Lagrange Multipliers and Couplings in Supersymmetric Field Theory

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In hep-th/0312098 it was argued that by extending the “a-maximization” of hep-th/0304128 away from fixed points of the renormalization group, one can compute the anomalous dimensions of chiral superfields along the flow, and obtain a better understanding of the irreversibility of RG flow in four dimensional supersymmetric field theory. According to this proposal, the role of the running couplings is played by certain Lagrange multipliers that are introduced in the construction. We show that one can choose a parametrization of the space of couplings in which the Lagrange multipliers can indeed be identified with the couplings, and discuss the consequences of this for weakly coupled gauge theory.
1. Introduction

In the last year there was significant progress in the study of four dimensional super-symmetric field theories. One interesting result was the demonstration [1] that in an $N = 1$ superconformal field theory, the R-current that appears in the superconformal multiplet (and thus can be used to determine the scaling dimensions of chiral operators) locally maximizes the combination of triangle anomalies

$$a = 3 \text{tr} R^3 - \text{tr} R$$

(1.1)

over the set of all possible R-currents. This observation is especially useful in theories in which the infrared R-current is preserved throughout the renormalization group (RG) flow. As discussed in [1-5], this is often the case in four dimensional supersymmetric field theories, and if it is, one can determine the IR $U(1)_R$ uniquely by using $a$-maximization. When the infrared R-current involves accidental symmetries which only appear at the fixed point, these techniques are less useful.\footnote{Although a class of accidental symmetries associated with the decoupling of gauge invariant chiral superfields that reach the unitarity bound [2,3], as well as some other cases [4], can still be treated this way.}

The work of [1] also contributed to the quest for an analog of the c-theorem [6] in four dimensional supersymmetric field theory. The central charge $a$ (1.1) is proportional to the coefficient of the Euler density in the conformal anomaly of the theory on a curved spacetime manifold [7,8]. The latter was conjectured many years ago to be an analog of the Virasoro central charge in four dimensions [9]. According to this conjecture, $a$ should be always positive, and smaller at the infrared fixed point of an RG flow than at the corresponding UV fixed point. The fact that this quantity plays a role in the solution of a different but related problem came as a pleasant surprise.

The authors of [1] proposed an argument based on $a$-maximization, that supports the validity of the conjectured "c-theorem", at least in cases where the infrared R-symmetry is preserved throughout the RG flow. Since in these cases the infrared $U(1)_R$ is a subgroup of the symmetry group of the full theory, it is obtained by maximizing (1.1) over a subset of the R-currents that exist at the ultraviolet fixed point. Hence, the corresponding maximum of $a$ should be lower.

An important loophole in this argument (which was pointed out in [1]) is that the maximum of (1.1) that gives rise to the superconformal $U(1)_R$ is a local one. For example,
if the R-charges in the infrared are driven to values that are significantly larger than those in the UV, a \((1.1)\) can, in principle, grow in the process. This can only happen when the IR fixed point is sufficiently far from the UV one (i.e., non-perturbatively), but since the techniques of [1] are expected to be valid in a finite region in coupling space [1-4], this possibility needs to be considered. While gauge interactions tend to reduce the R-charges, superpotentials often have the opposite effect. Hence, in the presence of both gauge interactions and superpotentials, the above argument needs to be sharpened.

This difficulty was overcome in [4]. The main idea was to extend \(\alpha\)-maximization away from fixed points of the renormalization group. It was shown in [4] that one can define a function \(a(\lambda_i)\) on a space of certain interpolating parameters \(\lambda_i\), with the following properties:

1. The number of different interpolating parameters \(\lambda_i\) is the same as the number of independent couplings (gauge and superpotential) in the Lagrangian of the theory in question. Each of the \(\lambda_i\) is associated to a specific coupling.
2. The function \(a(\lambda_i)\) smoothly interpolates between the UV and IR fixed points of all the possible RG flows in the theory.
3. There exist trajectories in the space of \(\lambda_i\) along which \(a(\lambda_i)\) monotonically decreases as one goes from the UV to the IR. Furthermore, it satisfies the gradient flow property

\[
\partial_\tau a = G_{ij} \beta_j .
\]  

These results together with those of [1] establish that in all RG flows in which the superconformal \(U(1)_R\) is a subgroup of the full symmetry group of the theory, the \(\alpha\)-theorem is satisfied. They also lead to the natural question what is the precise relation between the interpolating parameters \(\lambda_i\) which appear in the construction, and the gauge and superpotential couplings of the theory.

It was proposed in [4] that the \(\lambda_i\) provide a parametrization of the space of couplings of the quantum field theory. Evidence for this claim was provided by matching certain results in weakly coupled gauge theory with the analysis coming from \(\alpha\)-maximization. However, the question whether such an identification is fully consistent was left open.

There are a number of reasons why the identification of the interpolating parameters \(\lambda_i\) with the field theory couplings merits further investigation. First, the \(\lambda_i\) are introduced into the theory in a very different way than the couplings, and it is not obvious apriori that the two are related by a reparametrization. In the usual description of a four dimensional field
theory, the dependence of the anomalous dimensions and the $\beta$ function on the couplings is determined by performing loop calculations, which rapidly become intractable as one goes to higher orders. On the other hand, in [4] exact expressions were given for all the anomalous dimensions as a function of $\lambda_i$, without performing any loop calculations.\footnote{One such calculation is needed for each coupling, to normalize the interpolating parameters $\lambda_i$.} It is natural to ask what predictions these results provide for perturbative gauge theory.

Also, it is usually said that the perturbative expansions for quantities like anomalous dimensions and $\beta$-functions in QFT have a vanishing radius of convergence. The radius of convergence of the perturbative expansions in the $\lambda_i$ is finite, and it is interesting to see how that comes about.

The purpose of this note is to clarify the relation between the interpolating parameters $\lambda_i$ of [4], and the field theory couplings. In section 2 we briefly review the construction of [4], focusing on simplicity on the case of gauge theories with vanishing superpotential. In section 3 we show that the interpolating parameter $\lambda$ is related to the gauge coupling by an analytic transformation of the general form

$$\lambda = \alpha + A_2 \alpha^2 + A_3 \alpha^3 + \cdots$$  \hspace{1cm} (1.3)

as proposed in [4]. In section 4 we discuss the constraints placed by our discussion on the perturbative expansion of anomalous dimensions in supersymmetric gauge theory, and comment on other aspects of the analysis. Some related results were independently found in [10].

2. $\alpha$-maximization away from fixed points

The setup for our discussion is an $N = 1$ supersymmetric gauge theory with gauge group $G$ and chiral superfields $\Phi_i$ in the representations $r_i$ of the gauge group. The quadratic Casimir of $G$ in the representation $r$ will be denoted by $C_2(r)$. It is given in terms of the generators of $G$ in that representation, $T^a$, by

$$T^a T^b = C_2(r) |r| \delta^{ab} .$$ \hspace{1cm} (2.1)

Here and below, $|r|$ denotes the dimension of the representation $r$. Another useful object is $T(r)$ defined via the relation

$$\operatorname{Tr}_r(T^a T^b) = T(r) \delta^{ab} .$$ \hspace{1cm} (2.2)
The two objects (2.1), (2.2) are related by (|G| is the dimension of the gauge group):

\[
C_2(r) = \frac{|G|}{|r|} T(r) .
\]  

(2.3)

The gauge theory is asymptotically free when

\[
Q \equiv 3T(G) - \sum_i T(r_i) > 0 ,
\]

(2.4)

and we will restrict the discussion to this case.

At short distances, the gauge coupling \( a = g^2/4\pi \) goes to zero, and the theory approaches a free fixed point. As the distance scale increases, the gauge coupling grows, and the superfields \( \Phi_i \) develop anomalous dimensions \( \gamma_i(a) \). Our task is to understand this RG flow.

At the IR fixed point, the anomalous dimensions satisfy the constraint

\[
3T(G) - \sum_i T(r_i)(1 - \gamma_i(a^*)) = Q + \sum_i T(r_i)\gamma_i(a^*) = 0 ,
\]

(2.5)

that follows from the NSVZ \( \beta \)-function [11,12]. This constraint has a natural interpretation in terms of the R-charges of the fields \( \Phi_i, R_i \), which are related to the anomalous dimensions via the superconformal algebra:

\[
\Delta(\Phi_i) = 1 + \frac{1}{2} \gamma_i = \frac{3}{2} R_i .
\]

(2.6)

Plugging this into (2.5) we find

\[
T(G) + \sum_i T(r_i)(R_i(a^*) - 1) = 0 ,
\]

(2.7)

which can also be thought of as the condition for the R-symmetry with \( R(\Phi_i) = R_i(a^*) \) to be anomaly free at non-zero gauge coupling.

As we flow from short to long distances, the R-charges \( R_i(a) \) (2.6) flow from 2/3 in the UV to a solution of (2.7) in the IR. In general, this equation has many solutions. The one that gives the values of \( R_i \) at the IR fixed point locally maximizes \( a \) (1.1), subject to the constraint (2.7) [1].
As was shown in [4], a useful way of solving for $R_i(\alpha^*)$ is to take the constraint into account by means of a Lagrange multiplier:

$$a(R_i, \lambda) = 2|G| + \sum_i |r_i| \left[ 3(R_i - 1)^3 - |R_i - 1| \right] - \frac{2|G|}{\pi} \left[ T(G) + \sum_i T(r_i)(R_i - 1) \right].$$  \hspace{1cm} (2.8)

If we first locally maximize (2.8) with respect to the $R_i$, for fixed $\lambda$, solve for $R_i(\lambda)$, and substitute back into (2.8), we get a central charge that depends only on $\lambda$, $a(R_i(\lambda), \lambda)$ which has the following properties [4]:

(1) At $\lambda = 0$, one finds the R-charges and central charge (1.1) of the free UV fixed point.

(2) The infrared fixed point of the gauge theory corresponds to a local minimum of $a$ at a positive value of $\lambda$, $\lambda = \lambda^*$. $R_i(\lambda^*)$ are the IR R-charges and $a(R_i(\lambda^*), \lambda^*)$ gives the value of the central charge (1.1) at the infrared fixed point.

(3) As one varies $\lambda$ between 0 and $\lambda^*$, the central charge $a$ monotonically decreases. It satisfies the relation

$$\frac{da}{d\lambda} = -\frac{2|G|}{\pi} \left[ T(G) + \sum_i T(r_i)(R_i(\lambda) - 1) \right].$$  \hspace{1cm} (2.9)

The right hand side is proportional to the $\beta$-function for the gauge coupling.

(4) The R-charges along the flow are given by

$$R_i(\lambda) = 1 - \frac{1}{3} \left[ 1 + \frac{2\lambda}{\pi} C_2(r_i) \right]^{\frac{1}{3}}.$$  \hspace{1cm} (2.10)

As emphasized in [4], these properties strongly suggest that $\lambda$ should be identified with the gauge coupling in some scheme (or parametrization of the space of gauge theories).

We next show that this is indeed the case.

3. Lagrange multipliers and gauge couplings in 4d SYM theory

In order to understand the relation between the gauge coupling $\alpha$ and the Lagrange multiplier $\lambda$, we have to take a closer look at the freedom of reparametrizing the space of theories.

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3 We have rescaled $\lambda$ by a factor $2|G|/\pi$ relative to its normalization in [4].
Suppose we start in a scheme in which the running of the gauge coupling is governed by the NSVZ \( \beta \)-function [11,12]. The latter can be written as

\[
\beta(a) = f(a) \left[ Q + \sum_i T(r_i) \gamma_i(a) \right] \tag{3.1}
\]

with

\[
f(a) = -\frac{\alpha^2}{2\pi} \frac{1}{1 - \frac{\alpha}{2\pi} T(G)} . \tag{3.2}
\]

We will assume that throughout the RG flow, the gauge coupling never becomes large enough so that \( f(a), \) (3.2), becomes singular. For the purposes of the present discussion, this assumption is not restrictive, since we will work in the infinitesimal vicinity of \( a = 0 \) (to all orders in \( a \)), where the denominator of (3.2) is harmless.

Any other scheme will in general differ from this one by a reparametrization of the coupling, and by a coupling-dependent rescaling of the chiral superfields:

\[
a \to \tilde{a}(a) \\
\Phi_i \to \tilde{\Phi}_i = \sqrt{F_i(a)} \Phi_i . \tag{3.3}
\]

As usual, we would like all schemes to agree at weak coupling, so that:

\[
\tilde{a} = a (1 + a_1 a + a_2 a^2 + \cdots) \\
F_i(a) = 1 + k_i^{(1)} a + k_i^{(2)} a^2 + \cdots . \tag{3.4}
\]

The anomalous dimensions transform as follows under (3.3), (3.4):

\[
\tilde{\gamma}_i(\tilde{a}) = \gamma_i(a) - \beta(a) \frac{d}{da} \log F_i(a) . \tag{3.5}
\]

A short calculation shows that after the transformation (3.3), (3.5), the \( \beta \)-function takes the form

\[
\tilde{\beta}(\tilde{a}) = \tilde{f}(\tilde{a}) \left[ Q + \sum_i T(r_i) \tilde{\gamma}_i(\tilde{a}) \right] \tag{3.6}
\]

with

\[
\tilde{f}(\tilde{a}) = f(a) \frac{d\tilde{a}}{da} \frac{1}{1 - f(a) \frac{d}{da} \sum_i T(r_i) \log F_i(a)} . \tag{3.7}
\]

Thus, we see that the fact that the \( \beta \)-function is proportional to \( Q + \sum_i T(r_i) \gamma_i(a) \) is preserved under reparametrizations. The only thing that changes is the prefactor \( f(a) \) in (3.1). This fact implies that as long as the prefactor \( \tilde{f}(\tilde{a}) \) in (3.6) is non-singular along the
RG flow, the infrared fixed point corresponds to a solution of (2.5), in any parametrization of the gauge coupling and of the superfields $\Phi_i$.

Naively, one might be tempted at this point to argue as follows. The freedom (3.4), (3.5) allows one to bring the anomalous dimensions to a form linear in $\alpha$ (recall that the $\beta$-function (3.1) starts like $\alpha^2$ for small $\alpha$),

$$\gamma_i(\alpha) = -\frac{\tilde{\alpha}}{\pi} C_2(r_i) .$$

To find the IR fixed point it is natural to set the term in square brackets in (3.6) to zero,

$$Q + \sum_i T(r_i) \gamma_i(\alpha^*) = 0 ,$$

as in (2.5).

Substituting (3.8) into (3.9), we find

$$\tilde{\alpha}^* = \frac{\pi Q}{\sum_i T(r_i) C_2(r_i)} ,$$

$$\gamma_i(\tilde{\alpha}^*) = -\frac{QC_2(r_i)}{\sum_i T(r_i) C_2(r_i)} .$$

Interestingly, this answer is incorrect! Recall [4] that the correct answer is obtained by solving (2.7), (2.10); the resulting anomalous dimensions disagree with (3.9) beyond leading order in $\lambda^*$ (or more precisely, in $Q$; see discussion in section 4).

What went wrong? Evidently, the prefactor $\tilde{f}(\tilde{\alpha})$ in (3.6) is not harmless in this case. The transformation to the coordinates in which (3.8) is valid must have the property that it pushes the solution of (3.9), (3.10) beyond the regime in which (3.7) is well behaved. The lesson from this is that when we make coordinate transformations (3.3), we have to make sure that the solution of eq. (2.5), (3.9) is not pushed outside of the regime of validity of (3.6).

Now, suppose the anomalous dimensions $\gamma_i(\alpha)$ have been calculated in some scheme (e.g. that of [13,14]) with the $\beta$-function given by (3.1) (with $f(\alpha) = -\alpha^2/\pi(1 + O(\alpha))$, not necessarily given exactly by (3.2)). Perform a reparametrization (3.4) such that

$$\gamma_i(\tilde{\alpha}) = 1 - \left[ 1 + \frac{2\tilde{\alpha}}{\pi} C_2(r_i) \right]^+. \tag{3.11}$$

---

4 This expression is equivalent to (2.10), using $g(2\tilde{\alpha})$, with $\lambda$ replaced by $\tilde{\alpha}$.
By construction, after the reparametrization the $\beta$-function (3.6) vanishes when the expression in square brackets (3.9) is set to zero. Thus, it is natural to expect the dangerous prefactor $\tilde{f}(\tilde{\alpha})$ in (3.6) to be well behaved in this case, so that the transformation $(a, \Phi_i) \to (\tilde{\alpha}, \tilde{\Phi}_i)$ is non-singular throughout the RG flow for sufficiently weak coupling. We will assume this to be the case and will verify it in the next section, when we study the theory perturbatively.

The assumption that $\tilde{f}(\tilde{\alpha})$ is regular throughout the RG flow implies that the map $a \to \tilde{\alpha}(a)$ is monotonic: as $a$ increases from 0 to $a^*$, $\tilde{\alpha}$ increases from 0 to $\tilde{\alpha}^*$. Any turning points of the map would give rise to zeroes of $\frac{d\tilde{\alpha}}{da}$, and thus of $\tilde{f}$ (the other factors in (3.7) can not cancel such zeroes).

To recapitulate, in this section we have presented an explicit prescription for going from a general parametrization of the space of $N = 1$ SYM theories with matter, to the one naturally given by the Lagrange multiplier construction of [4]. Key elements are the fact that the NSVZ $\beta$-function, (3.1), transforms covariantly under analytic reparametrizations (3.3), and the fact that under the specific reparametrization that leads to (3.11) we expect the prefactor in (3.6) to be analytic at weak coupling.

4. Weakly coupled SYM theory

As mentioned in the introduction, the construction of [4] and the previous section gives rise to predictions for the perturbative expansion of anomalous dimensions in gauge theory. In order to see what those predictions are, we start with the prefered scheme in which the anomalous dimensions are given by (3.11), and perform a general coordinate transformation (3.4), (3.5).

We find the following structure:

$$\gamma_i(a) = 1 - \left[ 1 + \frac{2\tilde{\alpha}}{\pi} C_2(r_i) \right]^{\frac{1}{2}} + \beta(a) \frac{F_i'(a)}{F_i(a)} .$$

(4.1)

Expanding

$$\gamma_i(a) = \sum_{n=1}^{\infty} \gamma_i^{(n)} a^n ,$$

(4.2)

the leading contribution to $\gamma_i^{(n)}$ comes from replacing $\tilde{\alpha}$ by $a$ in the square root in (4.1), and expanding to $n$'th order. The second term in (4.1) can be neglected at this order, due to the overall factor of the $\beta$-function in it.\footnote{This statement will be made more precise below.}

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The expansion of the square root gives

\[ \gamma_i^{(n)} = \left[ -\frac{C_2(r_i)}{2\pi} \right]^{n/2} \frac{2(2n-2)!}{n!(n-1)!} + \cdots \]  

(4.3)

There are two types of corrections to (4.3). One comes from substituting subleading terms in the expansion of \( \tilde{a}(a) \) (3.4) in (4.1). This completes (4.3) to

\[ \gamma_i^{(n)} = a_0^{(n)} (C_2(r_i))^n + a_{n-1}^{(n)} (C_2(r_i))^{n-1} + \cdots + a_1^{(n)} C_2(r_i) \],

(4.4)

where \( a_0^{(n)} \) is given by (4.3), and the lower terms can be expressed in terms of the expansion coefficients \( a_i \) in (3.4).

The second source of corrections comes from the expansion of the last term in (4.1). It is sensitive to the precise form of the prefactor \( f(a) \) in the \( \beta \)-function (3.1) (which may or may not be given by (3.2), depending on the scheme), and the expansion coefficients \( b_i^{(n)} \) in (3.4).

The above predictions can be compared to “data.” Results for the anomalous dimensions up to three loop order appear in [13,14]:

\[ \gamma_i(a) = \frac{\alpha}{\pi} C_2(r_i) + \]

\[ \frac{\alpha^3}{4\pi^2} \left[ \frac{2}{(C_2(r_i))^2} - Q C_2(r_i) \right] - \]

\[ \frac{\alpha^3}{8\pi^3} \left[ \frac{4}{(C_2(r_i))^3} - Q (C_2(r_i))^2 + A C_2(r_i) \right] + O(\alpha^4) \]

(4.5)

The coefficient \( A \) is also given in [13,14]; we will not need its form here, and so do not quote it.

The first thing to note is that the leading term at each order in \( \alpha \) is in exact agreement with (4.3). The agreement of the one and two loop coefficients was checked before in [4]; the three loop check is new. This check was independently made in [10], where the prediction (4.3), (4.4) was also derived, assuming the relation between the Lagrange multiplier of [4] and the gauge coupling.

Comparing the term that goes like \( C_2(r_i) \) on the second line of (4.5) we conclude that the relation between the gauge couplings (3.4) is

\[ \tilde{a} = a \left( 1 + \frac{Q \alpha}{4\pi} + O(\alpha^2) \right) \]

(4.6)
Using this redefinition at order $\tilde{a}^2$ in (4.1), we find a term in $\gamma_i$ that goes like

$$\frac{Q a^3}{4\pi^2} (C_2(r_i))^2 .$$

This is larger by a factor of two than the second term on the third line of (4.5). The difference can be accounted for by using the last term in (4.1). One finds:

$$F_i(\alpha) = 1 + \frac{a^2}{8\pi^2} (C_2(r_i))^2 + O(a^3) .$$  (4.7)

We see that at three loop order, the structure is as anticipated in [4] and section 3. The leading terms in $\gamma_i^{(n)}$ (4.4), which are scheme independent, sum up to the square root (3.11). Using the subleading terms one can determine the reparametrization expansion (3.4) perturbatively in $a$. One finds that, to this order, the coefficients $a_i, b_i^{(n)}$ in (3.4) are polynomials in the Casimirs. We expect this to persist to all orders in $a$.

Note that if we treat the quadratic Casimirs $C_2(r_i), T(r_i), Q, a$ as formal variables (ignoring the fact that they take discrete values), the problem has a scaling symmetry in any renormalization scheme. Under this symmetry, the quadratic Casimirs transform with weight one, $\alpha$ and the $\beta$-function have weight $-1$, and the anomalous dimensions have weight zero.

The above symmetry is respected by our Lagrange multiplier inspired expression (3.11), and by any gauge theory loop expansion, such as (4.5). The expansion coefficients $a_n, b_i^{(n)}$ in (3.4) have weight $n$, while the coefficients $a_i^{(n)}$ in (4.4) have weight $n - i$. If no inverse powers of Casimirs appear, they are polynomials of the relevant degree in the Casimirs. As mentioned above, we expect this to be the case to all orders in $\alpha$.

The notion of weakly coupled RG flows can be made precise as follows. If we treat the Casimirs as free continuous parameters (something that becomes precisely justified for large rank gauge groups, but is often a very good approximation for finite rank as well), it is well known that as $Q$ (2.4) goes to zero the theory loses asymptotic freedom, and if $Q$ is positive and small, the IR fixed point is weakly coupled and can be treated perturbatively [15,16]. As one flows from the UV to the IR in these situations, $\alpha$ varies between zero and a value of order $Q$. Therefore, it makes sense to expand the anomalous dimensions (4.2) and other quantities in a double series in $\alpha$ and $Q$.

The expansion coefficients $a_i^{(n)}$ in (4.4) are polynomials of degree $n - i$ in $Q$. In particular, $a_i^{(1)}$ is a constant (given by (4.3)), $a_i^{(n-1)}$ is a first order polynomial (examples of which appear in (4.5)), etc. Another way of thinking about the construction of section
3 is that starting with a set of $\gamma(\alpha)$ with the properties outlined above, we can perform a transformation of the form (3.4), (3.5) to bring the anomalous dimensions to a form where $Q$ does not appear, without introducing non-analyticity in $Q$ in the prefactor $\tilde{f}(\tilde{\alpha})$ in (3.7). One can then show that, up to a $Q$-independent reparametrization, the anomalous dimensions must be given by (3.11).

We finish with a few comments:

(1) We found that there is an analytic redefinition of the coupling $\alpha$ and of the superfields $\Phi_i$, after which the anomalous dimensions are given by a perturbative series with a finite radius of convergence (3.11). It would be nice if this parametrization arose from a natural renormalization scheme in perturbative SYM theory. We do not know whether this is the case.

(2) We expect the coefficients $a_i, b_i^{(n)}$ (3.4), which determine the above redefinition, to be polynomials in the Casimirs, whose rank was indicated earlier in this section. This would guarantee the regularity of $\tilde{f}(\tilde{\alpha})$ (3.7) to all orders in $Q$. We have not proven this property – it would be very nice to do that.

(3) While we focused here on the case of a gauge theory with a semi-simple gauge group and vanishing superpotential, the discussion can be easily extended to situations with multiple couplings, as in [4]. In order to achieve a quantitative understanding of the flow of couplings with the scale in this case, one needs to know the metric governing the gradient flow of the central charge $\alpha_i$ (1.2). Some perturbative results on this metric appear in [4, 10].

(4) The most important open problem remains to extend the discussion beyond the regime of validity of the NSVZ $\beta$-function, or more generally to the regime where the description in terms of the original degrees of freedom $W_\alpha, \Phi_i$ breaks down.

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