Notes on Stein–Sahi Representations
and some Problems of non $L^p$ Harmonic Analysis

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Notes on Stein-Sahi representations and some problems of non $L^2$ harmonic analysis

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We discuss one natural class of kernels on pseudo-Riemannian symmetric spaces.

Recently, Oshima [67] published his formula for $\alpha$-function for $L^2$ on pseudo-Riemannian symmetric spaces (see also works of Delorm [12] and van den Ban-Schlichtkrull [3], [4]). After this, there arises a natural question about other solvable problems of non-commutative harmonic analysis.

In the Appendix to the paper [57], the author proposed a series of non $L^2$-inner products in spaces of functions on pseudo-Riemannian symmetric spaces and conjectured that this object is reasonable and admits an explicit harmonic analysis.

In this work, we discuss the problem in more details, in particular, we obtain the Plancherel formula for these kernel for Riemannian symmetric spaces $U(n)$, $U(n)/O(n)$, $U(2n)/Sp(n)$. We also give a new proof of Sahi’s results [76].

0. Introduction

6.1. Inner products defined by kernels. Starting the famous works of Bargmann [5] and Gelfand-Naimark [21], various inner products having the form

$$\langle f_1, f_2 \rangle = \int_{G/H \times G/H} K(x, y) f_1(x) \overline{f_2(y)} \, dx \, dy$$  \hspace{1cm} (0.1)

are quite usual in the theory of unitary representations. Here $G/H$ is a homogeneous space and $K(x, y)$ is a distribution (a ‘kernel’) on $G/H \times G/H$. The group $G$ acts in a space of functions on $G/H$ by transformations having the form

$$\rho(g)f(x) = f(gx)\gamma(g, x),$$  \hspace{1cm} (0.2)

where $\gamma(g, x)$ is some function (‘multiplier’) on $G \times G/H$. We intend to obtain action a unitary irreducible representation; under this requirement, the kernel $K(x, y)$ is uniquely determined by the explicit expression for the multiplier $\gamma(g, x)$. An actual evaluation of the kernel is not difficult.

But the scalar square $\langle f, f \rangle$ of a function $f$, i.e., the integral

$$\langle f, f \rangle = \int_{G/H \times G/H} K(x, y)f(x)\overline{f(y)} \, dx \, dy$$

has no visible reasons to be positive. Usually, positive definiteness of a given inner product of the form (0.1) is a nontrivial problem.

Example. We consider group SU(1, 1) consisting of 2 times 2 matrices $\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$, where $|a^2| - |b^2| = 1$. It acts in the space of functions
on the circle $|z| = 1$ (or $z = \epsilon^{ip\chi}$) by the formula
\[
\rho_\sigma \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) f(z) = f \left( \frac{az + b}{cz + d} \right) \overline{\rho_\sigma(z)} \left| z \right|^{1+s},
\]
(0.3)

These operators preserve the inner product given by
\[
\langle f, g \rangle = \int \int |z_1 - z_2|^{-1-s} f_1(z_1) \overline{f_2(z_2)} \, d\varphi_1 \, d\varphi_2
\]
(0.4)

This inner product is positive definite iff $-1 < s < 1$.

The inner product (0.4) is a Sobolev inner product. Also, the Sobolev inner products in spaces of functions on spheres appears in the representation theory of $SO(1, n)$. Some anisotropic Sobolev spaces arise in the representation theory of rank 1 groups. But usually inner products (0.1) define functional spaces that are unknown in the analysis. The spaces discussed in this paper can be considered as some kind of Sobolev spaces of matrix variables.

**6.2. Stein–Sahi complementary series.** In 1967, E. Stein [81] constructed an extremely degenerated complementary series of unitary representations of $GL(2n, \mathbb{C})$. Recall his construction. Consider the space $Mat_n(\mathbb{C})$ consisting of complex $n \times n$ matrices. Consider the group $GL(2n, \mathbb{C})$ consisting of $(n + 1) \times (n + 1)$ invertible complex matrices $g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$. This group acts on $Mat_n(\mathbb{C})$ by linear-fractional transformations
\[
z \mapsto \left( a + zc \right)^{-1} (b + zd).
\]
(0.5)

Fix $\sigma \in \mathbb{R}$ Consider the action of $GL(2n, \mathbb{C})$ in the space of functions on $Mat_n(\mathbb{C})$ by the operators
\[
\rho_\sigma(g) f(z) = f \left( \left( a + zc \right)^{-1} (b + zd) \right) |\det(a + zc)|^{-2n-\sigma}.
\]

Define the following Hermitian form in the space $D(Mat_n(\mathbb{C}))$ of smooth functions on $Mat_n(\mathbb{C})$ with compact support
\[
\langle f, g \rangle_\sigma = \int \int_{Mat_n(\mathbb{C}) \times Mat_n(\mathbb{C})} |\det(x - y)|^{-2n+\sigma} f(x) \overline{g(y)} \, dx \, dy.
\]
(0.6)

For $\sigma > 2n$ this integral converges, further we consider its meromorphic continuation in $\sigma$ to the whole plane $\sigma \in \mathbb{C}$.

Stein proved that for $-1 < \sigma < 1$,

1. the Hermitian form $\langle \cdot, \cdot \rangle_\sigma$ is positive definite; denote by $\mathcal{H}_\sigma$ the completion of $D(Mat_n(\mathbb{C}))$ with respect to $\langle \cdot, \cdot \rangle_\sigma$.
2. the operators $\rho_\sigma(g)$ are unitary in $\mathcal{H}_\sigma$.

The Stein-type constructions exist for all the series of classical groups, precisely for the groups

\[
GL(2n, \mathbb{C}), GL(2n, \mathbb{R}), GL(2n, \mathbb{R}),
\]
(0.7)

\[
O(2n, 2n), U(n, n), Sp(n, n),
\]
(0.8)

\[
SO(4n, \mathbb{C}), Sp(2n, \mathbb{C}), SO^*(4n, \mathbb{C}), Sp(2n, \mathbb{R}).
\]
(0.9)

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The cases (0.7) were considered in the classification work of Vogan ([87], Section 2), see also [75]. Sahi [76]-[77] constructed analogs of the Stein representations in the remaining cases.


The explicit Plancherel formula for $L^2$ on Riemannian noncompact symmetric spaces was obtained by Gindikin and Karpelevich [23] in 1964. In 1978, Berezin [7] observed that the space $L^2$ on a classical Hermitian symmetric space admits a natural deformation.

For definiteness, consider the Hermitian symmetric space

$$G/K = U(p,q)/U(p) \times U(q), \quad p \leq q$$

Following E. Cartan, we realize this space as the space $\mathcal{B}_{p,q}$ of $p \times q$ complex matrices with norm < 1, the group $U(p,q)$ acts on this space by the linear-fractional transformations $z \mapsto (a + z c)^{-1}(b + zd)$. Consider (see [57] for details) the space $\mathcal{D}(\mathcal{B}_{p,q})$ of smooth function on $\mathcal{B}_{p,q}$ with compact support. Denote the representation of the group $U(p,q)$ in this space by the shifts

$$\rho_\sigma \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) F(z) = F \left( (a + z c)^{-1}(b + zd) \right).$$

For $\sigma \in \mathbb{R}$, define the inner product in $\mathcal{D}(\mathcal{B}_{p,q})$ by

$$\langle F_1, F_2 \rangle_\sigma = \iint_{\mathcal{B}_{p,q} \times \mathcal{B}_{p,q}} \det(1 - z z^*)^\sigma \det(1 - u u^*)^\sigma \det(1 - z w^*)^\sigma \frac{F_1(z)}{F_2(u)} \frac{d\sigma(z)}{d\sigma(u)},$$

where

$$d\sigma(z) = \det(1 - z z^*)^{-\sigma - p} \prod_{1 \leq k \leq p, 1 \leq \ell \leq q} \frac{1}{2i} dz_k \overline{dz_{\ell}}$$

is the $U(p,q)$-invariant measure on $\mathcal{B}_{p,q}$. A simple calculation shows that the form (0.11) is $U(p,q)$-invariant.\footnote{The first work that can be attributed to this subject is Vershik, Gelfand, Graev, [84], [20]. These authors apply the representations $\rho_\sigma$ of the groups $U(1,q)$ to construct representations of current groups for a collection of other early references, see [56].}

It turns out to be that for

$$\sigma = 0, 1, \ldots, p - 1, \quad \text{or} \quad \sigma > p - 1,$$

our inner product is non-negative definite.

The latter statement is a reformulation of the well-known theorem (Berezin–Gindikin–Rossi–Vergne–Wallach) on unitarizability of scalar highest weight representations ([6], [83], see a relatively recent exposition in [18]).
Consider the completion $\mathcal{H}_\sigma$ of the space $\mathbb{D}(\mathbb{B}_{p,q})$ with respect to the inner product $(0,1)$, and the unitary representation $(0,10)$ of $U(p,q)$ in this space.\textsuperscript{3}

At first sight, the representation $\rho_\sigma$ given by the formula $(0.10)$ does not include $\sigma$. Really, for sufficiently large $\sigma$, all the representations $\rho_\sigma$ are equivalent to the standard representation of $U(p,q)$ in $L^2(U(p,q)/U(p) \times U(q))$. Moreover, the natural limit of the spaces as $\sigma \to \infty$ is $L^2[U(p,q)/U(p) \times U(q)]$

But the Plancherel formula for $\rho_\sigma$ depend on $\sigma$. For sufficiently small $\sigma > 0$, the spectrum of $\rho_\sigma$ undergo several bifurcations.\textsuperscript{4} It is possible to obtain an explicit decomposition of the Hilbert space $\mathcal{H}_\sigma$ into pieces with uniform spectra, see [55].

Next, $\rho_\sigma$ admits a natural analytic continuation to negative integer $\sigma$, and the corresponding limit as $\sigma \to -\infty$ is $L^2[U(p + q)/U(p) \times U(q)]$. Thus, we obtain some kind of interpolation between

$$L^2[U(p,q)/U(p) \times U(q)] \quad \text{and} \quad L^2[U(p + q)/U(p) \times U(q)].$$

In his short note [7], 1978, Berenstein gave (without proofs) the explicit Plancherel formula for representations $\rho_\sigma$ (in the case of Hermitian symmetric spaces) for sufficiently large $\sigma$ (before the start of bifurcations). Berenstein died soon after this, and the first known proof of his theorem was published by Unterberger and Upmeier [82], 1994.

In preprints [54], [56], 1999, the author considered the Berenstein representations for arbitrary classical Riemannian symmetric spaces, and obtained the explicit Plancherel formula for arbitrary $\sigma$.\textsuperscript{5} \textsuperscript{6}

Some steps in analysis after the Plancherel formula were undertaken in [57], [59], [58]. Also in [60] it was obtained a $p$-adic analog of Berenstein kernels.\textsuperscript{7}

Since there exists a reach non-$L^2$-analysis on Riemannian symmetric space, there arises the following question:

— are there other homogeneous spaces $G/H$ and other kernels $K(x, y)$ admitting an explicit and interesting harmonic analysis?

Since we discuss the harmonic analysis, a representation $\rho$ having the form (0.2) must be reducible, and hence the distribution $K(x, y)$ in the formula $(0.1)$ is not uniquely defined. At first sight, a choice is too large.

Indeed, for each function $K(z, u)$ on $\mathbb{B}_{p,q} \times \mathbb{B}_{p,q}$ satisfying

$$L[(a + zd)^{-1}(b + zd), (a + uc)^{-1}(b + ud)] = L[z, u],$$

\textsuperscript{3}This representation is equivalent to the tensor product of a scalar highest weight representation of $U(p,q)$ and the complex conjugate representation; it is natural to consider a similar and more general problem about the tensor product of a highest weight and lowest weight representation, in this case, the Plancherel formula was obtained by Zhang, [88].

\textsuperscript{4}In the work of Okland and the author [64] (see also [55], [58]), this phenomenon was applied to construction of exotic unitary representations of $O(p,q)$ and $U(p,q)$.

\textsuperscript{5}Some modifications of proofs of results of [54] were obtained later in [13] and [80], see also [13], [31].

\textsuperscript{6}Apparently the Plancherel Formula for all the groups can be reduced to some single identity with the Heckman-Opdam spherical hypergeometric transform (see, for instance, [28]), as far as I know, this possibility is yet not realized. Apparently also, that the Plancherel formula for Berenstein kernels can be interpolated as an exotic Plancherel formula for Heckman-Opdam spherical transform (this is correct in rank 1 case, see [58].

\textsuperscript{7}Only for Bruhat-Tits buildings of the series $A_n$. 
the expression

$$\langle F_1, F_2 \rangle = \iint_{\mathbb{R}^n \times \mathbb{R}^n} L(z, u) F_1(z) \overline{F_2(u)} \, d\sigma(z) \, d\sigma(u)$$

(0.13)

is invariant with respect to the translation operators

$$F(z) \mapsto F((a + z c)^{-1}(b + zd)).$$

A kernel $L$ satisfying (0.12) can be represented in the form

$$L(z, u) = \ell(\lambda_1, \ldots, \lambda_p),$$

where $\lambda_j$ are the eigenvalues of the matrix

$$(1 - zz^*)^{-1}(1 - z u^*)(1 - uu^*)^{-1}(1 - uz^*)$$

and $\ell$ is a symmetric function on the octant $\lambda_j > 0$. There are many such functions, and hence there are too much invariant kernels $L$.

Nevertheless, the existence of explicit harmonic analysis is a non-formal, but very strong restriction; some experience in the analysis on rank 1 symmetric spaces (or reading the book [70]) shows that a collection of possibilities is not large; for spaces of rank $> 1$ situation became strongly rigid and even an existence of examples is not obvious.

0.4. **Natural kernels on pseudo-Riemannian symmetric spaces.** In the first approximation, the problem that we formulate is a problem of restriction of Stein-Sahi representations to some special symmetric subgroups.

More precisely, for any classical symmetric space $G/H$ there exists an overgroup $\tilde{G} \supset G$ acting locally on $G/H$ ([52]). If $\tilde{G}$ is in the list (0.7–0.9) (and its action on $G/H$ is locally equivalent to a fractional-linear action on a matrix space $\mathcal{M}$, see below Section 4), then our problem is a problem of restriction of a Stein representation of $\tilde{G}$ to the subgroup $G$. In fact, we can forget about the group $G$ and consider only the restriction $K(\cdot, \cdot)$ of the Stein kernel to the symmetric space $G/H$ and action of $G$ in the Hilbert space defined by this kernel. For some value of the parameter $s$, we obtain the natural action of $G$ in $L^2(G/H)$.

It is possible to define the kernel $K(\cdot, \cdot)$ in the terms of the symmetric spaces $G/H$ themselves; this allows to extend our problem to the case, then $\tilde{G}$ is not in the list (0.7)–(0.9). But existence of an island of positivity of the form in these cases becomes a problem.

0.5. **Structure of the paper.** In Section 1, we discuss our kernels for the space $G/H \simeq U(n) \times U(n)/U(n)$. This also gives a realization of the Stein-Sahi representations of $U(n, n)$ (For this case, more detailed discussion including the unipotent representations is contained in [62]).

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*There is a lot of papers of Molchanov on this subject, for instance [39], [41], [14]*
In Section 2, we repeat the construction for spaces $U(n)/O(n)$, $U(2n)/Sp(n)$ and for their overgroups $Sp(2n,\mathbb{R})$, $SO^*(4n)$.

The proof of positivity is contained Section 3, it is based on the explicit expansions of the kernels in spherical functions. This done uniformly using Kadell's generalized Selberg integral.

In Section 4, we briefly discuss the remaining series of classical groups.

In Section 5, we formulate in details the problem discussed above in 0.5. We also try to explain why the problem looks as solvable.

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0.7. Notations.

Let $A = \{a_{ij}\}$ be a matrix. Then $A'$ is the transposed matrix, and $A^*$ is the adjoint matrix, i.e., matrix with elements $\overline{a_{ij}}$. Also, we denote by $\overline{A}$ the element-wise conjugate matrix, i.e., the matrix with the matrix elements $\overline{a_{ij}}$.

A matrix $x$ over $\mathbb{R}$, $\mathbb{C}$ is symmetric if $x = x^t$ and skew-symmetric if $x = -x$.

A matrix over $\mathbb{R}$, $\mathbb{C}$, or quaternions $\mathbb{H}$ is Hermitian if $x = x^*$ and anti-Hermitian if $x = -x^*$.

If $A$ is a Hermitian matrix, the notation $A > 0$ means that $A$ is positive definite, i.e., $h^*Ah > 0$ for each vector-row $h$.

The symbol $\|z\|$ denotes the norm of an operator $z$ in a Euclidean space. The value $\|z\|^2$ coincides with the maximal eigenvalue of $z^*z$.

We use the standard notation for the Pochhammer symbol

$$(a)_k := a(a + 1) \ldots (a + k - 1)$$

For a complex number $x$ we use the following notation for its powers

$$z^{\{r\hbar^\sigma\}} := z^r x^{r^\sigma} \quad \text{(0.14)}$$

The symbol $[a]$ denotes the integer part of $a$.

1. Sobolev kernels on the spaces $U(n)$ and unitary representations of $U(n, n)$.

On construction of this section, see also [62].

1.1. The group $U(n)$. The unitary group $U(n)$ is a compact Riemannian symmetric space

$$G/K = U(n) \cong U(n) \times U(n)/U(n).$$

Indeed, the group $G := U(n) \times U(n)$ acts on $U(n)$ by left and right multiplications

$$(h_1, h_2) : z \mapsto h_1^{-1}zh_2.$$
The stabilizer $K \simeq U(n)$ of the point $z = 1$ consists of elements $(h, h) \in U(n) \times U(n)$. Using transformations $z \mapsto h^{-1}zh$, each element of $U(n)$ can be reduced to the diagonal form
\[
\begin{pmatrix}
\epsilon^{i\varphi_1} & 0 & \cdots & 0 \\
0 & \epsilon^{i\varphi_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \epsilon^{i\varphi_n}
\end{pmatrix}.
\] (1.1)

1.2. Overgroup. The space $G/K$ admits a larger group of symmetries, namely the pseudounitary group $U(n, n)$. Recall that $U(n, n)$ is the group of $(n + n) \times (n + n)$ matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ preserving Hermitian form $H(\cdot, \cdot)$ with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, i.e.,
\[
H(v \oplus w, v' \oplus w') := \sum_{j=1}^{n} v_j \overline{v_j'} - \sum_{j=1}^{n} w_j \overline{w_j'}.
\]
In other words, the matrix $g$ satisfies the condition
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (1.2)

The group $U(n, n)$ acts on the space $U(n)$ by linear fractional transformations
\[
z \mapsto z^g := (a + ze^{-1})(b + zd).
\] (1.3) linear-fractional

**Lemma 1.1** *Let* $z \in U(n)$, $g \in U(n, n)$. *Then* $z^g \in U(n)$.

1.3. Proof of Lemma 1.1. Identification of $U(n)$ with a Grassmannian. For $z \in U(n)$ consider its graph $V_z \subset \mathbb{C}^n \oplus \mathbb{C}^n$. A vector $v \oplus w \in \mathbb{C}^n \oplus \mathbb{C}^n$ is an element of $V_z$ if $w = vz$.

For $v, v' \in \mathbb{C}^n$, we have
\[
H(v \oplus vz, v' \oplus vz) = \langle v, v' \rangle_{\mathbb{C}^n} - \langle vz, v'z \rangle_{\mathbb{C}^n} = \langle v, v' \rangle_{\mathbb{C}^n} - \langle v, v' \rangle_{\mathbb{C}^n} = 0,
\]
where $\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$ denotes the standard inner product in $\mathbb{C}^n$. Hence $V_z$ is a maximal $H$-isotropic subspace of $\mathbb{C}^n \oplus \mathbb{C}^n$.

Conversely, let $V$ be a maximal $H$-isotropic subspace in $\mathbb{C}^n \oplus \mathbb{C}^n$. Since the form $H$ is strictly negative on $0 \oplus \mathbb{C}^n$, we have $V \cap (0 \oplus \mathbb{C}^n) = 0$. But $\dim V = n$,

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\(^{\text{3Consider a linear space } L \text{ equipped with bilinear (sesquilinear) form } B(\cdot, \cdot). A subspace } M \subset L \text{ is isotropic if } B(h, h') = 0 \text{ for all } h, h' \in M.\)
and hence $V$ is a graph of some linear operator $z : \mathbb{C}^n \oplus 0 \to 0 \oplus \mathbb{C}^n$. By the isotropy condition,

$$0 = H(v \oplus vz, v' \oplus v'z) = H(v \oplus 0, v' \oplus 0) - H(0 \oplus vz, 0 \oplus v'z)$$

and hence $z \in U(n)$.

Thus we obtain the natural identification

$$\begin{align*}
\{ & \text{The Grassmannian of maximal} \} \\
\{ & H\text{-isotropic subspaces in } \mathbb{C}^n \oplus \mathbb{C}^n \} \leftrightarrow \begin{Bmatrix} & \text{The space } U(n) = \\
= & U(n) \times U(n)/U(n) \end{Bmatrix}
\end{align*}$$

The group $U(n, n)$ preserves the Hermitian form $H(\cdot, \cdot)$ and hence it transfer isotropic subspaces to isotropic subspaces. Thus $U(n, n)$ acts in a natural way on $U(n)$. It remains to write an explicit formula for this action.

**Lemma 1.2** We have $gV_z = V_u$, where $u = z^{[a]} := (a + zc)^{-1}(b + zd)$.

**Proof.** We have $v \oplus vz \in V_z$. Applying $g$, we obtain

$$v(a + zc) \oplus v(c + zd) \in gV_z.$$ Denoting $w = v(a + zc)$, we obtain the required statement. \hfill \square

1.4. Jacobians.

**Lemma 1.3** Denote by $\mu(z)$ the Haar measure on the group $U(n)$, denote by $\mu(z^{[a]})$ its image under the transformation $z \mapsto z^{[a]}$, given by (1.2). Then

$$\mu(z^{[a]}) = |\det(a + zc)|^{-2n} \mu(g) \quad (1.4)$$

**Proof.** Let $h \in U(n) \times U(n) \subset U(n, n)$ be an element of the form

$$h = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}. \quad (1.5)$$

Then $z^{[b]} = u^{-1}zv$. The Haar measure is invariant with respect to such transformations.

Denote

$$\gamma(g, z) := |\det(a + zc)|^{-2n}. \quad (1.6)$$

This expression satisfies the identity

$$\gamma(h_1gb_2, z^{[a]^{-1}}) = \gamma(g, z)$$

for $h_1, h_2$ having the form (1.5). Due the invariance of the Haar measure, the Jacobian (or Radon-Nykodem derivative)

$$\mu(z^{[a]})/\mu(z)$$

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satisfies the same condition.

Hence, without loss of generality, we can assume that \( z = 1, z^{[b]} = 1 \). The
tangent space to \( U(n) \) at this point consists of matrices \( \Delta \) satisfying \( \Delta + \Delta^* = 1 \).
We denote this space of all the anti-Hermitian matrices by AHerm.

It is easy to show (see, for instance, [54], Lemma 1.1), that the differential of a linear-fractional transformation \( z \mapsto z^{[b]} \) is given by

\[
\Delta \mapsto (a + zc)^{-1} \Delta (-cz^{[a]} + d),
\]

this formula is valid on whole space of \( n \times n \) matrices for \( g \in \text{GL}(2n, \mathbb{C}) \), in
particular it can be used in our case.

Now \( z = z^{[b]} = 1 \) and hence \( a + c = b + d \). This equation, together with
(1.2) allows to transform (1.7) to the form

\[
\Delta \mapsto (a + c)^{-1} \Delta [(a + c)^{-1}]^*.
\]

It remains to evaluate the determinant \( \delta(P) \) of the linear transformation
AHerm(\( n \)) \( \to \) AHerm(\( n \)) given by

\[
\Delta \mapsto P \Delta P^*, \quad P \in \text{GL}(n, \mathbb{C}).
\]

Obviously, \( \delta(P) \) is a homomorphism from \( \text{GL}(n, \mathbb{C}) \) to the multiplicative group
of positive numbers. Hence \( \delta(P) = 1 \) on \( U(n) \). Each \( P \in \text{GL}(n, \mathbb{C}) \) can be
represented in the form \( u \Lambda v \), where \( u, v \in U(n) \), and \( \Lambda \) is a diagonal matrix.
Obviously \( \delta(\Lambda) = |\det(\Lambda)|^{2n} \) and hence \( \delta(P) = |\det(P)|^{2n} \). This finishes the
proof. \( \square \)

1.5. Invariant kernels on \( U(n) \). Denote by \( \mathcal{B}_n \) the set of \( n \times n \) matrices
\( z \) with norm \( < 1 \); an equivalent condition is \( 1 - z^* z > 0 \) By \( \overline{\mathcal{B}}_n \) denote the set
of all \( n \times n \) matrices with norm \( \leq 1 \); an equivalent condition is \( 1 - z^* z \geq 0 \).

Fix \( \sigma \in \mathbb{C} \). For \( z \in \mathcal{B}_n \), we define

\[
(1 - z)^{\sigma} := \sum_{j=0}^{\infty} \frac{(-\sigma)_j}{j!} z^j,
\]

the series in the right-hand side is convergent.

Fix \( \sigma, \tau \in \mathbb{C} \). We define the function \( \det(1 - z)^{[\sigma, \tau]} \) depending in the
variable \( z \in \mathcal{B}_n \) by

\[
\det(1 - z)^{[\sigma, \tau]} = \det (1 - z^{\sigma}) \det (1 - z^{\tau}).
\]

This expression is a smooth \( \mathbb{C} \)-valued function on \( \mathcal{B}_n \). For \( z \in \overline{\mathcal{B}}_n \), we define

\[
\det(1 - z)^{[\sigma, \tau]} := \lim_{\varepsilon \to +0} \det (1 - (1 - \varepsilon) z)^{[\sigma, \tau]}.
\]

This limit exists outside the surface \( \det (1 - z) = 0 \), and the expression (1.11) is a smooth function on the set

\[ 1 - zz^* \geq 0, \quad \det (1 - z) \neq 0. \]
Obviously, for a unitary matrix \( h \in U(n) \),
\[
\det(1 - hzh^{-1})^{[\sigma, \tau]} = \det(1 - z)^{[\sigma, \tau]}.
\]

**Lemma 1.4** For \( z \in U(n) \), denote by \( e^{i\psi_1}, \ldots, e^{i\psi_n} \) its eigenvalues. Then
\[
\det(1 - z)^{[\sigma, \tau]} = \exp\left\{ (\sigma - \tau) \sum_{k=1}^{n} (\psi_k - \pi)/2 \right\} \cdot \prod_{k=1}^{n} |\sin(\psi_k/2)|^{\sigma + \tau}.
\]

This statement is more-or-less obvious, for a formal proof, see [62].

We define the kernel \( L_{\sigma, \tau}(z, u) \) on \( U(n) \) by
\[
L_{\sigma, \tau}(z, u) = \det(1 - zu^*)^{[\sigma, \tau]}, \quad z, u \in U(n). \tag{1.12}
\]

**Lemma 1.5** a) For \( \tau, \sigma \in \mathbb{R} \), the kernel \( L_{\sigma, \tau}(z, u) \) is Hermitian, i.e.,
\[
L_{\sigma, \tau}(u, z) = \overline{L_{\sigma, \tau}(z, u)}.
\]

b) The kernel \( L_{\sigma, \tau}(z, u) \) is \( U(n) \)-invariant, i.e.,
\[
L_{\sigma, \tau}(h_1zh_2, h_1uh_2) = L_{\sigma, \tau}(z, u), \quad h_1, h_2 \in U(n)
\]

The both statements are obvious.

We define the sesquilinear form \( I_{\sigma, \tau} \) on \( C^\infty(U(n)) \) given by
\[
I_{\sigma, \tau}(F_1, F_2) = \iint_{U(n) \times U(n)} L_{\sigma, \tau}(z, u) \overline{F_1(z)F_2(u)} \, d\mu(z) \, d\mu(u) \tag{1.13}
\]

The form is Hermitian. The integral is convergent if \( \text{Re}(\sigma + \tau) > -1 \). By general reasons, the integral admits a meromorphic continuation to the whole plane \( (\sigma, \tau) \in \mathbb{C}^2 \) (see [9], [2]). In fact, our proof of positivity given below is based on an the explicit construction of this meromorphic continuation.

**1.6. Action of \( U(n, n) \)**

**Lemma 1.6** The operators \( \rho_{\sigma, \tau} \) given by
\[
\rho_{\sigma, \tau} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) F(z) = F \left[ (a + ze)^{-1} (b + zd) \right] \det(a + ze)^{-[\sigma - \tau - n - \tau]} \tag{1.14}
\]

preserve the form \( I_{\sigma, \tau} \).

Before starting a proof, we give some preliminary comments on formula (1.14).

1. We must define these operators more carefully. First,
\[
\det(a + ze)^{-[\sigma - \tau - n - \tau]} = |\det(a + ze)|^{-2n}.
\]
and this expression is well-defined. Further

\[ \det(a + z\sigma)^{[-\sigma ||^{-\tau}]_a} = \det(a)^{[-\sigma ||^{-\tau}]_a} \det(1 + zca^{-1})^{[-\sigma ||^{-\tau}]} \]  

(1.15)

Since \( ||ca^{-1}|| < 1 \) (this easily follows from (1.2)), we have also \( ||zca^{-1}|| < 1 \). Thus the factor \( \det(1 + zca^{-1})^{[-\sigma ||^{-\tau}]} \) is well defined, see (1.9), (1.10). It remains to define

\[ \det(a)^{[-\sigma ||^{-\tau}]} = |\det(a)|^{-\sigma - \tau} \exp \{ i(\tau - \sigma) \arg(a) + 2\pi i(\tau - \sigma)k \} \],

where \( k \) ranges in \( \mathbb{Z} \). If \( (\sigma - \tau) \in \mathbb{Z} \), then the last expression is well defined, and \( \rho_{\sigma, \tau} \) is a linear representation of \( U(p, q) \).

Otherwise, the multi-valued function (1.15) on \( \mathcal{B}_u \) splits into a countable family of smooth branches\(^{10}\); these branches differ by constant factors \( \exp \{ 2\pi i(\tau - \sigma)k \} \). Thus the formula (1.14) for a given \( q \), defines a countable family of operators \( \rho_{\sigma, \tau} \), which differ one from another by constant factors. We can choose one such operator in an arbitrary way, and then we will obtain a unitary projective representation (see, for instance, [35], Section 14) representation of \( U(n, n) \),

\[ \rho_{\sigma, \tau}(g)\rho_{\sigma, \tau}(h) = \gamma(g, h)\rho_{\sigma, \tau}(gh), \quad \gamma(h, g) \in \mathbb{C}^* \]  

(1.16)

2. Since \( \det(a) \) does not vanish, the function \( \ln \det(a) \) is a well-defined function on the universal covering \( U(n, n)^\gamma \) of \( U(n, n) \). Hence the expression (1.15) is a well defined single-valued expression on \( U(n, n)^\gamma \); so we can consider \( \rho_{\sigma, \tau} \) as a linear representation of \( U(n, n)^\gamma \),

\[ \rho_{\sigma, \tau}(g)\rho_{\sigma, \tau}(h) = \rho_{\sigma, \tau}(gh). \]  

(1.17)

3. We must prove the identities (1.16), (1.17). A direct calculation is not difficult, but it is more reasonable to avoid it. Denote

\[ \nu(g, z) := \det(a + z\sigma)^{[-\sigma ||^{-\tau}]_a}. \]

The desired identities are reduced to the "cocycle identity"

\[ \nu(gh, z) = \nu(h, z)\nu(g, z^{[h]}). \]  

(1.18)

But \( \nu(g, h) \) is a power of the expression \( \gamma(g, z) \) given by (1.6), the latter expression is the Jacobian. For a Jacobian, the cocycle identity is obvious.

4. Let \( \tau = 0 \). Then our construction gives a representation of the Harish-Chandra holomorphic series. The kernel \( L_{\sigma, \beta} \) is the standard reproducing kernel for these representations, see, for instance, [6], [83], [64]. Representations of holomorphic series admit realizations (see [64]) in spaces of holomorphic functions, in spaces of distributions on matrix balls \( \mathcal{B}_n \), and in spaces of distributions

\(^{10}\)In particular, our operators preserve the space \( C^\infty (U(n)) \)
on Shilov boundary $U(n)$ of the matrix ball. The last variant corresponds to our realization.

For $\sigma = 0$, we obtain a lower weight representation.

5. Our construction is invariant with respect to the shift

$$(\sigma, \tau) \mapsto (\sigma + 1, \tau - 1).$$

We will not refer to this remark below, for discussion, see [62], 2.9.

1.7. Proof of Lemma 1.6. We use the following simple identity

$$\det (1 - z^{[\beta]} [u^{[\beta]}]^{*}) = \det (1 - z u^{*}) \det (a + z c)^{-1} \overline{\det (a + u c)^{-1}}$$

valid for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(n, n)$. Keeping it in mind and using the formula (1.4) for the Jacobian, we substitute $z \mapsto z^{[\beta]}$, $u \mapsto u^{[\beta]}$ to integral (1.13) and obtain

$$\int_{U(n) \times U(n)} L_{\sigma, \tau}(z, u) \det (a + z c)^{[-\sigma - \tau]} \overline{\det (a + u c)^{[-\sigma - \tau]}} \times$$

$$\times F_{1}(z^{[\beta]}) F_{2}[u^{[\beta]}] \det (a + z c)^{-2n} \det (a + u c)^{-2n} d\mu(z) d\mu(u)$$

Q.E.D.

1.8. Unitary representations of $U(n, n)$.

Theorem 1.7 Let $\sigma, \tau \notin \mathbb{Z}$. The Hermitian form $L_{\sigma, \tau}$ given by (1.13) is definite\footnote{Recall that $[x]$ denotes the integral part of $x$.}

$$[-\tau] = [\sigma + n].$$

Proof is given below in Section 3. More elementary proof is contained in [62]. This paper also contains a picture of a domain of positivity.

It is convenient to introduce new parameters $t, s$ by

$$\sigma = -n/2 + s, \quad \tau = -n/2 + t. \quad (1.19)$$

In this notation, the cases of even and odd $n$ are slightly different.

a) For an odd $n$ the condition of positivity is

$$|s - j| < 1/2, \quad |t - j| < 1/2, \quad \text{for some } j \in \mathbb{Z}.$$  

b) For an even $n$ the condition of positivity is

$$t \in [j, j + 1], \quad s \in [j - 1, j] \quad \text{for some } j \in \mathbb{Z}.$$
**Proposition 1.8** If \( s = 1 \), then the Hermitian form \( L_{\sigma, \tau} \) coincides with the \( L^2 \)-inner product

\[
\langle F_1, F_2 \rangle = \int_{\mathcal{U}(n)} F_1(z) \overline{F_2(z)} d\mu(z).
\]

See a proof in [62], this also can be easily derived from calculations of our Section 3.

Under the conditions of Theorem 1.7, we denote by \( \mathcal{A}_{\sigma, \tau} \) the completion of \( C^\infty(\mathcal{U}(n)) \) with respect to this form. We obtain that our representation \( \rho_{\sigma, \tau} \) in this case is unitary in \( H_{\sigma, \tau} \).

**2. Sobolev kernels on the spaces** \( U(n)/O(n), U(2n)/Sp(n) \) **and unitary representations of the groups** \( Sp(2n, \mathbb{R}), SO^*(4n) \)

Here we show that the space \( U(n)/O(n) \) can be realized as the space of unitary symmetric matrices and the space \( U(2n)/Sp(n) \) as the space of unitary skew-symmetric \( 2n \times 2n \)-matrices \(^{12}\). We show that the groups \( Sp(2n, \mathbb{R}) \) and \( SO^*(4n) \) respectively act on these spaces by linear-fractional transformations. Then we restrict the kernel \( L_{\sigma, \tau} \) defined by (1.12) and obtain unitary representations of \( Sp(2n, \mathbb{R}) \) and \( SO^*(4n) \).

**A. Spaces** \( U(n)/O(n) \).

**2.1. Symmetric spaces** \( U(n)/O(n) \). Now let \( \mathcal{X}_n \) be the space \( n \times n \) matrices \( z \) satisfying the conditions

\[ zz^* = 1, \quad z = z^t. \]

The group \( U(n) \) acts on \( \mathcal{X}_n \) by the transformations

\[ h : z \mapsto h^t z h. \]

The stabilizer of the point \( z = 1 \) is the real orthogonal group \( O(n) \). It can be easily verified that action of \( U(n) \) on \( \mathcal{X}_n \) is transitive. Thus,

\[ \mathcal{X}_n \cong U(n)/O(n). \]

Each element of \( \mathcal{X}_n \) can be reduced by the transformations \( z \mapsto h^t z h \), where \( h \in O(n) \), to the form

\[
\begin{pmatrix}
\epsilon^{i\varphi_1} & 0 & \cdots & 0 \\
0 & \epsilon^{i\varphi_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \epsilon^{i\varphi_n}
\end{pmatrix}, \tag{2.1}
\]

The collection \( \varphi_j \) is uniquely defined up to permutations.

\(^{12}\) A remark for experts. These spaces are realized as the Shilov boundaries of the bounded Cartan domains \( Sp(2n, \mathbb{R})/U(n), SO^*(4n)/U(2n) \), see [71].
A matrix $\Delta$ is an element of the tangent space to $X_n$ at $z = 1$ iff
\[\Delta = \Delta^t, \quad \text{and} \quad \frac{1}{\Delta} \text{ is a real matrix.}\]

2.2. **Overgroup.** Again, our space $X_n$ admits a larger group of symmetries, namely $\text{Sp}(2n, \mathbb{R})$.

To observe this, we realize the real symplectic group $\text{Sp}(2n, \mathbb{R})$ as a group of complex $(n + n) \times (n + n)$ matrices $g$ preserving the Hermitian form in $\mathbb{C}^n \oplus \mathbb{C}^n$ with the matrix \(\begin{pmatrix} 1 & 0 \\
0 & -1 \end{pmatrix}\) and the skew-symmetric bilinear form $B$ having the matrix
\[
\begin{pmatrix}
0 & 1 \\
-1 & 0 
\end{pmatrix}.
\] In other words,
\[
g^* \begin{pmatrix} 1 & 0 \\
0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\
0 & -1 \end{pmatrix} ; \quad g^t \begin{pmatrix} 0 & 1 \\
-1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\
-1 & 0 \end{pmatrix}.
\]

These two identities also imply that the matrix $g$ has the following block structure
\[
g = \begin{pmatrix}
\Phi & \Psi \\
\overline{\Psi} & \overline{\Phi} 
\end{pmatrix}.
\]

Equivalently $g$ preserves the real subspace of $\mathbb{C}^n \oplus \mathbb{C}^n$ consisting of vectors $h + \overline{t}$.

The group $\text{Sp}(2n, \mathbb{R})$ acts on $X_n$ by linear fractional transformations
\[
g : z \mapsto z^b : = (\Phi + z\overline{\Psi})^{-1}(\Psi + z\overline{\Phi}).
\]

**Lemma 2.1** Let $z \in X_n$, $g \in \text{Sp}(2n, \mathbb{R})$. Then $z^b \in X_n$.

**Proof.** As above, for a matrix $z \in X_n$, we consider its graph $V_z \subset \mathbb{C}^n \oplus \mathbb{C}^n$. Literally repeating the considerations of Subsection 1.2, we obtain that the subspace $V_z$ is maximal isotropic with respect to the Hermitian form $H$.

The condition $z = z^t$ means that $V_z$ is Lagrangian (=maximal isotropic) with respect to the skew-symmetric form $B$. Indeed,
\[
B(v \oplus vz, w \oplus wz) = vw - wzv,
\]
where $v, w$ are matrices-rows. Since $z = z^t$, we obtain zero in the right-hand side.

Thus, we obtain the identification
\[
\left\{ \begin{array}{l}
\text{Grassmannian of } n\text{-dimensional subspaces in } \mathbb{C}^n \oplus \mathbb{C}^n \\
\text{isotropic with respect to the both forms } H \text{ and } B
\end{array} \right\} \leftrightarrow
\left\{ \begin{array}{l}
The space $X_n = U(n)/O(n)$ of $n \times n$ matrices \\
z satisfying $z = z^t, z^* z = 1$
\end{array} \right\}.
\]

An element $g \in \text{Sp}(2n, \mathbb{R})$ preserves the both forms $H$, $B$ and hence it maps our isotropic Grassmannian to itself. \(\Box\).
We denote by $\mu$ the unique $U(n)$-invariant measure on $X_n$. The Jacobian of the transformation $z \mapsto z^{|z|}$ is given by the following formula

$$
\mu(z^{\bar{z}}) = \det (\Phi + z \Psi)^{-(n+1)} \mu(z) \quad (2.2)
$$

A proof is the same as above (Lemma 1.3).

2.3. Invariant kernels on $U(n)/O(n)$. Now fix $\sigma, \tau \in \mathbb{C}$. Consider the kernel $L_{\sigma, \tau}(z, u)$, where $z, u \in X_n$ the same kernel (1.12) restricted to $X_n$.

Obviously, the kernel $L_{\sigma, \tau}(z, u)$ is $U(n)$-invariant

$$
L_{\sigma, \tau}(h^T z h, h^T u h) = L_{\sigma, \tau}(z, u)
$$

We define the sesquilinear form on $C^\infty(X_n)$ by

$$
I_{\sigma, \tau}(F_1, F_2) = \iint_{X_n \times X_n} L_{\sigma, \tau}(z, u) \overline{F_1(z)} F_2(u) \, d\mu(z) \, d\mu(u). \quad (2.3)
$$

Again, we consider the meromorphic continuation of $I_{\sigma, \tau}$ to the domain $(\sigma, \tau) \in \mathbb{C}^2$.

As above, we define the linear operators

$$
\rho_{\sigma, \tau} \left( \begin{array}{c} \Phi \\ \Psi \\ \bar{\Phi} \end{array} \right) f(z) = f((\Phi + z \Psi)^{-1}(\Psi + z \bar{\Phi})) \det (\Phi + z \bar{\Psi})^{-(n+1)/2 - \sigma - (n+1)/2 - \tau}.
$$

As above, these operators preserve the form $I_{\sigma, \tau}$.

Theorem 2.2 Let $n > 1$. Let $2\sigma, 2\tau \in \mathbb{Z}$. The Hermitian form $I_{\sigma, \tau}$ is definite iff

$$
[-2\tau - n - 1] = [2\sigma].
$$

Proof is given below in Section 3.

It is convenient to define new parameters $s, t$ by

$$
\sigma = -(n + 1)/4 + s, \quad \tau = -(n + 1)/4 + t.
$$

The conditions of positivity are

a) For an even $n$,

$$
s, t \in [j/2 - 1/4, j/2 + 1/4] \quad \text{for some } j \in \mathbb{Z}.
$$

b) For an odd $n$

$$
s \in [j/2, j/2 + 1/4], \quad t \in [j/2 - 1/4, j/2] \quad \text{for some } j \in \mathbb{Z}.
$$

For $s = t$ the form $I_{\sigma, \tau}$ defines the $L^2$-inner product.

B. Spaces $U(2n)/Sp(n)$. 

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2.4. The space $U(2n)/Sp(n)$. Now we consider the space $Y_n$ of $2n \times 2n$ matrices $z$ satisfying the conditions

$$zz^* = 1, \quad z = -z^t.$$ 

An example of $z$ satisfying these conditions is

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{where} \quad 1 \text{ is the } n \times n \text{ unit matrix.} \quad (2.4)$$

The group $U(2n)$ acts on this space $Y_n$ by the transformations

$$h : \quad z \mapsto h^tzh.$$ 

The stabilizer of the point $J \in Y_n$ consists of matrices satisfying

$$hh^* = 1, \quad h^tJh = J.$$ 

These equations give one of the standard realizations of the compact symplectic group $Sp(n)$. It is easy to show that $Y_n$ is a homogeneous space

$$Y_n = U(2n)/Sp(n).$$

Using the transformations $z \mapsto h^tzh$, where $h \in Sp(n)$, we can reduce each element of $Y_n$ to the form

$$\Omega = \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix}, \quad \text{where} \quad \Lambda = \begin{pmatrix} e^{i\varphi_1/2} & 0 & \cdots & 0 \\ 0 & e^{i\varphi_2/2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\varphi_n/2} \end{pmatrix}. \quad (2.5)$$

The numbers are uniquely defined up to permutations and transformations $\varphi_i \mapsto -\varphi_i$.

2.5. Overgroup. We realize the group $SO^+(4n)$ as a group of complex $(2n+2n) \times (2n+2n)$ matrices $g$ preserving the Hermitian form in $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the symmetric bilinear form $B$ having the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

In other words,

$$g^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad g^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

These two identities also imply that the matrix $g$ has the following block structure

$$g = \begin{pmatrix} \Phi & \Psi \\ -\Psi^* & \Phi^* \end{pmatrix}.$$ 

The group $SO^+(4n)$ acts on $Y_n$ by the linear fractional transformations

$$g : \quad z \mapsto z^\omega := (\Phi - z\overline{\Psi})^{-1}(\Psi + z\overline{\Phi}). \quad (2.6)$$

linear-fractional-uω
Lemma 2.3 Let $z \in \mathcal{Y}_n, g \in SO^*(4n)$. Then $z^{[g]} \in \mathcal{Y}_n$.

Proof is the same as above. For a matrix $z \in \mathcal{Y}_n$, we consider its graph $V_z \subset \mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$. As it was shown in 1.3, the subspace $V_z$ is maximal isotropic with respect to the Hermitian form $H$. The condition $z = -z^*$ means that $V_z$ is maximal isotropic with respect to the symmetric form $B$.

Thus, we obtain the identification

$$\begin{align*}
\left\{ \text{Grassmannian of n-dimensional subspaces in } \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} \right\} & \longleftrightarrow \\
\left\{ \text{subspaces } V_z \right\} & \longleftrightarrow \\
\left\{ \text{the space } \mathcal{Y}_n = U(2n)/Sp(n) \text{ of } 2n \times 2n \text{ matrices} \right\} \\
& \quad \text{z satisfying } z = -z^*, z^* z = 1
\end{align*}$$

Now the statement became obvious.

2.6. Jacobian.

Lemma 2.4 For the linear-fractional transformations (2.6), we have

$$\mu(z^g) = |\det(\Phi - z\overline{\Psi})|^{-4n-2}\mu(z). \quad (2.7)$$

Proof. The both expressions,

$$\nu_1(g, z) := \det(\Phi - z\overline{\Psi}); \quad \nu_2(g, z) := \frac{\mu(z^g)}{\mu(z)}. \quad (2.8)$$

satisfy the cocycle identity (1.18) mentioned above.

Recall some standard facts about the solutions of the cocycle identity, see [35], 13.2. Let $G/R$ be a homogeneous space, let $u$ be a point fixed by the subgroup $R$. Let $\nu$ be a solution of the cocycle equation on $G/R$. Obviously, for $r_1, r_2 \in \mathbb{R}$,

$$\nu(r_1 r_2, u) = \nu(r_1, u) \nu(r_2, u),$$

i.e., the function $r \mapsto \nu(r, u)$ is a character of $R$. Moreover, this correspondence between solutions of (1.18) and characters of $R$ is a bijection.

In our case, $G = SO^*(4n)$ and $R$ is a maximal parabolic in $G$, the reductive part of $R$ is the group $GL(n, \mathbb{H})$. Each character of $GL(n, \mathbb{H})$ has the form $r \mapsto \det(r)^s$, and hence our question is reduced to an evaluation of $s$.

Since we must know the exponent $s$, it is sufficient to find (2.8) only for $z = J$ and some appropriate $g$ lying in the stabilizer of $J$.

We choose $g = r(t)$ being the block $(n + n + n + n) \times (n + n + n + n)$-matrix

$$r(t) := \begin{pmatrix}
\cosh t & 0 & 0 & \sinh t \\
0 & \cosh t & -\sinh t & 0 \\
0 & -\sinh t & \cosh t & 0 \\
\sinh t & 0 & 0 & \cosh t
\end{pmatrix}.$$
A direct calculation shows that $J^{[r]} = J$. Also

$$\det (\Phi - J \Psi)^{-1} = e^{2n t}.$$  \hspace{1cm} (2.9)

Applying the formula (1.7), we obtain that the differential of $z \mapsto z^{[r]}$ at $J$ is

$$\Delta \mapsto e^{4t} \Delta$$ \hspace{1cm} (2.10)

Since $\dim \mathfrak{y}_n = \dim U(2n) - \dim \mathfrak{sp}(n) = 2n^2 - n$, the determinant of the transformation (2.10) is $\exp(4n(2n-1)t)$. Comparing with (2.9), we obtain the exponent in (2.7). \quad \square

2.7. Unitary representations of $SO^{*}(2n)$.

We define the canonical invariant kernel on $\mathfrak{y}_n$ as the restriction of the kernel

$$L_{\sigma, \tau}(z, u) = \det(1 - z u^*)^{[\sigma, \tau]}$$

to $\mathfrak{y}_n$. We also define the Hermitian form on $C^\infty(\mathfrak{y}_n)$ by

$$L_{\sigma, \tau}(F_1, F_2) = \int_{\mathfrak{y}_n \times \mathfrak{y}_n} L_{\sigma, \tau}(z, u) F_1(z) \overline{F_2(u)} \, d\mu(z) \, d\mu(u).$$

**Theorem 2.5** Let $n > 1$. Let $\sigma, \tau \notin \mathbb{Z}$. The form $L_{\sigma, \tau}$ is sign definite iff

$$[-\tau] = [\sigma + 2n - 1].$$

The proof is contained in the next section.

We define the representations $\rho_{\sigma, \tau}$ of $SO^{*}(4n)$ in $C^\infty(\mathfrak{y}_n)$ by

$$\rho_{\sigma, \tau} \begin{pmatrix} \Phi & \Psi \\ -\overline{\Psi} & \overline{\Phi} \end{pmatrix} F(z) = F((\Phi - z \overline{\Psi})^{-1}(\Psi + z \overline{\Phi})) \det(\Phi - z \overline{\Psi})^{-1/2 - \sigma} \det(\Phi - z \overline{\Psi})^{-1/2 - \tau}.$$  \hspace{1cm}

These operators preserve the form $L_{\sigma, \tau}$. Under the conditions of Theorem 2.5, our representations are unitary.

It is also convenient to introduce new parameters $t, s$ by

$$\sigma = -n + 1/2 + s, \quad \tau = -n + 1/2 + t.$$  \hspace{1cm}

Then the condition of positivity is

$$|s - j| < 1/2, \quad |t - j| < 1/2, \quad \text{for some } j \in \mathbb{Z}.$$  \hspace{1cm}

3. Application of the Kadel integral

3.1. Jack polynomials. Preliminaries, for details see Macdonald, [37, VI.4, VI.10], and further references in this book. We consider $n$-dimensional torus $\mathbb{T}^n$ as a subset in $\mathbb{C}^n$ consisting of points

$$(x_1, \ldots, x_n), \quad |x_1| = 1, \ldots, |x_n| = 1.$$
It is also convenient to write
\[ x_1 = e^{i\psi_1}, \ldots, x_n = e^{i\psi_n}. \]

We denote by Sym\(_n\) the space of all polynomials in \(x_1, \ldots, x_n\) symmetric with respect to permutations of \(x_j\). We denote by LSym\(_n\) the space of all symmetric Laurent polynomials in \(x_1^{\pm 1}, \ldots, x_n^{\pm 1}\).

Fix \(\kappa > 0\). Consider the inner products in the spaces Sym\(_n\) and LSym\(_n\) given by
\[ \langle f, g \rangle_{\kappa} = \int_{\mathbb{T}^n} f(x) \overline{g(x)} \prod_{1 \leq k < \ell \leq n} |x_k - x_\ell|^{2\kappa} \prod_{k=1}^n d\varphi_k. \quad (3.1) \]

The Jack polynomials are orthogonal polynomials with respect to this scalar product. In a multivariate case, the Gramm–Schmidt orthogonalization of polynomials is not a canonical operation, and hence we must define them more carefully.

Let \(\lambda\) be a collection of integers (a Young diagram)
\[ \lambda: \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0. \quad (3.2) \]

Denote
\[ |\lambda| = \sum \lambda_j. \]

Let \(|\lambda| = |\mu|\). We say \(\lambda \geq \mu\) iff
\[ \lambda_1 + \cdots + \lambda_j \geq \mu_1 + \cdots + \mu_j \quad \text{for all } j. \]

We define the monomial symmetric function \(m_\lambda\) as
\[ m_\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n} + \ldots, \]
where \(\ldots\) is the sum of all pairwise distinct monomials \(x_{\sigma(1)}^{\lambda_{\sigma(1)}} \ldots x_{\sigma(n)}^{\lambda_{\sigma(n)}}\), where \(\sigma\) is a permutation.

The Jack polynomials \(P_\lambda(x) \in \text{Sym}_n\) are defined by two conditions:

1) \(P_\lambda\) are orthogonal with respect to the scalar product (3.1)
2) \(P_\lambda = m_\lambda + \sum_{\mu: |\mu| = |\lambda|, \mu < \lambda} u_\mu m_\mu \quad (3.3)\)

Uniqueness of the Jack polynomials is obvious but existence is a theorem.

The Jack polynomials satisfy the following identity
\[ P_{\lambda_1, \ldots, \lambda_n+1}(x) = x_1 \ldots x_n P_{\lambda_1, \ldots, \lambda_n}(x). \]

This allows to define Laurent Jack polynomials \(P_\lambda \in \text{LSym}_n\) for
\[ \lambda: \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n, \quad \lambda_j \in \mathbb{Z} \quad (3.4) \]

(we permit negative \(\lambda_j\)) by signatures


\[ P_{\lambda_1, \ldots, \lambda_n}(x) = (x_1 \ldots x_n)^{-m} P_{\lambda_1 + m, \ldots, \lambda_n + m}(x), \]

where \( m \) is sufficiently large (\( \lambda_n + m \geq 0 \)).

This system of polynomials forms an orthogonal basis in \( \mathrm{LSym}_n \) with respect to the scalar product (3.1).

3.2. Radial functions. Consider one of our symmetric spaces

\[ K/H = U(n) \times U(n)/U(n), \quad U(n)/O(n), \quad U(2n)/Sp(n). \tag{3.5} \]

Consider an \( H \)-invariant function \( F \) on \( G/H \). Obviously, this function can be considered as a symmetric function depending in the invariants \( \varphi_1, \ldots, \varphi_n \) defined respectively in (1.1), (2.1), (2.5).

Consider the map

\[ z \mapsto (\varphi_1, \ldots, \varphi_n) \tag{3.6} \]

that takes a matrix \( z \) to its collection of invariants. Since \( \varphi_j \) are defined up to a permutation, we must assume

\[ 0 \leq \varphi_1 \leq \varphi_2 \leq \ldots \leq \varphi_n < 2\pi. \]

The following variant of the Weyl integration formula holds.

Lemma 3.1 The push-forward\(^{3} \) of the \( K \)-invariant measure \( \mu \) under the map (3.6) is given by the formula

\[ \text{const}(\kappa) \cdot \prod_{1 \leq k < l \leq n} |e^{i \varphi_k} - e^{i \varphi_l}|^{2 \kappa} \prod_{j=1}^{n} d\varphi_j, \tag{3.7} \]

where

- \( \kappa = 1/2 \) for \( U(n)/O(n) \)
- \( \kappa = 1 \) for \( U(n) \times U(n)/U(n) \)
- \( \kappa = 2 \) for \( U(2n)/Sp(n) \)

See, for instance, [29], X.1, various calculations of this type are contained in [32].

Consider the space \( L^2(K/H)^H \) consisting of \( H \)-invariant \( L^2 \)-functions. To each function \( F \in L^2(K/H)^H \), we consider the corresponding function \( f \) in variables \( \varphi_j \). By Lemma 3.1, we have

\[
\int_{K/H} F_1(z) F_2(z) d\mu(z) = \text{const} \int_{\mathbb{T}^n} f_1(\varphi) f_2(\varphi) \prod_{1 \leq k < l \leq n} |e^{i \varphi_k} - e^{i \varphi_l}|^{2 \kappa} \prod_{j=1}^{n} d\varphi_j
\]

This is precisely the inner product (3.1).

3.3. Spherical functions. Preliminaries. Consider a symmetric space

\[ K/H = U(n) \times U(n)/U(n), \quad U(n)/O(n), \quad U(2n)/Sp(n) \]

\(^{3}\)Let \( A \) be a space with a measure \( \alpha \), and \( h : A \to B \) is a measurable map. The push-forward of the measure \( \alpha \) is the measure \( \beta \) on \( B \) defined by \( \beta(S) = \alpha(h^{-1}(S)) \), where \( S \subseteq B \).
The space $L^2(K/H)$ is a multiplicity-free direct sum of (finite-dimensional) $K$-modules $V_\lambda$,

$$L^2(K/H) \cong \oplus_{\lambda} V_\lambda$$  \hspace{1cm} (3.8)

An irreducible $K$-module $V_\lambda$ participates in this direct sum iff $V_\lambda$ contains a nonzero $H$-invariant vector $v_\lambda$.\footnote{This statement is a variant of the Frobenius reciprocity, see [36], 13.1, 13.5. For description of modules satisfying this property, see, for instance, [39].} By Gelfand’s theorem (see [30]), an $H$-invariant vector $v_\lambda \in V_\lambda$ is unique up to a scalar factor.

The spherical function of a module $V_\lambda$ is function on $K$ defined by

$$\xi_\lambda(k) := \langle k \cdot v_\lambda, v_\lambda \rangle_{V_\lambda}$$  \hspace{1cm} (3.9)

where $v_\lambda$ is normalized by the condition $\|v_\lambda\| = 1$. Obviously, for $h_1, h_2 \in H$ we have

$$\xi_\lambda(h_1 h_2) = \xi_\lambda(k)$$

This allows to consider the function $\xi_\lambda$ as a function on our symmetric space $G/H$ or as a function on the double coset space $H \backslash G/H$.

In expansion (3.8), we have $\xi_\lambda \in V_\lambda$. Hence, the functions $\xi_\lambda$ form an orthogonal basis in $L^2(K/H)^H$.

The following fact is well-known, see, for instance, [37], Chapter 7.

$H$-spherical functions on $K/H$ are the Jack polynomials (modulo some normalizing scalar factors.) the parameter $\pi$ was determined in Lemma 3.1.

**3.4. Reduction of our problem.** Thus we intend to investigate a positivity of the inner product

$$\langle F_1, F_2 \rangle_{\sigma, \tau} = \int_{K/H} L_{\sigma, \tau}(z) F_1(z) \overline{F_2(u)} \, d\mu(z) \, d\mu(u)$$

where $L_{\sigma, \tau} = \det(1 - z u^*)^{[\pi]}$ is the distribution defined above.

Let $\lambda$ be a collection (3.4), $P_\lambda(x; \pi)$ be the corresponding Jack polynomial, and $V_\lambda \subset C^\infty(K/H)$ be the $K$-invariant subspace containing $P_\lambda(x, \pi)$.

**Lemma 3.2** For $F \in C^\infty(K/H)$ consider its expansion $F = \sum_\lambda F_\lambda$, where $F_\lambda \in V_\lambda$. Then

$$\langle F, G \rangle_{\sigma, \tau} = \sum_\lambda c_\lambda(\sigma, \tau) \int_{K/H} F_\lambda(z) \overline{G_\lambda(z)} \, d\mu(z)$$

where $c_\lambda(\sigma, \tau)$ are some constants.

**Proof.** Consider the integral operator

$$U_{\sigma, \tau} F(z) = \int_{K/H} L_{\sigma, \tau}(z, u) F(u) \, d\mu(u)$$

$$\langle F_1, F_2 \rangle_{\sigma, \tau} = \int_{K/H} L_{\sigma, \tau}(z) F_1(z) \overline{F_2(u)} \, d\mu(z) \, d\mu(u)$$
Since the $K$-invariance of the kernel $L_{\sigma, \tau}$, the operator $U_{\sigma, \tau}$ is an intertwining operator for $K$. Since the action of $K$ in $C^\infty$ is multiplicity-free, the restriction of $U_{\sigma, \tau}$ to $\mathcal{V}_\lambda$ is a scalar operator (see [35]), i.e.,

$$U_{\sigma, \tau} F_\lambda(z) = \int_{K/H} L_{\sigma, \tau}(z, u) F_\lambda(u) \, d\mu(u) = c_\lambda(\sigma, \tau) F_\lambda(z)$$

(3.10)

and this implies the required statement. \hfill \Box

Next, we must evaluate the constants $c_\lambda(\sigma, \tau)$. For this purpose, we substitute $F_\lambda = P_\lambda$ and

- $z = 1$ in the case $K/H = U(n)$,
- $z = 1$ in the case $K/H = U(n)/O(n)$,
- $z = J$ (see (2.4)) in the case $K/H = U(2n)/Sp(n)$.

In all the cases we obtain the integral

$$\mathcal{L}_\lambda(\kappa; \sigma, \tau) =$$

$$= \int_{T^n} \prod_{k=1}^n (1 - e^{i\varphi_k})^{[\sigma||\tau]} P_\lambda(e^{i\varphi_1}, \ldots, e^{i\varphi_n}; \kappa) \prod_{1 \leq k < l \leq n} |e^{i\varphi_k} - e^{i\varphi_l}|^{2\kappa} \prod_{j=1}^n d\varphi_j$$

(3.11)

where $\kappa$ is the same as in Lemma 3.1.

In the first two cases, we obtain this integral immediately, for the case $U(2n)/Sp(n)$ we must write $\det(1 - J\Omega)$, where $J$ is (2.4), and $\Omega$ is (2.5). The matrix $\det(1 - J\Omega)$ is diagonal, and

$$\det(1 - J\Omega) = \prod_{k=1}^n (1 - e^{i\varphi_k/2})^{[\sigma||\tau]} (1 - e^{-i\varphi_k/2})^{[\sigma||\tau]} = \prod_{k=1}^n (1 - e^{i\varphi_k})^{[\sigma||\tau]}$$

3.5. The Kadell integral. Let $\kappa$ be a positive integer. Let $\lambda$ satisfy (3.2). The Kadell integral (see [34], [37], VI.10, Example 7) is given by

$$\mathcal{K}_\lambda(\kappa; r, s) := \frac{1}{n!} \int_{[0,1]^n} P_\lambda(x; \kappa) \prod_{k=1}^n x_k^{-1} (1-x_k)^{r-1} \prod_{1 \leq k < l \leq n} |x_k - x_l|^{2\kappa} \prod_{j=1}^n dx_j =$$

$$= a_\lambda(\kappa; r, s) v_\lambda(\kappa)$$

(3.12)

where

$$v_\lambda(\kappa) = \prod_{1 \leq k < l \leq n} \frac{\Gamma(\lambda_k - \lambda_l + \kappa(l-k+1))}{\Gamma(\lambda_k - \lambda_l + \kappa(l-k))}$$

(3.13)

$$a_\lambda(\kappa; r, s) = \prod_{j=1}^n \frac{\Gamma(\lambda_j + r + \kappa(n-j)) \Gamma(s + \kappa(n-j))}{\Gamma(\lambda_j + r + s + \kappa(2n-j+1))}$$

(3.14)

3.6. Transformation of the Kadell integral.
**Proposition 3.3** Let $\text{Re}(\sigma + \tau) > -1$, $\text{Re} \, \nu > 0$, and $\lambda$ satisfies (3.4). Then the integral $\mathcal{L}_\lambda(\nu; \sigma, \tau)$ given by (3.11) equals

\[
\mathcal{L}_\lambda(\nu; \sigma, \tau) = (2\pi)^n n! \, v_\lambda(\nu) \times \prod_{j=1}^n \frac{(-1)^j \, \Gamma(\sigma + \tau + 1 + \nu(n-j))}{\Gamma(-\lambda_j + \tau + 1 + \nu(n-j))} \frac{\Gamma(\lambda_j + \sigma + 1 + \nu(n-j))}{\Gamma(\lambda_j - \nu(j-1))} \Gamma(\lambda_j + \sigma + 1 + \nu(n-j)) (3.15)
\]

where $v_\lambda(\nu)$ is the same as above (3.13).

**Proof.** First, let $\nu$ be a positive integer. Assuming $x_k = e^{i\psi_k}$,

\[
|e^{i\psi_k} - e^{i\psi_l}|^2 = |x_k - x_l|^2 = (x_k - x_l)(x_k^{-1} - x_l^{-1}) = -(x_k - x_l)^2 x_k^{-1} x_l^{-1}
\]

Next, transform the expression (3.11) to a contour integral

\[
\mathcal{L}_\lambda(\nu; \sigma, \tau) = e^{i\pi \sigma} (-1)^{\nu(n-1)\nu/2} \int_{|z_k| < 1} P_\lambda(x; \nu) \prod_{k=1}^n x_k^{-\nu - \nu(n-1) - 1} (1 - x_k)^{\nu + \sigma} \prod_{1 \leq k < l \leq n} (x_k - x_l)^{2\nu} \prod_{j=1}^n dx_j
\]

Now the integrand is holomorphic in the polydisc $|z_k| < 1$ outside the cut $x_1 \in [0, 1], \ldots, x_n \in [0, 1]$. We define the branches of the factors by

\[
(1 - x)^\nu \bigg|_{x=0} = 1, \quad x^{\nu} \bigg|_{x=\e^{i\psi}, \psi \in [0, 2\pi]} = e^{i\nu \psi}
\]

The domain of convergence of this integral is

\[
\text{Re}(\sigma + \tau) > -1 \quad (3.16)
\]

Then we deform each one-dimensional contour $x_j = e^{i\psi_j}$ to the contour around the segment $x_j = [0, 1]$ lying in an infinitely thin strip $|\text{Im} x_j| < \e$.

Thus our expression converts to

\[
(-1)^{\nu(n-1)\nu/2} (-2\sin \pi \tau)^n \int_{[0,1]^n} \{\text{the same integrand}\}
\]

The domain of convergence is reduced to

\[
\text{Re}(\sigma + \tau) > -1, \quad \text{Re} \, \tau < -\nu(n-1) \quad (3.17)
\]

Applying the Kadell formula, we obtain

\[
\mathcal{L}_\lambda(\nu; \sigma, \tau) = 2^n n! (-\sin(\pi \tau))^n (-1)^{\nu(n-1)\nu/2} v_\lambda(\nu) \times \prod_{j=1}^n \frac{\Gamma(\lambda_j - \nu(j-1)) \Gamma(\sigma + \tau + 1 + \nu(n-j))}{\Gamma(\lambda_j + \sigma + 1 + \nu(n-j))} \frac{\Gamma(\lambda_j + \nu(j-1)) \Gamma(\sigma + \tau + 1 + \nu(n-j))}{\Gamma(\lambda_j + \sigma + 1 + \nu(n-j))} \quad (3.18)
\]
Then we write

$$- \sin(\pi \tau) \Gamma(\lambda_j - \tau - \kappa(j - 1)) =$$

$$= (-1)^{\lambda_j - \tau - \kappa(j - 1)} \sin(\lambda_j - \tau - \kappa(j - 1) \pi) \Gamma(\lambda_j - \tau - \kappa(j - 1))$$

(again, we use the condition $\kappa \in \mathbb{Z}$). Applying the reflection formula

$$\sin(\pi z) \Gamma(z) = \pi / \Gamma(1 - z)$$

we obtain (3.15).

Thus, we have the required identity (3.15) under the condition (3.17). But the both sides of (3.15) are analytic in the domain (3.16). Hence the identity is valid in this domain.

Thus the statement is proved for an integer positive $\kappa$.

Next, we intend to apply the following Carlson theorem, see [1], 2.8.1.

— Let $f(z)$ be an analytical function in the domain $\text{Re} \ z > 0$ and $f(z) = O(e^{a|z|})$ with some $a < \pi$. If $f(n) = 0$ for $n = 1, 2, \ldots$, then $f(z)$ is identically zero.

Fix real positive $\sigma$, $\tau$. The identity (3.11) is valid for positive integer $\kappa$. We intend to show that it is valid for $\text{Re} \ \kappa > 0$. First, we slightly change the factors of the integrand

$$|e^{i\psi_n} - e^{i\psi_k}| \to \frac{1}{2} |e^{i\psi_n} - e^{i\psi_k}|, \quad (1 - e^{i\psi_k}) \to \frac{1}{2} (1 - e^{i\psi_k})$$

Accordingly we multiply the right-hand side of (3.11) by $2^{-\sigma} (\sigma - n(\sigma + \tau))$. Now the expression $\prod_{n=1}^{\kappa} \ldots \prod_{1 \leq j < k \leq \kappa} \ldots$ in the integrand is $< 1$.

For coefficients of the Jack polynomials in formula (3.3), there exists a semi-explicit expression that is rational in the variable $\kappa$, see [37], VI.10, page 379. The poles of these expressions are some non-positive rational numbers. Hence, for a fixed $\lambda$, the expression $P_\lambda(x; \kappa)$ admits a holomorphic continuation to the domain $\text{Re} \ \kappa > 0$, and moreover $P_\lambda(x; \kappa)$ has at most a polynomial growth in $\kappa$.

Thus our integrand has (at most) a polynomial growth in $\kappa$. Hence the same statement is valid for the integral.

Now we estimate the growth of the right-hand side of 3.15. First, our formula is valid for $\lambda_1 = \cdots = \lambda_m = 0$, in this case, we obtain the Cauchy-type form of the Selberg integral, see [1]; see also [37], (10.38), for $\lambda = 0$. The integrand now is less than 1, and hence the right-hand side also is bounded.

Consider the ratio of right-hand sides

$$\mathcal{L}_\lambda(\kappa; \sigma, \tau) / \mathcal{L}_0(\kappa; \sigma, \tau)$$

This ratio is the product of expressions having the form $\Gamma(\mu + l\kappa) / \Gamma(\nu + l\kappa)$. This product has a polynomial growth in $\kappa$, since

$$\frac{\Gamma(a + z)}{\Gamma(b + z)} \sim z^{a - b}, \quad |z| \to \infty$$
(see, for instance [1], 1.4.3; the asymptotic is valid in the domain \(|\arg z| < \pi - \epsilon\).

Now we have two functions having a polynomial growth as \(|z| \to \infty\), they coincide in positive integers and by the Carlson theorem they coincide in the half-plane.

It remains to omit the condition \(\lambda_n \geq 0\). But the transformation

\[
\lambda_j \rightarrow \lambda_j - 1, \quad \tau \rightarrow \tau - 1, \quad \sigma \rightarrow \sigma + 1
\]

does not change the both sides of our integral, and hence our formula can be used for general \(\lambda\) satisfying (3.4).

3.7. Positivity. We must analyze, when (3.15) has constant signs for all \(\lambda\) satisfying (3.4). The factor \(v_\lambda\) given by (3.13) is positive. Hence we must analyze positivity of the factor \(\prod \ldots\) in (3.15). Equivalently we must trace the constancy of the sign of the factor

\[
\prod_{j=1}^{n} \frac{\Gamma(\lambda_j - \tau - \kappa(j - 1))}{\Gamma(\lambda_j + \sigma + 1 + \kappa(n - j))}
\]

in (3.18).

The case \(\kappa = 2\). We must analyze positivity of the expression

\[
\prod_{j=1}^{n} \frac{\Gamma(\lambda_j - \tau - 2j + 2)}{\Gamma(\lambda_j + \sigma + 1 + 2(n - j))}
\]

(3.19)

The function \(\Gamma(x)\) changes its sign at the points \(x = 0, -1, -2, \ldots\). Hence the product-for-signs condition

\[
[-\tau - 2j + 2] = [\sigma + 1 + 2(n - j)] \quad \iff \quad [-\tau] = [\sigma + 2n - 1] \quad (3.20)
\]

is sufficient for positivity.

Let us show that this condition is necessary. Assume that all the \(\lambda_j\) are negative integers having large absolute values, and assume that \(\lambda_i - \lambda_j\) also are large. Then the sign of (3.19) depends only on parities of \(\lambda_j\). Also, it is clear, that all the factors in (3.19) have to be positive. But this implies (3.20).

The case \(\kappa = 1\) is similar. We must examine the positivity of

\[
\prod_{j=1}^{n} \frac{\Gamma(\lambda_j - \tau - j + 1)}{\Gamma(\lambda_j + \sigma + n - j + 1)}
\]

The case \(\kappa = 1/2\). We must examine the positivity of

\[
\prod_{j=1}^{n} \frac{\Gamma(\lambda_j - \tau - j/2 + 1/2)}{\Gamma(\lambda_j + \sigma + 1 + (n - j)/2)}
\]

Considering even \(j\)-s, we obtain

\[
[-\tau + 1/2] = [\sigma + 1 + n/2]
\]
Considering odd $j$-s, we obtain
\[ [-\tau + 1/2 + 1/2] = [\sigma + 1 + n/2 + 1/2] \]

This implies Theorem 2.5.

4. Remarks on general case

Here we discuss Stein-Sahi representations of arbitrary classical groups.

4.1. Flat matrix spaces. We consider the following 10 types of spaces $\mathcal{M}$ of $n \times n$ matrices; in each case we write groups $G \supset K \supset H$, the sense of this notation will be explained below; if $n$ have to be odd, then we write $n = 2k$

- $n \times n$ matrices over $\mathbb{C}$;
- $n \times n$ Hermitian matrices over $\mathbb{C}$;
- $n \times n$ symmetric matrices over $\mathbb{C}$;
- $2k \times 2k$ skew-symmetric matrices over $\mathbb{C}$;
- $n \times n$ matrices over $\mathbb{R}$;
- $n \times n$ symmetric matrices over $\mathbb{R}$;
- $2k \times 2k$ skew-symmetric matrices over $\mathbb{R}$;
- $n \times n$ matrices over $\mathbb{H}$,
- $n \times n$ anti-Hermitian matrices over $\mathbb{H}$,
- $k \times k$ Hermitian matrices over $\mathbb{H}$;

The group $G$ is isomorphic to the group of linear-fractional transformations

$x \mapsto x^{bl} = (a + xc)^{-1}(b + xd)$

preserving the symmetry condition. The Jacobian of such transformation is

\[ J(g, x) = \det |(a + xc)|^{-h} \left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|^2 \]

where

\[ h = \frac{2}{n} \dim \mathcal{M}, \]

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$n$ is size of the matrices, and $\beta = n \text{dim}\mathbb{K}$ in 3 cases then we consider all the matrices without conditions of symmetry ($i.e.$, $G = \text{GL}(n, \mathbb{K})$); in all the remaining cases $|\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}| = 1$, see [54], Lemma 1.3.

The group $K$ is the maximal compact subgroup in $G$, and $H \subset K$ is the stabilizer of 0. In all the cases, $K/H$ is a compact symmetric space, and $M$ is an open dense subset in $K/H$.

Remark. In all the cases, there exists some Grassmannian type realisation of $K/H$ in spirit of our Subsections 1.3, 2.2, 2.5, see a table in [51]).

We define the Stein–Sahi inner product in $C^\infty(M)$ by

$$\langle F_1, F_2 \rangle = \int_M \int_M |\det (x - y)|^\beta F_1(x) \overline{F_2(y)} dx \, dy$$

The group $G$ acts in the space of functions on $M$ by formula

$$\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} F(x) = F((a + xc)^{-1} (b + xd)) |\det(a + xc)^{-\beta - \theta}$$

these transformations preserve the Hermitian form $\theta$.

4.2. Existence of an island of positivity. First, consider the representation of $G$ in functions on $M$ given by

$$F(x) \mapsto f((a + xc)^{-1} (b + xd)) |\det(a + xc)^{-\beta/2}$$

It is a representation of a degenerated unitary principal series.

Consider the maximal parabolic subgroup $P \subset G$ consisting of matrices having the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. This subgroup acts on functions by affine transformations

$$F(x) \mapsto F(a^{-1} (b + xd)) |\det(a)^u \det(d)^v,$n

where $u, d$ are some explicit real numbers (non-interesting for us).

Lemma 4.1 If $G \neq \text{Sp}(2n, \mathbb{R}), \text{U}(n, n), \text{SO}^*(4n)$, then the representation (4.4) of $P$ is irreducible.

Proof. Consider the subgroup $N \subset P$ consisting of matrices $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, it acts by shifts $f(x) \mapsto f(x + b)$.

Also consider the subgroup $L \subset P$ consisting of matrices $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, it acts by the transformations $f(x) \mapsto f(axd^{-1})$.

After the Fourier transform

$$F(x) \mapsto \int_M F(x) \exp(i \text{Re} \text{ tr} \xi^*) dx$$

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the group $N$ acts via multiplications by functions

$$
\Phi(\xi) \mapsto \Phi(\xi) \exp(i \operatorname{Re} \operatorname{tr} \xi^{\dagger})
$$

Assume that the representation (4.4) is reducible. Then there exists an intertwining projector $\Pi$. In the Fourier model, it commutes with all the operators (4.5) and hence $\Pi$ is a multiplication by a function $\chi$ taking values 0 and 1. But this operator also must commute with the subgroup $L$, which acts by the transformations having the form

$$
\Phi(\xi) \mapsto \Phi(A \xi D^{-1}) |\det A|^{\nu} |\det D|^{\nu}^{-1}
$$

But in all the cases, the group of transformations $\xi \mapsto A \xi D^{-1}$ has an open orbit on $\mathcal{M}$ and hence $\chi$ is a constant. \hfill \Box

**Corollary 4.2** If $G \neq U(n, n), \text{Sp}(2n, \mathbb{R}), \text{SO}^*(4n)$, then the representation (4.3) is irreducible.

Since (4.3) is irreducible, it admits a unique invariant Hermitian form and this form is the $L^2$-inner product. Representations (4.2) lying near (4.3) admit the invariant Hermitian form (4.1), and since this Hermitian form is a minor deformation of the $L^2$-inner product, this Hermitian form also is positive definite. It is not difficult to justify these heuristic arguments in this case (there are abstract theorems of this type, see [36], Theorems 6, 8; unfortunately they do not cover our cases).

Description of the poles of the distribution $|\det(x)|^\mu$ that appear in (4.1) is a relatively standard problem, see Sato, Shintani [80]. Knowledge of these poles allows to find precisely the interval of positivity. We omit details.

**4.3. Other possible ways of proof.** Existence of Stein–Sahi series is a relatively simple fact, and apparently it can be proved uniformly in various ways.

One possibility is to use the work of Branson, Olafsson, Ørsted, [11].

Another way is to use our method of Sections 1-3, but a necessary identity for multivariate Jacobi polynomials (see [28]) in my knowledge is not known yet. But for the groups $G = O(2n, 2n), \text{Sp}(n, n), \text{GL}(2n, \mathbb{C})$ this way can easily be realized (the first two cases are discussed in [62]) due existence of explicit elementary formulas (Weyl's and Berezin–Karpelivich's) for spherical functions.

5. One problem of non-$L^2$ harmonic analysis.

A. Construction of kernels

**5.1. Uniform realizations of symmetric spaces.** Consider the following classical groups

$$
Q = \text{GL}(n, \mathbb{C}), \text{GL}(n, \mathbb{R}), \text{GL}(n, \mathbb{H}), \text{O}(p, q), \text{U}(p, q), \text{Sp}(p, q), \text{O}(n, \mathbb{C}), \\
\text{Sp}(2n, \mathbb{C}), \text{Sp}(2n, \mathbb{R}), \text{SO}^*(2n)
$$

5.1
Remark. Our list contains $\GL(n, \mathbb{K})$ and not $\SL(n, \mathbb{K})$, also $\U(p, q)$ and not $\SU(p, q)$. Modulo this remark, our list contains all the classical groups.

Consider an affine symmetric space having the form $Q/Y$, where $Y$ is a symmetric subgroup in $Q$ (according the Berger classification, there are 54 series of classical affine symmetric spaces). It turns out to be (see [53]) that the space $Q/S$ admits a realization as a set, whose points are pairs of complementary subspaces $(V, W)$ in some linear space $\mathbb{K}^n$, these subspaces satisfy some simple conditions (as an isotropy with respect to some form, orthogonality, or existing of a given involution transposing two subspaces).

Example 1. The unitary group $U(n)$. Consider the space $\mathbb{C}^n \oplus \mathbb{C}^n$ equipped with the Hermitian form $H$ having the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as above (Subsection 1.2). Fix the linear operator $J$ in $\mathbb{C}^n \oplus \mathbb{C}^n$ having the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Consider the set $\mathcal{U}$, whose points are pairs of subspaces $(V, W)$ such that

1. $\mathbb{C}^n \oplus \mathbb{C}^n = V \oplus W$
2. $V$ and $W$ are $H$-isotropic.
3. $W = JV$

Now let $z : \mathbb{C}^n \to \mathbb{C}^n$ be an unitary operator. Let $V$ be a graph of $z$, and $W$ be the graph of $(-z)$. Then $V, W$ satisfy to the conditions 1-3. It can be readily checked that all the pairs satisfying 1-3 have this form.

Thus $\mathcal{U} \cong U(n)$.

Example 2. The space $\U(n, n)/\GL(n, \mathbb{C})$. Consider the same space $\mathbb{C}^n \oplus \mathbb{C}^n$ equipped with the same Hermitian form $H$. Consider the set $\mathcal{H}$, whose points are pairs $V, W$ of subspaces satisfying the conditions $0^* - 1^*$ from the previous example. Obviously, $\mathcal{H} \cong U(n, n)/\GL(n, \mathbb{C})$.

Example 3. The spaces $\U(n, n)/\U(p, q) \times \U(n - p, n - q)$. Consider the same space $\mathbb{C}^n \oplus \mathbb{C}^n$ with the same form $H$. Consider the set $\mathcal{G}$, whose points are pairs of subspaces $(V, W)$ satisfying the conditions $0^*$ and $-\dim V = a$

- $W$ is the orthocomplement of $V$ with respect to $H$.

Open orbits of the group $U(n, n)$ on $\mathcal{G}$ are enumerated by the inertia indices of $H$ on $V$. Thus

$$\mathcal{G} \cong \bigcup_{p+q=a} \U(n, n)/\U(p, q) \times \U(n - p, n - q).$$

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\textsuperscript{15}For more information on determinant distributions (explicit expressions at poles, supports, etc.) see works Rais [73], Ricci, Stein [74], Muro [45].

\textsuperscript{16}Of course, the subspace $W$ in our construction seems artificial and can be forgotten (as it was done above in Sections 1-2). But it take part in a general construction of the kernel below.

\textsuperscript{17}Again, it seems that $W$ is an artificial element of the construction (since $W$ is determined by $V$), but the transversality condition $0^*$ involves $W$. 

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list-of-groups

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It appears that enumerating carefully all the possible conditions of this type, we obtain precisely the classical part of the Berger list.\footnote{We must consider all the fields }\footnote{ types \( K = \mathbb{R}, \mathbb{C}, \mathbb{H} \), all possible natural forms and all possible involutions. For a table, see [33].}

5.2. Overgroup of the symmetric space. In Examples 1 and 3, the subspace \( W \) is determined by the subspace \( V \), and hence we have an embedding of the symmetric space to some Grassmannian \( G/P \); the same phenomenon holds for 44 series of symmetric spaces.

In Example 2, the subspaces \( V, W \) are 'independent', and we obtain an open embedding of a symmetric space to a product of two Grassmannians \( G/P_1 \times G/P_2 \)\footnote{In both examples \( G = U(n, n) \) for \( n \).\footnote{In our example \( G = U(n, n) \) for \( n \).}} (and this happens for remaining 10 series).

In particular, we obtain that for each classical symmetric space \( Q/Y \) there exists\footnote{Emphasis that \( Q \) is contained in the list (5.1). For instance, \( U(n, n) \) acts on the space \( U(n) \), but there is no group \( G \supseteq SU(n) \times SU(n) \) acting on the hypersurface \( SU(n) \subset U(n) \).}\footnote{Only few exceptional symmetric spaces admit such overgroup; see [10].} larger group \( G \) acting locally on \( Q/Y \).

Remark. It seems that this (trivial) observation was firstly claimed in [53]. But overgroups of symmetric spaces were discussed by different reasons by many authors, in particular Hua Loo Keng, Nagano, Takeuchi, Goncharov, Gindikin, Kaneyuki. Nagano [46] gave a complete analysis of such overgroups in Grassmannian case (including exceptional cases). Makarevich [38] in 1973 classified open orbits of reductive groups in Grassmannians. In fact, the right-hand side of long Makarevich's tables includes the whole classical part of Berger's list, and this implies our observation. As far as I know, nobody actually compared these Bersgs's and Makarevich's lists during 25 years.\footnote{Only few exceptional symmetric spaces admit such overgroup; see [10].}

5.3. Double ratio of 4 subspaces. Consider a linear space \( \mathbb{F}^{p+q} \). Let \( V_1, W_1, V_2, W_2 \) be four subspaces in \( \mathbb{F}^{p+q} \), and

\[ \dim V_1 = \dim V_2 = p, \quad \dim W_1 = \dim W_2 = q \]

For a quadruple being in general position, we define a canonical operator (the Hua Loo Keng double ratio) \( \mathcal{D} : V_1 \rightarrow V_1 \) by the following rule. Since \( \mathbb{F}^{p+q} = V_1 \oplus W_1 \), the subspace \( V_2 \) is a graph of an operator \( A : V_1 \rightarrow W_1 \), and \( B \) is a graph of an operator \( B : W_1 \rightarrow V_1 \). We assume

\[ \mathcal{D}(V_1, W_1, V_2, W_2) = BA \]

The operator is canonically defined, in particular, its eigenvalues are invariants of a quadruple of subspaces.

By Subsection 5.1, the double ratio is well defined in each classical symmetric space.

5.4. A problem. Different formulations. The author think, that for the following problems of the harmonic analysis, the explicit Plancherel formula can be obtained. The arguments for support of this point of view are discussed below.
We use the notation of Section 4.

a) Tensor products. For a given group $G$ from the list (0.7-line2), decompose a tensor product of two Stein-Sahi representations.

b) Restrictions. Let $Q/Y$ be a symmetric space. Assume $G$ that its overgroup has the Stein–Sahi representations. Assume that the local action of $G$ on $Q/Y$ is locally isomorphic to the action $G$ on the matrix space, see 4.1. Decompose the restriction of a Stein–Sahi representation to $G$.

**Remark.** The problem about tensor products is a problem of the restriction of a representation $\rho \otimes \rho'$ of the group $G \times G$ to the diagonal subgroup $G$.

c) Natural kernels on symmetric spaces. Let $Q/Y$ be the same as above. Let restrict the Stein kernel $L(\cdot, \cdot)$ to $Q/Y$. Consider the inner product in $C^\infty(Q/Y)$ defined by the kernel $L$ (see (4.1)) and the unitary representation of $Q$ in this space. To find the Plancherel formula for this representation.

**Remark.** This question slightly differs from the previous one in the case, then $Q$ has several open orbits $Q/Y_j$ on the Grassmannian, see Example 3 in 5.1.

d) More general kernels on symmetric spaces. Consider a classical symmetric space $Q/Y$ realized as in 5.1, i.e., points of $Q/Y$ are pairs $r = [V, W]$ of complementary subspaces. We define (see Addendum to [57]) the kernel $K(\cdot, \cdot)$ on $Q/Y$ by

$$K([V_1, W_1], [V_2, W_2]) = \left| \frac{\det D(V_1, W_1, V_2, W_2)}{1 - D(V_1, W_1, V_2, W_2)} \right|^d$$

and the inner product on $C^\infty(Q/Y)$ given by

$$\langle f_1, f_2 \rangle = \int_{Q/Y \times Q/Y} K(r_1, r_2)f_1(r_1)\overline{f_2(r_2)}\,d\mu(r_1)\,d\mu(r_2)$$

where $\mu$ is the $Q$-invariant measure on $Q/Y$.

If the overgroup $G$ of $Q/Y$ admits Stein–Sahi representations\footnote{To avoid ambiguity, we also must require that the local action of $G$ on $Q/Y$ is equivalent to the action on the matrix spaces for Grassmannians from 4.1. This slip in speech is important only for $G = \text{GL}(n, \mathbb{C})$.}, then our construction is equivalent to the previous one.

Otherwise, we obtain a problem of positivity of the inner product. There are spaces $Q/Y$, for which islands of positivity are absent ([63]), and the cases, when such islands exist ([41], [63]). Generally, the problem is open.

Again, our question is to find the Plancherel measure.

I do not know has this question sense if there is no positivity, see a discussion in Addendum to [57].

e) More tensor products. Our Stein representations of $G$ are induced from some parabolic $P$. We ask about the tensor products of arbitrary two representations induced from this parabolic.
Discussion

Below we discuss the variant b) of the problem, i.e., the restriction of a Stein representation to a symmetric subgroup.

5.5. Approximation of $L^2$ on symmetric spaces. First, for all the groups $G$, Stein–Sahi series includes some representation (or representations) of a principal series (in the notations of 1.8, 2.3, 2.7, this corresponds to $s = t$, in the notation of section 4, this corresponds to $\theta = -\hbar/2$.

In this case, our problem is equivalent to decomposition of $L^2(Q/Y)$.

Hence our representations are some kind of deformation of $L^2(Q/Y)$.

5.6. Modern picture of $L^2$ on a pseudo-Riemannian symmetric space. The problem of decomposition of $L^2$ on an arbitrary pseudo-Riemannian symmetric space was considered as important after the Flensted-Jensen’s work on discrete series [19], 1980.

After large efforts, the problem is not completely solved up to now.

First, in the following partial cases, the explicit solution is known.

— Riemannian symmetric spaces $G/K$ (Gelfand–Naimark, [22] for complex classical groups, Gindikin–Karpelevich, [23], for general case).

— Semisimple Lie groups (Gelfand–Naimark, [22] for complex classical groups, Harish-Chandra, [27], for general case

— Rank 1 spaces, Molchanov ([42], [43]).

The spaces of the form $G_C/G_{\mathbb{R}}$, where $G_C$ is a complex group, and $G_{\mathbb{R}}$ is a real form (Harinck, [25], [26]).

Second, there was a long story of search of the general Plancherel formula. Author do not intend discuss it and fix modern situation.

There are two groups of authors, van den Ban – Schlichtkrull [3], [4] and Delorme – Carmona [12] proposed variants of a Plancherel formula. Proofs are very heavy and occupy large collections of papers; also these formulae are nonexplicit and the $\epsilon$-function is not evaluated.

Oshima [67] recently announced (with sketches of proofs) an explicit formula for $\epsilon$-function for the "most continuous part of spectrum". Apparently, Oshima's formula together with van den Ban–Schlichtkrull's residue calculus allow to receive a final solution.

A more important problem is to make this branch of analysis available to a wider mathematical community.

5.7. Degenerated cases of our problem. Consider the representations $\rho_{\sigma, \tau}$ of $U(n, n)$ defined in 1.6. Consider a tensor product $\rho_{\sigma, \tau} \otimes \rho_{\sigma', \tau'}$. By definition, this representation acts in a space of functions on $U(n) \times U(n)$. The diagonal group $U(n, n) \subset U(n, n) \times U(n, n)$ has an open orbit $U(n, n)/GL(n, \mathbb{C})$ on $U(n, n)\mathbb{C}^d$.

Thus, $U(n, n)$ acts in the space of functions on $U(n, n)/GL(n, \mathbb{C})$.

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24 Indeed, a point of $U(n)$ can be considered as a pair of maximal isotropic subspaces in $\mathbb{C}^n \oplus \mathbb{C}^n$, see Example 2 from 5.1. A stabilizer of a pair of maximal isotropic subspaces is $GL(n, \mathbb{C})$. 

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Representations $\rho_{\sigma,\beta}$ are highest weight representations of $U(n, n)$, and $\rho_{0,\tau}$ are lowest weight representations. The problems of decomposition

$$\rho_{\sigma,\beta} \otimes \rho_{\sigma',\beta}, \quad \rho_{\sigma,\beta} \otimes \rho_{0,\tau}$$

are degenerated cases of our problem.

In the first case, we have purely discrete spectrum, and problem has combinatorial nature (see, for instance [33]), in the second case we obtain 'Berezin representations' discussed in Introduction.

Hence our problem admits a degeneration into 2 important problems.

Restriction of a highest weight representation $\rho_{\sigma,\beta}$ of $G$ to a symmetric subgroup $Q$ leads to a similar alternative, i.e., we have purely discrete spectrum\(^{25}\) or Berezin representation\(^{26}\).

But generally such restriction problems leads to harmonic analysis on some class of spaces that is more general than symmetric spaces.\(^{27}\)

5.8. Rank 1 cases.

a) Tensor products of unitary representations $SL(2, R) = SU(1, 1)$. For linear representations of $SL(2, R)$, the spectra were determined by Pukanzky [72] and the Plancherel formula by Molchanov [39], [40].

Even this relatively simple problem is unexpectedly non-trivial and not perfectly understood up to now\(^{28-29-30}\), see recent works [59], [61], [24].

Other rank one cases. There are many explicit calculations in the rank one case, that can be attributed to our subject, see [47]-[48], [44], [41], [14], our list do not pretend to be complete.

5.9. Compact symmetric spaces. Let our space $Q/Y$ is compact. For instance, let $Q/Y$ is $U(n)$. Then we have the space of functions on $U(n)$ with the inner product (1.15). The group $U(n) \times U(n)$ acts in this space by transformations $F(z) \mapsto F(a^{-1}zb)$. Search of the Plancherel formula is equivalent to expansion of $L_{\sigma,\tau}$ in spherical functions (modulo some correction factors).

\(^{25}\)An example is $G = U(n, n), \ Q = U(p, q) \times U(n - p, n - q)$, the corresponding symmetric spaces are described in Example 3 of 5.1.

\(^{26}\)An example is $U(n, n) \supset O(n, n)$. This restriction problem can be interpreted as a problem of analysis on the symmetric space $O(n, n)/O(n, C)$.

\(^{27}\)Some counterexamples are the restriction problems $U(p, q) \supset O(p, q)$ for $p \neq q$ and $U(n, n) \supset Sp(2n, \mathbb{R})$ for odd $n$.

\(^{28}\)A unitary (projective) representation of $SU(1, 1)$ depend on two real parameters, and a finite-dimensional representation of $SU(1, 1)$ depends on one integer parameter. By this reason, formulæ for infinite-dimensional representations are not analytic continuations of formulæ for finite-dimensional representations.

\(^{29}\)This problem has a continuous spectrum of multiplicity 2 and also some discrete spectra. By this reason, it is a good proving ground for understanding of some general phenomena of harmonic analysis.

\(^{30}\)Let $G$ be a semisimple group and $\rho_1, \rho_2$ are its unitary representations. Usually the spectrum of the tensor product $\rho_1 \otimes \rho_2$ has infinite multiplicities (non-formal argument: 'functional dimension' of this product is too large). In a general situation, the problem of description of spectrum can be reasonable, but question about the Plancherel formula looks as dangerous. It seems that our problem of analysis on $U(n, n)/GL(n, C)$ is the closest multirank approximation to tensor products for $SU(1, 1)$.
Since we have obtained explicit expansions, we also know the Plancherel formula for the corresponding compact symmetric spaces.

5.10. Oshima's argument. In [67], Oshima shows that for two spaces $Q_1/Y_1$ and $Q_2/Y_2$ having the same complexification, their $c$-functions satisfy the same difference equations. In particular, their ratio a priori is a trigonometric function.

Apparently, this phenomenon survives in our case. Since in the compact case the Plancherel formula can be obtained, it is natural to believe that it can be obtained in a general case.\textsuperscript{31}

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\textsuperscript{31}The author had evaluated the Plancherel measure in some relatively simple cases, see [63].


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