Global Existence of Quasilinear, Nonrelativistic Wave Equations Satisfying the Null Condition

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GLOBAL EXISTENCE OF QUASILINEAR, NONRELATIVISTIC
WAVE EQUATIONS SATISFYING THE NULL CONDITION

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Abstract. We prove global existence of solutions to multiple speed, Dirichlet-wave
equations with quadratic nonlinearities satisfying the null condition in the exterior of
compact obstacles. This extends the result of our previous paper by allowing general
higher order terms. In the current setting, these terms are much more difficult to
handle than for the free wave equation, and we do so using an analog of a pointwise
estimate due to Kubota and Yokoyama.

1. Introduction

The purpose of this paper is to provide a proof of global existence of solutions to gen-
eral quasilinear, multiple speed systems of wave equations satisfying the null condition.
The techniques presented are sufficient to handle both Minkowski wave equations and
Dirichlet-wave equations in the exterior of certain compact obstacles.

For the latter case, fix a smooth, compact obstacle $\mathcal{K} \subset \mathbb{R}^3$. We, then, wish to examine
the quasilinear system

$$\begin{align*}
\square u &= F(u, du, \partial_t u), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathcal{K} \\
\partial_t u(t, \cdot)_{\mid_{\partial \mathcal{K}}} &= 0 \\
u(0, \cdot) &= f, \quad \partial_t u(0, \cdot) = g.
\end{align*}$$

(1.1)

Here

$$\square = (\square_{c_1}, \square_{c_2}, \ldots, \square_{c_D})$$

denotes a vector-valued multiple speed d’Alembertian where

$$\square_{c_i} = \partial_t^2 - c_i^2 \Delta$$

and $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the standard Laplacian. For clarity, we will assume that we
are in the nonrelativistic case. That is, we assume that the wave speeds $c_i$ are positive
and distinct. Straightforward modifications can be made to allow various components to
have the same speed. For convenience, we will take $c_0 = 0$ and

$$0 = c_0 < c_1 < c_2 < \cdots < c_D$$

(1.3)

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throughout.

We now describe our conditions on the nonlinearity $F$. First of all, $F$ is assumed to be linear in $d^2 u$. $F$ is also required to vanish to second order. That is,

$$\partial^n F(0,0,0) = 0, \quad |a| \leq 1.$$ 

Additionally, we assume

$$\partial^2_u F(0,0,0) = 0.$$ 

Thus, $F$ may be decomposed as

$$F(u, du, d^2 u) = B(du) + Q(du, d^2 u) + R(u, du, d^2 u) + P(u, du)$$

where, for $1 \leq I \leq D$,

$$B^I(du) = \sum_{1 \leq J, K \leq D \atop 0 \leq j, k \leq 3} A^{I,jk}_{J,K} \partial_j u^I \partial_k u^K,$$

$$Q^I(du, d^2 u) = \sum_{1 \leq J, K \leq D \atop 0 \leq j, k \leq 3} B^{I,jk}_{K,l} \partial_j u^K \partial_l \partial_k u^I,$$

$$R^I(u, du, d^2 u) = \sum_{1 \leq J, K \leq D \atop 0 \leq j, k \leq 3} C^{I,jk}_{J,K} u^I \partial_j \partial_k u^J,$$

with $C^{I,jk}_{J,K}(u, du) = O(|u|^2 + |du|^2)$, and $P(u, du) = O(|u|^3 + |du|^3)$ near $(u, du) = 0$. Here and throughout, we use the notation $x_\circ = t$ and $\partial_\circ = \partial_t$ when convenient. Additionally, $du = d^2 u = \nabla_{t,x} u$ denotes the space-time gradient. The constants $B^{I,jk}_{K,l}$ are real, as are the $C^{I,jk}_{J,K}(u, du)$ terms. Moreover, the quasilinear terms are assumed to satisfy the symmetry conditions

$$B^{I,jk}_{K,l} = B^{I,kj}_{K,l} = B^{I,jk}_{K,l},$$


In order to establish global existence, we require that the quadratic terms satisfy the following null condition:

$$\sum_{0 \leq j, k \leq 3} A^{j,k}_{j,k} \xi_j \xi_k = 0, \quad \text{whenever } \frac{\xi_j^3}{c_j^3} - \xi_j^2 - \xi_j^3 = 0, \quad J = 1, 2, \ldots, D,$$

$$\sum_{0 \leq j, k, l \leq 3} B^{j,k}_{j,l} \xi_j \xi_k \xi_l = 0, \quad \text{whenever } \frac{\xi_j^3}{c_j^3} - \xi_j^2 - \xi_j^3 = 0, \quad J = 1, 2, \ldots, D.$$ 

This null condition guarantees that the self-interaction of each wave family is nonresonant and is the natural one for systems of quasilinear wave equations with multiple speeds. It is equivalent to the requirement that no plane wave solution of the system is genuinely nonlinear. This follows from an observation of John and Shatah, and we refer the reader to John [11] (p. 23) and Agemi-Yokoyama [1]. Additionally, in the setting of elasticity, Tahwilday-Zadeh [39] (see also Sideris [33]) observed that (1.9), (1.10) removed the physically unrealistic restrictions on the growth of the stored energy imposed by the null conditions used, for example, in [28], [34], and [38]. While general global existence
of solutions to (1.1) is only known (even in the Minkowski setting) under the assumption of (1.9), (1.10), recent works of Lindblad-Rodnianski [24, 25] suggest that a weak form of the null condition may be sufficient.

We now wish to describe our assumptions on the obstacle \( \mathcal{K} \subset \mathbb{R}^3 \). As mentioned above, we assume that \( \mathcal{K} \) is smooth and compact, but not necessarily connected. By shifting and scaling, we may take

\[
0 \in \mathcal{K} \subset \{ |x| < 1 \}
\]

with no loss of generality. The only additional assumption is that there is exponential decay of local energy. Specifically, if \( u \) is a solution to the homogeneous wave equation

\[
\begin{cases}
\Box u = 0 \\
u(t, \cdot) |_{\partial \mathcal{K}} = 0
\end{cases}
\]

and the Cauchy data \( u(0, \cdot), \partial_t u(0, \cdot) \) are supported in \( \{ |x| < 4 \} \), then we assume that there are constants \( c, C > 0 \) so that

\[
(1.11) \quad \left( \int_{|x|<4} |u'(t,x)|^2 \, dx \right)^{1/2} \leq Ce^{-ct} \sum_{|\alpha| \leq 1} \| \partial_x^\alpha u'(0, \cdot) \|_2.
\]

If the obstacle is nontrapping, a stronger version of (1.11) holds with \( |\alpha| = 0 \) (no loss of derivative). See, e.g., Morawetz-Ralston-Strauss [30]. In the presence of trapped rays, Ralston [31] observed that this stronger version could not hold, and Ikawa [9, 10] showed that (1.11) holds for certain finite unions of convex obstacles.

In order to solve (1.1), we must also require that the data satisfies certain compatibility conditions. Briefly, if we let \( J_k u = \{ \partial^\alpha u : 0 \leq |\alpha| \leq k \} \) and fix \( m \), we can write \( \partial_t^k u(0, \cdot) = \psi_k (J_k f, J_{k-1} g) \), \( 0 \leq k \leq m \) for any formal \( H^m \) solution of (1.1). Here, \( \psi_k \) is called a compatibility function and depends on \( F, J_k f, \) and \( J_{k-1} g \). The compatibility condition for (1.1) with \( (f, g) \in H^0 \times H^{m-1} \) states that the \( \psi_k \) vanish on \( \partial \mathcal{K} \) when \( 0 \leq k \leq m - 1 \). Additionally, we say that \( (f, g) \in C^\infty \) satisfy the compatibility condition to infinite order if this holds for all \( m \). See, e.g., [15] for a more detailed description of the compatibility conditions.

We can now state our main result.

**Theorem 1.1.** Let \( \mathcal{K} \) be a fixed compact obstacle with smooth boundary satisfying (1.11). Assume that \( F(u, du, d^2 u) \) and \( \Box \) are as above and that \( (f, g) \in C^\infty (\mathbb{R}^3 \setminus \mathcal{K}) \) satisfy the compatibility conditions to infinite order. Then, there is an \( \varepsilon_0 > 0 \) and an integer \( N > 0 \) so that for all \( \varepsilon < \varepsilon_0 \), if

\[
(1.12) \quad \sum_{|\alpha| \leq N} \| (x)^{|\alpha|} \partial_x^\alpha f \|_2 + \sum_{|\alpha| \leq N-1} \| (x)^{1+|\alpha|} \partial_x^\alpha g \|_2 \leq \varepsilon,
\]

then (1.1) has a unique global solution \( u \in C^\infty ([0, \infty) \times \mathbb{R}^3 \setminus \mathcal{K}) \).

As mentioned above, we will also handle the Minkowski case. Assuming that \( F \) and \( \Box \) are as above, we show that solutions of

\[
(1.13) \quad \begin{cases}
\Box u = F(u, du, d^2 u), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 \\
u(0, \cdot) = f, \quad \partial_t u(0, \cdot) = g
\end{cases}
\]
exist globally for small data. Specifically, we will prove

**Theorem 1.2.** Assume that $F$ and $\Box$ are as above. Then, there are constants $\varepsilon_0, N > 0$ so if $f, g$ are smooth functions satisfying

$$
(1.14) \quad \sum_{|\alpha| \leq N} \|\langle x \rangle^{|\alpha|} \partial_x^\alpha f \|_2 + \sum_{|\alpha| \leq N-1} \|\langle x \rangle^{1+|\alpha|} \partial_x^\alpha g \|_2 \leq \varepsilon,
$$

for all $\varepsilon < \varepsilon_0$, then the system (1.13) has a unique global solution $u \in C^\infty([0, \infty) \times \mathbb{R}^3)$.

We note that during preparation of this paper it was discovered that Theorem 1.2 was proven independently by Katayama [12] using different techniques. Additionally, in [13], Katayama explored the possibility of allowing $F$ to contain certain terms of the form $u^l \beta u^k$ if you assume the null condition of [34, 38] rather than (1.9), (1.10). The obstacle result, Theorem 1.1, is new.

By allowing general higher order terms, Theorem 1.2 extends the previously known results on multiple speed wave equations due to Sideris-Tu [35], Agemi-Yokoyama [1], Kubota-Yokoyama [21], and Katayama [14]. In a similar way, Theorem 1.1 extends the previous result of the authors [27].

In studying both the Minkowski setting and the exterior domain, we will be using modifications of the method of commuting vector fields due to Klainerman [19]. We will restrict to the class of vector fields $\Gamma = \{Z, L\}$ that seem “admissible” for boundary value problems and studies of multiple speed wave equations. Here, $Z$ denotes the generators of space-time translations and spatial rotations

$$
(1.15) \quad Z = \{\partial_i, x_j \partial_k - x_k \partial_j, \quad 0 \leq i \leq 3, 1 \leq j \leq k \leq 3\}
$$

and $L$ is the scaling vector field

$$
(1.16) \quad L = t \partial_t + r \partial_r.
$$

Additionally, we will write $r = |x|$ and

$$
(1.17) \quad \Omega_{jk} = x_j \partial_k - x_k \partial_j
$$

for the generators of spatial rotations. The generators of the Lorentz rotations, $x_i \partial_i + t \partial_t$, when $c_T = 1$, have an associated speed and have unbounded normal components on the boundary of our compact obstacle, and thus seem ill-suited to the problems in question. Katayama [12, 13] has shown that these hyperbolic rotations can be used in a limited fashion in the study of multiple speed wave equations, but we do not require those techniques here.

The most significant new difficulty in this case versus the one considered in [27] is the cubic terms not involving derivatives. Those involving derivatives can generally be handled using energy methods. In the approaches of Christodoulou [3] and Klainerman [19], such terms not involving derivatives were handled with a certain adapted energy inequality that resembles, e.g., the work of Morawetz [29]. This method relies on the use of the Lorentz rotations, and it is not clear how to adapt it to the current setting.

The new argument that we utilize uses an analog of a pointwise estimate that was established by Kubota-Yokoyama [21]. When combined with the pointwise estimates of Keel-Smith-Sogge [17] and sharp Huygens’ principle, we are able to establish low regularity decay of our solution $u$. These improved estimates allow us to handle the
cubic terms without derivatives discussed in the previous paragraph. In [27], using only the estimates of [17], the authors were only able to get such decay for the gradient of the solution \( u' \).

As in Keel-Smith-Sogge [16, 17], we will utilize a class of weighted \( L^2_\mu L^2_\nu \)-estimates where the weight is a negative power of \( \langle x \rangle = \langle r \rangle = \sqrt{t + r^2} \). Such estimates permit us to use the \( O(\langle x \rangle^{-1}) \) decay that is obtained from Sobolev inequalities rather than the more standard \( O(t^{-1}) \) decay which is difficult to prove without the use of the Lorentz rotations. Additionally, such estimates allow us to handle the boundary terms that arise in the energy estimates of nonlinear wave equations if we no longer have the convenient assumption of star-shapedness on the obstacle. This was one of the main innovations of Metcalfe-Sogge [28].

As in our previous work [27], we will require a class of weighted Sobolev estimates. The weights involve powers of \( r \) and \( \langle t - r \rangle \). In the Minkowski setting, these estimates are originally due to Klainerman-Sideris [20] and Hidano-Yokoyama [6].

This paper is organized as follows. In the next section, we gather our preliminary estimates that will be needed to show global existence in Minkowski space. In particular, we collect the pointwise estimates of Keel-Smith-Sogge [17] and Kubota-Yokoyama [21]. In Section 3, we prove Theorem 1.2. In Section 4, we gather the estimates that we will require to prove Theorem 1.1. Finally, in Sections 5-7, we prove our main theorem, Theorem 1.1.

2. Preliminary estimates in Minkowski space

In this section we gather the estimates for the free wave equation that we will require in order to prove global existence.

2.1. Energy estimates. We begin with the standard energy estimates for perturbed wave equations

\[
(\Box_\gamma u)' = (\partial_t^2 - \gamma^2 \Delta)u' + \sum_{K=1}^{D} \sum_{0 \leq j, k \leq 3} \gamma^{IK,jk} \partial_j \partial_k u^K = G^I, \quad I = 1, \ldots, D
\]

satisfying the symmetry condition

\[
\gamma^{IK,jk} = \gamma^{IK,kj} = \gamma^{KI,jk}, \quad 0 \leq j, k \leq 3, \quad 1 \leq I, K \leq D.
\]

As is standard, we let \( \epsilon_0 = \sum_{I=1}^{D} \epsilon_0^I \) be the associated energy form where

\[
(\Box_\gamma u, t) = (\partial_t u^I)^2 + \sum_{k=1}^{3} \gamma^2 (\partial_k u^I)^2 + 2 \sum_{J=1}^{D} \sum_{k=0}^{3} \gamma^{IJ,jk} \partial_0 \partial_k u^I \partial_k u^J
\]

\[\quad - \sum_{J=1}^{D} \sum_{0 \leq j, k \leq 3} \gamma^{IJ,jk} \partial_j \partial_k u^I \partial_k u^J.\]

If we assume that

\[
\max_{1 \leq I, K \leq D} \| \gamma^{IK,jk} \|_{\infty}
\]

\[\forall 0 \leq j, k \leq 3\]
is sufficiently small, then it follows that

\begin{equation}
(2.4) \quad \frac{1}{2} \sum_{1 \leq I \leq D} \min(1, c_I^2) \langle \nabla_{t,x} u \rangle^2 \leq \epsilon_0(u) \leq 2 \sum_{1 \leq I \leq D} \max(1, c_I^2) \langle \nabla_{t,x} u \rangle^2.
\end{equation}

If we set \( E(u, t) = \int_{\mathbb{R}} \epsilon_0(u, t) \, dt \) to be the associated energy, then we have the energy inequality

\begin{equation}
(2.5) \quad \sum_{|\alpha| \leq M} \partial_t E(\Gamma^\alpha u, t) \leq C \sum_{|\alpha| \leq M} \| \Gamma^\alpha G(t, \cdot) \|_2 + \sum_{|\alpha| \leq M} \| \left( \partial_{t} \Gamma^\alpha u(t, \cdot) \right) \|_2

+ C \sum_{|\alpha| \leq M} E(\Gamma^\alpha u, t) \sum_{0 \leq j, k \leq 3} \sum_{1 \leq I, K \leq D} \| \partial_{t}^{j} \partial_{x}^{k} \Gamma^\alpha (t, \cdot) \|_{\infty}.
\end{equation}

In addition to the energy estimate (2.5), we will need the following \( L_2^2 \, L_2^1 \) estimate of Keel-Smith-Sogge [16] (Proposition 2.1).

**Lemma 2.1.** Suppose that \( u \in C^\infty(\mathbb{R} \times \mathbb{R}^3) \) vanishes for large \( x \) for every \( t \). Then, there is a uniform constant \( C \) so that

\begin{equation}
(2.6) \quad (\log(2 + t))^{-1/2} \langle x \rangle^{-1/2} \langle t \rangle u' \lesssim L_2^2(\mathbb{R} \times \mathbb{R}^3) \leq C \langle x \rangle^{-1/2} \langle t \rangle u' + C \int_0^t \| \Box u(s, \cdot) \|_2 \, ds.
\end{equation}

2.2. **Pointwise estimates.** In this section, we will gather the pointwise estimates that will be needed in the sequel. The estimates that are presented are variants of those in Keel-Smith-Sogge [17], Sogge [38], and Kubota-Yokoyama [21]. The key innovation in our approach to Theorem 1.2 is the use of both of these pointwise estimates and sharp Huygens’ principle to allow us to get good pointwise bounds for \( u \) (not just \( u' \) as in [27]). This pointwise bound allows us to handle the higher order terms without having to strengthen the null condition as in [21]).

In our first estimate, we will concentrate on the scalar wave equation \( \Box = (\partial_t^2 - \Delta) \). The transition to vector valued, multiple speed wave equations is straightforward.

**Lemma 2.2.** Let \( u \) be the solution of \( \Box u(t, x) = F(t, x) \) with initial data \( u(0, \cdot) = f \), \( \partial_t u(0, \cdot) = g \) for \( (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 \). Then,

\begin{equation}
(2.7) \quad (1 + t + |x|)|u(t, x)| \leq C \sum_{|\alpha| \leq 4} \| \langle x \rangle |\alpha| \partial_x f \|_2 \leq C \sum_{|\alpha| \leq 3} \| \langle x \rangle^{1 + |\alpha|} \partial_x g \|_2

+ C \sum_{\mu \leq |\alpha| \leq 3} \int_0^t \int_{\mathbb{R}^3} \left| L_\mu Z^\alpha F(s, y) \right| \frac{dy \, ds}{(y)}.
\end{equation}

**Proof of Lemma 2.2:** For vanishing Cauchy data, (2.7) can be found in Keel-Smith-Sogge [17] and Sogge [38]. Thus, it will suffice to show the estimate for \( \cos(t \sqrt{-\Delta}) f \) and \( (\sin(t \sqrt{-\Delta}) / \sqrt{-\Delta}) g \). The proof is similar to that in [17] for the inhomogeneous case. If
we assume that \( F = 0 \) above, we will show

\[
(2.8) \quad (1 + t + |x|)|u(t, x)| \leq C \sum_{|\alpha| + \mu \leq 3} \int_{\mathbb{R}^3} \left( |(r \partial_r)^\mu Z^\alpha f| \frac{dy}{|y|^2} + \int_{\mathbb{R}^3} |(r \partial_r)^\mu Z^\alpha f| \frac{dy}{|y|^2} \right)
\]

\[
\left(1 + \frac{t}{|x|} \right) \left| \frac{\sin(t \sqrt{-\Delta})}{\sqrt{-\Delta}} g \right| = \frac{t |x|}{4\pi} \int_{|\theta| = 1} g(x - t \theta) \, d\sigma(\theta)
\]

\[
\leq \frac{1}{2} \int_{|\tau| \leq |x|} \|sg(\tau \cdot)\|_{L^\infty(S^2)} \, d\tau.
\]

By the embedding \( H^2_\theta \rightarrow L^\infty_\theta \), it follows that

\[
(2.10) \quad \left| \frac{\sin(t \sqrt{-\Delta})}{\sqrt{-\Delta}} g \right| \leq C \sum_{|\alpha| \leq 2} \int_{|\cdot| \leq |x|} |\Omega^\alpha g(y)| \frac{dy}{|y|}
\]

For \( t \geq 10 |x| \), we apply the relation \( s g(\theta t) = -\int_{s}^{\infty} \partial_t (\tau g(\tau \theta)) \, d\tau \) to (2.9) to see that

\[
| \frac{\sin(t \sqrt{-\Delta})}{\sqrt{-\Delta}} g | \leq C \frac{t}{|x|} \int_{|\tau| \leq |x|} \int_{s}^{\infty} |\tau \partial_t (\tau g(\tau \theta))|_L^\infty(S^2) \, d\tau \, ds
\]

\[
\leq C \int_{s}^{\infty} \|\tau g(\tau \cdot)|_{L^\infty(S^2)} + \|\tau (\tau \partial_t \theta) g(\tau \cdot)|_{L^\infty(S^2)} \, d\tau
\]

\[
\leq C \sum_{|\alpha| \leq 2} \int_{|\cdot| \leq |x|} \left| \frac{\partial |\partial_y|^\mu g(y)}{|y|} \right| \frac{dy}{|y|}.
\]

By (2.10) and (2.11), we obtain

\[
(2.12) \quad (1 + t + |x|) \left| \frac{\sin(t \sqrt{-\Delta})}{\sqrt{-\Delta}} g \right| \leq C \sum_{|\alpha| \leq 2} \int_{|\cdot| \leq |x|} \left| \frac{\partial |\partial_y|^\mu g(y)}{|y|} \right| \frac{dy}{|y|}.
\]

We now wish to show that

\[
(2.13) \quad (1 + t + |x|) \left| \frac{\sin(t \sqrt{-\Delta})}{\sqrt{-\Delta}} g \right| \leq C \sum_{|\alpha| + \mu \leq 3} \int_{\mathbb{R}^3} \left| \frac{\partial |\partial_y|^\mu g(y)}{|y|} \right| \frac{dy}{|y|}.
\]

For \( t + |x| \geq 1 \), (2.13) clearly follows from (2.12). For \( t + |x| \leq 1 \), let \( \chi \) denote a smooth function with \( \chi(x) \equiv 1 \) for \( |x| \leq 1 \) and \( \chi(x) \equiv 0 \) for \( |x| > 2 \), and let \( v \) be the solution to the shifted wave equation

\[
(2.14) \quad \Box v(t, x) = 0, \quad v(0, \cdot) = 0, \quad \partial_t v(0, x) = (\chi g)(x_1 - 10, x_2, x_3).
\]

By finite propagation, we have that \( \frac{\sin(t \sqrt{-\Delta})}{\sqrt{-\Delta}} g = v(t, x_1 + 10, x_2, x_3) \) for \( t + |x| \leq 1 \), and (2.13) follows by applying (2.12) to \( v \).
Finally, we turn to the task of showing that our desired result

\begin{equation}
(1 + t + |x|) \left| \frac{\sin(t \sqrt{-\Delta})}{\sqrt{-\Delta}} g \right| \leq C \sum_{|\alpha| + \mu \leq 3} \int_{\mathbb{R}^n} \left| (|y| \partial_{|y|^2})^\mu Z^\alpha g(y) \right| \frac{dy}{|y|^2}
\end{equation}

follows from (2.13). For \( \chi \) as above, write \( g = \chi g + (1 - \chi) \tilde{g} \). When \( g \) is replaced by \((1 - \chi)g\), (2.15) follows directly from (2.13). When \( g \) is replaced by \( \chi g \), we instead apply (2.13) to the shifted function \( r \). It is this use of the shifted function that introduces the translations on the right sides of (2.13) and (2.15).

Next, we consider \( \cos(t \sqrt{-\Delta}) f \). We have

\begin{equation}
\left| \cos(t \sqrt{-\Delta}) f \right| = \left| \partial_t \left( \frac{t}{4 \pi} \int_{|x| = 1} f(x - t \theta) \, d\sigma(\theta) \right) \right| \leq \left| \frac{\sin t \sqrt{-\Delta}}{t \sqrt{-\Delta}} f \right| + \left| \frac{\sin t \sqrt{-\Delta}}{t \sqrt{-\Delta}} |\nabla f| \right|.
\end{equation}

By (2.15), the \( |\nabla f| \) part is bounded by the right side of (2.8). For the first part, repeating the arguments of (2.9) and (2.11), we have

\begin{equation}
(t + |x|) \left| \frac{\sin(t \sqrt{-\Delta})}{t \sqrt{-\Delta}} f \right| \leq \frac{t + |x|}{2|x|} \int_{|y| = |x|} ||s f(s \cdot)||_{L^\infty_s(s^2)} \, ds \leq \frac{t + |x|}{2|x|} (t + |x| - |t - |x||) \int_0^\infty ||\partial_t (\tau f(\tau \cdot))||_{L^\infty(\tau^2)} \, d\tau \leq C \sum_{\mu \leq 1, |\alpha| \leq 2} \int_{|y| \geq |t - |x||} \left| (|y| \partial_{|y|^2})^\mu \Omega^\alpha f(y) \right| \frac{dy}{|y|^2}.
\end{equation}

Using the shifted function as in (2.13) and (2.15), it follows that

\begin{equation}
(1 + t + |x|) \left| \frac{\sin(t \sqrt{-\Delta})}{t \sqrt{-\Delta}} f \right| \leq C \sum_{|\alpha| + \mu \leq 3} \int_{|y| > |t - |x||} \left| (|y| \partial_{|y|^2})^\mu Z^\alpha f(y) \right| \frac{dy}{|y|^2}
\end{equation}

as desired. \( \square \)

We now wish to explore the version of the pointwise estimate of Kubota-Yokoyama [21] that we will use. We define the “neighborhoods” of the characteristic cones \( r = |x| = c_J t \) for \( \square_{c_J} \). That is, with the \( c_J \) as in (1.5), set

\begin{equation}
\Lambda_J = \{ (t, |x|) \in [1, \infty) \times [1, \infty) : |r - c_J t| \leq \delta t \}
\end{equation}

where \( \delta = \frac{1}{\pi} \min_{t \in D} (c_J - c_{J-1}) \) and \( I = 1, 2, \ldots, D \). Note that for \( (t, x) \notin \Lambda_J, \ c_J t - |x| \approx t + |x| \). Additionally, define

\begin{equation}
z(s, \lambda) = \begin{cases} (1 + |\lambda - c_J s|), & \text{for } (s, \lambda) \in \Lambda_J, \ J = 1, 2, \ldots, D, \\ (1 + \lambda), & \text{otherwise.} \end{cases}
\end{equation}

With this notation, we then have
Lemma 2.3. Let \( l = 1, 2, \ldots, D \), and assume that \( G^l(t, x) \) is a continuous function of \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^3\). Let \( w^l \) be the solution of \((\partial_t^2 - c_l^2 \Delta)w^l = G^l\) with vanishing Cauchy data at time \( t = 0\). Then,

\[
(2.21) \quad (1 + r + t) \left( 1 + \log \frac{1 + r + c_l t}{1 + |r - c_l t|} \right)^{-1} |w^l(t, x)| \leq C \sup_{(s, y) \in D^l(t, r)} |y|(1 + s + |y|^{1+\mu})^{1-\mu} \Gamma^l(s, y) \]

for any \( \mu > 0 \) and

\[
(2.22) \quad D^l(t, r) = \{(s, y) \in \mathbb{R} \times \mathbb{R}^3 : 0 \leq s < t, |r - c_l(t - s)| \leq |y| \leq r + c_l(t - s)\}.
\]

The above estimate is due to Kubota-Yokoyama [21] (Theorem 3.4). If we combine (2.7) and (2.21) and use the fact that \([\Box, Z] = 0\) and \([\Box, L] = 2\Box\), we get our main pointwise estimates.

Theorem 2.4. Let \( l = 1, 2, \ldots, D \), and assume that \( F^l(t, x), G^l(t, x) \) are smooth functions of \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^3\). Let \( w^l \) be the solution of \((\partial_t^2 - c_l^2 \Delta)w^l = F^l + G^l\). Then, there is a uniform constant \( \hat{C}_1 > 0 \) so that

\[
(2.23) \quad (1 + r + t)|\Gamma^\alpha w^l(t, x)| \leq \hat{C}_1 \sum_{|\beta| \leq 4} \|\langle x \rangle^{1+|\beta|} (\partial^\beta \Gamma^\alpha w^l)(0, \cdot)\|_2 + \hat{C}_1 \sum_{|\beta| \leq 1, |\alpha| + 3} \int_0^t \left[ \int \Gamma^\beta F^l(s, y) \right] \frac{dy ds}{\langle y \rangle} \sum_{|\beta| \leq |\alpha|} \sup_{(s, y) \in D^l(t, r)} |y|(1 + s + |y|^{1+\mu})^{1-\mu} \Gamma^\beta G^l(s, y) \]

for any multiindex \( \alpha, \mu > 0 \), and \( D^l \) as in (2.22).

Using strong Huygens’ principle, we can establish the following variant of the previous theorem.

Theorem 2.5. Fix \( l = 1, 2, \ldots, D \), and assume that \( F^l(t, x), G^l(t, x) \) are smooth functions of \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^3\). Moreover, assume that \( F^l(t, x) \) is supported in \( \Lambda_l \) for some \( J \neq l \). Let \( w^l \) be the solution of \((\partial_t^2 - c_l^2 \Delta)w^l = F^l + G^l\). Then, there are uniform constants \( c, C_1 > 0 \) depending on the wavespeeds \( c_1, c_2 \) so that

\[
(2.24) \quad (1 + r + t)|\Gamma^\alpha w^l(t, x)| \leq C_1 \sum_{|\beta| \leq 3} \|\langle x \rangle^{1+|\beta|} (\partial^\beta \Gamma^\alpha w^l)(0, \cdot)\|_2 + C_1 \sum_{|\beta| \leq |\alpha| + 2} \sup_{0 \leq s \leq t} \left[ \int \Gamma^\beta F^l(s, y) \right] dy \sum_{|\beta| \leq |\alpha|} \sup_{(s, y) \in D^l(t, r)} \left[ \int \Gamma^\beta G^l(s, y) \right] \frac{dy ds}{\langle y \rangle} \]

for any multiindex \( \alpha, \mu > 0 \), and \( D^l \) as in (2.22).
Here, and throughout, \(|y| \approx s| is used to denote that there is a positive constant \(\tilde{c}| so that 
\(1/\tilde{c}| \leq s \leq \tilde{c}| y\). 

**Proof of Theorem 2.5:** By (2.21), we may take \(G^f \equiv 0\) without restricting generality. We then note that there is a constant \(c| so that the intersection of the backward light cone through \((t, x)| with speed \(c_t|, \{c_t| t - s| \leq |x - y|\}, \text{and } \Lambda| \text{ is contained in } |c| c_t| - |x|, |t| \times \{ |y| \approx s\}.\) With this in mind, we fix a smooth cutoff function \(\rho| so that \(\rho(s) \equiv 1\) for \(s \geq c| c_t| - |x|\) and \(\rho(s) \equiv 0\) for \(s \leq c| c_t| - |x| - 1.\) Notice that by strong Huygens’ principle, we have \(\Gamma^a| u^f(t, x) = \Gamma^a| \hat{u}| where \(\hat{u}| is the solution to 
\(\Box, \Gamma^a| \hat{u}|(s, y) = \rho(s)| \Gamma^a| F^f(s, y) + \rho(s)| [\Box, \Gamma^a]| F^f(s, y)\) 
and \(\Gamma^a| \hat{u}| \text{ has the same Cauchy data as } \Gamma^a| u.| 

The result now follows from an application of (2.7) to \(\Gamma^a| \hat{u}|.\) So long as the scaling vector field \(L| in the third term on the right of (2.7) does not hit \(\rho| the bound (2.24) follows and the third term on the right is unnecessary. If the \(L| in (2.7) is applied to \(\rho| we get an additional term which is bounded by 
\(C| \sum_{\beta| \leq |\alpha| + 2} \int_{|y| \approx s} s| |\beta| F(s, y)| \frac{dy}{\langle y\rangle}.\) 
Since \(|y| \approx s| and the time integral is taken over an interval of length at most one, this term is easily seen to be dominated by the third term in (2.24) which completes the proof. 

\(\square\)

2.3. Null form estimates and Sobolev-type estimates. In this section, we gather our bounds on the null forms and some weighted Sobolev-type estimates. The first of these is the null form estimate. See, e.g., [35], [38].

**Lemma 2.6.** Suppose that the quadratic parts of the nonlinearity \(Q(du, d^2u), B(du)\) satisfy the null conditions (1.9) and (1.10). Then, 
\(\sum_{0 \leq j, k \leq 3} B_{jk}^{K\ell} \partial_u \partial_j \partial_k v \leq C(r)^{-1} (|\Gamma u| |\partial^2 v| + |\partial u| |\partial \Gamma v|) + \frac{C(\ell K t - r)}{(t + r)} |\partial u| |\partial^2 v|,\) 
and 
\(\sum_{0 \leq j, k \leq 3} A_{jk}^{K\ell} \partial_j \partial_k v \leq C(r)^{-1} (|\partial \Gamma u| |\partial v| + |\partial u| |\partial \Gamma v|) + \frac{C(\ell K t - r)}{(t + r)} |\partial u| |\partial v|.)\)

For the Sobolev-type results, we begin with

**Lemma 2.7.** Suppose that \(h \in C^\infty(\mathbb{R}^3).\) Then, for \(R > 1,\) 
\(|h|_{L^\infty(R/2 < |r| < R)} \leq CR^{-1} \sum_{|\alpha| + |\beta| \leq 2} |\Omega^\alpha \partial^\beta h|_{L^2(R/4 < |r| < 2R)}.\)

This has become a rather standard result. See Klainerman [18]. A proof can also be found, e.g., in [16].

Additionally, we have the following space-time weighted Sobolev results:
Lemma 2.8. Let $u \in C^\infty_0(\mathbb{R}_+ \times \mathbb{R}^3)$. Then,

\begin{equation}
\langle r \rangle^{1/2} |u(t, x)| \leq C \sum_{|\alpha| \leq 1} \| \partial_\alpha u(t, \cdot) \|_2,
\end{equation}

\begin{equation}
\| (\epsilon t - r)^2 u(t, \cdot) \|_2 \leq C \sum_{|\beta| \leq 1} \| \Gamma^{\beta} u(t, \cdot) \|_2 + C \| (t + r) \partial_x u(t, \cdot) \|_2,
\end{equation}

\begin{equation}
\langle r \rangle^{1/2} \langle \epsilon t - r \rangle^2 |u'(t, x)| \leq C \sum_{|\beta| \leq 1} \| \Gamma^{\beta} u'(t, \cdot) \|_2 + C \sum_{|\beta| \leq 1} \| \langle \epsilon t - r \rangle \Gamma^{\beta} \partial^2 u(t, \cdot) \|_2,
\end{equation}

\begin{equation}
\langle r \rangle^{1/2} \langle \epsilon t - r \rangle^2 |u'(t, x)| \leq C \sum_{|\beta| \leq 2} \| \Gamma^{\beta} u'(t, \cdot) \|_2 + C \sum_{|\beta| \leq 1} \| \langle \epsilon t - r \rangle \Gamma^{\beta} \partial^2 u(t, \cdot) \|_2.
\end{equation}

The estimates (2.28) and (2.31) are shown in Sideris [33] (Proposition 3.3). (2.29) is due to Klainerman-Sideris [20] (Lemma 2.3 and Lemma 3.1). (2.30) is from Hidano-Yokoyama [6] (Lemma 4.1) and follows from (2.28).

Lastly, by interpolating between (2.30) and (2.31), it is easy to see that

\begin{equation}
\langle r \rangle^{1/2+\mu} \langle \epsilon t - r \rangle^{1-\mu} |\Gamma^\alpha u(t, x)| \leq C \sum_{|\beta| \leq |\alpha|+2} \| \Gamma^{\beta} u(t, \cdot) \|_2 + C \sum_{|\beta| \leq |\alpha|+1} \| \langle \epsilon t - r \rangle \Gamma^{\beta} \partial^2 u(t, \cdot) \|_2
\end{equation}

for any $0 \leq \mu \leq 1/2$.

3. Global existence in Minkowski space

Here we prove Theorem 1.2. We will take $N = 71$ in (1.14). This, however, is not optimal.

To proceed, we shall require a standard local existence theorem.

Theorem 3.1. Let $f \in H^{71}(\mathbb{R}^3)$ and $g \in H^{70}(\mathbb{R}^3)$. Then, there is a $T > 0$ dependent on the norm of the data so that the initial value problem (1.13) has a $C^2$ solution satisfying

\begin{equation}
\begin{array}{c}
\langle r \rangle^{1/2} |u(t, x)| \leq C \sum_{|\alpha| \leq 1} \| \partial_\alpha u(t, \cdot) \|_2, \\
\| (t + r) \partial_x u(t, \cdot) \|_2.
\end{array}
\end{equation}

The supremum of all such $T$ is equal to the supremum of all $T$ such that the initial value problem has a $C^2$ solution with $\partial^\alpha u$ bounded for all $|\alpha| \leq 2$.

This result is a multi-speed analog of Theorem 6.4.11 in [7] (which is stated only for scalar wave equations). Since the proof is based only on energy inequalities, the same argument yields Theorem 3.1 provided we assume the symmetry conditions (1.7) and (1.8).
We are now ready to set up our continuity argument. If $\varepsilon$ is as above, we will assume that we have a solution of our equation (1.13) for $0 \leq t \leq T$ satisfying the following:

\[
\begin{align*}
(3.2) & \quad \sum_{|\alpha| \leq 5^0} \|\Gamma^\alpha u'(t, \cdot)\|_2 \leq A_0 \varepsilon \\
(3.3) & \quad (1 + t + |x|) \sum_{|\alpha| \leq 4^0} |\Gamma^\alpha u'(t, x)| \leq A_1 \varepsilon \left(1 + \log \frac{1 + t + |x|}{1 + \varepsilon t} \right) \\
(3.4) & \quad (1 + t + |x|) \sum_{|\alpha| \leq 6^0} |\Gamma^\alpha u'(t, x)| \leq A_2 \varepsilon (1 + t)^{1/10} \log(2 + t) \left(1 + \log \frac{1 + t + |x|}{1 + \varepsilon t} \right) \\
(3.5) & \quad (1 + t + |x|) \sum_{|\alpha| \leq 6^0} |\Gamma^\alpha u(t, x)| \leq B_1 \varepsilon \\
(3.6) & \quad \sum_{|\alpha| \leq 7^0} \|\Gamma^\alpha u'(t, \cdot)\|_2 \leq B_2 \varepsilon (1 + t)^{1/40} \\
(3.7) & \quad \sum_{|\alpha| \leq 7^0} \|\langle x \rangle^{1/2} \Gamma^\alpha u'(t, \cdot)\|_{L^2(S_t)} \leq B_3 \varepsilon (1 + t)^{1/20} (\log(2 + t))^{1/2}.
\end{align*}
\]

Here $S_t$ denotes the time strip $[0, t] \times \mathbb{R}^3$.

By (1.14), we have the estimate

\[
\sum_{l=1}^D \sum_{|\alpha| \leq 57} (1 + C_1 + \hat{C}_1) \left\{ \sum_{|\beta| \leq 4^0} \|\langle x \rangle^{\beta} (\partial^\beta \Gamma^\alpha u^l)(0, x)\|_2 \right. \\
+ \left. \sum_{|\beta| \leq 3^0} \|\langle x \rangle^{1 + |\beta|} (\partial^\beta \Gamma^\alpha \tilde{u}^l)(0, x)\|_2 \right\} \leq C_2 \varepsilon
\]

for some constant $C_2 > 0$. Here $\hat{C}_1$ and $C_1$ are the constants occurring in (2.23) and (2.24) respectively. In our estimates above, we choose $A_0 = A_1 = A_2 = A \geq 10 \max(1, C_2)$.

We shall then prove that for $\varepsilon$ sufficiently small,

i.) (3.2) holds with $A_0$ replaced by $A_0/2$.

ii.) (3.3), (3.4) hold with $A_1, A_2$ replaced by $A_1/2, A_2/2$ respectively.

iii.) (3.2)-(3.4) imply (3.5)-(3.7) for a suitable choice of constants $B_1, B_2, B_3$.

We will prove items (i.)-(iii.) in the next three subsections respectively.

Before we begin with the proof of (i.), we will set up some preliminary results under the assumption of (3.2)-(3.7). Let us first prove

\[
\sum_{|\alpha| \leq 5^8} \langle r \rangle^{1/2 \times \mu} \langle c_t - r \rangle^{1 - \mu} \|\Gamma^\alpha \partial u^l(t, x)\| \leq C \varepsilon (1 + t)^{1/40}, \quad 0 \leq \mu \leq 1/2.
\]

Indeed, by (2.32) and (2.29), we have that the left side of (3.8) is controlled by

\[
C \sum_{|\alpha| \leq 5^8} \|\Gamma^\alpha u^l(t, \cdot)\|_2 + C \sum_{|\alpha| \leq 5^9} \|(t + r)\Gamma^\alpha \nabla_{c_t} u^l(t, \cdot)\|_2.
\]
By (3.6), the first term is controlled by the right side of (3.8). Thus, it remains to show

\begin{equation}
\sum_{|\alpha| \leq 30} \| (t + r) \Gamma^\alpha \Box u' (t, \cdot) \|_2 \leq C \varepsilon (1 + t)^{1/40}.
\end{equation}

By our definition of \( \Box u \), we have that the left side of (3.9) is bounded by

\begin{align*}
C \sum_{|\alpha| \leq 30} \| (t + r) \Gamma^\alpha \Box u' (t, \cdot) \|_\infty & \sum_{|\alpha| \leq 59} \| \Gamma^\alpha u' (t, \cdot) \|_2 \\
+ C \sum_{1 \leq J, K \leq D} \| (t + r) \sum_{|\alpha| \leq 31} \| \Gamma^\alpha u' \|_\infty \sum_{|\alpha| \leq 60} \| \Gamma^\alpha u \|_2 \\
+ C \sum_{1 \leq J, K \leq D} \| (t + r) \sum_{|\alpha| \leq 31} | \Gamma^\alpha u' | \sum_{|\alpha| \leq 60} | \Gamma^\alpha u | \sum_{|\alpha| \leq 60} \| \Gamma^\alpha \Box u \|_2.
\end{align*}

By (3.5) and (3.6), we see that the first term is controlled by \( C \varepsilon^2 (1 + t)^{1/40} \) as desired. For the second term, we apply (3.3) to see that we have the bound

\begin{equation}
C \varepsilon^2 \left( 1 + \log \frac{1 + t + |x|}{1 + |c_H t - x|} \right) \left( 1 + \log \frac{1 + t + |x|}{1 + |c_K t - x|} \right) (1 + t + |x|)^{-1} \sum_{|\alpha| \leq 59} \| \Gamma^\alpha u (t, \cdot) \|_2.
\end{equation}

We, then, see that this is \( O (\varepsilon^3) \) using (3.4). The bound for the third term follows similarly from applications of (3.3) and (3.6).

If we argued similarly, using (3.2) instead of (3.6), it follows that

\begin{equation}
\sum_{|\alpha| \leq 48} \langle r \rangle^{1/2 + \mu} \langle c_H t - r \rangle^{1 - \mu} | \Gamma^\alpha \Box u' (t, x) | \leq C \varepsilon, \quad 0 \leq \mu \leq 1/2,
\end{equation}

and

\begin{equation}
\sum_{|\alpha| \leq 49} \| \langle c_H t - r \rangle \Gamma^\alpha \Box^2 u' (t, \cdot) \|_2 \leq C \varepsilon.
\end{equation}

Indeed, the latter follows from (2.29) and the proof of (3.9) where, as mentioned above, we use the lossless estimate (3.2) rather than (3.6).

3.1. **Proof of (i.)**: In this section, we will show that (3.2)-(3.7) allow you to prove (3.2) with \( A_0 \) replaced by \( A_0/2 \). By the standard energy inequality (see, e.g., [37]), the square of the left side of (3.2) is controlled by

\begin{equation}
\sum_{|\alpha| \leq 58} \| \Gamma^\alpha u' (0, \cdot) \|_2^2 + \sum_{|\alpha| \leq 58} \int_0^t \int \langle \eta \Gamma^\alpha u, \Box \Gamma^\alpha u \rangle \ dy \ ds.
\end{equation}

It follows from (1.14) and our choice of \( A_0 \) that the first term is controlled by \((A_0/10)^2 \varepsilon^2\). Thus, it will suffice to show that

\begin{equation}
\sum_{|\alpha| \leq 58} \int_0^t \int \langle \eta \Gamma^\alpha u, \Box \Gamma^\alpha u \rangle \ dy \ ds \leq C \varepsilon^3.
\end{equation}
The left side of (3.13) is dominated by

\[ C \int_0^t \int_{\mathbb{R}^3} \sum_{K=1}^D \left| \partial_t \Gamma^\alpha u^K \right| \sum_{|\alpha|+|\beta| \leq 50} \sum_{0 \leq j,k,l \leq 3} \hat{B}_{K,j}^{K,k,l} \partial_t \Gamma^\alpha u^K \partial_j \partial_k \Gamma^\beta u^K \, dy \, ds \]

\[ + C \int_0^t \int_{\mathbb{R}^3} \sum_{K=1}^D \left| \partial_t \Gamma^\alpha u^K \right| \sum_{|\alpha|+|\beta| \leq 50} \sum_{0 \leq j,k,l \leq 3} \hat{A}_{K,j}^{K,k,l} \partial_j \Gamma^\alpha u^K \partial_k \Gamma^\beta u^K \, dy \, ds \]

\[ + C \int_0^t \int_{\mathbb{R}^3} \sum_{1 \leq I,J,K \leq D} \sum_{|\alpha| \leq 50} \partial^I \Gamma^\alpha u^K \sum_{|\alpha| \leq 51} \partial^I \Gamma^\alpha v^K \sum_{|\alpha| \leq 51} \left| \Gamma^\alpha u^K \right| \, dy \, ds \]

\[ + C \int_0^t \int_{\mathbb{R}^3} \sum_{|\alpha| \leq 50} \left| \partial_t \Gamma^\alpha u^K \right| \left( \sum_{|\alpha| \leq 50} \left| \Gamma^\alpha u^K \right| \right)^2 \sum_{|\alpha| \leq 52} \left| \Gamma^\alpha u^K \right| \, dy \, ds. \]

Due to constants that are introduced when \( L^\nu Z^\alpha \) commutes with \( \partial_{j,k,l} \), the coefficients \( A_{K,j}^{K,k,l}, B_{K,j}^{K,k,l} \) become new constants \( \hat{A}_{K,j}^{K,k,l}, \hat{B}_{K,j}^{K,k,l} \). It is known, however, that \( \Gamma \) preserves the null forms. That is, since the original constants satisfy (1.9) and (1.10), so do the new ones \( \hat{A}_{K,j}^{K,k,l} \) and \( \hat{B}_{K,j}^{K,k,l} \). See, e.g., Sideris-Tu [35] (Lemma 4.1).

The first three terms are handled as in [27]. Let us begin with the null terms (i.e., the first two terms in (3.14)). By (2.25) and (2.26), these terms are dominated by

\[ C \int_0^t \int_{\mathbb{R}^3} \sum_{|\alpha| \leq 50} \left| \Gamma^\alpha u^K \right| \sum_{|\alpha| \leq 51} \left| \Gamma^\alpha u^K \right| \frac{dy \, ds}{|y|} \]

\[ + C \int_0^t \int_{\mathbb{R}^3} \sum_{K=1}^D \frac{\langle \xi_K s - r \rangle}{\langle s + r \rangle^2} \left( \sum_{|\alpha| \leq 51} \left| \Gamma^\alpha \partial u^K \right| \right)^2 \, dy \, ds. \]

In order to handle the contribution by the first term of (3.15), notice that by (3.4)

\[ \sum_{|\alpha| \leq 51} \left| \Gamma^\alpha u(s, y) \right| \leq C \varepsilon (s + |y|)^{-9/10}. \]

Thus, the first term in (3.15) has a contribution to (3.14) which is dominated by

\[ C \varepsilon \int_0^t \langle s \rangle^{-9/10+} \sum_{|\alpha| \leq 51} \left| \langle y \rangle^{-1/2} \Gamma^\alpha u(s, \cdot) \right|^2 \, ds \]

by the Schwarz inequality. By (3.7), it follows that this contribution is \( O(\varepsilon^3) \).

In order to show that the second term in (3.15) satisfies a similar bound, we apply (3.8) with \( \mu = 0 \) and the Schwarz inequality to see that it is controlled by

\[ C \varepsilon \int_0^t (1 + s)^{1/40} \int_{\mathbb{R}^3} \frac{1}{\langle r \rangle^{3/4} (s + r)} \sum_{|\alpha| \leq 51} \left| \Gamma^\alpha \partial u \right|^2 \, dy \, ds \]

\[ \leq C \int_0^t \langle s \rangle^{-13/40} \sum_{|\alpha| \leq 51} \left| \langle y \rangle^{-1/2} \Gamma^\alpha u(s, \cdot) \right|^2 \, ds. \]

It then follows from (3.7) that this term also has an \( O(\varepsilon^3) \) contribution to (3.14).
We now wish to show that the multi-speed terms

\begin{equation}
(3.18) \quad \int_0^1 \int_{\mathbb{R}^3} \sum_{|\alpha| \leq 50} |\partial^\alpha u^K| \sum_{|\alpha| \leq 51} |\partial^\alpha u^J| \, dy \, ds
\end{equation}

with \((I, K) \neq (J, K)\) have an \(O(\varepsilon^3)\) contribution to (3.14). For simplicity, let us assume that \(I \neq K, I = J\). A symmetric argument will yield the same bound for the remaining cases. If we set \(\delta < |c_I - c_K|/3\), it follows that \(\{|y| \in [(c_I - \delta)s, (c_I + \delta)s]\} \cap \{|y| \in [(c_K - \delta)s, (c_K + \delta)s]\} = \emptyset\). Thus, it will suffice to show the bound when the spatial integral is taken over the complements of each of these sets separately. We will show the bound over \(\{|y| \notin [(c_K - \delta)s, (c_K + \delta)s]\}\). The same argument will symmetrically yield the bound over the other set.

If we apply (3.8) with \(\mu = 0\), we see that over the indicated set, (3.18) is bounded by

\begin{equation}
(3.19) \quad C\varepsilon \int_0^1 \int_{\{|y| \notin [(c_K - \delta)s, (c_K + \delta)s]\}} \langle s + r \rangle^{-39/48} \langle r \rangle^{-1/2} \sum_{|\alpha| \leq 51} |\partial^\alpha u^J| \, dy \, ds
\end{equation}

\begin{equation}
\leq C\varepsilon \int_0^1 \langle s \rangle^{-9/48} \sum_{|\alpha| \leq 51} \|y\|^{-1/2} \Gamma^\alpha u^J(s, \cdot) \|\|^2 \, ds.
\end{equation}

Thus, it again follows from (3.7) that this term is \(O(\varepsilon^3)\).

Finally, it remains to bound the contribution to (3.14) by the cubic terms (the fourth term in (3.14)). If we apply (3.3) and (3.4), it is clear that this term is dominated by

\begin{equation}
C\varepsilon^3 \int_0^1 \int_{\{|y| \notin [(c_K - \delta)s, (c_K + \delta)s]\}} \frac{|\log(1 + s + |y|)|^4}{(1 + s + |y|)^{25/16}} \sum_{|\alpha| \leq 50} |\partial^\alpha \Gamma^\alpha u| \, dy \, ds.
\end{equation}

By Schwarz inequality and (3.6), we see that this term is \(O(\varepsilon^4)\) which completes the proof of (3.13).

### 3.2. Proof of (ii.): In this section, we wish to show that our pointwise estimates (3.3) and (3.4) hold with \(A_1, A_2\) replaced by \(A_1/2, A_2/2\) respectively. Let us begin with (3.3).

Fix a smooth cutoff function \(\eta_J\) satisfying \(\eta_J(s) \equiv 1, s \in [(c_J + (\delta/2))^{-1}, (c_J - (\delta/2))^{-1}]\) where, as in (2.19), \(\delta = (1/3)\min(c_I - c_{I-1})\), and \(\eta_J(s) \equiv 0, s \notin [(c_J + \delta)^{-1}, (c_J - \delta)^{-1}]\). We also set \(\beta\) to be a smooth function satisfying \(\beta(x) \equiv 1, |x| < 1\) and \(\beta(x) \equiv 0, |x| \geq 2\). Then, let \(\rho_J(x, t) = (1 - \beta)(x)\eta_J(|x|^{-1})\). By construction when \(|x| \geq 2\), \(\rho_J\) is identically 1 in a conic neighborhood of \(\{c_J t = |x|\}\) and is supported on \(\Lambda_J\).

We then set

\begin{equation}
\hat{F}^I = \sum_{1 \leq j < D} \sum_{0 \leq k, l, \ell \leq 3} B^{I,j,k}_{J,j,l} \rho_J \partial_t u^j \partial_j \partial_k u^\ell + \sum_{1 \leq j < D} \sum_{0 \leq k, l \leq 3} A^{I,j,k}_{J,j,l} \rho_J \partial_t u^j \partial_k u^\ell
\end{equation}
and $\hat{G}^I = \hat{F}^I - \hat{F}^I$. By (2.24) and our choice of $C_2$, we have that the left side of (3.3) is dominated by

$$
C_2 \varepsilon + C \left(1 + \log \frac{1 + r + c_I t}{1 + |r - c_I t|} \right) \sum_{|\beta| \leq 44} \sup_{(s, y) \in D'(t, r)} |y| |1 + s + |y||^{1 + \mu} z^{1 - \mu} (s, |y|) |\Gamma^\beta \hat{G}^I(s, y)|
$$

$$
+ C \sum_{|\beta| \leq 43} \int_{\max(0, e^{t - r} - |x| - 1)} \int_{|y| \leq s} |\Gamma^\beta \hat{F}^I(s, y)| \frac{dy}{y^2} ds + C \sum_{|\beta| \leq 42} \sup_{0 \leq \xi \leq t} \int |\Gamma^\beta \hat{F}^I(s, y)| dy.
$$

By construction, we have $C_2 \varepsilon \leq \langle A_1 / 10 \rangle \varepsilon$.

We now turn to the second to last term in (3.21). Since $|y| \approx s$ on the support of $\rho_I$, it follows that this term is controlled by

$$
C \sum_{1 \leq j \leq N} \int_{\max(0, e^{t - r} - |x| - 1)} \frac{1}{1 + s} \sum_{|\beta| \leq 44} |\Gamma^\beta \partial u^I| \frac{dy}{y^2} ds
$$

$$
\leq C \left(1 + \log \frac{1 + r + c_I t}{1 + |r - c_I t|} \right) \sum_{|\beta| \leq 44} \sup_{0 \leq \xi \leq t} \|\Gamma^\beta u'(s, \cdot)\|_2^2.
$$

The correct bound for the right side then follows from (3.2). If we apply the Schwarz inequality, it follows that the last term in (3.21) is dominated by

$$
C \sum_{|\beta| \leq 43} \sup_{0 \leq \xi \leq t} \|\Gamma^\beta u'(s, \cdot)\|_2^2.
$$

Thus, by (3.2), we get the desired bound for the $\hat{F}^I$ terms in (3.21).

It remains to examine the $\hat{G}^I$ term in (3.21). The proof of (3.3) will be complete if we can show that

$$
(3.22) \sum_{|\beta| \leq 40} |y| |1 + s + |y||^{1 + \mu} z^{1 - \mu} (s, |y|) |\Gamma^\beta \hat{G}^I(s, y)| \leq C \varepsilon^2.
$$

When $\hat{G}^I$ is replaced by the null forms

$$
A_{H, j}^I \partial_{\beta j} u^I \partial_{\beta k} u^I + \sum_{0 \leq \beta, \beta I, I \leq 3} B_{H, j}^{I, k} \partial_{\beta j} u^I \partial_{\beta k} u^I
$$

we apply (2.25) and (2.26) to bound this term by

$$
(3.23) C \sup_{(s, y) \in D'(t, r)} |y| |1 + s + |y||^{1 + \mu} z^{1 - \mu} (s, |y|) \sum_{|\beta| \leq 41} |\Gamma^\beta u^I| \sum_{|\beta| \leq 41} |\Gamma^\beta \partial u^I|
$$

$$
+ C \sup_{(s, y) \in D'(t, r)} |y| |1 + s + |y||^{1 + \mu} z^{1 - \mu} (s, |y|) \sum_{|\beta| \leq 41} |\Gamma^\beta \partial u^I| \sum_{|\beta| \leq 41} |\Gamma^\beta \partial u^I|.
$$

For the first term in (3.23), if we apply (3.4), we see that it is controlled by

$$
C \varepsilon \sup_{(s, y) \in D'(t, r)} |y| |1 + s + |y||^{1 + \mu} z^{1 - \mu} (s, |y|) \sum_{|\beta| \leq 41} |\Gamma^\beta \partial u^I|.
$$

It follows, then, that this is $O(\varepsilon^2)$ by (3.10). Indeed, if $(s, y) \in \Lambda_I$, then $s \approx |y|$ and $z(s, |y|) = \langle c_I s - y \rangle$. For $(s, y) \not\in \Lambda_I$, it follows that $s + |y| \approx |c_I s - y|$ and
\[ z^{1-\mu}(s, |y|) \leq \langle y \rangle^{1-\mu}. \] Similarly, by (3.10), it follows that the second term in (3.23) is bounded by
\[
C\varepsilon \sup_{(s, y) \in D \setminus (0, r)} \langle y \rangle^{1/2} (1 + s + |y|)^{\mu} z^{1-\mu}(s, |y|) \sum_{|\beta| \leq 41} |\Gamma^\beta \partial u^I|.
\]
By (3.10) and the same considerations as above, this is in turn \(O(\varepsilon^3)\) as desired.

When we replace \(\hat{G}^I\) by
\[
(3.24) \sum_{1 \leq J \leq \delta} \left( \sum_{0 \leq j, k \leq 3} \right) B_{JJK}^{\delta j k} (1 - \rho J) \partial u^J \partial \partial_k u^J + \sum_{0 \leq j, k \leq 3} A_{JJK}^{\delta j k} (1 - \rho J) \partial u^J \partial \partial_k u^J
\]
in the left side of (3.22), we see that it is bounded by
\[
C \sup_{(s, y) \in \supp(1 - \rho J)} \langle y \rangle (1 + s + |y|)^{1+\mu} z^{1-\mu}(s, |y|) \sum_{|\beta| \leq 41} |\Gamma^\beta \partial u^J|^\beta.
\]
Since \(\langle c_J s - |y| \rangle \gtrsim \langle s + |y| \rangle \gtrsim z(s, |y|)\) for \((s, |y|)\) in the support of \((1 - \rho J)\), it follows easily from (3.10) with \(\mu = 0\) that this term is \(O(\varepsilon^3)\) as desired.

Next, we shall examine (3.22) with \(\hat{G}^I\) replaced by the multi-speed terms
\[
\sum_{1 \leq JK \leq \delta} \sum_{|\alpha| \leq \delta} \partial \Gamma^\alpha u^K \sum_{|\alpha| \leq \delta} \partial \Gamma^\alpha u^J.
\]
Suppose that \((s, |y|) \in \Lambda_J\). Since \(J \neq K\), we have \(|c_K s - |y|| \gtrsim (s + |y|)\). Thus, if we apply (3.10) to the \(u^K\) piece (with \(\mu = 0\)), we see that the left side of (3.22) is controlled by
\[
C\varepsilon \sup_{(s, y) \in \Lambda_J} \langle y \rangle^{1/2} (1 + s + |y|)^{1+\mu} z^{1-\mu}(s, |y|) \sum_{|\beta| \leq 41} |\Gamma^\beta \partial u^J|.
\]
Since \(|y| \approx s\) on \(\Lambda_J\), we see that this term is also \(O(\varepsilon^3)\) by another application of (3.10).

A symmetric argument can be used when \((s, |y|) \in \Lambda_K\). If \((s, |y|) \not\in \Lambda_J \cup \Lambda_K\), then \(|c_J s - |y||, |c_K s - |y|\| \approx (s + |y|)\) and the bound follows from two applications of (3.10) with \(\mu = 0\).

Finally, we are left with proving (3.22) when \(\hat{G}^I\) is replaced by \(R^I + P^I\). In this case, the right side of (3.22) is bounded by
\[
(3.25) C \sum_{1 \leq JK \leq \delta} \sup_{(s, y) \in D \setminus (0, r)} \langle y \rangle (1 + s + |y|)^{1+\mu} z^{1-\mu}(s, |y|) \times \sum_{|\beta| \leq 22} |\Gamma^\beta u^J| \sum_{|\beta| \leq 22} |\Gamma^\beta u^K| \sum_{|\beta| \leq 40} |\Gamma^\beta \partial u^L| + C \sum_{1 \leq JK \leq \delta} \sup_{(s, y) \in D \setminus (0, r)} \langle y \rangle (1 + s + |y|)^{1+\mu} z^{1-\mu}(s, |y|) \times \sum_{|\beta| \leq 22} |\Gamma^\beta u^J| \sum_{|\beta| \leq 22} |\Gamma^\beta u^K| \sum_{|\beta| \leq 40} |\Gamma^\beta \partial u^L|.
\]
By the inductive hypothesis (3.3), the first term in (3.25) is controlled by
\[
C\varepsilon^3 \sup_{(s, y) \in D \setminus (0, r)} \frac{z^{1-\mu}(s, |y|)}{(1 + s + |y|)^{1-\mu} \left(1 + \log \frac{1 + s + |y|}{z(s, |y|)}\right)^3}.
\]
Since \((\log x)^{\beta}/x^{1-\mu}\) is bounded for \(x \geq 1\) and \(\mu < 1\), it follows that the first term in (3.25) is \(O(\varepsilon^3)\). For the second term in (3.25), if we apply (3.10), we see that it is bounded by
\[
C\varepsilon \sum_{1 \leq j, k \leq D} \sup_{(\tau, y) \in D^j} |y|^{1+\beta} \left(1 + \frac{|y|}{1 + |r - c_p|}ight) \sum_{|\beta| \leq 12} |\Gamma^{\beta} u_j| \sum_{|\beta| \leq 12} |\Gamma^{\beta} u^K|
\]
It then follows easily via (3.3) that this term is also \(O(\varepsilon^3)\) as desired. This completes the proof of (3.22), and thus, also (3.3).

We now wish to prove that (3.4) can be obtained with \(A_2\) replaced by \(A_2/2\). Here, we apply (2.23) with \(F^I\) replaced by \(B(\delta u) + Q(\delta u, \delta^2 u)\) and \(G^I\) replaced by \(R(u, du, \delta^2 u) + P(u, du)\) to see that the left side of (3.4) is bounded by
\[
(3.26) \quad C\varepsilon + C \sum_{|\beta| \leq 6} \int_0^1 \int \Gamma^{\beta} [B(\delta u) + Q(\delta u, \delta^2 u)](s, y) \frac{dy \, ds}{(y)}
+ C \left(1 + \log \frac{1 + r + c_I}{1 + |r - c_I|}\right) \sum_{|\beta| \leq 6} \sup_{(\tau, y) \in D^j} |y|(1 + s + |y|)^{1+\mu} z^{1-\mu}(s, |y|)
\times |\Gamma^{\beta} [R(u, du, \delta^2 u) + P(u, du)](s, y)|.
\]
By our choice of \(A_2\), it follows that the first term in (3.26) is controlled by \((A_2/10)\varepsilon\). To complete the proof of (ii.), it will suffice to show that the last two terms in (3.26) are bounded by \(C\varepsilon^2 (1 + t)^{1/10} \log(2 + t) \left(1 + \log \frac{1 + r + c_I}{1 + |r - c_I|}\right)\).

Since \(B(\delta u)\) and \(Q(\delta u, \delta^2 u)\) are quadratic, this is relatively easy for the second term. In fact, this term is bounded by
\[
C \int_0^1 \int \sum_{|\alpha| \leq 6} |\Gamma^{\alpha} \partial u(s, y)|^2 \frac{dy \, ds}{(y)}.
\]
Since this is controlled by the square of the left side of (3.7), the desired bound follows immediately.

To complete the proof of (ii.), it suffices to show that
\[
(3.27) \quad \sup_{(\tau, y) \in D^j} |y|(1 + s + |y|)^{1+\mu} z^{1-\mu}(s, |y|) \sum_{|\beta| \leq 6} \Gamma^{\beta} [R(u, du, \delta^2 u) + P(u, du)](s, y)
\leq C\varepsilon^3 (1 + t)^{1/10} \log(2 + t).
\]
The left side of (3.27) is controlled by
\[
(3.28) \quad C \sup_{(\tau, y) \in D^j} |y|(1 + s + |y|)^{1+\mu} z^{1-\mu}(s, |y|) \left(\sum_{|\beta| \leq 32} \Gamma^{\beta} u\right)^2 \left(\sum_{|\beta| \leq 6} \Gamma^{\beta} u\right)
+ C \sup_{(\tau, y) \in D^j} |y|(1 + s + |y|)^{1+\mu} z^{1-\mu}(s, |y|) \left(\sum_{|\beta| \leq 32} \Gamma^{\beta} u\right)^2 \left(\sum_{|\beta| \leq 6} \Gamma^{\beta} u\right).
\]
By (3.3) and (3.4), we see that the first term is dominated by
\[
C\varepsilon^3 \sup_{(\tau, y) \in D^j} \left(\frac{|z(s, |y|)}{1 + s + |y|}\right)^{1-\mu} \left(1 + \log \frac{1 + s + |y|}{z(s, |y|)}\right)^3 (1 + s)^{1/10} \log(2 + s).
\]
As above, since \((\log x)^2/x^{1-\mu}\) is bounded for \(x > 1\) and \(\mu\) small, we easily obtain the desired bound. For the second term in (3.28), applying (2.27) and (3.6) we see that it is dominated by

\[
C \varepsilon (1 + t)^{1/40} \sup_{(s,y)} (1 + s + |y|)^{1+\mu} z^{1-\mu} (s, |y|) \left( \sum_{|\beta| \leq 2} |\Gamma^\beta u|^2 \right).
\]

Applying (3.3) yields the desired bound (3.27) and finishes the proof of (ii.).

3.3. **Proof of (iii.):** In this section, we finish the continuity argument, and thus the proof of Theorem 1.2, by showing that (3.5)-(3.7) follow from (3.2)-(3.4).

We begin with (3.5). Outside of \(\Lambda_I\), \(\log \frac{1+2|x|}{1+|x|}\) is \(O(1)\), and (3.5) follows directly from (3.3). Within \(\Lambda_I\), we have \(t \approx |x|\), and (3.5) follows from (2.27) and (3.2).

Next, we want to show that the higher order energy bound (3.6) holds. We will apply (2.5) with

\[
\gamma^{IJ,jk} = -\sum_{1 \leq K \leq D \atop 0 < l \leq S} B_{K,l}^{IJ,jk} \partial_\ell u^K - C_{IJ,jk}^{I'} (u, u')
\]

and

\[
G^I = B^I (du) + P^I (u, du).
\]

In order to prove (3.6), by (2.4), (3.2), and an induction argument, it will suffice to prove the following.

**Lemma 3.2.** Assume that (3.2)-(3.5) hold and \(M \leq 70\). Additionally, suppose that

\[
\sum_{|\alpha| \leq M - 1} E(\Gamma^\alpha u, t) \leq C \varepsilon (1 + t)^{C \varepsilon + \sigma}
\]

with \(\sigma > 0\). Then, there is a constant \(C'\) so that

\[
\sum_{|\alpha| \leq M} E(\Gamma^\alpha u, t) \leq C' \varepsilon (1 + t)^{C' \varepsilon + C' \sigma}.
\]

**Proof of Lemma 3.2:** Since

\[
\sum_{|\alpha| \leq M} |[\Box, \Gamma^\alpha] u| \leq C \sum_{|\alpha| \leq M - 1} |\Gamma^\alpha \Box u| + C \sum_{|\alpha|, |\beta| \leq M} |\Gamma^\alpha \Gamma^\beta \partial^2 u|
\]

and since (3.3) and (3.5) imply that

\[
\sum_{|\alpha| \leq N} |\Gamma^\alpha \gamma| \leq \frac{C \varepsilon}{1 + t}
\]

(3.34)
for \( N \leq 39 \), it follows from (2.5) that
\[
\begin{align*}
(3.35) \quad &\sum_{|\alpha| \leq M} \partial_t E(\Gamma^\alpha u, t) \leq C \sum_{|\alpha| \leq M} \|\Gamma^\alpha B(du(t, \cdot))\|_2 + C \sum_{|\alpha| \leq M} \|\Gamma^\alpha P(du, du(t, \cdot))\|_2 \\
&+ C \sum_{|\alpha| \leq M-1} \|\Gamma^\alpha Q(du, d^2u(t, \cdot))\|_2 + C \sum_{|\alpha| \leq M-1} \|\Gamma^\alpha R(du, du, d^2u(t, \cdot))\|_2 \\
&+ C \sum_{|\alpha| + |\beta| \leq M, |\beta| \leq M-1} \|\Gamma^\alpha \Gamma^\beta \partial^\beta u\|_2 + \frac{C\varepsilon}{1 + t} \sum_{|\alpha| \leq M} E(\Gamma^\alpha u, t).
\end{align*}
\]

Note that it follows from (3.5) that
\[
(3.36) \quad \sum_{|\alpha| \leq M} \|\Gamma^\alpha B^T(du(t, \cdot))\|_2 + \sum_{|\alpha| \leq M-1} \|\Gamma^\alpha Q^T(du, d^2u(t, \cdot))\|_2 \\
\leq \frac{C\varepsilon}{1 + t} \sum_{|\alpha| \leq M} E(\Gamma^\alpha u, t).
\]

Additionally, by (3.3), we have
\[
\sum_{|\alpha| \leq M} \|\Gamma^\alpha P^T(du, du(t, \cdot))\|_2 + \sum_{|\alpha| \leq M-1} \|\Gamma^\alpha R^T(du, du, d^2u(t, \cdot))\|_2 \\
\leq C\varepsilon^2 \sum_{|\alpha| \leq M, |\beta| \leq 1} \left\| \frac{1 + \log(1 + t)}{1 + t + |x|} \right\| \Gamma^\alpha \partial^\beta u(t, \cdot)\|_2.
\]

Since the coefficients of \( \Gamma \) are \( O(1 + t + |x|) \), it follows from (3.3) that this is
\[
(3.37) \quad \leq C\varepsilon^3 (1 + t)^{-3/2} + C\varepsilon^2 \frac{(1 + \log(1 + t))^2}{1 + t} \sum_{|\alpha| \leq M-1} E(\Gamma^\alpha u, t) \\
+ \frac{C\varepsilon^2 (1 + \log(1 + t))^2}{(1 + t)^2} \sum_{|\alpha| \leq M} E(\Gamma^\alpha u, t).
\]

The first term on the right side corresponds to the case \( |\alpha| = |\beta| = 0 \) on the right side of the previous equation. Similarly, the second term is for the case \( |\beta| = 0 \), and the last term bounds the case \( |\alpha|, |\beta| \neq 0 \). By a similar argument, the fifth term on the right of (3.35) is also controlled by the right sides of (3.36) and (3.37). Thus, we see that
\[
(3.38) \quad \sum_{|\alpha| \leq M} \partial_t E(\Gamma^\alpha u, t) \leq \frac{C\varepsilon}{1 + t} \sum_{|\alpha| \leq M} E(\Gamma^\alpha u, t) \\
+ C\varepsilon^3 (1 + t)^{-3/2} + C\varepsilon^2 \frac{(1 + \log(1 + t))^2}{1 + t} \sum_{|\alpha| \leq M-1} E(\Gamma^\alpha u, t).
\]

Integrating both sides in \( t \), applying the smallness assumption on the data (1.14) and the inductive hypothesis (3.31), and using Gronwall’s inequality yields (3.32) as desired. \( \Box \)
We are, thus, left with the task of showing (3.7). Applying (2.6) with $u$ replaced by $\Gamma^\alpha u$, we see that the left side of (3.7) is controlled by

(3.39) \[ C(|\log(2 + t)|)^{1/2} \left( \sum_{|\alpha| \leq 5} \|\Gamma^\alpha f\|_2 + \sum_{|\alpha| \leq 5} \|\Gamma^\alpha g\|_2 + \int_0^t \|\Gamma^\alpha \Box u(s, \cdot)\|_2 \, ds \right). \]

By (1.14), the first two terms satisfy the desired bound. Since

(3.40) \[ \sum_{|\alpha| \leq 5} \|\Gamma^\alpha u(s, \cdot)\|_2 \leq C \left( \sum_{|\alpha| \leq 33} \|\Gamma^\alpha u'\|_2 \sum_{|\alpha| \leq 66} \|\Gamma^\alpha u'\|_2 \right)^{1/2} \sum_{|\alpha| \leq 67} \|\Gamma^\alpha u\|_2, \]

we may use (3.3), (3.5), and the fact that the coefficients of $\Gamma$ are $O(1 + t + |x|)$ to see that the right side of (3.40) is dominated by

(3.41) \[ \frac{C\varepsilon}{1 + s} \sum_{|\alpha| \leq 66} \|\Gamma^\alpha u'(s, \cdot)\|_2 + C\varepsilon \frac{1 + \log(2 + s)}{1 + s} \sum_{|\alpha| \leq 66} \|\Gamma^\alpha u'(s, \cdot)\|_2 \]

\[ + \varepsilon^{1/2} \frac{(1 + \log(2 + s))^2}{(1 + s)^{3/2}} \|1 + s + |\cdot|^{-1/2} u(s, \cdot)\|_2. \]

Plugging (3.40) and (3.41) into (3.39), we see that the third term of (3.39) is bounded by the right side of (3.7) by using (3.3) and (3.6).

This completes the proof of (iii.), and hence the proof of Theorem 1.2.

4. Preliminary estimates in the exterior domain

In this section, we will collect the exterior domain analogs of the estimates in Section 2. Many of these estimates were previously established in [17], [27], and [28]. The main new item will be the use of the pointwise estimates found in the second subsection.

4.1. Energy estimates. We begin by gathering the $L^2$ estimates that we will need in order to show global existence in the exterior domain. These estimates are from Metcalfe-Sogge [28] (see also [17]), and unless stated otherwise, their proofs can be found there. Specifically, we will be concerned with solutions $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3\setminus K)$ of the Dirichlet-wave equation

(4.1) \[
\begin{cases}
\Box_\gamma u = F \\
 u|_{\partial K} = 0 \\
 u|_{t=0} = f, \quad \partial_t u|_{t=0} = g
\end{cases}
\]

with $\Box_\gamma$ as in (2.1). We shall assume that the $\gamma^{IJ, jk}$ satisfy the symmetry conditions (2.2) as well as the size condition

(4.2) \[ \sum_{I,J=1}^3 \sum_{j,k=0}^3 \|\gamma^{IJ, jk}(t, x)\|_\infty \leq \delta \]
for $\delta$ sufficiently small (depending on the wave speeds). The energy estimate will involve bounds for the gradient of the perturbation terms

$$||\gamma'(t, \cdot)||_\infty = \sum_{I, J = 1}^{D} \sum_{j, k, l = 0}^{3} ||\partial_t \gamma^{I,J,k}(t, \cdot)||_\infty,$$

and the energy form associated with $\Box \gamma$, $e_0(u) = \sum_{I=1}^{D} e_I^l(u)$, where $e_I^l(u)$ is given by (2.3).

The most basic estimate will lead to a bound for

$$E_M(t) = E_M(u)(t) = \int \sum_{j=0}^{M} e_0(\partial_t^j u)(t, x) \, dx.$$

**Lemma 4.1.** Fix $M = 0, 1, 2, \ldots$, and assume that the perturbation terms $\gamma^{I,J,k}$ satisfy (2.2) and (4.2). Suppose also that $u \in C^\infty$ solves (4.1) and for every $t$, $u(t, x) = 0$ for large $x$. Then there is an absolute constant $C$ so that

$$\partial_t E_M^{1/2}(t) \leq C \sum_{j=0}^{M} ||\Box \gamma^j \partial_t^j u(t, \cdot)||_2 + C ||\gamma'(t, \cdot)||_\infty E_M^{1/2}(t). \quad (4.3)$$

Before stating the next result, let us introduce some notation. If $P = P(t, x, D_t, D_x)$ is a differential operator, we shall let

$$[P, \gamma^{k,l}]\partial_k \partial_l u = \sum_{1 \leq I, J \leq D} \sum_{0 \leq k, l \leq 3} [P, \gamma^{I,J,k,l}]\partial_k \partial_l^j u.$$

In order to generalize the above energy estimate to include the more general vector fields $L, Z$, we will need to use a variant of the scaling vector field $L$. We fix a bump function $\eta \in C^\infty(\mathbb{R}^3)$ with $\eta(x) = 0$ for $x \in K$ and $\eta(x) = 1$ for $|x| > 1$. Then, set $L = \eta(x) r \partial_r + t \partial_t$. Using this variant of the scaling vector field and an elliptic regularity argument, one can establish

**Proposition 4.2.** Suppose that the constant in (4.2) is small. Suppose further that

$$||\gamma'(t, \cdot)||_\infty \leq \delta/(1 + t), \quad (4.4)$$

and

$$\sum_{j + \mu \leq N_e + \nu_0 \atop \mu \leq \nu_0} \left( ||\tilde{L}^\mu \partial_t^j \Box \gamma u(t, \cdot)||_2 + ||\tilde{L}^\mu \partial_t^j \gamma^{k,l} \partial_k \partial_l^j u(t, \cdot)||_2 \right) \leq \frac{\delta}{1 + t} \sum_{j + \mu \leq N_e + \nu_0 \atop \mu \leq \nu_0} ||\tilde{L}^\mu \partial_t^j u'(t, \cdot)||_2 + H_{\nu_0, N_e}(t). \quad (4.5)$$
where \( N_0 \) and \( \nu_0 \) are fixed. Then

\[
\sum_{|\alpha| + \mu \leq N_0 + \nu_0} \| I^\mu \partial^\alpha u'(t, \cdot) \|_2
\]

\[
\leq C \sum_{|\alpha| + \mu \leq N_0 + \nu_0 - 1} \| I^\mu \partial^\alpha \Box u(t, \cdot) \|_2 + C(1 + t)^A \sum_{\mu \leq \nu_0} \left( \int_0^t \int |I^\mu \partial^\alpha \Box u(s, \cdot)\|_2^2 + \int_0^t H_{\nu_0, N_0}(s) ds \right)^{1/2}
\]

\[
+ C(1 + t)^A \int_0^t \sum_{|\alpha| + \mu \leq N_0 + \nu_0 - 1} \| I^\mu \partial^\alpha u'(s, \cdot) \|_{L^2(|x| < 1)}^2 ds,
\]

where the constants \( C \) and \( A \) are absolute constants.

In practice \( H_{\nu_0, N_0}(t) \) will involve \( L^2_\nu \) norms of \( I^\mu \partial^\alpha u' \) with \( \mu + |\alpha| \) much smaller than \( N_0 + \nu_0 \), and so the integral involving \( H_{\nu_0, N_0} \) can be dealt with using an inductive argument and the weighted \( L^2_\nu L^2_\nu \) estimates that will be presented at the end of this subsection.

In proving our existence results for (1.1), the key step will be to obtain a priori \( L^2 \)-estimates involving \( I^\mu \partial^\alpha u' \). Begin by setting

\[
Y_{N_0, \nu_0}(t) = \int \sum_{|\alpha| + \mu \leq N_0 + \nu_0} \epsilon_0 |I^\mu \partial^\alpha Z^\nu u'(t, x)| dx.
\]

We, then, have the following proposition which shows how the \( I^\mu \partial^\alpha Z^\nu u' \) estimates can be obtained from the ones involving \( I^\mu \partial^\alpha u' \).

**Proposition 4.3.** Suppose that the constant \( \delta \) in (4.2) is small and that (4.4) holds. Then,

\[
\partial_t Y_{N_0, \nu_0} \leq CY_{N_0, \nu_0}^{1/2} \sum_{|\alpha| + \mu \leq N_0 + \nu_0} ||\square \gamma I^\mu \partial^\alpha Z^\nu u(t, \cdot)\|_2 + C||\gamma' \|_\infty Y_{N_0, \nu_0}
\]

\[
+ C \sum_{|\alpha| + \mu \leq N_0 + \nu_0 + 1} \| I^\mu \partial^\alpha u'(t, \cdot) \|_{L^2(|x| < 1)}^2.
\]

As in [16] and [17] we shall also require some weighted \( L^2_\nu L^2_\nu \) estimates. They will be used, for example, to control the local \( L^2 \) norms such as the last term in (4.8). For convenience, for the remainder of this subsection, allow \( \square = \partial^2_t - \Delta \) to denote the unit speed, scalar d’Alembertian. The transition from the following estimates to those involving (1.2) is straightforward. Also, allow

\[
S_T = \{0, T] \times \mathbb{R}^3 \setminus K
\]

to denote the time strip of height \( T \) in \( \mathbb{R} \times \mathbb{R}^3 \setminus K \).
We, then, have the following proposition which is an exterior domain analog of (2.6).

**Proposition 4.4.** Fix $N_0$ and $\nu_0$. Suppose that $\mathcal{K}$ satisfies the local exponential energy decay (1.11). Suppose also that $u \in C^\infty$ satisfies $u(t, x) = 0$, $t < 0$. Then there is a constant $C = C_{N_0, \nu_0, \mathcal{K}}$ so that if $u$ vanishes for large $x$ at every fixed $t$

\begin{equation}
(\log(2 + T))^{-1/2} \sum_{|\alpha| + \mu \leq N_0 + \nu_0 \atop \mu \leq \nu_0} \|\langle x \rangle^{-1/2} L^\mu \partial^\alpha u'\|_{L^2(S_T)}
\leq C \int_0^T \sum_{|\alpha| + \mu \leq N_0 + \nu_0 \atop \mu \leq \nu_0} \|\partial^\alpha u(t, \cdot)\|_2 \, ds + C \sum_{|\alpha| + \mu \leq N_0 + \nu_0 \atop \mu \leq \nu_0} \|\partial^\alpha u\|_{L^2(S_T)}
\end{equation}

and

\begin{equation}
(\log(2 + T))^{-1/2} \sum_{|\alpha| + \mu \leq N_0 + \nu_0 \atop \mu \leq \nu_0} \|\langle x \rangle^{-1/2} L^\mu Z^\alpha u'\|_{L^2(S_T)}
\leq C \int_0^T \sum_{|\alpha| + \mu \leq N_0 + \nu_0 \atop \mu \leq \nu_0} \|\partial^\alpha u(s, \cdot)\|_2 \, ds + C \sum_{|\alpha| + \mu \leq N_0 + \nu_0 \atop \mu \leq \nu_0} \|\partial^\alpha u\|_{L^2(S_T)}
\end{equation}

We end this subsection with a couple of results that follow from the local energy decay (1.11).

**Lemma 4.5.** Suppose that (1.11) holds and that $\Box u(t, x) = 0$ for $|x| > 4$. Suppose also that $u(t, x) = 0$ for $t \leq 0$. Then, if $N_0$ and $\nu_0$ are fixed and if $c > 0$ is as in (1.11), the following estimate holds:

\begin{equation}
\sum_{|\alpha| + \mu \leq N_0 + \nu_0 \atop \mu \leq \nu_0} \|L^\mu \partial^\alpha u'(t, \cdot)\|_{L^2(\mathbb{R}^n \setminus \mathcal{K} : |x| < 4)}
\leq C \sum_{|\alpha| + \mu \leq N_0 + \nu_0 \atop \mu \leq \nu_0 - 1} \|L^\mu \partial^\alpha \Box u(t, \cdot)\|_2
\end{equation}

\begin{equation}
+ C \int_0^t e^{-(c/2)(t-s)} \sum_{|\alpha| + \mu \leq N_0 + \nu_0 + 1 \atop \mu \leq \nu_0} \|L^\mu \partial^\alpha \Box u(s, \cdot)\|_2 \, ds.
\end{equation}

To be able to handle the last term in (4.6), we shall need the following.

**Lemma 4.6.** Suppose that (1.11) holds, and suppose that $u \in C^\infty$ satisfies $u(t, x) = 0$ for $t < 0$. Then, for fixed $N_0$ and $\nu_0$ and $t > 2$,

\begin{equation}
\sum_{|\alpha| + \mu \leq N_0 + \nu_0 \atop \mu \leq \nu_0} \int_0^t \|L^\mu \partial^\alpha u'(s, \cdot)\|_{L^2(|x| < 2)} \, ds
\leq C \sum_{|\alpha| + \mu \leq N_0 + \nu_0 + 1 \atop \mu \leq \nu_0} \int_0^t \left( \int_0^s \|L^\mu \partial^\alpha \Box u(\tau, \cdot)\|_{L^2(|x| - (t - \tau)| < 1)} \, d\tau \right) \, ds.
\end{equation}
4.2. **Pointwise estimates.** Here, we will describe the various pointwise estimates that we shall require. These include variants of those of Keel-Smith-Sogge [17] and Metcalfe-Sogge [28] and exterior domain analogs of the estimates of Kubota-Yokoyama [21].

Let us begin with the former. We will need analogs of the pointwise estimates of [17] and [28] that allow Cauchy data that vanishes in a neighborhood of the obstacle. That is, we will estimate solutions of the scalar wave equation with boundary \( (\p_t^2 - \Delta) u(t, x) = F(t, x) \). Additionally, we will require that \( u(0, x) = \partial_t u(0, x) = 0 \) if \( |x| \leq 6 \), and \( F(t, x) = 0 \) if \( |x| \leq 6 \) and \( 0 \leq t \leq 1 \). With these assumptions, we can greatly reduce the technical details involving the compatibility conditions. In the sequel, we will reduce our study to this case. Assuming, as we do throughout, that \( \mathcal{K} \subset \{ x \in \mathbb{R}^3 : |x| < 1 \} \), we have

**Theorem 4.7.** Suppose that the local energy decay bounds (1.11) hold for \( \mathcal{K} \). Additionally, assume that \( u(t, x) = 0 \) for \( x \in \partial \mathcal{K} \), \( u(0, x) = \partial_t u(0, x) = 0 \) for \( |x| \leq 6 \), and \( F(t, x) = 0 \) if \( 0 \leq t \leq 1 \) and \( |x| \leq 6 \). Then, if \( \alpha = M \),

\[
(1 + t + |x|)|L^\nu Z^\alpha u(t, x)| \leq C \sum_{j + |\beta| + \kappa \leq \nu + M + 8} \|
(\langle x \rangle)^{j+|\beta|} \partial_x^\beta \partial_x^j u(0, x)\|_2 \\
+ C \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} |L^\nu Z^\beta F(s, y)| \frac{dy}{|y|} \\
+ C \int_0^t \sum_{|\beta| \leq \nu + 4} \|L^\nu \partial^\beta F(s, \cdot)\|_{L^2(\{|x| \leq 2 \})} ds.
\]

**Proof of Theorem 4.7:** If \( u \) has vanishing Cauchy data with \( F(t, x) = 0 \) for \( 0 \leq t \leq 1 \) and \( x \in \mathbb{R}^3 \setminus \mathcal{K} \), (4.13) follows from Theorem 3.1 in [28]. We, thus, may assume \( F(t, x) = 0 \) for \( 0 \leq t < \infty \) and \( |x| \leq 6 \) and that the Cauchy data is as stated above. The proof follows from the arguments of [28] for the inhomogeneous case very closely. We include a sketch of the proof for completeness.

We first note that if we argue as in [17] (Lemma 4.2) we have

\[
(1 + t + |x|)|L^\nu Z^\alpha u(t, x)| \leq C \sum_{j + |\beta| + \kappa \leq \nu + M + 4} \|
(\langle x \rangle)^{j+|\beta|} \partial_x^\beta \partial_x^j u(0, x)\|_2 \\
+ C \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} |L^\nu Z^\beta F(s, y)| \frac{dy}{|y|} \\
+ C \sum_{|\beta| \leq \nu + 1} \sup_{0 \leq s \leq t} |L^\nu \partial^\beta u(s, y)|.
\]

While the arguments in [17] are given for vanishing Cauchy data, straightforward modifications allow the current setting.
It remains to prove bound in the region $|x| < 2$. We show

\[(4.15)\]

\[
\sum_{|\beta| + \mu \leq M + \nu + 1} \sup_{|y| \leq 2} (1 + s) |L^\mu \partial^\beta w(s, y)| \leq C \sum_{j + |\alpha| + k \leq M + \nu + 8} \|\langle x \rangle^j + \alpha \partial_x^\alpha \partial_x^j w(0, x)\|_2
\]

\[+ C \sum_{|\beta| + \mu \leq M + \nu + 7} \int_0^\delta \int_{\mathbb{R}^N} |L^\mu Z^\beta F(s, y)| \frac{dy \, ds}{|y|}.
\]

To see this, write $w = w_0 + w_r$ where $w_0$ solves the boundaryless wave equation $(\partial_t^2 - \Delta)w_0 = F$ with initial data $w_0(0, \cdot) = w(0, \cdot)$ and $\partial_\nu w_0(0, \cdot) = \partial_\nu w_0(0, \cdot)$. If we fix $\eta \in C_0^\infty (\mathbb{R}^n)$ with $\eta(x) \equiv 1$ for $|x| < 2$ and $\eta(x) \equiv 0$ for $|x| \geq 3$ and set $\tilde{w} = \eta w_0 + w_r$, it follows that $w = \tilde{w}$ for $|x| < 2$. Thus, it will suffice to show (4.15) with $w$ replaced by $\tilde{w}$.

Notice that $\tilde{w}$ solves the Dirichlet-wave equation

\[
(\partial_t^2 - \Delta)\tilde{w} = -2 \nabla \eta \cdot \nabla x w_0 - (\Delta \eta)w_0
\]

with vanishing initial data since the support of $\eta$ does not intersect the supports of $F$, $w_0(0, \cdot)$ and $\partial_\nu w_0(0, \cdot)$ and that this forcing term vanishes unless $2 \leq |x| \leq 3$.

In order to complete the proof, we begin by noting the following consequence of the Fundamental Theorem of Calculus:

\[
\sup_{|y| \leq 2} (1 + s) |L^\mu \partial^\beta \tilde{w}(s, y)| \leq C \sum_{j = 0, 1} \sup_{|y| \leq 2} \int_0^\delta (\tau \partial_\nu)^j L^\mu \partial^\beta \tilde{w}(\tau, y) \, d\tau.
\]

Using Sobolev’s lemma and the fact that the Dirichlet condition allows us to control $\tilde{w}$ locally by $\tilde{w}'$, we see that the left hand side of (4.15) is bounded by

\[
C \sum_{|\beta| + \mu \leq M + \nu + 3} \sum_{j = 0, 1} \int_0^\delta \|\langle \tau d\tau \rangle^j L^\mu \partial^\beta \tilde{w}(\tau, y)\|_{L^2(\mathbb{R}^N; |x| \leq 4)} \, d\tau
\]

\[\leq C \sum_{|\beta| + \mu \leq M + \nu + 3} \int_0^\delta \|L^\mu \partial^\beta \tilde{w}'(\tau, y)\|_{L^2(\mathbb{R}^N; |x| \leq 4)} \, d\tau
\]

By (4.11), it follows that the right side of the above estimate is controlled by

\[
C \int_0^\delta \sum_{|\beta| + \mu \leq M + \nu + 5} \|L^\mu \partial^\beta w_0(s, \cdot)\|_{L^\infty(2 \leq |x| \leq 3)} \, ds.
\]
From (2.10), (2.16), and the fact that $1/t \leq 1/|y|$ on the domain of integration in (2.10), we have

\begin{align}
(4.16) \quad \| L^\beta \partial^\beta w_0 (s, \cdot) \|_{L^\infty (|y| \leq 4)} \leq C \sum_{|\alpha| \leq 2} \int_{|\tau - |y|| \leq 4} \| (\Omega^\alpha \nabla L^\beta \partial^\beta w_0) (0, y) \| \frac{dy}{|y|} \\
+ C \sum_{|\alpha| \leq 2} \int_{|\tau - |y|| \leq 4} \| (\Omega^\alpha L^\beta \partial^\beta w_0) (0, y) \| \frac{dy}{|y|^2} \\
+ C \sum_{|\alpha| \leq 2} \int_{|\tau - |y|| \leq 4} \| (\Omega^\alpha \partial_\tau L^\beta \partial^\beta w_0) (0, y) \| \frac{dy}{|y|} \\
+ C \sum_{|\alpha| \leq 2} \int_{0}^{t} \int_{|\tau - \tau_0||y| \leq 4} \| \Omega^\alpha (\partial_\tau^2 - \Delta) L^\beta \partial^\beta w_0 (\tau, y) \| \frac{dy \, d\tau}{|y|}. 
\end{align}

Since the sets $\Lambda_\epsilon = \{ y : |s - |y|| \leq 4 \}$ satisfy $\Lambda_\epsilon \cap \Lambda_\epsilon' = \emptyset$ if $|s - s'| \geq 10$, if we sum over $|\beta| + \mu \leq M + \nu + 5$, $\mu \leq \nu + 1$, and integrate over $s \in [0, t]$, we conclude that the left side of (4.15) is controlled by

\begin{align}
C \sum_{k + |\beta| \leq M + \nu + 7} \int \langle \chi \rangle |\beta| \| (\partial^\beta \partial_\tau^k \nabla w) (0, y) \| \frac{dy}{|y|} \\
+ C \sum_{k + |\beta| \leq M + \nu + 7} \int \langle \chi \rangle |\beta| \| (\partial^\beta \partial_\tau^k w) (0, y) \| \frac{dy}{|y|^2} \\
+ C \sum_{k + |\beta| \leq M + \nu + 7} \int \langle \chi \rangle |\beta| \| (\partial^\beta \partial_\tau^k \partial_\tau w) (0, y) \| \frac{dy}{|y|} \\
+ C \sum_{|\beta| + \mu \leq M + \nu + 7} \int_{0}^{t} \int_{\mathbb{R}^3 \setminus K} \| L^\beta Z^\beta F (s, y) \| \frac{dy \, ds}{|y|}. 
\end{align}

Using the Schwarz inequality, (4.15), and thus (4.13), follows. \qed

For the remainder of the estimates in this section, it will suffice to take $w$ to be a solution to the following Dirichlet-wave equation with vanishing initial data.

\begin{equation}
(4.17) \quad \begin{cases}
(\partial_\tau^2 - c_1^2 \Delta) w (t, x) = F (t, x), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 \setminus K \\
w (t, x) = 0, & x \in \partial K \\
w (t, x) = 0, & t \leq 0.
\end{cases}
\end{equation}

In the sequel, we will reduce showing that (1.1) has a global solution to showing that an equivalent system of nonlinear wave equations with vanishing data has a global solution. Since the previous theorem will suffice to make this reduction, it is unnecessary to consider nonvanishing Cauchy data in the subsequent estimates.

We will need the following version of (4.13) that does not require a loss of a scaling vector field on the right.
Theorem 4.8. Suppose that the local energy decay bound (1.11) holds for $\mathcal{K}$. Suppose that $w$ is a solution to (4.17) and $|\alpha| = M$. Then,

$$
(4.18) \quad (1 + |x|)|L^\nu Z^\alpha w(t, x)| \leq \int_0^t \int_{\mathbb{R}^N \setminus \mathcal{K}} |y|^\nu |\partial w_0(s, y)| \frac{dy \, ds}{|y|} + C \int_0^t \sum_{|\beta| + \nu \leq M + \nu_0 + 3} \|L^\nu \partial^\beta F(s, \cdot)\|_{L^2([x_0 \leq |r| < 4])} \, ds.
$$

Here, we refer the reader to similar arguments in the previous articles of Keel-Smith-Sogge [17] (Theorem 4.1), Metcalfe-Sogge [28] (Theorem 3.1), and the authors [27] (Lemma 3.3, Lemma 3.4). Since we are only requiring decay in $|x|$, the proof is based only on the Minkowski estimate

$$
|x||w_0(t, x)| \leq C \int_0^t \int_{|x_0 - |t-s|| < 1} \sup \{ |\partial w_0(s, r\theta)| r \, dr \, ds
$$

$$
(4.19) \quad \leq C \int_0^t \int_{y \in \mathbb{R}^N : |y| \leq |x_0 - |t-s||, |y| = 1} \sum_{|\nu| = 2} \|\Omega^\nu \partial^\nu w_0(s, y)\| \frac{dy \, ds}{|y|}.
$$

We, thus, do not require the additional $L$ that appears on the right side of the estimates in [17], [28], and [27].

Letting $A_j$ be the small conic neighborhood of the characteristic cone $|x| = c_j t$ for $c_j$ defined by (2.19), we also have the following estimate when the forcing term is localized to such a region. This is an analog of (2.24) for the Dirichlet-wave equation.

Theorem 4.9. Let $w$ be a solution to (4.17). Suppose that $F(t, x)$ is supported in some $A_j$ for $j \neq 1$. Then, there are constants $c, c', C > 0$ depending on $c_1, c_2$ so that for $t > 2$, $I, J = 1, 2, \ldots, D$,

$$
(4.20) \quad (1 + t + |x|)|L^\nu Z^\alpha w(t, x)|
$$

$$
\leq C \int_{\max(t-c, c_1 t - |x_0| - 1)}^{t} \int_{|y| \geq 1} \sum_{|\beta| + \nu \leq M + \nu_0 + 3} \|L^\nu Z^\beta F(s, y)\| \frac{dy \, ds}{|y|} + C \sum_{|\beta| + \nu \leq M + \nu_0 + 3} \sup_{x \leq s} \int \|L^\nu Z^\beta F(s, y)\| \, dy
$$

$$
+ C \sup_{0 \leq \tau \leq 1} \int_{|x| + \tau}^{s} \int_{|y| \geq \tau} \|L^\nu Z^\beta F(\tau, y)\| \frac{dy \, d\tau}{|y|} + C \sum_{0 \leq \tau \leq 1} \sup_{0 \leq \tau \leq 1} (1 + \tau) \sum_{|\beta| + \nu \leq M + \nu_0 + 3} \|L^\nu \partial^\beta F(s, \cdot)\|_{L^2(|x_0 - x| < 1)}.
$$

Here, as before, $|y| \approx s$ indicates that there is some positive constant $c$ so that $\frac{1}{c}s \leq |y| \leq cs$. 
We shall need an analog of Lemma 2.3, the result of Kubota-Yokoyama [21], for
Dirichlet-wave equations. With \( z \) as in (2.20), we have

**Theorem 4.10.** Let \( I = 1, 2, \ldots, D \), and let \( w \) be a solution to (4.17). Then, for any
\( \mu > 0 \),

\[
(4.21) \quad (1 + t + r)(1 + \log \frac{1 + t + r}{1 + |ct - r|})^{-1} L^\mu Z^\alpha u(t, x) \leq C \sup_{(t, y)} |y|(1 + s + |y|)^{1 + \mu} \lesssim (s, |y|) \sum_{|\beta + \nu| \leq |\alpha| + \nu_0} |L^\nu Z^\beta F(s, y)|
\]

\[+ C \sup_{(t, y)} |y|(1 + s + |y|)^{1 + \mu} \lesssim (s, |y|) \sum_{|\beta + \nu| \leq |\alpha| + \nu_0 + 3} |L^\nu \partial^\beta \partial F(s, y)|.
\]

The proofs of Theorem 4.9 and Theorem 4.10 are quite similar, and we will only provide the
proof of the latter. In order to prove Theorem 4.9, we need only replace the
applications of (2.21) by (2.24) which is the appropriate free space analog of (4.26).

**Proof of Theorem 4.10:** We begin by claiming that

\[
(4.22) \quad (1 + t + r)(1 + \log \frac{1 + t + r}{1 + |ct - r|})^{-1} L^\mu Z^\alpha u(t, x) \leq C \sup_{(t, y)} |y|(1 + s + |y|)^{1 + \mu} \lesssim (s, |y|) \sum_{|\beta + \nu| \leq |\alpha| + \nu_0} |L^\nu Z^\beta F(s, y)|
\]

\[+ C \sup_{(t, y) \atop |y| < 2} (1 + s) \sum_{|\beta + \nu| \leq |\alpha| + \nu_0 + 1} |L^\nu \partial^\beta w(s, y)|.
\]

Indeed, over \( |x| < 2 \), the left side is clearly bounded by the second term on the right
side since the coefficients of \( Z \) are \( O(1) \) on this set. To see the estimate on \( |x| \geq 2 \), we
fix a cutoff function \( \rho \in C^\infty \) where \( \rho(x) \equiv 0 \) for \( |x| < 3/2 \) and \( \rho(x) \equiv 1 \) for \( |x| > 2 \). If
we let \( w_j \) denote the solutions to the boundaryless wave equations \((\partial_t^2 - c_j^2 \Delta)w_j = G_j, j = 1, 2 \) where \( G_1 = \rho(\Delta^2 - c_1^2 \Delta)w \) and \( G_2 = -2c_1^2 \nabla \rho \cdot \nabla w - c_1^2 (\Delta \rho)w \), we see that
\( w = w_1 + w_2 \). Since \( |\Box, Z| = 0 \) and \( |\Box, L| = 2 \Box \), we can establish the bound for the \( w_1 \)
piece by applying (2.21). Arguing as in Lemma 4.2 of Keel-Smith-Sogge [17], we see that the
\( w_2 \) term is bounded by the second term on the right side of (4.22).

To finish the proof, it suffices to show

\[
(4.23) \quad \sup_{|y| \leq 1} (1 + s) \sum_{|\beta + \nu| \leq |\alpha| + \nu_0 + 1} |L^\nu \partial^\beta w(s, \cdot) \|_{L^\infty(|x| < 2)} \leq C \sup_{(t, y)} |y|(1 + s + |y|)^{1 + \mu} \lesssim (s, |y|) \sum_{|\beta + \nu| \leq |\alpha| + \nu_0} |L^\nu F(s, y)|
\]

\[+ C \sup_{(t, y)} |y|(1 + s + |y|)^{1 + \mu} \lesssim (s, |y|) \sum_{|\beta + \nu| \leq |\alpha| + \nu_0 + 3} |L^\nu \partial^\beta \partial F(s, y)|.
\]
When \(F(s, y) = 0\) for \(|y| > 10\), we can apply the following Lemma, which is essentially Lemma 3.3 from [27].

**Lemma 4.11.** Suppose that \(w\) is as above. Suppose further that \((\partial_t^2 - c_i^2 \Delta) w(s, y) = F(s, y) = 0\) if \(|y| > 10\). Then,

\[
(1 + t) \sup_{|x| < 2} |L^\nu \partial^\alpha w(t, x)| \leq C \sup_{0 \leq s \leq t} \sum_{0 \leq \mu \leq \nu} (1 + s) \|L^\mu \partial^\beta F(s, \cdot)\|_2.
\]

Since \(F\) is supported on \(|y| < 10\) and since \(|y|\) is bounded below on the complement of \(K\), it follows that this term is controlled by the right side of (4.23).

We also need an estimate for solutions whose forcing terms vanish near the obstacle. Assume now that \(F(s, y) = 0\) for \(|y| < 5\) and write \(w = w_0 + w_r\), where \(w_0\) solves the boundaryless wave equation \((\partial_t^2 - c_i^2 \Delta) w_0 = F\) with vanishing initial data. Fixing \(\eta \in C_0^\infty(\mathbb{R}^3)\) satisfying \(\eta(x) \equiv 1\) for \(|x| < 2\) and \(\eta(x) \equiv 0\) for \(|x| \geq 3\) and setting \(\hat{w} = \eta w_0 + w_r\), we see that \(w = \hat{w}\) on \(|x| < 2\). Since \(\hat{w}\) solves the Dirichlet-wave equation

\[
(\partial_t^2 - c_i^2 \Delta) \hat{w} = -2c_i \nabla \eta \cdot \nabla_x w_0 - c_i^2 (\Delta \eta) w_0 = G
\]

and \(G\) vanishes unless \(2 \leq |x| \leq 3\), we may apply Lemma 4.11 to see

\[
(1 + t) \sup_{|x| < 2} \sum_{\nu \leq \mu \leq \nu_0 + 1} |L^\nu \partial^\beta \hat{w}(t, x)| \leq (1 + t) \sup_{|x| < 2} \sum_{\nu \leq \mu \leq \nu_0 + 1} |L^\nu \partial^\beta \hat{w}(t, x)|
\]

\[
\leq C \sup_{0 \leq s \leq t} \sum_{\nu \leq \mu \leq \nu_0 + 1} (1 + s) \|L^\nu \partial^\beta \hat{w}(s, \cdot)\|_{L^\infty(|x| < 3)}
\]

\[
+ C \sup_{0 \leq s \leq t} \sum_{\nu \leq \mu \leq \nu_0} \|L^\nu \hat{w}(s, \cdot)\|_{L^\infty(|x| < 3)}.
\]

We thus see that (4.23) follows from an application of (2.21). \(\square\)

4.3. **Sobolev-type estimates.** In this subsection, we state the exterior domain analogs of Lemma 2.8 that we will require. The proofs of the relevant extensions to the exterior domain can be found in [27] (Lemma 4.2 and Lemma 4.3).

**Lemma 4.12.** Suppose that \(u(t, x) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3 \setminus K)\) vanishes for \(x \in \partial K\). Then, if \(|\alpha| = M\) and \(\nu\) are fixed

\[
(4.25) \quad \|c_l t - r \|L^\nu Z^\alpha \partial^\beta u(t, \cdot)\|_2 \leq C \sum_{|\beta| + \mu \leq M + \nu + 1} \|L^\mu Z^\beta u(t, \cdot)\|_2
\]

\[
+ C \sum_{|\beta| + \mu \leq M + \nu} \|t + r \|L^\mu Z^\beta (\partial_t^2 - c_i^2 \Delta) u(t, \cdot)\|_2 + C(1 + t) \sum_{\mu \leq \nu} \|L^\mu u(t, \cdot)\|_{L^2(|x| < 2)}.
\]
and
\[(4.26) \quad r^{1/2 + \theta} \leq \ell \sum_{|\beta| + \mu \leq M + \nu + 1} \beta L^\nu Z^\alpha u(t, x) \leq C \sum_{|\beta| + \mu \leq M + \nu + 1} \|L^\nu Z^\beta u(t, \cdot)\|_2 + C \sum_{\mu \leq \nu} \|L^\mu u'(t, \cdot)\|_2 + C(1 + t) \sum_{\mu \leq \nu} \|L^\mu u'(t, \cdot)\|_{L^\infty(|x| < 2)}
\]

for any $0 \leq \theta \leq 1/2$.

5. THE CONTINUITY ARGUMENT IN THE EXTERIOR DOMAIN

In this section, we will prove the main result, Theorem 1.1. We shall take $N = 322$ in the smallness hypothesis (1.12). This can be improved considerably, but here we will take such a liberty in order to avoid unnecessary technicalities.

Our global existence theorem will be based on the following local existence result.

**Theorem 5.1.** Suppose that $f$ and $g$ are as in Theorem 1.1 with $N \geq 7$ in (1.12). Then, there is a $T > 0$ so that the initial value problem (1.1) with this initial data has a $C^2$ solution satisfying

\[u \in L^\infty([0, T]; H^N(\mathbb{R}^3 \setminus \mathcal{K})) \cap C^{0, 1}([0, T]; H^{N-1}(\mathbb{R}^3 \setminus \mathcal{K})).\]

The supremum of such $T$ is equal to the supremum of all $T$ where the initial value problem has a $C^2$ solution with $\partial^\alpha u$ bounded for all $|\alpha| \leq 2$. Also, one can take $T \geq 2$ if $\|f\|_{H^\infty} + \|g\|_{H^{N-1}}$ is small enough.

This is essentially from Keel-Smith-Sogge [15] (Theorem 9.4 and Lemma 9.6). These were only stated for diagonal single-speed systems. Since the proofs relied only on energy estimates, the results extend to the current setting provided (1.7) and (1.8) hold.

Prior to setting up the continuity argument, it is convenient to reduce to an equivalent system of nonlinear equations with vanishing Cauchy data. By doing so, we will avoid complications related to the compatibility conditions. We first reduce to an equivalent system of nonlinear equations whose data vanish in a neighborhood of the obstacle. Initially, we note that if $\varepsilon$ in (1.12) is sufficiently small, then there is a constant $C$ so that

\[(5.1) \quad \sup_{0 \leq t \leq 2} \sum_{|\alpha| \leq 2} \|\partial^\alpha u(t, \cdot)\|_{L^2(|x| \leq 1)} \leq C\varepsilon.
\]

This, again, follows from the local existence theory (see, e.g., [15]). On the other hand, over $\{t \in [0, 2] \times \{|x| \geq 6\}$, by finite propagation speed, $u$ corresponds to a solution of the boundaryless wave equation $\Box u = F(u, d_u, d^2u)$. If we take $N = 322$ in (1.14), it is clear that the analogs of (3.4) and (3.6) yield

\[(5.2) \quad \sup_{0 \leq t \leq 2} \sum_{|\alpha| + \mu \leq 331} \|L^\mu Z^\alpha u(t, \cdot)\|_{L^2(|x| \geq 6)} + \sup_{0 \leq t \leq 2} \frac{1 + t + |x|}{|F| \geq 1} \sum_{|\alpha| + \mu \leq 311} \|L^\mu Z^\alpha u(t, x)\| \leq C\varepsilon.
\]

Here we have used our assumption that $\mathcal{K} \subset \{|x| < 1\}$.
We will use this local solution to set up our reduction. First, we fix a cutoff function \( \eta \in C^\infty (\mathbb{R} \times \mathbb{R}^3) \) satisfying \( \eta(t, x) \equiv 1 \) if \( t \leq 3/2 \) and \( |x| \leq 6 \), \( \eta(t, \cdot) \equiv 0 \) for \( t > 2 \), and \( \eta(\cdot, x) \equiv 0 \) for \( |x| > 8 \). If we set
\[
\psi_0 = \eta \psi,
\]
it follows that \( \Box \psi_0 = \eta F(u, du, d^2 u) + [\Box, \eta] \psi \). Thus, \( \psi \) solves (1.1) for \( 0 < t < T \) if and only if \( \psi = \psi - \psi_0 \) solves
\[
\begin{align*}
\Box \psi &= (1 - \eta) F(u, du, d^2 u) - [\Box, \eta] \psi \\
\psi|_{\partial \mathcal{K}} &= 0 \\
\psi(0, \cdot) &= (1 - \eta)(0, \cdot) f \\
(\partial \psi)(0, \cdot) &= (1 - \eta)(0, \cdot) g - \eta f(0, \cdot) f
\end{align*}
\]
for \( 0 < t < T \).

We now fix a smooth cutoff function \( \beta \) with \( \beta(t) \equiv 1 \) for \( t \leq 1 \) and \( \beta(t) \equiv 0 \) for \( t \geq 3/2 \). If we let \( v \) be the solution of the linear equation
\[
\begin{align*}
\Box v &= \beta(1 - \eta) F(u, du, d^2 u) - [\Box, \eta] \psi \\
v|_{\partial \mathcal{K}} &= 0 \\
v(0, \cdot) &= (1 - \eta)(0, \cdot) f \\
(\partial v)(0, \cdot) &= (1 - \eta)(0, \cdot) g - \eta f(0, \cdot) f
\end{align*}
\]
we will show that there is an absolute constant so that
\[
(1 + t + |x|) \sum_{\mu + |\alpha| \leq |3|} |L^\mu Z^\alpha v(t, x)| + \sum_{\mu + |\alpha| \leq |3|} \| L^\mu Z^\alpha v'(t, \cdot) \|_2
\]
\[
+ (\log(2 + t))^{-1} \sum_{\mu + |\alpha| \leq |3|} \| \langle x \rangle^{-1/2} L^\mu Z^\alpha v' \|_{L^2(S^t)} \leq C\varepsilon
\]
where, as above, \( S_t = [0, t] \times \mathbb{R}^3 \setminus \mathcal{K} \) denotes the time strip of height \( t \).

Indeed, by (4.13), the first term on the left side of (5.5) is bounded by
\[
(1 + t + |x|) \sum_{j + |\beta| + |\alpha| \leq |3|} \| \langle x \rangle^{|j| + |\beta|} \partial_x^{|\alpha|} v(0, \cdot) \|_2
\]
\[
+ C \int_0^t \int_{|\alpha| + \mu \leq |3|} |L^\mu Z^\alpha (\beta(s)(1 - \eta)(s, y) F(u, \partial_u, \partial^2 u)(s, y)) \frac{dy}{|y|}
\]
\[
+ C \int_0^t \int_{|\beta| + |\mu| \leq |3|} \| L^\mu Z^\alpha \Box, \eta \psi \|_2 \frac{dy}{|y|}
\]
\[
+ C \int_0^t \sum_{|\beta| + |\mu| \leq |3|} \| L^\mu \partial_x^{|\beta|} \Box, \eta \psi \|_{L^2(\{x \in \mathbb{R}^3 \setminus \mathcal{K} : |x| < 2 \})} ds.
\]
It follows from (1.12) that the first term in (5.6) is \( O(\varepsilon) \). Since \([\Box, \eta] \psi \) vanishes unless \( t \leq 2 \) and \( |x| \leq 8 \), the last two terms in (5.6) are also \( O(\varepsilon) \) by (5.1). Thus, it remains to
study the second term in (5.6). This term is bounded by

\[ C \int_0^{3/2} \int_0^t \sum_{|\alpha| + \mu \leq 310} |L_{\mu} Z^{\alpha} u'(s, y)|^2 \frac{dy \, ds}{|y|} + C \int_0^{3/2} \int_0^t \sum_{|\alpha| + \mu \leq 311} |L_{\mu} Z^{\alpha} u(s, y)|^3 \frac{dy \, ds}{|y|}. \]

This is also clearly \( O(\varepsilon) \) by (5.2).

For the second term on the left of (5.5), we use the standard energy integral method (see, e.g., Sogge [37], p.12) to see that

\[ \partial_t \sum_{|\alpha| + \mu \leq 300} \| L_{\mu} Z^{\alpha} v'(t, \cdot) \|_{L^2}^2 \leq C \left( \sum_{|\alpha| + \mu \leq 300} \| L_{\mu} Z^{\alpha} v(t, \cdot) \|_{L^2}^2 \right)^{1/2} \left( \sum_{|\alpha| + \mu \leq 300} \| L_{\mu} Z^{\alpha} v(t, \cdot) \|_{L^2}^2 \right)^{1/2} + C \sum_{|\alpha| + \mu \leq 300} \left| \int_{\partial K} \partial_\nu (L_{\mu} Z^{\alpha} v(t, \cdot) \nabla L_{\mu} Z^{\alpha} v(t, \cdot) \cdot n \, d\sigma) \right|, \]

where \( n \) is the outward normal at a given point on \( \partial K \). Since \( K \subset \{|x| < 1\} \) and since \( \Box v = \beta(t)(1 - \eta) \Box u - [\Box, \eta] u \), it follows that

\[ \left( \sum_{|\alpha| + \mu \leq 300} \| L_{\mu} Z^{\alpha} v'(s, \cdot) \|_{L^2}^2 \right) \leq C \sum_{|\alpha| + \mu \leq 300} \| L_{\mu} Z^{\alpha} v'(0, \cdot) \|_{L^2}^2 \]

\[ + C \left( \int_0^t \sum_{|\alpha| + \mu \leq 300} \| L_{\mu} Z^{\alpha} (1 - \eta)(s, \cdot) F(u, du, d^2 u)(s, \cdot) \|_{L^2} \right)^2 \]

\[ + C \left( \int_0^t \sum_{|\alpha| + \mu \leq 300} \| L_{\mu} Z^{\alpha} (\Box, \eta) u(s, \cdot) \|_{L^2} \right)^2 \]

\[ + C \left( \int_0^t \sum_{|\alpha| + \mu \leq 311} \| L_{\mu} \partial_\nu v'(s, \cdot) \|_{L^2(|x| < 1)} \right)^2. \]

The first term is \( O(\varepsilon) \) by (1.12). Since \( [\Box, \eta] u \) is compactly supported in both \( t \) and \( x \), the third term in the right of (5.7) is also \( O(\varepsilon) \) by (5.1). Using the bound that we just obtained for the first term in the left of (5.5), it follows that the last term in (5.7) also satisfies the desired bound. We are left with studying the second term in (5.7). This is clearly controlled by

\[ C \left( \int_0^{3/2} \sum_{|\alpha| + \mu \leq 301} \| L_{\mu} Z^{\alpha} u'(s, \cdot) \|_{L^2(|x| > 6)} \right)^2 \]

\[ + C \left( \int_0^{3/2} \sum_{|\alpha| + \mu \leq 302} \| L_{\mu} Z^{\alpha} u(s, \cdot) \|_{L^2(|x| > 6)} \right)^2. \]

These terms are also easily seen to be \( O(\varepsilon) \) by (5.2), which establishes the estimate for the second term in (5.5).
Finally, it remains to show that the third term on the left side of (5.5) is $O(\varepsilon)$. To do so, we first notice that by (4.26) we have

$$
(5.8) \quad \sum_{|\beta|+|\alpha| \leq 298} \sum_{0 \leq t \leq \varepsilon/2} \| \partial_t^\alpha \partial x^\beta \partial^t v(t, x) \|_2 + C \sum_{|\beta|+|\alpha| \leq 298} \sum_{0 \leq t \leq 2} \| \partial_t^\alpha \partial x^\beta (\partial^t - c_1^2 \Delta) v(t, \cdot) \|_2 + C(1+t) \sum_{|\alpha| \leq 2} \| \partial^\alpha v(t, \cdot) \|_{L^\infty(\mathbb{R}^3)}.$$

for $\mu + |\alpha| \leq 298$. The first and last term on the right side of (5.8) are clearly $O(\varepsilon)$ by the bounds for the first two terms in the left side of (5.5). Since $\square v = \beta(1-\eta) \Box u - \Box \eta u$, the second term on the right of (5.8) is controlled by

$$
 C \sum_{|\beta|+|\alpha| \leq 298} \sup_{0 \leq t \leq \varepsilon/2} \| (\partial_t) \partial^\alpha \partial x^\beta v(t, \cdot) \|_{L^2(|x| \geq \varepsilon)} \sup_{0 \leq t \leq \varepsilon/2} \| \partial^\alpha v(t, \cdot) \|_{L^2(|x| \leq \varepsilon/2)} + C \sum_{|\alpha| \leq 2} \sup_{0 \leq t \leq \varepsilon/2} \| \partial^\alpha v(t, \cdot) \|_{L^2(|x| \leq \varepsilon/2)} - \quad (5.9)
$$

This is also $O(\varepsilon)$ by (5.1) and (5.2). Thus, we have

$$
(5.9) \quad \sum_{\mu + |\alpha| \leq 298} r(\partial_t^\alpha \partial x^\beta (\partial^t - c_1^2 \Delta) v(t, x)) \leq C\varepsilon.
$$

In order to use this to bound the last term on the left of (5.5), notice that we can write

$$
(5.10) \quad \sum_{\mu + |\alpha| \leq 298} \| \partial^\alpha v(t, x) \|_{L^2(|x| \leq \varepsilon/2)}^2 \leq \sum_{\mu + |\alpha| \leq 298} \int_0^\varepsilon \frac{1}{1+s} \| L^\mu \partial x^\alpha v(s, \cdot) \|_{L^2(|x| \leq \varepsilon/2)}^2 ds + C \sum_{\mu + |\alpha| \leq 298} \int_0^\varepsilon \| \partial^\alpha v(t, \cdot) \|_{L^2(|x| \leq \varepsilon/2)}^2 ds.
$$

By the bound for the second term on the right side of (5.5), the first term in (5.10) is clearly controlled by $C\varepsilon^2 \log(2+t)$. If we apply (5.9) to the second term in (5.10), assuming as in §4 that the wave speeds satisfy $0 < c_1 < c_2 < \cdots < c_D$, we see that it is controlled by

$$
C\varepsilon^2 \int_0^\varepsilon \frac{1}{1+s} \| \partial^\alpha v(t, \cdot) \|_{L^2(|x| \leq \varepsilon/2)}^2 ds.
$$

This is easily seen to be bounded by $C\varepsilon^2 (\log(2+t))^2$, which completes the proof of (5.5).

The bounds (5.5) will allow us in many instances to restrict our study to $u - v$ which is the solution of

$$
(5.11) \quad \begin{cases}
\Box (u - v) = (1-\beta)(1-\eta)F(u, du, d^2u), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathcal{K} \\
(u - v)(t, x) = 0, & x \in \partial \mathcal{K} \\
(u - v)(t, x) = 0, & t \leq 0.
\end{cases}
$$

Here, as mentioned earlier, we have vanishing Cauchy data, which allows us to avoid technical details involving the compatibility conditions.

Depending on the linear estimates we employ, at times we shall use certain $L^2$ and $L^\infty$ bounds for $u$ while at other times we shall use them for $u - v$ or $v$. Since $u = (u - v) + v + u_0$,
and \(u_0, v\) satisfy the bounds (5.1), (5.5) respectively, it will always be the case that bounds for \(w - v\) will imply those for \(w\) which in turn imply the same bounds for \(u\) and vice versa.

We are now ready to set up the continuity argument. If \(\varepsilon > 0\) is as above, we shall assume that we have a solution of our equation (1.1) for \(0 \leq t \leq T\) satisfying the following dispersive estimates

\[
(5.12) \quad (1+t + |x|) \sum_{|\alpha| \leq 2M} |Z^\alpha w^I(t,x)| \leq A_0 \varepsilon \left(1 + \log \frac{1 + t + |x|}{1 + |ct - |x|}\right)
\]

\[
(5.13) \quad (1+t + |x|) \sum_{|\alpha| + \nu \leq M} |L^\nu Z^\alpha w^I(t,x)| \leq A_1 \varepsilon (1+t)^{\delta M + 1}\varepsilon \left(1 + \log \frac{1 + t + |x|}{1 + |ct - |x|}\right)
\]

\[
(5.14) \quad (1+t + |x|) \sum_{|\alpha| + \nu \leq 2M} |L^\nu Z^\alpha w^I(t,x)| \leq A_2 \varepsilon (1+t)^{\beta M + 1}\varepsilon \left(1 + \log \frac{1 + t + |x|}{1 + |ct - |x|}\right)
\]

\[
(5.15) \quad (1+t + |x|) \sum_{|\alpha| + \nu \leq 180, \nu \leq N} |L^\nu Z^\alpha w^I(t,x)| \leq A_3 \varepsilon (1+t)^{\gamma N}\varepsilon \left(1 + \log \frac{1 + t + |x|}{1 + |ct - |x|}\right)
\]

\[
(5.16) \quad (1+t + |x|) \sum_{|\alpha| + \nu \leq 255, \nu \leq N} |L^\nu Z^\alpha w^I(t,x)| \leq A_4 \varepsilon (1+t)^{\gamma N}\varepsilon \left(1 + \log \frac{1 + t + |x|}{1 + |ct - |x|}\right)
\]

\[
(5.17) \quad (1+t + |x|) \sum_{|\alpha| \leq 300} |Z^\alpha w^I(t,x)| \leq B_1 \varepsilon
\]

for \(M = 0, 1, 2\) and \(N = 0, 1, 2, 3\), and the following energy estimates

\[
(5.18) \quad \sum_{|\alpha| + \nu \leq 250, \nu \leq N} \|L^\nu Z^\alpha w^I(t, \cdot)\|_2 \leq A_5 \varepsilon
\]

\[
(5.19) \quad \sum_{|\alpha| \leq 300} \|\delta^\alpha w^I(t, \cdot)\|_2 \leq B_2 (1+t)^{\gamma N}
\]

\[
(5.20) \quad \sum_{|\alpha| + \nu \leq 250, \nu \leq N} \|L^\nu Z^\alpha w^I(t, \cdot)\|_2 + \sum_{|\alpha| + \nu \leq 250, \nu \leq 300} \|\langle x \rangle^{-1/2} L^\nu Z^\alpha w^I\|_{L^2(S_t)} \leq B_3 \varepsilon (1+t)^{\gamma N}
\]

\[
(5.21) \quad \sum_{|\alpha| + \nu \leq 250, \nu \leq N} \|L^\nu Z^\alpha w^I(t, \cdot)\|_2 + \sum_{|\alpha| + \nu \leq 250, \nu \leq 300} \|\langle x \rangle^{-1/2} L^\nu Z^\alpha w^I\|_{L^2(S_t)} \leq B_4 \varepsilon (1+t)^{\gamma N}.
\]

As before, the \(L^2\) norms are taken over \(\mathbb{R}^3 \setminus \mathcal{K}\), and the weighted \(L^2\) norms are taken over \(S_t = [0,t] \times \mathbb{R}^3 \setminus \mathcal{K}\).
In (5.19), \( \hat{C} \) is independent of the losses \( \hat{a}_M, \hat{b}_M, \hat{c}_M, a_M, b_M, \) and \( c'_M \). The other associated losses satisfy

\[
(5.22) \quad \hat{a}_M \ll \hat{b}_M \ll \hat{c}_M \ll a_M \ll b_M \ll c'_M \ll \hat{a}_{M+1}
\]

for \( M = 1, 2, 3 \), and

\[
\hat{c} \ll \hat{a}_0 = \hat{c}_0 = a_0 \ll c'_0 \ll \hat{a}_1.
\]

It is worth noting that (5.14), (5.17), (5.18), (5.19), and (5.21) are the estimates that made up the simpler argument in the preceding paper [27]. (5.12) is the main new estimate required in order to handle the higher order terms that do not involve derivatives. The remaining estimates are technical pieces that are needed (or convenient) to make the argument work.

In the estimates (5.12)-(5.16) and (5.18), we take \( A_j = 4C_2 \) where \( j = 0, 1, \ldots, 5 \) and \( C_2 \) is the uniform constant appearing in the bounds (5.5) for \( v \). If \( \varepsilon > 0 \) is small, all of these estimates are valid for \( T = 2 \) by Theorem 5.1. With this in mind, we shall prove that for \( \varepsilon > 0 \) sufficiently small depending on \( B_1, \ldots, B_4 \)

(i.) (5.12)-(5.16) and (5.18) are valid with \( A_j \) replaced by \( A_j/2 \);

(ii.) (5.17) and (5.19)-(5.21) are a consequence of (5.12)-(5.16) and (5.18) for suitable constants \( B_j \).

By the local existence theorem, it will follow that a solution exists for all \( t > 0 \) if \( \varepsilon > 0 \) is sufficiently small. We now explore (i.) and (ii.) in the next two sections respectively.

6. Proof of (i.)

In this section, we will show step (i.) of the proof of Theorem 1.1. Specifically, we must show that (5.12)-(5.16) and (5.18) hold with \( A_j \) replaced by \( A_j/2 \) under the assumption of (5.12)-(5.21).

6.1. Preliminaries: We begin with some preliminary estimates that follow from (5.12)-(5.21).

First, we shall prove that if \( |\alpha| + \nu \leq 270, \nu \leq 2 \), then there is a constant \( \hat{b} \) so that

\[
(6.1) \quad \langle r \rangle^{1/2+\theta} \langle \xi t - r \rangle^{1-\theta} |\xi'' Z^\alpha \partial_u \hat{u}(t, x)| \leq C\varepsilon(1 + t)^{\delta_\varepsilon}, \quad 0 \leq \theta \leq 1/2
\]

for any \( 0 \leq \theta \leq 1/2 \). Additionally,

\[
(6.2) \quad \|\langle t + r \rangle L'' Z^\alpha \partial u(t, \cdot)\|_2 \leq C\varepsilon(1 + t)^{\delta_\varepsilon}.
\]

for \( |\alpha| + \nu \leq 271 \) and \( \nu \leq 2 \).
By (4.26), (6.1) follows from (5.13), (5.21), and (6.2). It, thus, suffices to show (6.2). To do so, notice that the left side can be controlled by

\[ C\varepsilon (1 + t)^{5/2} \left( 1 + \log(1 + t) \right) \sum_{|\alpha| \leq 218} \|L^\alpha u(t, \cdot)\|_2 \]

(6.3)

\[ + C\varepsilon (1 + t)^2 \left( \sum_{|\alpha| \leq 218} \|L^\alpha u(t, \cdot)\|_2^2 \right)^{1/2} \sum_{|\alpha| \leq 218} \|L^\alpha u(t, \cdot)\|_2 \].

For the first term, if we apply (5.13), we establish the bound

\[ C\varepsilon (1 + t)^{5/2} (1 + \log(1 + t)) \sum_{|\alpha| \leq 218} \|L^\alpha u(t, \cdot)\|_2. \]

The desired estimate for the first term, thus, follows from (5.21).

For the second term in (6.3), we will again apply (5.13). Since the coefficients of \( \Gamma = \{L, Z\} \) are \( O(t + r) \), it follows that this term is controlled by

\[ C\varepsilon^2 (\log(2 + t)) (1 + t)^{25/2} \left[ \sum_{|\alpha| \leq 218} \|L^\alpha u(t, \cdot)\|_2 + \|(t + r)^{-1} u(t, \cdot)\|_2 \right]. \]

The desired bound then follows from (5.12) and (5.21), thus completing the proof of (6.2).

We will argue similarly to show a lossless version of (6.1) and (6.2) that does not involve the scaling vector field \( L \). In particular, we shall prove, for \( |\alpha| \leq 218 \) and any \( 0 \leq \theta \leq 1/2 \),

(6.4)

\[ r^{1/2} \langle \ell t - r \rangle^{1 - \theta} |Z^\alpha \partial u(t, \cdot)| \leq C\varepsilon, \]

and for \( |\alpha| \leq 219 \),

(6.5)

\[ \|(t + r)^{\alpha} u(t, \cdot)\|_2 \leq C\varepsilon. \]

As before, (6.4) follows from (6.5) by (4.26), (5.17), and (5.18).

In order to show (6.5), we again expand the left side to get the bound

\[ C\varepsilon^2 (1 + \log(1 + t))^2 \sum_{|\alpha| \leq 218} \|L^\alpha u(t, \cdot)\|_2. \]

By (5.1), (5.17) and (5.18), the first term is \( O(\varepsilon) \) as desired. Applying (5.1) and (5.12) to the second term in (6.6), we see that it is dominated by

\[ C\varepsilon^2 (1 + \log(1 + t))^2 \sum_{|\alpha| \leq 218} \|L^\alpha u(t, \cdot)\|_2. \]

By (5.1) and (5.14), it follows that this term is \( O(\varepsilon) \) if \( \varepsilon > 0 \) is sufficiently small. This completes the proof of (6.5).
Notice that (6.1) and (6.4) hold when \( u \) is replaced by \( w - v \). Indeed, since \(|L^\alpha Z^\alpha \Box (w - v)| \lesssim \sum \beta_1 \| u \|_{H^{\alpha_0}} \| L^\nu Z^\alpha \Box u \|_t \), (6.2) and (6.5) hold with \( w - v \) substituted for \( u \). Thus, the appropriate versions of (6.1) and (6.4) are consequences of (4.26), (5.1), (5.5), (5.13), (5.17), (5.18) and (5.21).

6.2. Proof of (5.12): Assuming (5.12)-(5.21), we must show that (5.12) holds with \( A_0 \) replaced by \( A_0/2 \). Since the better bounds (5.5) hold for \( v \), it will suffice to show

\[
(6.7) \quad (1 + t + |x|) \sum_{|\alpha| \leq 2\theta_1} |Z^\alpha (w - v)^t(t,x)| \leq C \varepsilon^2 \left( 1 + \log \frac{1 + t + |x|}{1 + |c_{\alpha} t - |x||} \right).
\]

Fix a smooth cutoff function \( \eta_J \) satisfying \( \eta_J(s) \equiv 1 \) for \( s \in [(c_J + (\delta/2))^{-1}, (c_J - (\delta/2))^{-1}] \) with \( \delta = (1/3) \min(c_t - c_{t-1}) \), and \( \eta_J(s) \equiv 0 \) for \( s \not\in [(c_J + \delta)^{-1}, (c_J - \delta)^{-1}] \). Then, set \( \rho_J(t, x) = \eta_J(|x|^{-1} t) \). Since we may assume that \( 0 \in K \), we have that \( |x| \) is bounded below on the complement of \( K \), and the function \( \rho_J \) is smooth and homogeneous of degree 0 in \((t, x)\). Clearly, \( \rho_J \) is identically one on a conic neighborhood of \( \{|x| = c_J t\} \), and its support does not intersect any \( \{|x| = c_J t\} \) for \( J \neq I \). Let

\[
(6.8) \quad \tilde{F}^I = \sum_{1 \leq J < D, J \neq I} \rho_J \sum_{0 \leq j, k \leq 3} B_{LJ}^{IK} \partial^j u^I \partial^k u^J + \sum_{1 \leq J < D, J \neq I} \rho_J \sum_{0 \leq j, k \leq 3} A_{LJ}^{IK} \partial^j u^I \partial^k u^J,
\]

and set \( \hat{G}^I = F^I - \tilde{F}^I \).

By (4.20) and (4.21), we have that the left side of (6.7) is bounded by

\[
(6.9) \quad C \int_{\max(0, c_J t - |\tau||-1)}^{(1 + s)} \int_{|y| \leq \tau, |x| \leq 2\theta_1} |L^\nu Z^\alpha \hat{F}^I(s, y)| \frac{dy \, ds}{|y|} + C \sum_{|\alpha| \leq 2\theta_1} \sup_{0 \leq \nu \leq 1} \int_{|y| \leq \tau} |Z^\alpha \hat{F}^I(s, y)| \, dy
\]

\[
+ C \sum_{|\alpha| \leq 2\theta_1} \sup_{0 \leq \nu \leq 1} \int_{|y| \leq \tau, |x| \leq 2\theta_1} |L^\nu Z^\alpha \hat{F}^I(\tau, y)| \frac{dy \, d\tau}{|y|} + C \sum_{0 \leq \nu \leq 1} (1 + s) \sum_{|\alpha| \leq 2\theta_1} \| \partial^\alpha \hat{F}^I(s, \cdot) \|_{L^\infty(|x| \leq 1)}
\]

\[
+ C \left( 1 + \log \frac{1 + t + |x|}{1 + |c_{\alpha} t - |x||} \right) \sup_{(\tau, y)} |y|(1 + s + |y|)^{1 + \nu - 1 - \nu} (s, |y|) \sum_{|\alpha| \leq 2\theta_1} |Z^\alpha \hat{G}^I(s, y)|
\]

\[
+ C \left( 1 + \log \frac{1 + t + |x|}{1 + |c_{\alpha} t - |x||} \right) \sup_{(\tau, y)} |y|(1 + s + |y|)^{1 + \nu - 1 - \nu} (s, |y|) \sum_{|\alpha| \leq 2\theta_1} |\partial^\alpha \partial \hat{G}^I(s, y)|.
\]

We need to show that each of these terms is bounded by \( C \varepsilon^2 \left( 1 + \log \frac{1 + t + |x|}{1 + |c_{\alpha} t - |x||} \right) \).
For the first term in (6.9), it follows immediately that we have the bound
\[
C \int_{\max(0,|\xi_{y_{y_t}} - |x||^{-1})}^{t} \frac{1}{1 + s} \int_{|y| \in \Omega} \sum_{\nu \leq 205, \nu \leq 1} |L^\nu Z^\alpha \partial u^J|^2 \, dy \, ds
\]
\[
\leq C \left(1 + \log \frac{1 + t}{1 + |x_t - |x||} \right) \sup_{0 \leq s \leq t} \|L^\nu Z^\alpha u^J(s, \cdot)\|_2^2.
\]

The desired bound follows from (5.1) and (5.18). The third term in (6.9) can be handled quite similarly. The second term above is easily seen to be \(O(\varepsilon^2)\) by the Schwarz inequality, (5.1), and (5.18). The fourth term above is bounded by
\[
\sup_{0 \leq s \leq t} (1 + t) \sum_{|\alpha| \leq 103} |\partial^\alpha \partial u^J| \sum_{|\alpha| \leq 205} |\partial^\alpha \partial u^J| \|_\infty.
\]

If we apply (5.1) and (5.17), this is controlled by
\[
C \varepsilon \sup_{0 \leq s \leq t} \|\partial^\alpha u^J\|_\infty.
\]

Thus, by Sobolev’s lemma, (5.1), and (5.18), we see that this term is also \(O(\varepsilon^2)\).

It remains to show that
\[
(6.10) \sup_{(s, y)} |y|(1 + s + |y|)^{1+\mu} z^{1-\mu} |y| \sum_{|\alpha| \leq 3} Z^\alpha \hat{G}^I(s, y) + \sum_{|\alpha| \leq 204} \|\partial^\alpha \partial \hat{G}^I(s, y)\|
\]
is \(O(\varepsilon^2)\).

When \(\hat{G}^I\) is replaced by the null forms
\[
\sum_{0 \leq j, k \leq 3} A_{ij}^{jk} \partial_j u^I \partial_k u^J + \sum_{0 \leq j, k, \ell \leq 3} B_{ij}^{jk, \ell} \partial_j u^I \partial_k \partial_{\ell} u^J,
\]
we can apply (2.25) and (2.26) to see that (6.10) is controlled by
\[
(6.11) \sup_{(s, y)} |y|(1 + s + |y|)^{1+\mu} z^{1-\mu} |y| \sum_{|\alpha| \leq 207} |L^\nu Z^\alpha u^J(s, y)| \sum_{|\alpha| \leq 206} |Z^\alpha \partial u^J(s, y)| + \varepsilon \sup_{(s, y)} |y|(1 + s + |y|)^{1+\mu} z^{1-\mu} |y| \sum_{|\alpha| \leq 203} |Z^\alpha \partial u^J(s, y)| \sum_{|\alpha| \leq 206} |Z^\alpha \partial u^J(s, y)|.
\]

For the first term, if we apply (5.1) and (5.14), we get the bound
\[
C \varepsilon \sup_{(s, y)} |y|(1 + s + |y|)^{1+\mu} z^{1-\mu} |y| \sum_{|\alpha| \leq 206} |Z^\alpha \partial u^J|.
\]

If \(\mu\) and \(\varepsilon\) are sufficiently small, the desired \(O(\varepsilon^2)\) bound follows from (6.1). Indeed, if \((s, |y|) \in \Lambda_I\), then \(z(s, |y|) = \langle c_I s - |y| \rangle\) and \(|y| \geq (1 + s + |y|)\). On the other hand, if \((s, |y|) \not\in \Lambda_I\), then \(\langle c_I s - |y| \rangle \geq (s + |y|)\) and \(|y|^{1-\mu} \geq z^{1-\mu}(s, |y|)\).
For the second term in (6.11), we apply (6.4) to obtain the bound
\[ C \varepsilon \sup_{(t,y)} |y|^{1/2} \left( 1 + s + |y| \right) (1 + s, |y|) \sum_{|\alpha| \leq 2^{06}} |Z^\alpha \partial u^j|. \]

Using considerations as above, this is \( O(\varepsilon^2) \) by a subsequent application of (6.4).

When \( \tilde{G}^I \) is replaced by
\[
\sum_{1 \leq J \leq D} \sum_{J \neq \tilde{A}} \left( 1 - \rho_J \right) B_{J,\lambda} I^I_{j,k} \partial_{\lambda} \partial_{j,k} u^I + \sum_{1 \leq J \leq D} \sum_{J \neq \tilde{A}} \left( 1 - \rho_J \right) A_{J,\lambda} I^I_{j,k} \partial_{j} \partial_{k} u^I,
\]
(6.10) is dominated by
\[ C \sum_{1 \leq J \leq D} \sup_{(t,y) \in \Lambda_J} |y| \left( 1 + s + |y| \right) (1 + s, |y|) \left( \sum_{|\alpha| \leq 2^{06}} |Z^\alpha \partial u^I| \right)^2. \]

Since \( \langle e_J s - |y| \rangle^{1 - \mu} \geq \langle s + |y| \rangle^{1 - \mu} \geq z^{1 - \mu}(s, |y|) \) on the support of \( (1 - \rho_J) \), these terms are \( O(\varepsilon^2) \) by two applications of (6.4) with \( \theta = 0 \).

If \( \tilde{G}^I \) in (6.10) is replaced by the remaining quadratic terms
\[
\sum_{1 \leq J \leq D} \sum_{J \neq \tilde{A}} B_{J,\lambda} I^I_{j,k} \partial_{\lambda} \partial_{j,k} u^I + \sum_{1 \leq J \leq D} \sum_{J \neq \tilde{A}} A_{J,\lambda} I^I_{j,k} \partial_{j} \partial_{k} u^I,
\]
we see that it is bounded by
\[ C \sum_{1 \leq J \leq D} \sup_{(t,y) \in \Lambda_J} |y| \left( 1 + s + |y| \right) (1 + s, |y|) \left( \sum_{|\alpha| \leq 2^{06}} |Z^\alpha \partial u^I| \right) \left( \sum_{|\alpha| \leq 2^{06}} |Z^\alpha \partial u^K| \right). \]

If \( (s, y) \notin \Lambda_J \cup \Lambda_K \), then we can argue as in the previous case to see that this is \( O(\varepsilon^2) \). Thus, let us assume that \((s, y) \in \Lambda_J \), and hence \( (s, y) \notin \Lambda_K \). The reverse case will follow symmetrically. For such \((s, y)\), we have \( z^{1 - \mu}(s, |y|) = \langle e_J s - |y| \rangle^{1 - \mu} \). Thus, by (6.4), we see that in this case (6.12) is controlled by
\[ C \varepsilon \sum_{1 \leq K \leq D} \sup_{(t,y) \in \Lambda_J} |y|^{1/2} \left( 1 + s + |y| \right) (1 + s, |y|) \sum_{|\alpha| \leq 2^{06}} |Z^\alpha \partial u^I|. \]

Since \( \langle e_K s - |y| \rangle \gtrsim (s + |y|) \) and \( |y| \approx s \) on \( \Lambda_J \), the desired \( O(\varepsilon^2) \) bound follows from (6.4).

Finally, when \( \tilde{G}^I \) is replaced by the cubic terms \( R^I + P^I \), (6.10) is dominated by
\[ C \sup_{(t,y)} |y| \left( 1 + s + |y| \right) (1 + s, |y|) \left( \sum_{|\alpha| \leq 2^{04}} |Z^\alpha u(s, y)| \right)^3 + C \sup_{(t,y)} |y| \left( 1 + s + |y| \right) (1 + s, |y|) \left( \sum_{|\alpha| \leq 2^{04}} |Z^\alpha u(s, y)| \right)^2 \sum_{|\alpha| \leq 2^{06}} |Z^\alpha \partial u(s, y)|. \]

By the inductive hypothesis (5.12), the first term in (6.13) is controlled by
\[ C \varepsilon^3 \sup_{(t,y)} \frac{z^{1 - \mu}(s, |y|)}{1 + s + |y|} \left( 1 + \log \frac{1 + s + |y|}{z(s, |y|)} \right)^3. \]
Since \((\log x)^3/x^{1-\mu}\) is bounded for \(x \geq 1\) and \(\mu < 1\), it follows that this term is \(O(\varepsilon^3)\).

For the second term in (6.13), by (6.4), we have the bound

\[
C\varepsilon \sup_{(t,y)} |y|^{1/2-\mu}(1 + s + |y|)^{1+\mu} \left( \sum_{|a| \leq 104} |Z^a u(s, y)| \right)^2.
\]

This term is then easily seen to be \(O(\varepsilon^3)\) by (5.1) and (5.12) which completes the proof.

6.3. **Proof of (5.13):** In this section, we show that if you assume (5.12)-(5.21), then you can prove (5.13) with \(A_1\) replaced by \(A_1/2\). By the arguments in the previous section, this clearly holds when \(M = 0\). As before, by (5.5), it suffices to show

\[
(6.14) \quad \left( 1 + \log \frac{1 + t + |x|}{1 + c(t - |x|)} \right)^{-1} \sum_{|a| + \nu \leq 190} \frac{|L'' Z^a (w - v)^I(t, x)|}{|y|} \leq C\varepsilon^2 (1 + t^{\delta_1 M + \varepsilon}).
\]

Since \(\Box (w - v) = (1 - \beta)(1 - \eta)\Box u = (1 - \beta)(1 - \eta)(B + Q + R + P)\), by (4.13) and (4.21), we see that the left side of (6.14) is dominated by

\[
(6.15) \quad C \int_0^t \int \sum_{|a| + \nu \leq 190} \frac{|L'' Z^a (B^I + Q^I)(s, y)|}{|y|} \frac{dy\, ds}{|y|} + C \sup_{(t,y)} |y(1 + s + |y|)^{1+\mu} z^{-\mu}(s, |y|) \sum_{|a| + \nu \leq M} |L'' Z^a (R^I + P^I)(s, y)|
\]

\[
+ C \sup_{(t,y)} |y(1 + s + |y|)^{1+\mu} z^{-\mu}(s, |y|) \sum_{|a| + \nu \leq 195} |L'' \partial^\alpha \partial (R^I + P^I)(s, y)|.
\]

Here we have used the fact that the last term in (4.13) is controlled by the second term in the right of (4.13) using Sobolev estimates and the fact that we may assume \(0 \in \mathcal{K}\) without loss of generality.

By (1.4) and (1.5), we have that the first term in (6.15) is dominated by

\[
C \sum_{|a| + \nu \leq 198} ||(x)^{-1/2} L'' Z^a u' ||_{L^2(S_1)}^2.
\]

It, thus, follows from (5.20) that these terms are bounded by \(C\varepsilon^2 (1 + t)^{\delta_2 M + \varepsilon}\) as desired.
The last two terms of (6.15) are controlled by a constant times
\begin{equation}
(6.16) \sup_{(s,y)} \left| y \right| (1 + s + |y|)^{1+\mu} z^{1-\beta}(s, |y|) \left( \sum_{|\alpha| + \nu < 190} \left| Z^\alpha u(s, y) \right| \right)^2 \sum_{|\alpha| + \nu \leq M} \left| L^\nu Z^\alpha u(s, y) \right|
\end{equation}
\begin{align*}
+ \sup_{(s,y)} y(1 + s + |y|)^{1+\mu} z^{1-\beta}(s, |y|) \left( \sum_{|\alpha| + \nu < 190} \left| L^\nu Z^\alpha u(s, y) \right| \right)^3 \\
+ \sup_{(s,y)} y(1 + s + |y|)^{1+\mu} z^{1-\beta}(s, |y|) \left( \sum_{|\alpha| + \nu < 190} \left| L^\nu Z^\alpha u(s, y) \right| \right) \sum_{|\alpha| + \nu < 195} \left| L^\nu Z^\alpha \partial u(s, y) \right|.
\end{align*}

For the first term in (6.16), we apply (5.1), (5.12), and (5.13) to obtain the bound
\begin{equation}
C\varepsilon^3 (1 + t)^{\delta_{M+\varepsilon}} \sup_{(s,y)} \left| z^{1-\beta}(s, |y|) \left( 1 + \log \frac{1 + s + |y|}{z(s, |y|)} \right)^3 \right| \leq C\varepsilon^3 (1 + t)^{\delta_{M+\varepsilon}}.
\end{equation}

If we apply (5.13) and argue similarly, it follows that the second term is controlled by
\begin{equation}
C\varepsilon^3 (1 + t)^{\delta_{M+\varepsilon}}. \text{ Finally, for the third term in (6.16), we first apply (6.1) to see that it is controlled by}
\end{equation}
\begin{equation}
C\varepsilon \sup_{(s,y)} \left| y \right|^{1/2-\beta}(1 + s + |y|)^{1+\mu} (1 + s)^{\delta_{\varepsilon}} \left( \sum_{|\alpha| + \nu \leq M} \left| L^\nu Z^\alpha u(s, y) \right| \right)^2.
\end{equation}

It, thus, follows from the inductive hypothesis (5.13) that this term is $O(\varepsilon^3)$ if $\varepsilon > 0$ is sufficiently small, which completes the proof of (5.13).

6.4. Proof of (5.14): In this section, by proving
\begin{equation}
(6.17) \left( 1 + \log \frac{1 + t + x}{1 + |x|} \right)^{-1} (1 + t + |x|) \sum_{|\alpha| + \nu \leq 255} \left| L^\nu Z^\alpha (w - v)^I(t, x) \right| \leq C\varepsilon^3 (1 + t)^{\delta_{M+\varepsilon}},
\end{equation}
we show that (5.14) holds with $A_2$ replaced by $A_2/2$ for $M = 0, 1, 2$.

Here, again, we apply (4.13) and (4.21) to see that the left side of (6.17) is controlled by
\begin{equation}
(6.18) \quad C \sum_{|\alpha| + \nu < 256} \int_0^1 \int \left| L^\nu Z^\alpha (Q^I + B^I)(s, y) \right| \frac{dy \, ds}{|y|} \\
+ C \sup_{(s,y)} \left| y \right| (1 + s + |y|)^{1+\mu} z^{1-\beta}(s, |y|) \sum_{|\alpha| + \nu < 255} \left| L^\nu Z^\alpha (R^I + P^I)(s, y) \right| \\
+ C \sup_{(s,y)} \left| y \right| (1 + s + |y|)^{1+\mu} z^{1-\beta}(s, |y|) \sum_{|\alpha| + \nu < 258} \left| L^\nu \partial \partial (R^I + P^I)(s, y) \right|.
\end{equation}
The first term is controlled by

\[ C \sum_{|\alpha|+\nu \leq 263} \| (x)_{1/3}^{1/3} L^\nu Z^\alpha u' \|_{L^2(S_i)}^2 \leq C \varepsilon^2 (1 + t)^{2M+\varepsilon} \]

by (5.21). For the last two terms in (6.18), which involve the cubic nonlinearities, we have the bound

\[ C \sup_{(s, y)} |y| (1 + s + |y|)^{1+\mu} z^{1-\mu} (s, |y|) \left( \sum_{|\alpha| \leq 131} |Z^\alpha u(s, y)|^2 \right)^{1/2} \sum_{|\alpha|+\nu \leq 255} |L^\nu Z^\alpha u(s, y)| + C \sup_{(s, y)} |y| (1 + s + |y|)^{1+\mu} z^{1-\mu} (s, |y|) \left( \sum_{|\alpha|+\nu \leq 131} |L^\nu Z^\alpha u(s, y)|^2 \right)^{1/2} \sum_{|\alpha|+\nu \leq 255} |L^\nu Z^\alpha u'(s, y)|.

Using the fact that \((\log x)^3 / x^{1-\mu}\) is bounded for \(x > 1\) and \(\mu < 1\), the first term is dominated by \(C \varepsilon^3 (1 + t)^{2M+\varepsilon}\) by (5.12) and (5.14). Similarly, using (5.13) and (5.14), the second term is controlled by \(C \varepsilon^3 (1 + t)^{2M+\varepsilon+\delta M\varepsilon}\). Again arguing as in the proof of (5.13), the final term in (6.19) is easily seen to be \(O(\varepsilon^3)\) if \(\varepsilon > 0\) is sufficiently small using (6.1) and (5.13). This completes the proof of (6.17), and hence, that of (5.14).

6.5. Proof of (5.15): In the proof of part (iii.), we will require pointwise estimates that allow up to three occurrences of the scaling vector field \(L\). This is not the case for the previous estimates due to the loss of an \(L\) associated to (4.13). Here, we may argue as in the proofs of the previous estimates (in particular, that of (5.13)) replacing (4.13) by (4.18).

Clearly (5.15) holds when \(N = 0\) by (5.12). Thus, by (5.5), in order to show that (5.15) holds with \(A_3\) replaced by \(A_3/2\), it suffices to show

\[ (6.20) \left( 1 + \log \frac{1 + t + |x|}{1 + |ct - x|} \right)^{-1} (1 + |x|) \sum_{|\alpha|+\nu \leq 188} |L^\nu Z^\alpha (w - v)'(t, x)| \leq C \varepsilon^2 (1 + t)^{2N\varepsilon} \]

for \(N = 1, 2, 3\).
Since \( \Box (u - v) = (1 - \beta)(1 - \eta)\Box u = (1 - \beta)(1 - \eta)(B + Q + R + K) \), by (4.18) and (4.21), we see that the left side of (6.20) is controlled by

\[
\begin{align*}
&\int_0^t \int \sum_{|\alpha| + \nu \leq N} |L^\nu Z^\alpha (B^I + Q^I) (s, y)| \left\| \frac{d y}{d s} \right\| \\
&\quad + C \sup_{(s, y)} |y| (1 + s + |y|)^{1 + \mu} z^{1 - \mu} (s, |y|) \sum_{|\alpha| + \nu \leq N} |L^\nu Z^\alpha (R^I + P^I) (s, y)| \\
&\quad + C \sup_{(s, y)} |y| (1 + s + |y|)^{1 + \mu} z^{1 - \mu} (s, |y|) \sum_{|\alpha| + \nu \leq N} |L^\nu u^I (s, y)|.
\end{align*}
\]

As above, using (5.20), the first term is bounded by

\[
C \sum_{|\alpha| + \nu \leq N} \left\| \langle x \rangle^{-1/2} L^\nu Z^\alpha u^I \right\|_{L^2 (S_t)}^2 \leq C \varepsilon^2 (1 + t)^{\delta_N}. \tag{6.21}
\]

The cubic terms require a little additional care. To begin, we have that the last two terms of (6.21) are controlled by

\[
\begin{align*}
&\sup_{(s, y)} |y| (1 + s + |y|)^{1 + \mu} z^{1 - \mu} (s, |y|) \left( \sum_{|\alpha| + \nu \leq N} |Z^\alpha u (s, y)| \right)^2 \sum_{|\alpha| + \nu \leq N} |L^\nu Z^\alpha u (s, y)| \\
&\quad + C \sup_{(s, y)} |y| (1 + s + |y|)^{1 + \mu} z^{1 - \mu} (s, |y|) \left( \sum_{|\alpha| + \nu \leq N} |Z^\alpha u (s, y)| \right)^2 \sum_{|\alpha| + \nu \leq N} |L^\nu Z^\alpha u' (s, y)| \\
&\quad + C \sup_{(s, y)} |y| (1 + s + |y|)^{1 + \mu} z^{1 - \mu} (s, |y|) \left( \sum_{|\alpha| + \nu \leq N} |L^\nu Z^\alpha u (s, y)| \right)^3.
\end{align*}
\]

Applying (5.1) and (5.15) to the first term and using (2.27) and (5.20) in the second, we see that the first two terms of (6.22) are controlled by

\[
C \varepsilon (1 + t)^{\tilde{\delta}_N} \sup_{(s, y)} |y| (1 + s + |y|)^{1 + \mu} z^{1 - \mu} (s, |y|) \left( 1 + \log \frac{1 + s + |y|}{z (s, |y|)} \right) \left( \sum_{|\alpha| + \nu \leq N} |Z^\alpha u (s, y)| \right)^2.
\]

Here, we have used that (5.22) gives \( \tilde{\delta}_N \leq \delta_N \). If we in turn apply (5.12), we see that this is bounded by the right side of (6.20) as desired. When \( N = 1 \), this is sufficient to complete the proof. When \( N = 2, 3 \), we must also consider the last term in (6.22). The bound here, however, follows quite simply from three applications of (5.13) (and (5.1)). Doing so, we see that this last term is controlled by \( C \varepsilon^3 (1 + t)^{3\delta_N} \). Since we may choose \( \tilde{\delta}_N > 3 \delta_N \) (see (5.22)), this is sufficient to complete the proof of (6.20).

6.6. Proof of (5.16): In this section, we will argue much as in the previous section to establish the higher order pointwise estimate that permits three occurrences of \( L \). Here,
we must establish (5.16) with $A_4$ replaced by $A_4/2$. This is accomplished by showing

\[ (6.23) \quad \left( 1 + \log \frac{1+|t|}{1+|ct-t|-|x|} \right)^{-1} (1 + |x|) \sum_{|\alpha|+\nu \leq 255 \atop \nu \leq N} |L^n Z^\alpha (w-v)'(t, x)| \leq C\varepsilon^2 (1+|t|)^{3}\varepsilon\]

for $N = 0, 1, \ldots, 4$ and using (5.5).

Applying (4.18) and (4.21) and arguing as in the proof of (5.15), we see that the left side of (6.23) is dominated by

\[ (6.24) \quad C \sum_{|\alpha|+\nu \leq 255 \atop \nu \leq N} \| \langle x \rangle^{-1/2} L^n Z^\alpha u' \|^2_{L^2(S_1)} \]

\[ + C \sup_{(s,y)} |y| (1 + s + |y|) \sum_{|\alpha| \leq 132} |Z^\alpha u(s,y)| \left( \sum_{|\alpha|+\nu \leq 255 \atop \nu \leq N} |L^n Z^\alpha u(s,y)| \right)^2 \sum_{|\alpha|+\nu \leq 255 \atop \nu \leq N} |L^n Z^\alpha u(s,y)| \]

\[ + C \sup_{(s,y)} |y| (1 + s + |y|) \sum_{|\alpha| \leq 132} |Z^\alpha u(s,y)| \left( \sum_{|\alpha|+\nu \leq 255 \atop \nu \leq N-1} |L^n Z^\alpha u(s,y)| \right)^2 \sum_{|\alpha|+\nu \leq 255 \atop \nu \leq N-1} |L^n Z^\alpha u(s,y)| \]

\[ + C \sup_{(s,y)} |y| (1 + s + |y|) \sum_{|\alpha| \leq 132} |Z^\alpha u(s,y)| \sum_{|\alpha|+\nu \leq 255 \atop \nu \leq N-1} |L^n Z^\alpha u(s,y)| \sum_{|\alpha|+\nu \leq 255 \atop \nu \leq N-1} |L^n Z^\alpha u(s,y)|. \]

Choosing $\sigma_N > \max(2a_N, 2b_N + \hat{e}_N)$ as we may, we see that this is bounded by the right side of (6.23). Indeed, the bound for the first term in (6.24) follows directly from (5.21). For the second term, we apply (5.1), (5.12), and (5.16) as before. This suffices to handle the $N = 0$ case. In order to complete the proof for $N = 1, 2, 3$, we similarly, bound the third term using applications of (2.27), (5.1), (5.12), and (5.21). To get control over the fourth term, we apply (5.1), (5.14), and (5.15). Using (5.1), (5.13), (5.15), and (6.1), one can see that the last term is $O(\varepsilon^3)$ for small $\varepsilon$ which completes the proof of (6.23).

6.7. Proof of (5.18): In order to complete the proof of part (i.), it remains to show that the low order, lossless energy inequality (5.18) with $A_5$ replaced by $A_5/2$ follows from (5.12)-(5.21). Since $v$ satisfies the better bound (5.5), it suffices to establish

\[ (6.25) \quad \sum_{|\alpha|+\nu \leq 258 \atop \nu \leq 1} \| L^n Z^\alpha (w-v)'(t, \cdot) \|^2_{L^2} \leq C\varepsilon^3. \]
By the standard energy integral method, we have that the left side of (6.25) is bounded by

$$
C \sum_{|\alpha|+\nu \leq 220} \int_0^T \int_{\mathbb{R}^N \setminus K} |\partial_0 L^\nu Z^\alpha (w - v), \Box L^\nu Z^\alpha (w - v)| \, dy \, ds
$$

$$
+ C \sum_{|\alpha|+\nu \leq 220} \left| \int_0^T \int_{\partial K} \partial_0 L^\nu Z^\alpha (w - v) \nabla_x L^\nu Z^\alpha (w - v) \cdot \mathbf{n} \, d\sigma \, ds \right|
$$

where \( \mathbf{n} \) is the outward normal at a given point on \( K \) and \( \langle \cdot, \cdot \rangle \) is the standard Euclidean inner product on \( \mathbb{R}^D \). Since \( K \subseteq \{|x| < 1\} \) and since the coefficients of \( Z \) are \( O(1) \) on \( \partial K \), it follows that the last term in controlled by

$$
C \int_0^T \int_{\{x \in \mathbb{R}^N \setminus K: |x| < 1\}} \sum_{|\alpha|+\nu \leq 221} |L^\nu Z^\alpha (w - v) (s, y)|^2 \, dy \, ds.
$$

Additionally, by the commutation properties of \( \Gamma \) with \( \Box \) and the fact that \( \Box (w - v) = (1 - \beta)(1 - \eta) \Box u \), we see that the left side of (6.25) is dominated by

$$
C \int_0^T \int_{\mathbb{R}^N \setminus K} \sum_{|\alpha_j + \nu_j \leq 220 \atop \nu_j \leq 1; j = 1, 2} \left| \langle \partial_0 L^\nu \alpha Z^\alpha (w - v), L^\nu Z^\alpha (w - v) \rangle \right| \, dy \, ds
$$

$$
+ C \int_0^T \int_{\mathbb{R}^N \setminus K: |x| < 1} \sum_{|\alpha|+\nu \leq 221} |L^\nu Z^\alpha (w - v) (s, y)|^2 \, dy \, ds.
$$
If we expand using the definition of \( F(u, du, \hat{d}u) \), the preceding equation is controlled by

\[
(6.26) \quad C \int_0^t \int_{\mathbb{R}^n} \sum_{K=1}^D \sum_{\alpha+\nu \leq 220}^{\nu < 1} \left| \partial_0 L^\nu Z^\alpha (w - v)^K \right| \\
\times \sum_{\alpha+\nu \leq 220}^{\nu < 1} \left| \sum_{\beta+\mu \leq 220}^{\mu < 1} \tilde{B}_{K,\beta} \partial_0 L^\nu Z^\alpha u^K \partial_j \partial_k L^\mu Z^\beta u^K \right| dy \, ds \\
+ C \int_0^t \int_{\mathbb{R}^n} \sum_{K=1}^D \sum_{\alpha+\nu \leq 220}^{\nu < 1} \left| \partial_0 L^\nu Z^\alpha (w - v)^K \right| \\
\times \sum_{\alpha+\nu \leq 220}^{\nu < 1} \left| \sum_{\beta+\mu \leq 220}^{\mu < 1} \partial_0 L^\nu Z^\alpha u^K \right| dy \, ds \\
+ C \int_0^t \int_{\mathbb{R}^n} \sum_{K=1}^D \sum_{\alpha+\nu \leq 220}^{\nu < 1} \left| L^\nu Z^\alpha \partial(w - v)^K \right| \\
\times \sum_{\alpha+\nu \leq 220}^{\nu < 1} \left| L^\nu Z^\alpha \partial u^K \right| dy \, ds \\
+ C \int_0^t \int_{\mathbb{R}^n} \sum_{K=1}^D \sum_{\alpha+\nu \leq 220}^{\nu < 1} \left| L^\nu Z^\alpha \partial(w - v)^K \right| \\
\times \sum_{\alpha+\nu \leq 220}^{\nu < 1} \left| L^\nu Z^\alpha \partial u^K \right|^3 dy \, ds \\
+ C \int_0^t \int_{|x| < 1} \left| L^\nu Z^\alpha \partial(w - v)^K \left( \sum_{\alpha+\nu \leq 220}^{\nu < 1} \left| L^\nu Z^\alpha u^K \right| \right)^3 \right| dy \, ds \\
+ C \int_0^t \int_{|x| < 1} \left| L^\nu Z^\alpha \partial(w - v)^K \left( \sum_{\alpha+\nu \leq 220}^{\nu < 1} \left| L^\nu Z^\alpha u^K \right| \right)^2 \right| dy \, ds.
\]

By Lemma 4.1 of Sideris-Tu [35], the constants \( \tilde{A}_{K,K}^{\nu,j} \) and \( B_{K,j}^{
u,K} \) satisfy (1.9) and (1.10).

The first two terms in (6.26) satisfy the bounds of Lemma 2.6. The third term involves quadratic interactions between waves of different speeds, and the fourth term is the cumulative effect of the nonlinearities of higher order. The arguments to bound the first three terms and the final term follow from those in [27]. For completeness, we sketch the argument.

Let us first handle the null terms. By (2.25) and (2.26), the first two terms in (6.26) are controlled by

\[
(6.27) \quad C \int_0^t \int_{\mathbb{R}^n} \sum_{K=1}^D \sum_{\alpha+\nu \leq 220}^{\nu < 3} \left| L^\nu Z^\alpha \partial(w - v)^K \right| \\
\times \sum_{\alpha+\nu \leq 220}^{\nu < 1} \left| L^\nu Z^\alpha \partial u^K \right| \frac{dy \, ds}{|y|} \\
+ C \int_0^t \int_{\mathbb{R}^n} \sum_{K=1}^D \sum_{\alpha+\nu \leq 220}^{\nu < 1} \left| L^\nu Z^\alpha \partial(w - v)^K \right| \\
\times \sum_{\alpha+\nu \leq 220}^{\nu < 1} \left| L^\nu Z^\alpha \partial u^K \right|^2 dy \, ds.
\]
To handle the first term of (6.27), notice that by (5.1), (5.5), and (5.14), we have
\[ \sum_{a, \nu \leq 221} |L^\nu Z^a u(s, y)| \leq C \varepsilon (s + |y|)^{-1 + b_2 \varepsilon} \log(2 + s), \]
which means that the first term of (6.27) has a contribution to (6.26) which is dominated by
\[
C \varepsilon \int_0^1 \frac{\log(2 + s)}{(s)^{1-b_2 \varepsilon}} \sum_{a, \nu \leq 221} \|\langle y \rangle^{-1/2} L^\nu Z^a u'(s, \cdot)\|_2 \times \sum_{|a|+\nu \leq 221} \|\langle y \rangle^{-1/2} L^\nu Z^a (w - v)'(s, \cdot)\|_2 \, ds
\]
by the Schwarz inequality. Thus, if we subsequently apply the Schwarz inequality, (5.1), (5.5), and (5.21), we see that this contribution is \( O(\varepsilon^3) \) for \( \varepsilon > 0 \) small.

We now want to show that the second term of (6.27) satisfies a similar bound. If we apply (6.1), we see that the second term of (6.27) is controlled by
\[
C \varepsilon \int_0^1 (1 + s)^{(1/2) + b_2} \int_{\mathbb{R}^3 \setminus \mathbb{K}} \frac{1}{(s + |y|)^{1/2}} \sum_{a, \nu \leq 221} |L^\nu Z^a u'|^2 \, dy \, ds.
\]
For \( \varepsilon \) sufficiently small, it follows similarly that this term is \( O(\varepsilon^3) \) by the \( L^2 \) estimates of (5.21).

For the multi-speed terms (i.e., the third term in (6.26)), let us for simplicity assume that \( I \neq K, I \neq J \). A symmetric argument will yield the same bound for the remaining cases. If we set \( \delta < |c_I - c_K| / \beta + \beta, \) it follows that \( \{ |y| \in [(c_I - \delta)s, (c_I + \delta)s] \} \cap \{ [(c_K - \delta)s, (c_K + \delta)s] \} = \emptyset. \) Thus, it will suffice to show the bound when the spatial integral is taken over the complements of each of these sets separately. We will show the bound over \( \{ |y| \notin [(c_K - \delta)s, (c_K + \delta)s] \}. \) A symmetric argument will yield the bound over the other set.

If we apply (6.1), we see that over \( \{ |y| \notin [(c_K - \delta)s, (c_K + \delta)s] \} \) the third term in (6.26) is bounded by
\[
C \varepsilon \int_0^1 \frac{1}{(s)^{(1/2) + b_2}} \int_{\mathbb{R}^3 \setminus \mathbb{K}} \frac{1}{(s + |y|)^{1/2}} \sum_{a, \nu \leq 221} |L^\nu Z^a \partial u'|^2 \, dy \, ds.
\]
Arguing as above, it is easy to see that these multiple speed quadratic terms are also \( O(\varepsilon^3) \) by (5.21).

Next, we need to show that the last term in (6.26) enjoys an \( O(\varepsilon^4) \) contribution. This is clear, however, since this term is bounded by
\[
\int_0^1 \sum_{a, \nu \leq 221} \|L^\nu \partial^a (w - v)'(s, \cdot)\|_\infty^2 \, ds,
\]
and an application of (6.17) yields the desired bound.
In order to finish the proof of (6.25), and hence that of part (i), it remains to bound the cubic terms in (6.26). By (5.1) and (5.14), it follows that this fourth term of (6.26) is controlled by

\[ C \varepsilon^3 \int_0^t \int_{B \setminus B_{\varepsilon^2}} \frac{(\log(2 + s))^3 (1 + s)^{3b} \varepsilon}{(1 + s + |y|)^3} \sum_{|\alpha + \nu| \leq 2|\alpha|} \int B^{\alpha, \nu} \partial (w - v) \, dy \, ds. \]

By the Schwarz inequality, (5.5), and (5.18), this last term is also \( O(\varepsilon^4) \) if \( \varepsilon \) is sufficiently small. Thus, we have shown (6.25) and have finished the proof of (i).

7. Proof of (ii.)

We now begin part (ii.) of the continuity argument. In particular, we need to show that (5.17), (5.19), (5.20), and (5.21) follow from (5.12)-(5.16) and (5.18). This will complete the proof of Theorem 1.1.

7.1. Proof of (5.17): In this subsection, we briefly prove that (5.17) holds. Indeed, if \((t, x) \notin A_I\), it follows that \( \langle \eta(t - |x|) \rangle \gtrsim (t + |x|) \). Thus, away from the associated light cone, we have that \( 1 + \log \frac{1 + t + |x|}{1 + |t| - |x|} \) is \( O(1) \). Hence, by (5.12), we have

\[ (1 + t + |x|) \sum_{|\alpha| \leq 2|\alpha|} |Z^{\alpha} \partial w^I(t, x)| \leq C \varepsilon \]

when \((t, x) \notin A_I\). For \((t, |x|) \in A_I\), it follows that \( |x| \approx t \). Thus, by (2.27), we have

\[ (1 + t + |x|) \sum_{|\alpha| \leq 2|\alpha|} |Z^{\alpha} \partial w^I(t, x)| \leq C \sum_{|\alpha| \leq 2|\alpha|} \|Z^{\alpha} w^I(t, \cdot)\|_2 \]

provided \((t, x) \in A_I\). Since the right side of this equation is \( O(\varepsilon) \) by (5.18), we have established (5.17) as desired.

7.2. Proof of (5.19): The next step is to show that we have the higher order, lossy energy estimates and mixed norm estimates when the scaling vector field does not occur. Here, we modify the arguments of [28] to allow the general higher order terms in the nonlinearity.

In the notation of §2, we have \( \Box_s u = B(du) + P(u, du) \) where \( B(du) + P(u, du) \) is the semilinear part of the nonlinearity and

\[ \gamma^{IJ, jk} = \gamma^{IJ, jk}(u, du) = - \sum_{1 < K \leq D} B^{IJ, jk}_{K, l} \partial_l u^K - C^{IJ, jk}(u, du). \]

Also, note that by (5.12) and (5.17)

\[ \sum_{|\alpha| \leq 1} \|\partial^\alpha \gamma(s, \cdot)\|_\infty \leq \frac{C \varepsilon}{1 + s}. \]
We begin by proving (5.19). To do so, we first estimate the energy of $\partial^j u$ for $j \leq M \leq 300$. Notice that by (4.3) and (7.1), we have

\begin{equation}
\partial_t E_M^{1/2} (u)(t) \leq C \sum_{j \leq M} \| \Box_j \partial^j u(t, \cdot) \|_2 + \frac{C \varepsilon}{1 + t} E_M^{1/2} (u)(t).
\end{equation}

Note also that for $M = 1, 2, \ldots$

$$
\sum_{j \leq M} |\Box_j \partial^j u| \leq C \left( \sum_{j \leq M} |\partial^j u| + \sum_{j \leq M-1} |\partial^j \partial^j u| \right) \sum_{|\alpha| \leq 200} \| \partial^\alpha u \|_1 + C \left( \| u \|_1 + \sum_{j \leq M} |\partial^j u| + \sum_{j \leq M-1} |\partial^j \partial^j u| \right) \left( \sum_{|\alpha| \leq 200} \| \partial^\alpha u \|_1 \right)^2.
$$

Using (5.1), (5.12), and (5.17), it follows that this is bounded by

$$
\frac{C \varepsilon}{1 + t} \left( \sum_{j \leq M} |\partial^j u| + \sum_{j \leq M-1} |\partial^j \partial^j u| \right) + \frac{C \varepsilon}{1 + t + |x|^{\gamma/2}}.
$$

If we use elliptic regularity and repeat this argument, we get

$$
\sum_{j \leq M-1} \| \partial^j u(t, \cdot) \|_2 \leq C \sum_{j \leq M} \| \partial^j u(t, \cdot) \|_2 + C \sum_{j \leq M-1} \| \partial^j \Box u(t, \cdot) \|_2 \leq C \sum_{j \leq M} \| \partial^j u(t, \cdot) \|_2 + \frac{C \varepsilon}{1 + t} \sum_{j \leq M-1} \| \partial^j \partial^j u(t, \cdot) \|_2 + \frac{C \varepsilon^3}{(1 + t)^{3/2}}.
$$

If $\varepsilon$ is small, we can absorb the second term into the left side of the preceding inequality. Therefore, if we combine the last two estimates, we conclude that

$$
\sum_{j \leq M} \| \Box_j \partial^j u(t, \cdot) \|_2 \leq \frac{C \varepsilon}{1 + t} \sum_{j \leq M} \| \partial^j u(t, \cdot) \|_2 + \frac{C \varepsilon^3}{(1 + t)^{3/2}}.
$$

If we use this in (7.2), we get that for small $\varepsilon > 0$

$$
\partial_t E_M^{1/2} (u)(t) \leq \frac{C \varepsilon}{1 + t} E_M^{1/2} (u)(t) + \frac{C \varepsilon^3}{(1 + t)^{3/2}}
$$

since $\frac{1}{2} E_M^{1/2} (u)(t) \leq \sum_{j \leq M} \| \partial^j u(t, \cdot) \|_2 \leq 2 E_M^{1/2} (u)(t)$ when $\varepsilon$ is small. Since (1.12) implies that $E_{200}^{1/2} (u)(0) \leq C \varepsilon$, Gronwall’s inequality yields

\begin{equation}
\sum_{j \leq 300} \| \partial^j u(t, \cdot) \|_2 \leq 2 E_{300}^{1/2} (u)(t) \leq C \varepsilon (1 + t)^{\hat{C} \varepsilon},
\end{equation}

for some constant $\hat{C} > 0$. By elliptic regularity, this leads to the bound (5.19) if $\varepsilon > 0$ is sufficiently small.

7.3. Proof of the base case, $N = 0$, of (5.20) and (5.21): We begin by showing

\begin{equation}
(\log(2 + t))^{-1/2} \sum_{|\alpha| \leq 258} \| \partial^\alpha u(t, \cdot) \|_{L^2(S_t)} \leq C (1 + t)^{\tilde{C} \varepsilon}
\end{equation}
where \( \hat{C} \) is the constant appearing in (5.19). By (4.9), (5.1), and (5.5), we have

\[
(\log(2 + t))^{-1/2} \sum_{|\alpha| \leq 298} \| \langle x \rangle^{-1/2} \partial^\alpha u' \|_{L^2(S_t)}
\leq C \varepsilon (\log(2 + t))^{1/2} + (\log(2 + t))^{-1/2} \sum_{|\alpha| \leq 298} \| \langle x \rangle^{-1/2} \partial^\alpha (w - t)' \|_{L^2(S_t)}
\]

(7.5)

\[
\leq C \varepsilon (\log(2 + t))^{1/2} + C \sum_{|\alpha| \leq 298} \int_0^t \| \partial^\alpha \Box (w - t)(s, \cdot) \|_2 \, ds
+ C \sum_{|\alpha| \leq 298} \| \partial^\alpha \Box (w - t) \|_{L^2(S_t)}.
\]

Since \( \partial^\alpha \Box (w - t) = \partial^\alpha (1 - \beta)(1 - \eta)\Box u \), the right side is

\[
\leq C \varepsilon (\log(2 + t))^{1/2} + C \sum_{|\alpha| \leq 299} \int_0^t \| \partial^\alpha \Box u(s, \cdot) \|_2 \, ds + C \sum_{|\alpha| \leq 298} \| \partial^\alpha u \|_{L^2(S_t)}.
\]

If we apply (5.1), (5.12), and (5.17) as in the proof of (5.19), it is easy to see that

\[
\sum_{|\alpha| \leq 299} \| \partial^\alpha \Box u(s, \cdot) \|_2 \leq \frac{C \varepsilon}{1 + s} \sum_{|\alpha| \leq 299} \| \partial^\alpha u'(s, \cdot) \|_2 + \frac{C \varepsilon^3}{(1 + s)^{3/2}}.
\]

If we plug this into the previous equation and apply (5.19), (7.4) follows immediately.

We next wish to show

(7.6)

\[
\sum_{|\alpha| \leq 297} \| Z^\alpha u'(t, \cdot) \|_2 \leq C \varepsilon (1 + t)^{a \varepsilon}
\]

for some \( a_0 > \hat{C} \). In order to show this, we will argue inductively. That is, for \( M \leq 297 \), we will assume that

(7.7)

\[
\sum_{|\alpha| \leq M - 1} \| Z^\alpha u'(t, \cdot) \|_2 \leq C \varepsilon (1 + t)^{C \varepsilon},
\]

and we will use this to show

(7.8)

\[
\sum_{|\alpha| \leq M} \| Z^\alpha u'(t, \cdot) \|_2 \leq C \varepsilon (1 + t)^{C \varepsilon + \sigma}
\]

where \( \sigma > 0 \) can be chosen arbitrarily small. Notice that the base case follows trivially from (5.18).

In order to control the left side of (7.8), we use (4.8). To do so, we must estimate the first term in its right side. We have

(7.9)

\[
\sum_{|\alpha| \leq M} \| \Box_{\alpha} Z^\alpha u(t, \cdot) \|_2 \leq C \sum_{|\beta| \leq 299} \| Z^\beta u'(t, \cdot) \|_{\infty} \sum_{|\alpha| \leq M} \| Z^\alpha u'(t, \cdot) \|_2
+ C \sum_{|\beta|, |\gamma| \leq 299, |\alpha| \leq M} \| Z^\beta u(t, \cdot) Z^\gamma u(t, \cdot) Z^\alpha u'(t, \cdot) \|_2
+ C \sum_{|\beta|, |\gamma| \leq 299, |\alpha| \leq M} \| Z^\beta u(t, \cdot) Z^\gamma u(t, \cdot) Z^\alpha u(t, \cdot) \|_2
\]
By (5.1), (5.12), and (5.17), the first two terms are controlled by
\[
\frac{C\varepsilon}{1 + t} \sum_{|\alpha| \leq M} \|Z^\alpha u'(t, \cdot)\|_2 \leq \frac{C\varepsilon}{1 + t} Y_{M,\beta}^{1/2}(t)
\]
where $Y_{M,\beta}(t)$ is as in (4.7). Since the coefficients of $Z$ are $O(|x|)$, we can apply (5.1) and (5.12) to see that the last term in (7.9) is dominated by
\[
\frac{C\varepsilon^3}{(1 + t)^{3/2}} + \frac{C\varepsilon^2 (1 + \log(1 + t))^2}{1 + t} \sum_{|\alpha| \leq M-1} \|Z^\alpha u'(t, \cdot)\|_2^2.
\]
The first term here corresponds to the case $|\alpha| = |\beta| = |\gamma| = 0$ in the last term of (7.9).

Plugging these bounds for (7.9) into (4.8) and applying (7.1) yields
\[
(7.10) \quad \partial_t Y_{M,\beta}(t) \leq \frac{C\varepsilon}{1 + t} Y_{M,\beta}(t)
+ Y_{M,\beta}^{1/2}(t) \left( \frac{C\varepsilon^3}{(1 + t)^{3/2}} + \frac{C\varepsilon^2 (1 + \log(1 + t))^2}{1 + t} \sum_{|\alpha| \leq M-1} \|Z^\alpha u'(t, \cdot)\|_2^2 \right) + C \sum_{|\alpha| \leq M+1} \|\partial^\alpha u'(t, \cdot)\|_{L^2(|x| < 1)}^2
\]
if $\varepsilon$ is sufficiently small. Therefore, by Gronwall’s inequality, (1.12), and the inductive hypothesis (7.7), we have
\[
Y_{M,\beta}(t) \leq C(1 + t)^{C\varepsilon} \left[ \varepsilon^2 + \left( \sup_{s \leq t} Y_{M,\beta}^{1/2}(s) \right) \varepsilon^2 (1 + t)^{C\varepsilon + \sigma} \right]
+ \sum_{|\alpha| \leq M+1} \|\langle x \rangle^{-1/2} \partial^\alpha u'\|_{L^2(S_t)}^2.
\]
If we apply (7.4) to the last term, we see that (7.8) follows. By induction, this yields (7.6).

Using (4.10), this in turn implies
\[
(7.11) \quad (\log(2 + t))^{-1/2} \sum_{|\alpha| \leq 2^{95}} \|\langle x \rangle^{-1/2} Z^\alpha u'\|_{L^2(S_t)} \leq C\varepsilon(1 + t)^{A\varepsilon + \sigma}
\]
which completes the proof of $N = 0$ cases of (5.20) and (5.21).

7.4. **Proof of (5.20) and (5.21) for $N > 0$:** In order to complete the proof of Theorem 1.1, we must show that (5.20) and (5.21) hold for $N = 1, 2, 3$. To do so, we argue inductively in $N$. We fix an $N$ and assume that (5.21) holds with $N$ replaced by $N - 1$. It then remains to show (5.20) and (5.21) for that $N$.

We begin with the task of showing (5.20). The first step will be to show that
\[
(7.12) \quad \sum_{|\alpha| + |\beta| \leq 2^{95} \atop k \leq N} \|L^k \partial^\alpha u'(t, \cdot)\|_2 \leq C\varepsilon(1 + t)^{A\varepsilon + \sigma}.
\]
For this, we shall want to use (4.6). We must first establish an appropriate version of (4.5) for \( N_0 + \nu_0 \leq 205, \nu_0 \leq N \). For this, we note that for \( M \leq 205, \)
\[
\sum_{j+\mu \leq M, \mu \leq N} \left( \| \tilde{L}^\mu \partial_j^\mu \Box_\gamma u \| + \| \tilde{L}^\mu \partial_j^\mu (\Box - \Box_\gamma) u \| \right)
\leq C \left( \sum_{j \leq M-N} \| \tilde{L}^N \partial_j^\mu \partial_j^\mu u \| + \sum_{j \leq M-N-1} \| \tilde{L}^N \partial_j^\mu \partial_j^\mu u \| \right) \left( \sum_{|\alpha| \leq 200} \| \partial^\alpha u' \| + \left( \sum_{|\alpha| \leq 200} \| \partial^\alpha u \| \right)^2 \right)
+ C \sum_{|\alpha|+\mu \leq M-200, \mu \leq N} \| L^\mu \partial^\alpha u' \| \sum_{|\alpha|+\mu \leq M-200, \mu \leq N-1} \| L^\mu \partial^\alpha u \|
+ C \sum_{|\alpha|+\mu \leq \max(M/2, M-200), \mu \leq N-1} \| L^\mu \partial^\alpha u' \| \sum_{|\alpha|+\mu \leq \max(M/2, M-200), \mu \leq N-1} \| L^\mu \partial^\alpha u \|
+ C \left( \sum_{|\alpha|+\mu \leq M+1, \mu \leq N-1} \| L^\mu \partial^\alpha u \| \right)^2 \sum_{|\alpha|+\mu \leq N} \| L^\mu \partial^\alpha u \|.
\]
Using elliptic regularity, (2.27), (5.1), (5.12), and (5.18), we conclude that
\[
\sum_{j+\mu \leq M, \mu \leq N} \left( \| \tilde{L}^\mu \partial_j^\mu \Box_\gamma u(t, \cdot) \|_2 + \| \tilde{L}^\mu \partial_j^\mu (\Box - \Box_\gamma) u(t, \cdot) \|_2 \right) \leq \frac{C\varepsilon}{1+t} \sum_{j+\mu \leq M, \mu \leq N} \| \tilde{L}^\mu \partial_j^\mu u'(t, \cdot) \|_2
+ C \sum_{|\alpha|+\mu \leq M-200, \mu \leq N} \| \langle x \rangle^{-1/2} L^\mu \partial^\alpha u' \| \| \langle x \rangle^{-1/2} L^\mu Z^\alpha u' \|_2 \sum_{|\alpha|+\mu \leq \max(M/2, M-200), \mu \leq N-1} \| \langle x \rangle^{-1/2} L^\mu Z^\alpha u' \|_2
+ C \sum_{|\alpha|+\mu \leq \max(M/2, M-200), \mu \leq N-1} \| \langle x \rangle^{-1/2} L^\mu Z^\alpha u' \|_2
+ C \left( \sum_{|\alpha|+\mu \leq M+1, \mu \leq N-1} \| L^\mu \partial^\alpha u \| \right)^2 \sum_{|\alpha|+\mu \leq N} \| L^\mu \partial^\alpha u \|.
\]
Based on this, (4.5) holds with \( \delta = C\varepsilon \) and
\[
H_{N, M-N}(t) = C \sum_{|\alpha|+\mu \leq M-200, \mu \leq N} \| \langle x \rangle^{-1/2} L^\mu \partial^\alpha u' \|_2^2 \sum_{|\alpha|+\mu \leq \max(M/2, M-200), \mu \leq N-1} \| \langle x \rangle^{-1/2} L^\mu Z^\alpha u' \|_2^2 \sum_{|\alpha|+\mu \leq \max(M/2, M-200), \mu \leq N-1} \| \langle x \rangle^{-1/2} L^\mu Z^\alpha u' \|_2^2
+ C \sum_{|\alpha|+\mu \leq \max(M/2, M-200), \mu \leq N-1} \| \langle x \rangle^{-1/2} L^\mu Z^\alpha u' \|_2^2 \sum_{|\alpha|+\mu \leq \max(M/2, M-200), \mu \leq N-1} \| \langle x \rangle^{-1/2} L^\mu Z^\alpha u' \|_2^2
+ C \left( \sum_{|\alpha|+\mu \leq M+1, \mu \leq N-1} \| L^\mu \partial^\alpha u \| \right)^2 \sum_{|\alpha|+\mu \leq N} \| L^\mu \partial^\alpha u \|.
\]
Since \( M+1 \leq 206, (5.1), (5.14), \) and (5.15) imply that this last term is controlled by
\[
C\varepsilon^3 (1 + \log(1+t))^3 (1+t)^{205/2} \| 1/(1+t + |x|)^2 \|_2 \leq \frac{C\varepsilon^3}{(1+t)^{1+\gamma}}
\]
if \( \varepsilon \) is sufficiently small.

Since the conditions on the data give \( \int e^{\beta(t)} \delta^j u(0, x) \, dx \leq C \varepsilon^2 \) if \( j + \nu \leq 300 \), it follows from (4.6) and the inductive hypothesis ((5.21) with \( N \) replaced by \( N - 1 \) 1) that for \( M \leq 205 \)

\[
(7.13) \quad \sum_{|\alpha| + |\rho| \leq M \atop \rho \leq N} \| L^\rho \partial^\alpha u'(t, \cdot) \|_{L^2} \leq C \varepsilon (1 + t)^{C \varepsilon + \sigma} + C (1 + t)^{C \varepsilon} \sum_{|\alpha| + |\rho| \leq M - 200 \atop \rho \leq N} \| (x)^{-1/2} L^\rho \partial^\alpha u' \|_{L^2(S_t)}^{2}

+ C(1 + t)^{C \varepsilon} \int_0^t \sum_{|\alpha| + |\rho| \leq M \atop \rho \leq N - 1} \| L^\rho \partial^\alpha u'(s, \cdot) \|_{L^2(|x| < 1)} \, ds
\]

for some constant \( \sigma > 2a_{N-1} \varepsilon \).

If we apply (4.12), (5.1), and (5.5), we get that the last integral is dominated by

\( C \varepsilon \log(2 + t) \) plus

\[
\int_0^t \sum_{|\alpha| + |\rho| \leq M \atop \rho \leq N - 1} \| L^\rho \partial^\alpha (w - v)'(s, \cdot) \|_{L^2(|x| < 1)} \, ds
\]

\[
\leq C \sum_{|\alpha| + |\rho| \leq M + 1 \atop \rho \leq N - 1} \int_0^t \left( \int_0^s \| L^\rho \partial^\alpha \Box (w - v)(\tau, \cdot) \|_{L^2(|x| - (s - \tau) < 1)} \, d\tau \right) \, ds.
\]

Since \( \Box (w - v) = (1 - \beta)(1 - \eta)\Box u \), we conclude that this last term is bounded by

\[
(7.14) \quad \sum_{|\alpha| + |\rho| \leq M + 1 \atop \rho \leq N - 1} \int_0^t \left( \int_0^s \| L^\rho \partial^\alpha \Box u(\tau, \cdot) \|_{L^2(|x| - (s - \tau) < 1)} \, d\tau \right) \, ds.
\]

As in [28], when \( \Box u \) is replaced by the quadratic terms \( B(du) + Q(du, d^2 u) \) in (7.14), we see from an application of (2.27) that the integrand is bounded by

\[
\sum_{|\alpha| + |\rho| \leq 200 \atop \rho \leq N - 1} \| (x)^{-1/2} L^\rho Z^\alpha u'(\tau, \cdot) \|_{L^2(|x| - (s - \tau) < 20)}^{2}.
\]

Since the sets \( \{ (\tau, x) : |x| - (j - \tau) < 20 \}, j = 0, 1, 2, \ldots \) have finite overlap, we conclude that, in this case, the last integral in (7.13) is bounded by

\[
C \varepsilon \log(2 + t) + C \sum_{|\alpha| + |\rho| \leq 200 \atop \rho \leq N - 1} \| (x)^{-1/2} L^\rho Z^\alpha u' \|_{L^2(S_t)} \leq C \varepsilon (1 + t)^{2a_{N-1} \varepsilon}.
\]

The last inequality follows from the inductive hypothesis.
When $\Box u$ is replaced by the higher order terms $P(u, du) + R(u, du, d^2u)$, we see that (7.14) is bounded by
\[
C \left( \int_0^T \left( \sum_{|\alpha| + \rho \leq 208} |L^\rho \partial^\alpha u(\tau, \cdot)| \right)^2 \sum_{|\alpha| + \rho \leq 208} |L^\rho \partial^\alpha u(\tau, \cdot)| \right) \, d\tau \, ds
\]
By (5.1), (5.14), and (5.16), we have that
\[
\left( \sum_{|\alpha| + \rho \leq 208} |L^\rho \partial^\alpha u(\tau, \cdot)| \right)^2 \sum_{|\alpha| + \rho \leq 208} |L^\rho \partial^\alpha u(\tau, \cdot)| \leq C \varepsilon^3 (1 + t)^{2N-1+\varepsilon_N+\varepsilon_N+1}(1 + t)^{-1}(1 + |x|)^{-2}.
\]
Since the norm is taken over $|x| \approx (t - s)$, it follows that (7.15) is bounded by $C \varepsilon^3 (1 + t)^{2N-1+\varepsilon_N+\varepsilon_N+1}$. Since we may take $\tilde{a}_N > 2b_{N-1} + \varepsilon_{N-1}$, this will be sufficient for $N \geq 2$. When $N = 1$, there are no occurrences of $L$ in (7.15), and appropriate bounds follow simply from (5.1), (5.12), and (5.16).

Plugging this into (7.13), we see that
\[
(7.16) \sum_{|\alpha| + \rho \leq M} |L^\rho \partial^\alpha u(t, \cdot)|_2 \leq C \varepsilon (1 + t) \varepsilon + \sigma
\]
\[
+ C (1 + t)^\varepsilon \sum_{|\alpha| + \rho \leq M-2} \|\langle x \rangle^{-1/2} L^\rho \partial^\alpha u'\|_{L^2(S_1)}^2
\]
which yields (7.12) for $M \leq 200$.

For $M > 200$, (7.12) will follow from a simple induction argument using the following lemma.

**Lemma 7.1.** Under the above assumptions, if $M \leq 205$, $1 \leq N \leq 4$, and
\[
(7.17) \sum_{|\alpha| + \rho \leq M} |L^\rho \partial^\alpha u(t, \cdot)|_2 + \sum_{|\alpha| + \rho \leq M-3} \|\langle x \rangle^{-1/2} L^\rho \partial^\alpha u'\|_{L^2(S_1)}
\]
\[
+ \sum_{|\alpha| + \rho \leq M-4} \|L^\rho Z^\alpha u(t, \cdot)|_2 + \sum_{|\alpha| + \rho \leq M-6} \|\langle x \rangle^{-1/2} L^\rho Z^\alpha u'\|_{L^2(S_1)} \leq C \varepsilon (1 + t)^\varepsilon + \sigma
\]

with $\sigma > 0$, then there is a constant $C'$ so that
\[
(7.18) \sum_{|\alpha| + \rho \leq M-2} \|\langle x \rangle^{-1/2} L^\rho \partial^\alpha u'\|_{L^2(S_1)} + \sum_{|\alpha| + \rho \leq M-3} \|L^\rho Z^\alpha u'(t, \cdot)|_2
\]
\[
+ \sum_{|\alpha| + \rho \leq M-5} \|\langle x \rangle^{-1/2} L^\rho Z^\alpha u'\|_{L^2(S_1)} \leq C' \varepsilon (1 + t)^{C' \varepsilon + \sigma}.
\]
Proof of Lemma 7.1: Let us start with the first term on the left side of (7.18). Using (4.9), (5.1), and (5.5) as in (7.5), we see that
\[
(\log(2 + t))^{-1/2} \sum_{|\alpha| + \nu \leq M-2} \| (x)^{-1/2} L^\nu \partial^\alpha u' \|_{L^2(S_t)}
\]
is controlled by \(C(\log(2 + t))^{1/2}\) plus
\[
(7.19) \quad C \sum_{|\alpha| + \nu \leq M-1} \int_0^t \left\| \sum_{\nu \leq \frac{M}{2}} \partial^\nu u(s, \cdot) \right\|_2 ds + C \sum_{|\alpha| + \nu \leq M-2} \| L^\nu \partial^\alpha u \|_{L^2(S_t)}.
\]

When \(\Box u\) is replaced by the quadratic terms \(B(du) + Q(du, d^2 u)\), the first term in (7.19) is controlled by
\[
C \int_0^t \left\| \sum_{|\alpha| \leq 200} \partial^\alpha u'(s, \cdot) \right\|_2 ds + \sum_{|\alpha| + \nu \leq M} \left( \sum_{\nu \leq N} \left\| L^\nu \partial^\alpha u'(s, \cdot) \right\|_2^2 \right) ds
\]

\[
+ C \int_0^t \left( \sum_{|\alpha| + \nu \leq M-200} \left\| L^\nu \partial^\alpha u'(s, \cdot) \right\|_2^2 \right) ds.
\]

Notice that by (5.17) and (7.17), the desired bound holds for the first term. By (2.27), the last term is bounded by
\[
C \sum_{|\alpha| + \nu \leq M+2} \left\| (x)^{-1/2} L^\nu Z^\alpha u' \right\|_{L^2(S_t)}^2,
\]
and the appropriate bound follows from the (5.21) with \(N\) replaced by \(N - 1\). This is sufficient to show that the result holds for this case when \(M \leq 200\). When \(M \geq 201\), we must also handle the second term above. By (2.27), this is controlled by
\[
C \sum_{|\alpha| \leq M} \left\| (x)^{-1/2} \partial^\alpha u' \right\|_{L^2(S_t)}^2 + C \sum_{|\alpha| + \nu \leq M-198} \left\| (x)^{-1/2} L^\nu Z^\alpha u' \right\|_{L^2(S_t)}^2,
\]
and the bounds follow from (7.4) and (7.17). When \(\Box u\) is quadratic, the desired estimates for the last term in (7.19) follow from very similar arguments.

It remains to bound (7.19) when \(\Box u\) is replaced by the higher order terms \(P(u, du) + R(u, du, d^2 u)\). Since \(M + 1 \leq 206\), by (5.1), (5.14) and (5.16), we have
\[
\left\| \sum_{|\alpha| + \nu \leq M+1} \left( \sum_{\nu \leq N} L^\nu \partial^\alpha u(s, \cdot) \right)^2 \right\|_2 \leq C \varepsilon^2 (1 + t)^{-1-
}
\]
for \(\varepsilon > 0\) sufficiently small. Upon integration, it is easy to see that these terms satisfy the desired bounds, and this finishes the proof that the first term on the left side of (7.18) is dominated by its right side.
To control the second term in (7.18), we will use (4.8). This means that we must estimate the first term in its right side, which satisfies

\[
\sum_{|\alpha|+\nu \leq M-3} \| \Box \alpha L^\nu \alpha u(s, \cdot) \|_2 \leq C \sum_{|\alpha| \leq 2M-3} \left( \sum_{|\alpha|+\nu \leq M-3} \| L^\nu \alpha u(s, \cdot) \|_2 \right)^2 + C \sum_{|\alpha| \leq 2M-3} \left( \sum_{|\alpha|+\nu \leq M-1} \| L^\nu \alpha u(s, \cdot) \|_2 \right)^2
\]

Applying (5.1) and (5.17) to the first term, (2.27) to the second and third terms, and (5.1), (5.14), and (5.16) to the cubic term, we have that this is dominated by

\[
(7.20) \quad \frac{C \varepsilon}{1 + s} Y_{M-3-N,N}^1 \langle s \rangle + C \sum_{|\alpha|+\nu \leq M-2} \| \langle \rangle^{-1/2} L^\nu \alpha u(s, \cdot) \|_2^3 + C \varepsilon \langle 1 + s \rangle^{-1/2} \sum_{|\alpha|+\nu \leq M-2} \| \langle \rangle^{-1/2} L^\nu \alpha u(s, \cdot) \|_2^3 + C \varepsilon^3 \langle 1 + s \rangle^{-1/2}
\]

with \( Y_{M-3-N,N}(t) \) as in (4.7).

Plugging this into (4.8), applying the inductive hypothesis ((5.21) with \( N \) replaced by \( N - 1 \), using Gronwall’s inequality, and arguing as in the proof of (7.6), we see that the bound for the second term in (7.18) follows for \( M \leq 203 \). When \( M \geq 203 \), we must also deal with the third term in (7.20), but this is done trivially by applying (7.17).

Using (4.10) and the arguments that preceed, this in turn implies that the third term in (7.18) is bounded by the right side, which completes the proof. \( \square \)

From (7.16) and the lemma, one gets (7.12) and (5.20).

In order to complete the proof of Theorem 1.1, one must show that (5.21) follows from (5.12), (5.20) and (5.21) with \( N \) replaced by \( N - 1 \). The first step is to show that

\[
(7.21) \quad \sum_{|\alpha|+\nu \leq M} \| L^\nu \alpha u(t, \cdot) \|_2 \leq C \varepsilon (1 + t)^{A_N \varepsilon}
\]

for some \( A_N \). For this, as in the proof of (7.12), we will use (4.6) once we are able to establish an appropriate version of (4.5) for \( N_0 + \nu_0 \geq 300 - 8N, \nu_0 \leq N \). Notice that
for $M \leq 300 - 8N$, we have

\[
\sum_{j + \mu \leq M \atop \mu \leq N} \left( \hat{L}^\mu \partial_j^\mu \square_\gamma u + [[\hat{L}^\mu \partial_j^\mu, \square - \square_\gamma]]u \right)
\]

\[
\leq C \left( \sum_{j + \mu \leq M \atop \mu \leq N} |\hat{L}^\mu \partial_j^\mu u| + \sum_{j + \mu \leq M \atop \mu \leq N} \left| \hat{L}^N \partial_j^N \partial_3 u \right| \left( \sum_{|\alpha| \leq 200 \atop \rho \leq N-1} |\partial_\alpha u| + \sum_{|\alpha| \leq 190 \atop \rho \leq N-1} |L^\rho \partial_\alpha u| \right)^2 \right) + C \sum_{|\alpha| + \mu \leq M \atop \mu \leq N-1} |L^\rho \partial_\alpha u| \sum_{|\alpha| + \mu \leq M \atop \mu \leq N-1} |L^\rho \partial_\alpha u|'
\]

\[
+ C \sum_{|\alpha| + \mu \leq M \atop \mu \leq N-1} |L^\rho \partial_\alpha u|' \sum_{|\alpha| + \mu \leq M \atop \mu \leq N-1} |L^\rho \partial_\alpha u|'
\]

\[
+ C \left( \sum_{|\alpha| + \mu \leq 255 \atop \mu \leq N-1} |L^\rho \partial_\alpha u| \right)^2 \sum_{|\alpha| + \mu \leq 180 \atop \mu \leq N} |L^\rho \partial_\alpha u|
\]

By this, (2.27), (5.1), (5.14), and (5.17), and elliptic regularity, we get that for $M \leq 300 - 8N$

\[
\sum_{j + \mu \leq M \atop \mu \leq N} \left( \left\| \hat{L}^\mu \partial_j^\mu u(t, \cdot) \right\|_2 + \left\| [[\hat{L}^\mu \partial_j^\mu, \square - \square_\gamma]]u(t, \cdot) \right\|_2 \right) \leq \frac{C\tau}{1 + t} \sum_{j + \mu \leq M \atop \mu \leq N} \left\| \hat{L}^\mu \partial_j^\mu u(t, \cdot) \right\|_2
\]

\[
+ C \sum_{|\alpha| + \mu \leq M - 200 \atop \mu \leq N} \left\| (x)^{-1/2} L^\mu \partial^\alpha u(t, \cdot) \right\|_2^2 + C \sum_{|\alpha| + \mu \leq M + 2 \atop \mu \leq N-1} \left\| (x)^{-1/2} L^\mu Z^\alpha u(t, \cdot) \right\|_2^2
\]

\[
+ C \left( \sum_{|\alpha| + \mu \leq 255 \atop \mu \leq N-1} |L^\rho \partial_\alpha u| \right)^2 \sum_{|\alpha| + \mu \leq 180 \atop \mu \leq N} |L^\rho \partial_\alpha u|
\]

\[
+ C \left( \sum_{|\alpha| + \mu \leq M \atop \mu \leq N-1} |L^\rho \partial_\alpha u|' \sum_{|\alpha| + \mu \leq M \atop \mu \leq N-1} |L^\rho \partial_\alpha u|' \right) \sum_{|\alpha| + \mu \leq 180 \atop \mu \leq N} |L^\rho \partial_\alpha u| \sum_{|\alpha| + \mu \leq 180 \atop \mu \leq N} |L^\rho \partial_\alpha u|
\]

Based on this, (4.5) holds with

(7.22) \( H_{N,M-N}(t) = C \sum_{|\alpha| + \mu \leq M - 200 \atop \mu \leq N} \left\| (x)^{-1/2} L^\mu Z^\alpha u(t, \cdot) \right\|_2^2 \)

\[
+ C \sum_{|\alpha| + \mu \leq M + 2 \atop \mu \leq N-1} \left\| (x)^{-1/2} L^\mu Z^\alpha u(t, \cdot) \right\|_2^2 + C \left( \sum_{|\alpha| + \mu \leq 255 \atop \mu \leq N-1} |L^\rho \partial_\alpha u| \right)^2 \sum_{|\alpha| + \mu \leq 180 \atop \mu \leq N} |L^\rho \partial_\alpha u|
\]

\[
+ C \left( \sum_{|\alpha| + \mu \leq M \atop \mu \leq N-1} |L^\rho \partial_\alpha u|' \sum_{|\alpha| + \mu \leq M \atop \mu \leq N-1} |L^\rho \partial_\alpha u|' \right) \sum_{|\alpha| + \mu \leq 180 \atop \mu \leq N} |L^\rho \partial_\alpha u| \sum_{|\alpha| + \mu \leq 180 \atop \mu \leq N} |L^\rho \partial_\alpha u|
\]
Notice that by (5.1), (5.14), and (5.15), the third term on the right of (7.22) is controlled by $C\varepsilon^2(1 + t)^{-1}$. Also notice that by (5.1), (5.13), and (5.15), the last term in (7.22) is dominated by

$$
\frac{C\varepsilon^2(1 + t)^{2N^2 + 3N^1 + 1}}{1 + t} \sum_{|\alpha| + \mu \leq M \atop \mu \leq N-1} \|x\|^{-1/2} L^\mu \partial^\alpha u(t, \cdot) \|_2.
$$

From this, we see that

$$
(7.23) \quad \int_0^t H_{N, M - N}(s) \, ds \leq C\varepsilon^2 + C \sum_{|\alpha| + \mu \leq M - 200 \atop \mu \leq N} \|\langle x\rangle\|^{-1/2} L^\mu \partial^\alpha u'\|_{L^2(S_t)}^2

+ C \sum_{|\alpha| + \mu \leq M - 202 \atop \mu \leq N} \|\langle x\rangle\|^{-1/2} Z^\alpha u'\|_{L^2(S_t)}^2

+ C\varepsilon^2 \int_0^t (1 + s)^{-1 + \delta N^2 + 3N^1 + 1} \sum_{|\alpha| + \mu \leq M \atop \mu \leq N-1} \|\langle x\rangle\|^{-1/2} L^\mu \partial^\alpha u'(s, \cdot) \|_2 \, ds.
$$

If one applies the Schwarz inequality and uses (5.21) (with $N$ replaced by $N - 1$), the last term above is $O(\varepsilon^3)$ for sufficiently small $\varepsilon$.

If we use this in (4.6) and apply the inductive hypothesis to handle terms that involve $N - 1$ or fewer occurrences of $L$, we see that

$$
(7.24) \quad \sum_{|\alpha| + \mu \leq M \atop \mu \leq N} \|L^\mu \partial^\alpha u'(t, \cdot)\|_2 \leq C\varepsilon(1 + t)^{C\varepsilon + \sigma}

+ C(1 + t)^{C\varepsilon} \sum_{|\alpha| + \mu \leq M - 200 \atop \mu \leq N} \|\langle x\rangle\|^{-1/2} L^\mu \partial^\alpha u'\|_{L^2(S_t)}^2

+ C(1 + t)^{C\varepsilon} \int_0^t \sum_{|\alpha| + \mu \leq M \atop \mu \leq N-1} \|L^\mu \partial^\alpha u'(s, \cdot)\|_{L^2(|x| < 1)} \, ds
$$

since the conditions on the data give $\sum c_0 \langle \tilde{L}^{\nu} \partial^j u \rangle(t, x) \, ds \leq C\varepsilon^2$ if $\nu + j < 300$.

As before, if we apply (4.12), (5.1), and (5.5), the last integral is dominated by $C\varepsilon \log(2 + t) t$ plus

$$
(7.25) \quad \sum_{|\alpha| + \mu \leq M + 1 \atop \mu \leq N-1} \int_0^t \left( \int_0^t \|L^\mu \partial^\alpha \Box u(\tau, \cdot)\|_{L^2(|x| - (j - \tau)| < 10)} \, d\tau \right) \, ds.
$$

When $\Box u$ is replaced by $B(du) + Q(du, d^2 u)$, as in the proof of (7.12), we can apply (2.27) and finite overlap of the sets $\{(\tau, x) : |x| - (j - \tau)| < 20\}$, $j = 0, 1, 2, \ldots$ to see that this is bounded by

$$
C \sum_{|\alpha| + \mu \leq M + 3 \atop \mu \leq N-1} \|\langle x\rangle\|^{-1/2} Z^\alpha u'\|_{L^2(S_t)}^2 \leq C\varepsilon^2(1 + t)^{2N^2 - 1}.\]
The last inequality follows from the inductive hypothesis (5.21).

We must now examine the case when \( \Box u \) in (7.25) is replaced by the cubic terms \( P(u, du) + R(u, du, \partial u) \). Here, we see that (7.25) is bounded by

\[
(7.26) \quad C \int_0^T \left( \int_0^T \left( \sum_{|\alpha| + \mu \leq M+1} |L^\mu \partial^\alpha u(\tau, \cdot)| \right)^2 \right. \\
\times \sum_{|\alpha| + \mu \leq M+2} |L^\mu \partial^\alpha u'(\tau, \cdot)| \left\| L^2(\| x -(s-\tau) \| < 10) \right\| dt \right) ds \\
+ C \int_0^T \left( \int_0^T \left( \sum_{|\alpha| + \mu \leq N-1} |L^\mu u(\tau, \cdot)| \right)^3 \left\| L^2(\| x -(s-\tau) \| < 10) \right\| ds \right) ds.
\]

Since the norm is taken over \( |x| \approx (s-\tau) \), we can apply (5.1) and (5.13) to bound the first term by

\[
C \varepsilon^2 \int_0^T \left( \int_0^T \frac{(1 + \tau)^{2\nu} + (1 + s-\tau)}{(1 + \tau)(1 + (s-\tau))} \sum_{|\alpha| + \mu \leq M+2} |L^\mu \partial^\alpha u'(\tau, \cdot)| \left\| L^2(\| x -(s-\tau) \| < 10) \right\| dt \right) ds.
\]

By the inductive hypothesis (5.21), it follows that this term is dominated by \( C \varepsilon^3 (1 + t)^{3 \delta + \delta - \varepsilon} \). Using three applications of (5.1) and (5.13) we see that

\[
\left( \sum_{\mu \leq N-1} |L^\mu u(\tau, x)| \right)^3 \leq C \varepsilon^3 (1 + t)^{3 \delta + \delta - \varepsilon + 1}(1 + |x|)^{-2},
\]

and thus it follows that the last term in (7.26) is controlled by \( C \varepsilon^3 (1 + t)^{3 \delta + \delta} \).

Plugging these bounds in (7.24), it follows that

\[
(7.27) \quad \sum_{|\alpha| + \mu \leq M} \| L^\mu \partial^\alpha u'(t, \cdot) \|_2 \leq C \varepsilon (1 + t)^{C \varepsilon + \delta} \\
+ C(1 + t)^{C \varepsilon} \sum_{|\alpha| + \mu \leq M-200} \| \langle x \rangle^{-1/2} L^\mu \partial^\alpha u' \|_L^2(S_1)
\]

which yields (7.21) for \( M \leq 200 \). For \( M > 200 \), similar to what we have seen previously, (7.21) will follow from a simple induction argument using the following lemma.

**Lemma 7.2.** Under the above assumptions, if \( M \leq 300 - 8N \) and

\[
(7.28) \quad \sum_{|\alpha| + \nu \leq M} \| L^\nu \partial^\alpha u'(t, \cdot) \|_2 + \sum_{|\alpha| + \nu \leq M-3} \| \langle x \rangle^{-1/2} L^\nu \partial^\alpha u' \|_L^2(S_1) \\
+ \sum_{|\alpha| + \nu \leq M-4} \| L^\nu Z^\alpha u'(t, \cdot) \|_2 + \sum_{|\alpha| + \nu \leq M-6} \| \langle x \rangle^{-1/2} L^\nu Z^\alpha u' \|_L^2(S_1) \leq C \varepsilon (1 + t)^{C \varepsilon + \delta}
\]


with $\sigma > 0$, then there is a constant $C'$ so that

\begin{equation}
(7.29) \quad \sum_{|\alpha| + |\nu| \leq M-2} \| x^{-1/2} L^\nu \delta^\alpha u' \|_{L^2(S_1)} + \sum_{|\alpha| + |\nu| \leq M-3} \| L^\nu Z^\alpha u'(t, \cdot) \|_2 \\
+ \sum_{|\alpha| + |\nu| \leq M-5} \| \langle x \rangle^{-1/2} L^\nu Z^\alpha u' \|_{L^2(S_1)} \leq C \varepsilon (1 + t)^{C' + C \varepsilon}.
\end{equation}

Proof of Lemma 7.2: Here we use arguments similar to those applied to prove Lemma 7.1. We begin by showing that the first term on the left side of (7.29) satisfies the bound. Using (4.9), (5.1), and (5.5) as in (7.5), we see that

\begin{equation}
(7.30) \quad (\log(2 + t))^{-1/2} \sum_{|\alpha| + |\nu| \leq M-2} \| x^{-1/2} L^\nu \delta^\alpha u' \|_{L^2(S_1)} \leq C (\log(2 + t))^{1/2}
+ C \int_0^t \| \sum_{|\alpha| + |\nu| \leq M-1} \langle L^\nu \delta^\alpha \Box u(s, \cdot) \rangle \|_2 \, ds + C \sum_{|\alpha| + |\nu| \leq M-2} \| L^\nu \delta^\alpha \Box u \|_{L^2(S_1)}.
\end{equation}

Notice that the second term in the right side of (7.30) is

\begin{equation}
(7.31) \quad \leq C \int_0^t \| \sum_{|\alpha| + |\nu| \leq M-1} \langle L^\nu \delta^\alpha u'(s, \cdot) \rangle \sum_{|\alpha| + |\nu| \leq M} \langle L^\nu \delta^\alpha u'(s, \cdot) \rangle \|_2 \, ds \\
+ C \int_0^t \| \sum_{|\alpha| + |\nu| \leq M-1} \langle L^\nu \delta^\alpha u(s, \cdot) \rangle \sum_{|\alpha| + |\nu| \leq M} \langle L^\nu \delta^\alpha u(s, \cdot) \rangle \|_2 \, ds \\
+ C \int_0^t \| \sum_{|\alpha| + |\nu| \leq M-1} \langle L^\nu u(s, \cdot) \rangle \sum_{|\alpha| + |\nu| \leq M} \langle L^\nu \delta^\alpha u(s, \cdot) \rangle \|_2 \, ds \\
+ C \int_0^t \left( \sum_{|\alpha| + |\nu| \leq M} \langle L^\nu u(s, \cdot) \rangle \right)^2 \sum_{|\alpha| + |\nu| \leq M} \| L^\nu u(s, \cdot) \|_2 \, ds.
\end{equation}

By (5.1), (5.13), and (5.15), it follows that the last term is $O(c^3)$. Applying (5.1), (5.13), and (5.15) to the first and third terms and using (2.27) and the Schwarz inequality on the second, we see that (7.31) is

\begin{equation}
\leq C \varepsilon^3 + C \varepsilon \int_0^t (1 + s)^{-1 + 5 \varepsilon} \varepsilon + \sum_{|\alpha| + |\nu| \leq M} \| L^\nu \delta^\alpha u' \|_2 \, ds \\
+ C \sum_{|\alpha| + |\nu| \leq M} \| \langle x \rangle^{-1/2} L^\nu Z^\alpha u'(s, \cdot) \|_{L^2(S_1)} \sum_{|\alpha| + |\nu| \leq M-1} \| \langle x \rangle^{-1/2} L^\nu \delta^\alpha u' \|_{L^2(S_1)}.
\end{equation}

When $N \leq 190$, the last term above is unnecessary, and the bound follows from (7.28). For $N > 190$, one uses (7.28) and the inductive hypothesis (5.21) to bound the additional
term. Since the same arguments can be employed to bound the last term in (7.30), this finishes the proof of the bound for the first term in the left side of (7.29).

To control the second term on the left side of (7.29), we will use (4.8). The main step is to estimate the first term on its right. Here, we have

\begin{equation}
\sum_{[\alpha] + \nu \leq M-3} \norm{\nabla_\gamma \Delta^\nu \Delta^\mu u(t, \cdot)}_2 \leq C \sum_{[\alpha] + \nu \leq M-3} \norm{\Delta^\mu u(t, \cdot)}_2 \sum_{[\alpha] + \nu \leq M-3} \norm{\Delta^\nu \Delta^\mu u(t, \cdot)}_2 \\
+ C \left( \sum_{[\alpha] + \nu \leq M-3} \norm{\Delta^\nu \Delta^\mu u(t, \cdot)}_2 \right)^2 \\
+ C \sum_{[\alpha] + \nu \leq M-3} \norm{\Delta^\nu \Delta^\mu u(t, \cdot)}_2 \\
+ C \sum_{[\alpha] + \nu \leq M-3} \norm{\Delta^\nu \Delta^\mu u(t, \cdot)}_2 \
\end{equation}

With \( Y_{M-N-3, N}(t) \) as in (4.7), we can apply (5.1) and (5.17) to bound the first term in the right by

\begin{equation}
\frac{C \varepsilon}{1 + t} \varepsilon_{M-N-3, N}(t).
\end{equation}

By applying (5.1) and (5.13), the same bound holds for the fourth term in the right side of (7.32). Using (2.27), the second and third terms in the right of (7.32) are controlled by

\begin{equation}
C \sum_{[\alpha] + \nu \leq M-3} \norm{(x)^{-1/2} \Delta^\nu \Delta^\mu u(t, \cdot)}_2 + C \sum_{[\alpha] + \nu \leq M-3} \norm{(x)^{-1/2} \Delta^\nu \Delta^\mu u(t, \cdot)}_2.
\end{equation}

Since the coefficients of \( \Delta \) are \( O([x]) \), one may use (5.1), (5.13) and (5.15) to bound the fifth and sixth terms in the right side of (7.32) by

\begin{equation}
C \varepsilon^2 (1 + t)^{-1 + \frac{\beta}{2} + \varepsilon N} + \left( \sum_{[\alpha] + \nu \leq M-4} \norm{\Delta^\nu \Delta^\mu u(t, \cdot)}_2 + \sum_{[\alpha] + \nu \leq M-3} \norm{\Delta^\nu \Delta^\mu u(t, \cdot)}_2 \right)
\end{equation}

Finally, the last term in (7.32) is easily seen to be \( \leq C \varepsilon^2 (1 + t)^{-1} \) by (5.1), (5.13), and (5.15).
If we use these bounds for (7.32) in (4.8), we see that

\[
(7.33) \quad \partial_t Y_{M - N, N}(t) \leq \frac{C \varepsilon}{1 + t} Y_{M - N, N}(t) \\
+ C Y_{M - N, N}(t) \left[ \sum_{|\alpha| + \nu \leq M - 1, \nu \leq N - 1} \| (x)^{-1/2} L^\nu Z^\alpha u'(t, \cdot) \|^2 \right]^{3/2} \\
+ \sum_{|\alpha| + \nu \leq M - 3, \nu \leq N} \| (x)^{-1/2} L^\nu Z^\alpha u'(t, \cdot) \|^2 \\
+ \varepsilon^2 (1 + t)^{-1 + \delta N \varepsilon + \varepsilon_N + \varepsilon + \varepsilon} \left[ \sum_{|\alpha| + \nu \leq M - 4, \nu \leq N} \| L^\nu Z^\alpha u'(t, \cdot) \|^2 + \sum_{|\alpha| + \nu \leq M - 3, \nu \leq N - 1} \| L^\nu Z^\alpha u'(t, \cdot) \|^2 \right] \\
+ \varepsilon^2 (1 + t)^{-1 - \varepsilon} \right] + C \sum_{|\alpha| + \nu \leq M - 2, \nu \leq N} \| L^\nu \partial^\alpha u'(t, \cdot) \|^2\|_{L^2(|x| < 1)}.
\]

Thus, by Gronwall’s inequality, we have

\[
(7.34) \quad Y_{M - N, N}(t) \leq C (1 + t)^{2C \varepsilon} \left[ \varepsilon + \sum_{|\alpha| + \nu \leq M - 1, \nu \leq N - 1} \| (x)^{-1/2} L^\nu Z^\alpha u' \|^2_{L^2(S_t)} \right]^{3/2} \\
+ \sum_{|\alpha| + \nu \leq M - 203, \nu \leq N} \| (x)^{-1/2} L^\nu Z^\alpha u' \|^2_{L^2(S_t)} \\
+ \varepsilon^2 \int_0^t (1 + s)^{-1 + \delta N \varepsilon + \varepsilon_N + \varepsilon + \varepsilon} \| L^\nu Z^\alpha u'(s, \cdot) \|^2 ds \\
+ \varepsilon^2 \int_0^t (1 + s)^{-1 + \delta N \varepsilon + \varepsilon_N + \varepsilon + \varepsilon} \| L^\nu Z^\alpha u'(s, \cdot) \|^2 ds \right]^{3/2} \\
+ C(1 + t)^{C \varepsilon} \sum_{|\alpha| + \nu \leq M - 2, \nu \leq N} \| (x)^{-1/2} L^\nu \partial^\alpha u' \|^2_{L^2(S_t)}
\]

since \( Y_{M - N, N}(0) \leq C \varepsilon^2 \) by (1.12).

For \( M \leq 203 \), the third term on the right does not appear, and the proof of the bound

\[
\sum_{|\alpha| + \nu \leq M - 3, \nu \leq N} \| L^\nu Z^\alpha u'(t, \cdot) \|^2 \leq CY_{M - N, N}(t) \leq C \varepsilon^2 (1 + t)^{C \varepsilon + C \varepsilon}
\]

is completed by applying the inductive hypothesis (5.21) (with \( N \) replaced by \( N - 1 \)) to the second and fifth terms on the right, applying (7.28) to the fourth term on the right, and using the bound for the first term on the left of (7.29) to control the last term in (7.34). For \( M > 203 \), a subsequent application of (7.28) to the third term in (7.34) completes the proof of the bound for the second term in the left of (7.29).
Using (4.10) and the arguments above, this in turn implies that the third term in (7.29) is controlled by its right side, which completes the proof of the lemma. \qed

By (7.27) and the lemma, we get (7.21). The inductive argument using the lemma also yields (5.21) which completes the proof of Theorem 1.1.

References


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