such that the following is true for any $r$.

$$\exp(-C_{81} \exp(-C_{82} r) + \exp(-C_{83} r^2) + \exp(-C_{84} r^{(1-\theta)/2})) \leq \frac{\mathbb{P}(\omega[|1, r|] = \mathfrak{C}_1 \omega[2r^{2}+2r = \mathfrak{C}_3]}{\mathbb{P}(\mathfrak{C}_1)} \leq \exp(C_{81} \exp(-C_{82} r) + \exp(-C_{83} r^2) + \exp(-C_{84} r^{(1-\theta)/2})).$$

Moreover, in view of mixing, Proposition 16, and Proposition 17, the same estimate, up to a constant, takes place for any $n \geq r^2$.  

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Proposition 25 There exist constants $C_{85}, C_{86}, C_{87}, C_{88}$ such that the following is true for all $r$ and all $n \geq r^2$.

$$\exp\left(-C_{85}\left(\exp(-C_{86}r) + \exp(-C_{87}r^{\theta}) + \exp(-C_{88}r^{(1-\theta)/2})\right)\right) \leq \frac{\mathbb{P}[\omega|_{[1,r]} = C_1 \omega_{[r+1,n]} \in G(n-r), \omega_{[n,n+r]} = C_2]}{\mathbb{P}(C_1)} \leq \exp(C_{85}\exp(-C_{86}r) + \exp(-C_{87}r^{\theta}) + \exp(-C_{88}r^{(1-\theta)/2}))].$$

14 Approximation of Hölder Functions and Completion of the Proof of Theorems 4, 7, 8.

We shall prove the decay of correlations for a slightly more general class of functions on $\Delta(R)$ than Hölder functions. (we shall need this slightly more general class in the proof of the Central Limit Theorem).

Namely, we shall only require that a function be Hölder in restriction to cylinders of some given length and we shall also allow a moderate growth of the Hölder constant at infinity.

Formally, say that a function $\phi : \Delta(R) \to \mathbb{R}$ is weakly $l, \alpha$-Hölder if the following holds. Let $k$ be a positive integer, and let $w \in W_{A,B}, |w| \leq l$ be such that $\Delta(w) \subset \Delta_{A^0(-k)}$. Then there exists a constant $C(\phi)$ such that for any $(\lambda, \pi), (\lambda', \pi) \in \Delta(w)$, we have

$$|\phi(\lambda, \pi) - \phi(\lambda', \pi)| \leq Ck \theta(|\lambda, \lambda'|)^\alpha.$$

The smallest such $C$ for a given $\phi$ will be denoted $C_{l,\alpha}^{weak}(\phi)$. Clearly, if $\phi$ is Hölder with exponent $\alpha$, then it is also weakly $l, \alpha$-Hölder for any $l$ and $C_{l,\alpha}^{weak}(\phi) \leq C(\phi)$.

Recall that $B_n$ is the $\sigma$-algebra of sets of the form $G^{-n}(A), A \subset \Delta(R)$.

To prove the decay of correlations, it suffices to estimate the $L_2$-norm of $E(\phi|B_{2n})$ for a given weakly $l, \alpha$-Hölder $\phi$.

It will be convenient to assume that $\phi \geq 1$ (by linearity, it suffices to consider that case).

Proposition 26 Let $\theta \in \mathbb{R}, 0 < \theta < 1$. Let $p > 2$ and $\alpha > 0$. There exist constants $C_{31}, C_{32}, C_{33}$ such that the following is true for any $r$ and any $n \geq r^2$.

Let $l \leq r$. Let $\phi \in L_p(\Delta(R)^+, \nu)$ be weakly $l, \alpha$-Hölder and satisfy $\phi \geq 1$.

Then $\phi = \phi_1 + \phi_2 + \phi_3$ where

1. $\phi_1 \geq 1$ on $G(n)$ and $\phi_1 = \phi_2 = 0$ on $\Delta(B(n))$.

2. for any $(\lambda, \pi) \in G(n)$, we have

$$\left|\frac{E(\phi_1|F_n)(\lambda, \pi)}{E(\phi_1)} - 1\right| \leq \exp(-C_{31}(r^{(1-\theta)/2} + r^{\theta}).$$

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3. for \((λ, π) \in G(n)\), we have \(|φ_2| ≤ C_{l, α}^{\nu, var}(φ) \exp(−C_{l, 2} r^\theta)\).

4. \(||φ_3||_{L^2} ≤ \exp(−C_{l, 2} r^{(1−θ)/2})||φ||_{L^2}\).

Proof: For any good word \(w = w_1 \ldots w_{n+r}\), consider its beginning \(w_1 \ldots w_r\) and choose a point \(x_{w_1 \ldots w_r} \in Δ(w_1 \ldots w_r)\).

Denote by \(χ_{Δ(w)}\) the characteristic function of \(Δ(w)\) and set

\[φ_1 = \sum_{w \in G(n+r)} φ(x_{w_1 \ldots w_r}) χ_{Δ(w)}.\]

Proposition 25 yields the required properties of \(φ_1\) (note that we sum over all good words of length \(n + r\) in order to be able to apply the Proposition).

We set \(φ_2 = (φ - φ_1)χ_{G(n+r)}\) and \(φ_3 = φχ_{Δ(B(n+r))}\). The estimate for \(φ_2\) is satisfied by the definition of a Hölder function.

Finally, we have

\[||φ_3||_{L^2} ≤ E(|φχ_{Δ(B(n+r))}|^2).\]

whence, by Hölder’s inequality, using the estimate (28), we obtain the desired estimate for \(φ_3\), and the Proposition is proved completely.

Proposition 26 with \(θ = 1/3\) yields Theorem 4.

We now complete the proof of Theorem 7.

For a word \(w \in W_{A,B}\), \(|w| = 2n+1\), \(w = w_1 \ldots w_{2n+1}\), denote \(C^{[-n,n]}(w) = \{ω \in Ω_{A,B}^\infty : [ω_n = w_1, \ldots, ω_{n} = w_{2n+1}\}\) and set \(Σ(w) = Φ^{-1}C^{[-n,n]}(w)\).

Denote by \(B_{[-n,n]}\) the sigma-algebra generated by \(Σ(w)\) for all \(w \in W_{A,B}\).

Also, for \(ε > 0\), denote

\[Σ_ε = \{(λ, h, a, π) \in Σ(R) : λ \in Δ_ε\}.\]

Again, we shall prove the Theorem for a slightly larger class of functions.

Say that a function \(φ : Σ(R) \rightarrow \mathbb{R}\) is weakly \(l, α\)-Hölder if the following holds. Let \(k\) be a positive integer, and let \(w \in W_{A,B}\), \(|w| ≤ 2^l + 1\) be such that \(Σ(w) \subset Δ_{exp(−k)}\). Then there exists a constant \(C(φ)\) such that for any \((λ, h, a, π), (λ', h', a', π) \in Σ(w)\), we have

\[|φ(λ, h, a, π) - φ(λ', h', a', π)| ≤ Cd((λ, h, a, π), (λ', h', a', π))^α.\]

The smallest such \(C\) for a given \(φ\) will be denoted \(C_{l, α}^{\nu, var}(φ)\). Clearly, if \(φ\) is Hölder with exponent \(α\), then it is also weakly \(l, α\)-Hölder for any \(l\) and \(C_{l, α}^{\nu, var}(φ) ≤ C_α(φ)\).

Denote by \(G(2n+1)\) the union of all \(Σ(w)\) for good \(w\), by \(\overline{B}(2n+1)\) the complement of \(G(2n+1)\).

**Proposition 27** Let \(θ \in \mathbb{R}\), \(0 < θ < 1\). Let \(p > 2\) and \(α > 0\). There exist constants \(C_{101,1}, C_{102,2}\) such that the following is true for any \(r\) and any \(n ≥ r^2\).

Let \(l ≤ r\). Let \(φ \in L_p(Σ(R)\overline{B})\) be weakly \(l, α\)-Hölder and satisfy \(φ ≥ 1\). Then there exist functions \(φ_1, φ_2, φ_3\) such that

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1. \( \phi = \phi_1 + \phi_2 + \phi_3. \)

2. \( \phi_1 \) is \( \mathcal{B}_{[-\eta, \eta]} \)-measurable and supported on \( \Sigma(2n + 1) \).

3. \( |\phi_2| \leq C_{101} C_\alpha \phi \exp(-r^{(1-\theta)/2} + r^\theta). \)

4. \( |\phi_3|_{L_2} \leq C_{102} \exp(-r^{(1-\theta)/2}) ||\phi||_{L_2}. \)

For any good \( w \), \( |w| = 2n + 1 \), take an arbitrary point \( x_w \) in \( \Sigma(w) \). Set

\[ \phi_1 = \sum_{w \in G(2n + 1)} \phi(x_w) \chi_{\Sigma(w)}, \]

\[ \phi_2 = (\phi - \phi_1) \cdot \chi_{G(2n + 1)}, \]

\[ \phi_3 = \phi \cdot \chi_{B(2n + 1)}, \]

and the Proposition is proved.

Proposition 27 with \( \theta = 1/3 \) yields Theorem 7.

It remains to establish the Central Limit Theorem for the flow \( P^t \). Consider the special function \( \hat{\tau} \) of the flow \( P^t \) over the transformation \( \mathcal{F} \). Note that \( \hat{\tau}(\lambda, h, a, \pi) \) only depends on \( (\lambda, \pi) \). Consider the restriction of \( \hat{\tau} \) on a cylinder of the form \( \Delta(w_1), w_1 \in \mathcal{A} \). Then there exist distinct \( j(1), \ldots, j(l) \in \{1, \ldots, m\} \) such that

\[ \hat{\tau}(\lambda, \pi) = \log(\lambda_{j(1)} + \lambda_{j(2)} + \cdots + \lambda_{j(l)}), \]

which shows that the function \( \hat{\tau} \), restricted to an arbitrary \( \Delta(w_1) \) is Lipshitz with respect to the Hilbert metric on \( \Delta(\mathcal{R}) \).

Now for a Hölder \( \phi \) consider the function

\[ \tilde{\phi}(x) = \int_{\mathbb{R}} \phi(P^t x). \]

For any \( k > 1 \), if \( (\lambda, \pi) \in \Delta_{exp(-k)} \), then, by definition, \( \hat{\tau}(\lambda, \pi) \leq k \). Therefore, if \( \phi \) is Hölder of exponent \( a \), then \( \tilde{\phi} \) is weakly \( 1, a \)-Hölder.

It is easy to see that \( \hat{\tau}(\lambda, \pi) \in L_r(\Delta(\mathcal{R}), \nu) \) for any \( r > 1 \), whence, if \( \phi \in L_p(\Omega(\mathcal{R}), \mu) \) for some \( p > 2 \), then there exists \( p' > 2 \) such that the function

\[ \tilde{\phi}(x) = \int_{\mathbb{R}} \phi(P^t x) \]

satisfies \( \tilde{\phi} \in L_{p'}(\mathcal{Y}^{\hat{\tau}}, \mathcal{P}) \).

Therefore, the Theorem of Melbourne and Török [15] implies Theorem 8, the Central Limit Theorem for the flow \( P^t \).

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