Some Remarks on Stable Densities and Operators of Fractional Differentiation

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Some remarks on stable densities and operators of fractional differentiation

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To A.M. Vershik in his 70 birthday

Let $D(s)$ be a fractional derivation of order $s$. For real $\alpha \neq 0$, we construct an integral operator $A(\alpha)$ in an appropriate functional space such that $A(\alpha)D(s)A(\alpha)^{-1} = D(\alpha s)$ for all $s$. The kernel of the operator $A(\alpha)$ is expressed in terms of a function similar to the stable densities.

0.1. Definition of functions $\mathbb{L}_{\alpha, \beta}$. This paper contains several simple observations concerning the special function

$$\mathbb{L}_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\alpha n + \beta)}{n!} z^n$$

where $0 < \alpha < 1$, $\Re \beta > 0$, and $z \in \mathbb{C}$. We can also represent this function in the form

$$\mathbb{L}_{\alpha, \beta}(z) = \int_{0}^{\infty} x^{\beta-1} \exp(-zt^n - t) \, dt$$

$$\mathbb{L}_{\alpha, \beta}(z) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \Gamma(\beta - \alpha s) \Gamma(s) z^{-s} \, ds$$

The integrals (0.2), (0.3) also make sense for $\alpha > 1$. The definition of functions $\mathbb{L}_{\alpha, \beta}$ is discussed in details below in Section 1.

The function $\mathbb{L}_{\alpha, \beta}$ is one of the simplest examples of the so-called H-functions (or Fox functions), see [14]. In a strange way, the function $\mathbb{L}_{\alpha, \beta}$ has no official name. Obviously, for rational $\alpha = p/q$ the function $\mathbb{L}_{\alpha, \beta}$ can be expressed in the terms of higher hypergeometric functions. But for $q > 4$ such expressions do not seem very useful.

0.2. Results of the paper. Integral operators with functions $\mathbb{L}_{\alpha, \beta}$ in kernels.

A) We consider the space $\mathcal{K}$ of functions holomorphic in the half-plane $\Re z > 0$, smooth up to the line $\Re z = 0$, and satisfying the following condition

$$|f^{(k)}(z)| \leq M (1 + |z|)^{-N}$$

We define the operators of fractional differentiation $D_h$ in the space $\mathcal{K}$ by

$$D_h f(z) = \frac{\Gamma(h+1)}{2\pi} \int_{-\infty}^{\infty} \frac{f(it) \, dt}{(-it + z)^{h+1}}$$
For a positive integer \( n \), we have \( D_n = (-1)^n d^n/dz^n \); the operator \( D_{\pm n} \) is the indefinite integration iterated \( n \) times. Also \( D_{h+\tau} = D_h D_{\tau} \). See Section 2 below for details.

Next, for \( a > 0 \) we define the kernel

\[
K_a(u, v) = \int_0^\infty \exp(-ux^\alpha - vx) \, dx = v^{-1} \Pi_a(u/v^\alpha),
\]

and the operator in the space \( \mathcal{K} \) given by

\[
A_\alpha f(v) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} K_a(-it, v) f(it) \, dt \quad (0.5)
\]

The operators \( A_\alpha \) form an one-parameter group (see Subsection 3.5),

\[
A_\alpha A_\beta = A_{\alpha \beta}; \quad (0.6)
\]

We also show that they satisfy the property

\[
A_\alpha D_h A_\alpha^{-1} = D_{ah} \quad (0.7)
\]

**Remark.** Emphasis the following particular cases of (0.7):

\[
A_{m/n} \frac{d^n}{dz^n} A_{m/n}^{-1} = (-1)^{m-n} \frac{d^n}{dz^n};
\]

\[
A_k \frac{d}{dz} A_k^{-1} = (-1)^{k-1} \frac{d^k}{dz^k}
\]

for integer \( m, n, k \).

**Remark.** Also

\[
A_\alpha z A_\alpha^{-1} f(z) = \frac{1}{\alpha} D_{1-\alpha} (zf(z)) \quad (0.8)
\]

**Remark.** It seems that (0.7), (0.8) and formula (0.12) below give a possibility to strange transformations of partial differential equations and their solutions.

Further, the operators of dilatation

\[
R_\alpha g(z) = a^{-1} g(z/a), \quad a > 0
\]

satisfy

\[
A_\alpha^{-1} R_\alpha A_\alpha = R_{\alpha^a} \quad (0.9)
\]

The generator of the one-parameter group \( R_\alpha \) is \((z d/dz + 1)\). Hence (0.9) can be written in the form

\[
A_\alpha^{-1} \left( z \frac{d}{dz} + 1 \right) A_\alpha = \alpha \left( z \frac{d}{dz} + 1 \right)
\]

**B)** In Section 3, we consider the group \( G \) of operators in \( \mathcal{K} \) generated by the operators \( A_\alpha \), the fractional derivations \( D_h \), and the dilatations \( R_\alpha \). We
observe that \( G \) is a 6-dimensional solvable Lie group with 2-dimensional center and kernels of all elements of this group admit simple expressions in the terms of the functions \( \mathbb{L}_{\alpha,\beta} \) (Theorems 3.1, 3.2).

C) In Section 4, we consider the usual Riemann–Liouville fractional integrations \( J_\beta \) in the space of functions on the half-line \( x \geq 0 \), see below (4.1). For \( 0 < \alpha < 1 \), we consider the Zolotarev operators \([25]–[26]\) defined by the formula

\[
B_\alpha f(x) = \frac{1}{\pi x} \int_0^\infty \text{Im}\left\{ \mathbb{L}_{\alpha,1}(x^{-\alpha}y^e^{i\pi \alpha}) \right\} f(y) \, dy
\]

We have

\[
B_\alpha B_\beta = B_{\alpha \beta}
\]

\[
B_\alpha J_\beta = J_{\alpha \beta} B_\alpha
\]

(but we cannot represent the identity (0.12) in the form (0.7), since the operators \( B_\alpha \) are not invertible).

These operators can be included to a 7-dimensional semigroup of integral operators on the half-line, this semigroup has a 2-dimensional center (Theorem 4.2).

0.3. Some references on functions \( \mathbb{L}_{\alpha,\beta} \).

1) Barnes in 1906 [2] evaluated asymptotics of several \( H \)-functions and, in particular, for \( \mathbb{L}_{\alpha,\beta} \). But it seems that he had no reasons to investigate \( \mathbb{L}_{\alpha,\beta} \) in details; in the sequel years this function (as far as I know) had not attracted specialists in special functions.

2) The functions

\[
\mathcal{W}_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{n! \Gamma(\alpha n + \beta)}, \quad \alpha > 0
\]

('Wright functions', 'Bessel–Maitland functions')\(^1\) were discussed more, see [24], [1], and references in "Higher transcendental functions" [6] (Section "Mittag-Leffler function"). The functions \( \mathbb{L}_{\alpha,\beta}(z) \), \( \mathcal{W}_{\alpha,\beta}(z) \) quite often appear in formulas in similar cases (for instance, see below Subsection 1.5). Another 'relative' of the function \( \mathbb{L}_{\alpha,\beta} \) is the Mittag-Leffler function \( \sum z^n / \Gamma(n + 1) \), that appear in literature quite often.

3) The functions \( \mathbb{L}_{\alpha,\beta} \) appear (see Feller [7]) if we solve the Cauchy problem for the partial pseudo-differential equation

\[
\left[ \frac{d}{dt} - \frac{d^\alpha}{dx^\alpha} \right] f(x, t) = 0, \quad f(x, 0) = \psi(x)
\]

where \( d^\alpha / dx^\alpha \) is some fractional derivative. Sometimes it is possible to write

\[
f(x, t) = \int K(t, x, y) f(y) \, dy
\]

\(^1\) Apparently, Wright and Maitland are coinciding persons (Edward Maitland Wright). He also coincides with the author of the well-known book Hardy, Wright "An introduction to the number theory".
where the kernel $K$ can be expressed in the terms of the function $\mathbb{L}_{\alpha,\beta}$. The work of Feller generated a wide literature on diffusions generated by pseudo-differential operators.

4) Now, we recall the most important situation, where the functions $\mathbb{L}_{\alpha,\beta}$ arise in a natural way.

Consider a sequence $\xi_j$ of independent random variables and its partial sums sums $S_n = \xi_1 + \cdots + \xi_n$. Consider the distribution $\mu_n$ of $S_n$. Let us center and normalize $\mu_n$ in some way, $\bar{\mu}_n(t) := \mu_n(a_n t + b_n)$, where $a_n > 0, b_n \in \mathbb{R}$ are some constants. Which distributions can appear as limits of sequences $\bar{\mu}_n$? In the most common cases, we obtain a normal (Gauss) distribution. Nevertheless, there are other possible limits ([12], see also [8]), they are named by stable distributions. Densities of these distributions admit a simple expression (0.15) in terms of the functions $\mathbb{L}_{\alpha,1}$.

A logical possibility of non-normal distributions in limit theorems of this kind was observed by Cauchy in 1853, see [4]. He claimed that the distribution, whose densities are given by

$$\varphi_n(x) = \int_0^\infty \exp(-x^2) \cos(t x) \, dt$$

(0.14)
can appear in limit theorems for sums of independent random variables. Firstly, it was necessary to verify positivity of the functions $\varphi_n$. They are is really positive for $0 < \alpha \leq 2$, but Cauchy could not prove this except several simple cases ($\alpha = 1, 1/2, 1$). In 1922 P.Levy\footnote{Some references between Cauchy and Levy can be found in [23]. Also, there was work of Holtsmark [1919] on distribution of gravitation force in Universe, see its exposition in [8], [26]} attracted attention to the problem ([10]), and in 1938 Polya [19] proved positivity of (0.14) for $0 < \alpha < 1$.

After appearance of Kolmogorov–Levy–Hinchin integral representation for infinitely divisible laws, a complete description of stable distributions became solvable problem, the final result is present in the books of P.Levy [11], 1937, and A.Hinchin (another spelling is 'Khinchine'), [9], 1938. The stable densities can be represented in the form

$$p(x; \alpha, \gamma) = \frac{1}{\pi x} \text{Im} \mathbb{L}_{\alpha,1}(x^{-\alpha} e^{i(\gamma-\alpha)\pi/2})$$

(0.15)

where $0 < \alpha < 2, \gamma \in \mathbb{R}$, and $|\gamma| < \min(\alpha, 2-\alpha)$ (in this formula, we omit the exceptional and simple case $\alpha = 1$).

It was clear that the integrals of the form (0.2), (0.14) have no expression in terms of classical special functions, but they were important for probabilists and attracted their interest, see [7], [8], [25]. The basic text on this subject is Zolotarev’s book [26], 1986, see also bibliography in this book.

Levy also introduced stable stochastic processes (see [12]). Non-explicitness of stable densities make stable processes difficult for investigations; nevertheless some collection of explicit formulae is known, see Dynkin [5], Neretin [16], Pitman, Yor [18].
In this paper, the expression (0.15) appears in the formulae (0.10), (1.10). Also the formulae (0.11), (0.6) are variants of the "multiplication theorem for the stable laws "[26], Theorem 3.3.1; there are many other places, where we touch formulae from Zolotarev’ book [26], I do not try to fix all similarities in formulae.

5) The functions \( \mathbb{L}_{\alpha,\beta} \) arise in a relatively natural way in the theory of the Laplace transform (the 'operation calculus'), see below. The tables of McLachlan, Humbert, Poli [13], 1950, contain 18 partial cases of the integral transformations defined below; also the transformations (0.10) are contained in Zolotarev [25], and a similar construction with Wright functions is a subject of Agarwal [1].

6) It is known (see [15]) that pseudodifferential equations with constant coefficients of the type

\[
\left( \sum_{k=0}^{n} a_k D^{k\alpha} \right) f = 0
\]

admit explicit analysis. Apparently this phenomena is related to the identities (0.7), (0.12).

0.4. Structure of the paper. In Section 1, we discuss various definitions of the functions \( \mathbb{L}_{\alpha,\beta} \), theirs integral representations, and also some integrals containing products of two functions \( \mathbb{L}_{\alpha,\beta} \).

In Section 2, we discuss the space \( \mathcal{K} \) of holomorphic functions defined above; also we introduce a standard scale \( H_{\beta} \) of Hilbert spaces of holomorphic functions in a half-plane. The latter spaces are well-known in representation theory of \( SL_2(\mathbb{R}) \).

For our purposes, the space \( \mathcal{K} \) and the Hardy space \( H^2 \) are almost sufficient. In Section 3, we introduce a simple construction in a spirit of the Vilenkin-Klimyk book [22]. We consider the 6-dimensional solvable Lie group of operators

\[
f(x) \mapsto \lambda x^h f(ax^a); \quad \lambda \in \mathbb{C}^*, \quad h \in \mathbb{C}, \quad a > 0, \quad a \in \mathbb{R} \setminus 0
\]

acting in a space of functions on a half-line and consider the image of this group under the Laplace transform. As a result, we obtain a group of continuous operators, whose kernels are expressed in terms of \( \mathbb{L}_{\alpha,\beta} \). The most interesting property of these operators is the identity (0.7) given above.

In Section 4, we consider a similar construction. We start from a 7-dimensional semigroup (4,2) of operators acting in a space of holomorphic function on half-plane and consider its image under the inverse Laplace transform. As a result, we obtain a semigroup of integral operators acting in an appropiate space of functions on half-line.

1. Some properties of the functions \( \mathbb{L}_{\alpha,\beta} \).

1.1. Definition. We define the function \( \mathbb{L}_{\alpha,\beta} \) as the Barnes integral

\[
\mathbb{L}_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \Gamma(s) \Gamma(\beta - s) z^{-s} \, ds \quad (1.1)
\]
We must explain a meaning of elements of this formula.

1) Our indices are in the domain \( \alpha \in \mathbb{R}, \beta \in \mathbb{C} \). Assume also
\[
\beta + \alpha m + n \neq 0 \quad \text{for all } n, m = 0, 1, 2, \ldots
\] (1.2)

2) Our integral is convergent if \(|\arg z| < (1 + \alpha)\pi/2\).

3) Our integrand has poles at the points
\[
s = 0, -1, -2, \ldots \quad \text{and} \quad s = \beta/\alpha, (\beta + 1)/\alpha, (\beta + 2)/\alpha, \ldots
\]

Now, we consider two cases \( \alpha > 0 \) and \( \alpha < 0 \).

First, let \( \alpha > 0 \). For \( \beta > 0 \), we can assume that the contour of the integration is the imaginary axis \( i\mathbb{R} \) and we leave the pole \( s = 0 \) on the left side from the contour (denote this contour by \( +0 + i\mathbb{R} \)). Otherwise, we consider a contour \( L \) coinciding with \( i\mathbb{R} \) near \( \pm \infty \) and separating the left series of poles \( s = -n \) and the right series \( s = (\beta + n)/\alpha \) of poles. Such contour exists due the condition (1.2).

We also can transform this contour integral to
\[
\int_L = \int_{+0+i\mathbb{R}} - \sum_{n: (\beta+n)/\alpha < 0} \text{res}_{s=(\beta+n)}
\]

Second, let \( \alpha < 0 \). Then we consider an arbitrary contour \( L \) coinciding with the imaginary axis near \( \pm \infty \) and leaving all the poles of the integrand on the left side. If \( \beta > 0 \), then we can choose \( L \) being \( +0 + i\mathbb{R} \).

Remark. For fixed \( \beta \), \( z > 0 \), the function \( \mathbb{L}_{\alpha,\beta}(z) \) as a function of the parameter \( \alpha \) is \( C^\infty \)-smooth at \( \alpha = 0 \) but it is not real analytic in \( \alpha \) at this point (compare (1.4) and (1.5)). Thus it is not quite clear, is natural to consider \( \mathbb{L}_{\alpha,\beta} \) as one function or as two functions defined for \( \alpha > 0 \) and \( \alpha < 0 \). For local purposes of this paper, the first variant is more convenient.

1.2. Expansion of \( \mathbb{L}_{\alpha,\beta} \) into power series. We write expansions of \( \mathbb{L}_{\alpha,\beta} \) in series applying the standard Barnes method, see [21], [14].

a) Let \( 0 < \alpha < 1 \). Then the integral (1.1) is the sum of residues at the points \( s = -n \), i.e.,
\[
\mathbb{L}_{\alpha,\beta}(z) = \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(\alpha n + \beta)}{n!} z^n
\] (1.3)
This function is well defined on the whole complex plane \( z \in \mathbb{C} \). Due (1.2), the \( \Gamma \)-functions in numerators have no poles.

b) Let \( \alpha > 1 \). Then (1.1) is the sum of residues at the points \( s = (\beta + n)/\alpha \), i.e.,
\[
\mathbb{L}_{\alpha,\beta}(z) = -\frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma((n + \beta)/\alpha)}{n!} z^{(n+\beta)/\alpha}
\] (1.4)

\(^3\)See the integral representation (1.9), we differentiate it in \( \alpha \) and apply the Lebesgue dominant convergence theorem.
Here we assume $z' = \exp(\nu \ln z)$ and $\ln z \in \mathbb{R}$ for $z > 0$. The series is convergent in the domain $|\arg z| < \infty$ (i.e., our function is defined on the universal covering surface $\mathbb{C}^*$ of the punctured complex plane $\mathbb{C}^* = \mathbb{C} \setminus 0$).

c) For $a < 0$, the integral (1.1) is the sum of residues at all the poles, i.e.,

$$\mathbb{L}_{a,\beta}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(a n + \beta)}{n!} z^n - \frac{1}{a} \sum_{n=m}^{\infty} \frac{(-1)^n \Gamma(n + \beta)}{n!} z^{(n+\beta)/a} \quad (1.5)$$

This expression is valid if poles are simple, i.e., $\beta + a n + m \neq 0$ for $n, m = 0, 1, 2, 3, \ldots$. But the points $\beta = -a n - m$ are not really singular, in these cases some of poles of the integrand have order 2, and we must apply a formula for a residue in a non-simple pole (or remove singularities in (1.5)).

The domain of convergence of (1.5) is $|\arg z| < \infty$.

1.3. A symmetry.

**Lemma 1.1.** a) For $a > 0$, $\beta > 0$,

$$\mathbb{L}_{a,\beta}(z) = a^{-1} z^{-\beta/a} \mathbb{L}_{1/a,\beta/a}(z^{-1/a}) \quad (1.6)$$

b) For $0 < a < 1$,

$$\mathbb{L}_{a,\alpha}(z) = -\frac{1}{a z} [\mathbb{L}_{1/a,\alpha}(z) - 1]$$

c) For $a > 0$,

$$\mathbb{L}_{a,1}(z) = 1 - \mathbb{L}_{1/a,1}(z^{-1/a}) \quad (1.7)$$

**Proof.** a) Substituting $t = \beta - a s$ to (1.1), we obtain

$$\mathbb{L}_{a,\beta}(z) = \frac{1}{2\pi i a} \int_{-\infty}^{+\infty} \Gamma((\beta - t)/a) \Gamma(t) z^{-t - 1/a} dt \quad (1.8)$$

as it was required.

Statement b) follows from

$$\mathbb{L}_{a,\alpha}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(a n + \alpha)}{n!} z^n = \frac{1}{a} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(a n + \alpha)}{(n + 1)!} z^{(n+1)/a}$$

c) Substitute $\beta = 1$ to (1.6) and assume $a > 1$. Applying b), we obtain the required statement for $a > 1$. But the identity (1.6) is symmetric with respect to the transformation $a \mapsto 1/a$, $z \mapsto z^{-1/a}$.

**Remark.** The statement c) is a well-known symmetry in the theory of stable distributions, see [8], (17.6.10), see also [26], Section 2.3.

1.4. Some integral representations of $\mathbb{L}_{a,\beta}$.

**Lemma 1.2.** Let $a \in \mathbb{R}$, $Re u > 0$, $Re v > 0$, $Re h > 0$. Then

$$\int_{0}^{\infty} x^{h-1} \exp(-u x^a - v x) \, dx = v^{-h} \mathbb{L}_{a,h}(u/v^a) \quad (1.9)$$
Proof. It is very easy to verify this for \( \alpha > 0 \).

1) for \( 0 < \alpha < 1 \): we expand the factor \( \exp(-ux^\alpha) \) in (1.9) in Taylor series and integrate term-wise

\[
\sum_{n=0}^{\infty} \frac{(-u)^n}{n!} \int_{\|}^{\infty} x^{\alpha n + h - 1} e^{-v x} dx = \sum_{n=0}^{\infty} \frac{(-u)^n}{n!} \cdot \frac{\Gamma(h + \alpha n)}{v^h}
\]

2) Similarly, for \( \alpha > 1 \), we expand the factor \( \exp(-ux) \) into a Taylor series

\[
\sum_{n=0}^{\infty} \frac{(-u)^n}{n!} \int_{\|}^{\infty} x^{\alpha n + h - 1} \exp(-ux^\alpha) dx = \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{(-u)^n}{n!} \cdot \frac{\Gamma(h + n + \alpha)}{u^{\alpha n + h}} = \frac{1}{\alpha} \frac{u^{-h/\alpha + 1} / \Gamma(h + n + \alpha)}{\alpha u^{\alpha}}
\]

and apply the symmetry (1.6)

3) The case \( \alpha < 0 \) is not obvious, and we give a calculation that is valid for all \( \alpha \in \mathbb{R} \). Consider the space \( L^2 \) on \( \mathbb{R}^+ \) with respect to the measure \( dx / x \). The left hand side of (1.9) is the \( L^2 \)-inner product of the functions \( \Phi_1, \Phi_2 \) given by

\[
\Phi_1(x) = \exp(-ux^\alpha); \quad \Phi_2(x) = x^\alpha \exp(-ux^\alpha)
\]

The Mellin transform of \( \Phi_1 \) is

\[
\Phi_1(\lambda) = \int_{\|}^{\infty} x^{\lambda - 1} \exp(-ux^\alpha) dx = \frac{\text{sgn}(\alpha)}{\alpha} \int_{\|}^{\infty} \exp(-uy) y^{\lambda - 1} dy = \frac{\text{sgn}(\alpha) \Gamma(\lambda / \alpha)}{\alpha u^{\lambda / \alpha}}
\]

The Mellin transform of \( \Phi_2 \) is

\[
\Phi_2(\lambda) = \int_{\|}^{\infty} x^{\lambda + \alpha - 1} \exp(-ux^\alpha) dx = e^{-\lambda h} \Gamma(h)
\]

By the Plancherel formula for the Mellin transform, we have

\[
\int_{\|}^{\infty} \Phi_1(x) \Phi_2(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_1(is) \Phi_2(is) ds
\]

i.e.,

\[
\int_{\|}^{\infty} x^{\alpha n + h - 1} \exp(-ux^\alpha - v x) dx = \frac{\text{sgn}(\alpha)}{2\pi \alpha v} \int_{-\infty}^{\infty} \Gamma(is / \alpha) \Gamma(h + is) u^{-\alpha / v} v^{is} ds
\]

Then we introduce the new variable \( t = s / \alpha \).

1.5. Integral representations. Variants. Now let \( x, y > 0 \). Let \( \alpha < 1, \theta > 0 \). Then

\[
\frac{1}{2i} \int_{-i\infty}^{+i\infty} p^{-\alpha - \theta} \exp(-p^\alpha x + py) dp = -\text{Im} \left[ \Gamma(\alpha + \theta) \Gamma(h + \alpha \theta) \int_0^1 \left( xy^{-\theta} e^{\pi i \theta} \right) dx \right] = -\sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n + \theta)}{n!} x^n y^{-\theta} \sin(n \alpha + \theta) \quad (1.10)
\]
where the integration is given over the imaginary axis, and $p^\alpha$ is positive real for $p > 0$.

Indeed, we represent the integral in the form

$$\frac{1}{2i}e^{\pi i \theta /2} \int_{0}^{\infty} \exp(-t^\alpha x e^{\pi i \theta /2} + ity) t^{\theta - 1} dt = -\frac{1}{2i}e^{-\pi i \theta /2} \int_{0}^{\infty} \exp(-t^\alpha x e^{-\pi i \theta /2} - ity) t^{\theta - 1} dt$$

and apply (1.9) with $u = x \exp(\pm \pi i \theta /2)$, $v = y \exp(\mp \pi i /2)$.

**Remark.** For $0 < \alpha < 1$ the expression (1.10) is a density of a stable subordinator.

**Remark.** The calculation given above survives for the case $\alpha < 0$. The factor $\exp(-t^\alpha x e^{\pi i \theta /2})$ is flat at $t = 0$ and hence we can consider $\theta < 0$. We transform

$$\Gamma(\alpha n + \theta) \sin[(\alpha n + \theta)\pi] = \pi / \Gamma(1 - \theta - \alpha n)$$

and we reduce (1.10) to the form $y^{-\theta} \mathbb{W}_{-\alpha, 1-\theta}(x/y^\alpha)$, where $\mathbb{W}$ is the Wright function.

1.6. **Remark.** An $L^2(\mathbb{R})$-inner product. Let $\alpha > 0$, $\beta > 0$. Let $x$, $y > 0$. Consider the function

$$\Psi_{\alpha, \beta, y}(x) := x^{\beta - 1} \exp(-yx^\alpha)$$

(1.11)

By (1.9), its Laplace transform is

$$\hat{\Psi}_{\alpha, \beta, y}(\xi) = \int_{0}^{\infty} x^{\beta - 1} \exp\{-yx^\alpha - \xi x\} dx = \xi^{-\beta} \mathbb{W}_{\alpha, \beta}(y/\xi^\alpha)$$

(1.12)

Evaluating the $L^2(\mathbb{R})$-inner product of $\Psi_{\alpha_1, \beta_1, y_1}$ and $\Psi_{\alpha_2, \beta_2, y_2}$, we obtain

$$\int_{0}^{\infty} x^{\beta_1 - 1} \exp(-y_1 x^{\alpha_1}) x^{\beta_2 - 1} \exp(-y_2 x^{\alpha_2}) dx = a_2^{-1} y_2^{-(\beta_1 + \beta_2 - 1)/\alpha_2} \mathbb{W}_{\alpha_1, \beta_1, \beta_2, y_2} = a_2^{-1} y_2^{-(\beta_1 + \beta_2 - 1)/\alpha_2} \mathbb{W}_{\alpha_1, \beta_1, \beta_2, y_2}$$

(1.13)

(we substitute $t = x^{\alpha_2}$ and apply Lemma 1.2). The expression in the right-hand side is symmetric with respect to $(\alpha_1, \beta_1) \leftrightarrow (\alpha_2, \beta_2)$ by Lemma 1.1.

By the Plancherel formula for the Fourier transform, the same expression can be written in the form

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (it)^{-\beta_1} \mathbb{W}_{\alpha_1, \beta_1} (y_1/(it)^{\alpha_1}) (-it)^{-\beta_2} \mathbb{W}_{\alpha_2, \beta_2} (y_2/(-it)^{\alpha_2}) dt$$

(1.14)

Thus, the expression (1.14) equals (1.13).

**Remark.** The integral (1.14) looks like a kernel of a product of two integral operators; moreover (1.13) shows that this product has the same form, i.e., we
obtain a family of integral operators closed with respect to multiplication. Below we propose two ways to give a precise sense for this observation; apparently, there are other possibilities.

1.7. Remark. Convolutions. Preserve notation (1.11), (1.12). We have
\[ \Psi_{\alpha, \beta, \gamma}(x) \Psi_{\alpha', \beta', \gamma'}(x) = \Psi_{\alpha, \beta + \beta', \gamma + \gamma'}(x) \]

Hence
\[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \hat{\Psi}_{\alpha, \beta, \gamma}(u) \hat{\Psi}_{\alpha', \beta', \gamma'}(z - u) du = \hat{\Psi}_{\alpha, \beta + \beta', \gamma + \gamma'}(z) \]

2. Spaces of holomorphic functions. Preliminaries

2.1. Spaces \( L^2_\mu(\mathbb{R}_+) \). Fix \( \mu > 0 \). Denote by \( L^2_\mu(\mathbb{R}_+) \) the space \( L^2 \) on the half-line \( \mathbb{R}_+, x > 0 \), with respect to the weight \( \Gamma(\mu)^{-1} x^\mu^{-1} dx \), i.e., the Hilbert space with the inner product
\[ \langle f, g \rangle_{[\mu]} := \frac{1}{\Gamma(\mu)} \int_0^\infty f(x) \overline{g(x)} x^\mu^{-1} dx \]

For instance,
\[ \langle \exp(-z x), \exp(-u x) \rangle_{[\mu]} = (z + u)^{-\mu} \tag{2.1} \]
for arbitrary complex \( u, z \) satisfying \( \text{Re} \, z > 0, \text{Re} \, u > 0 \).

2.2. Hilbert spaces of holomorphic functions on a half-plane. Let \( \Pi \) be the right half-plane \( \text{Re} \, z > 0 \) on the complex plane. We consider the Hardy space \( \mathcal{H}^2 \) on \( \Pi \). Recall that this space consists of functions holomorphic in the half-plane, whose boundary values on the imaginary axis \( \text{Re} \, z = 0 \) exist and are contained in \( L^2(\mathbb{R}_+) \).

The inner product in \( \mathcal{H}^2 \) is given by
\[ \langle f, g \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(it) \overline{g(it)} dt := \lim_{\varepsilon \to +0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\varepsilon + it) \overline{g(\varepsilon + it)} dt \]

The Hardy space is an element of the following one-parametric scale \( H_\mu \),
\( \mu > 0 \), of spaces of holomorphic functions.

Fix \( \mu > 1 \). Consider the space \( H_\mu = H_\mu(\Pi) \) consisting of functions \( f(z) \) holomorphic in \( \Pi \) and satisfying the condition
\[ \int_{\Pi} |f(z)|^2 (\text{Re} \, z)^{-\mu/2} d\sigma < \infty \]
where \( d\sigma \) denotes the Lebesgue measure on \( \Pi \).

We define an inner product in \( H_\mu \) by
\[ \langle f, g \rangle_\mu = \frac{\mu - 1}{\pi} \int_{\Pi} f(z) g(z) (\text{Re} \, z)^{-\mu/2} dz \, d\sigma \tag{2.2} \]

The space \( H_\mu \) is a Hilbert space with respect to this inner product.
The reproducing kernel\(^4\) of this space is
\[
K_\mu(z, u) = (z + \overline{u})^{-\mu}
\]
This means that the function \(\Xi_u(z)\) given by
\[
\Xi_u(z) = (z + \overline{u})^{-\mu}
\]
satisfies the reproducing property
\[
\langle f, \Xi_u \rangle_\mu = f(u) \quad \text{for all } f \in H_\mu. \quad (2.3)
\]
In particular,
\[
\langle \Xi_u, \Xi_w \rangle_\mu = (w + \overline{u})^{-\mu} \quad (2.4)
\]
The space \(H_\mu\) can be defined by (2.3), (2.4) without reference to explicit formula (2.2) for the inner product. Indeed, consider an abstract Hilbert space \(H\) with a system of vectors \(\Xi_u\), where \(u \in \Pi\), and assume that their inner products are given by (2.4). Such Hilbert space exists, see formula (2.1). Assume also that linear combinations of \(\Xi_u\) are dense in \(H\). Then for each \(h \in H\) we consider the holomorphic function on \(\Pi\) given by
\[
f_h(u) := \langle h, \Xi_u \rangle_H \quad (2.5)
\]
and thus we have identified our space \(H\) with some space of holomorphic functions on \(\Pi\).

But the last construction survives for arbitrary \(\mu > 0\) (since the existence of \(H_\mu\) is provided by formula (2.1) and this formula is valid for \(\mu > 0\)).

Remark. For \(\mu = 1\) we obtain the Hardy space \(H^2\).

Remark. For \(0 < \mu < 1\), it is possible to write an integral formula for the inner product in \(H_\mu\) involving derivatives. But it is more convenient to use the definition (2.3)-(2.4) or to consider the analytic continuation of the integral (2.2) with respect to \(\mu\).

We define the weighted Laplace transform \(\mathcal{L}_\mu\) by
\[
\mathcal{L}_\mu f(z) = \frac{1}{\Gamma(\mu)} \int_0^\infty f(x) \exp(-ux) x^{\mu - 1} dx
\]
For \(\mu = 1\) we obtain the usual Laplace transform \(\mathcal{L} = \mathcal{L}_1\). The following statement is well-known\(^5\).

Lemma 2.1. The weighted Laplace transform is a unitary operator
\[
\mathcal{L}_\mu : L^2_\mu(\mathbb{R}_+) \rightarrow H_\mu(\Pi)
\]

Proof. Consider a function
\[
\xi_u(x) = \exp(-ux)
\]
\(^4\) For machinery of reproducing kernels, see, for instance, [9], [17].
\(^5\) The case \(\mu = 1\) is a Paley-Wiener theorem.
in \( L^2_\mu \). Its image under \( \mathcal{L}_\mu \) is \( \Xi_\mu \). It remains to compare (2.1) and (2.4).

**Remark.** The transform \( \mathcal{L}_\mu \) is precisely the operator defined by the formula (2.5). Indeed, we can assume \( H = L_\mu \), then the corresponding space of functions \( f_\mu \) is \( H_\mu \).

### 2.3. Operators in the spaces \( H_\mu \).
Recall a standard general trick, apparently discovered by Berezin [3] (formulae (2.6), (2.9) given below are valid in arbitrary Hilbert space defined by a reproducing kernel).

Let \( A \) be a bounded operator \( H_\mu \to H_\mu \). Define the function

\[
M(z, u) = A\Xi_\mu(z)
\]

Then \( M(z, u) \) is the kernel of the operator \( A \). For \( \mu > 1 \) we can write literally

\[
Af(z) = \frac{\mu - 1}{\pi} \int_\Pi M(z, u) f(u) (\text{Re } u)^{\mu - 2} \, du \, d\theta
\]

Respectively, for \( \mu = 1 \),

\[
Af(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} M(z, -it) f(it) \, dt
\]

For general \( \mu > 0 \), we can write

\[
Af(z) = \langle f, M(z, u) \rangle_\mu
\]

the inner product is given in the space \( H_\mu \) of functions depending in the variable \( u \). Formulae (2.7)–(2.8) are partial cases of this formula. In particular, the integrals (2.7)–(2.8) are convergent for each \( z \in \Pi \) and \( f \in H_\mu \).

### 2.4. Space of rapidly decreasing functions.
We also consider the space \( \mathcal{K} = H^2 \cap \mathcal{S}(\mathbb{R}) \), where \( \mathcal{S}(\mathbb{R}) \) is the Schwartz space (consisting of functions on the imaginary axis rapidly decreasing with all derivatives).

We can say that \( \mathcal{K} \) is the space of functions holomorphic in \( \text{Re } z > 0 \) and continuous in \( \text{Re } z \geq 0 \) such that

a) \( f(it) \) is \( C^\infty \)-smooth

b) For each \( k > 0 \) and \( N > 0 \) there exists \( M \) such that

\[
|f^{(k)}(z)| \leq M (1 + |z|)^{-N}
\]

(2.10)

Consider the space \( \mathcal{S}(\mathbb{R}_+) \) consisting of smooth functions \( f \) on \([0, \infty)\) such that

a) \( f^{(k)}(0) = 0 \) for all \( k \geq 0 \)

b) \( \lim_{t \to +\infty} f^{(k)}(x)x^N = 0 \) for all \( N > 0 \), \( k \geq 0 \).

In other words, \( \mathcal{S}(\mathbb{R}_+) \) is the intersection of the Schwartz space \( \mathcal{S}(\mathbb{R}) \) on \( \mathbb{R} \) and the space \( L^2(\mathbb{R}_+) \).

**Lemma 2.2.** The space \( \mathcal{K} \) is the image of \( \mathcal{S}(\mathbb{R}_+) \) under the Laplace transform.
Proof. Let \( f \in \mathcal{S}(\mathbb{R}_+) \). Integrating by parts, we obtain
\[
\int_0^{\infty} f^{(k)}(x) e^{-px} \, dx = p^k \int_0^{\infty} f(x) e^{-px} \, dx
\]
The left-hand side is a bounded function in \( p \), looking to the right-hand side, we observe that \( (\mathcal{L} f)(p) \) is rapidly decreases for \( \Re p \geq 0 \).

Conversely, a function \( F \) satisfying (2.10) is an element of \( \mathcal{H}^2 \). Hence \( f = \mathcal{L}^{-1} F \) is supported by \( \mathbb{R}_+ \). Since (2.10) is valid for \( z \in \mathbb{R} \), we have \( f \in \mathcal{S}(\mathbb{R}) \).

2.5. Fractional derivations. We define the operators of fractional differentiation \( D_h \) in \( \mathcal{K} \) by
\[
D_h f(z) = \frac{\Gamma(h+1)}{2\pi} \int_{-\infty}^{+\infty} \frac{f(it)}{(-it + z)^{h+1}} \, dt
\]
A branch of \( \theta(z, \bar{t}) = (-it + z)^{h+1} \) is determined from the condition \( \theta(x, 0) > 0 \) for \( x > 0 \).

Lemma 2.3. a) \( D_h \) is an operator \( \mathcal{K} \to \mathcal{K} \) for each \( h \in \mathbb{C} \).

b) For integer \( n > 0 \),
\[
D_n f(z) = (-1)^n \frac{d^n}{dz^n} f(z)
\]
c) For positive integer \( m \),
\[
D_{-m} f(z) := \lim_{\lambda \to m} \frac{\Gamma(-s+1)}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(it)}{(-it + z)^{s+1}} \, dt = (-1)^m \int_{-\infty}^{+\infty} dz_1 \int_{-\infty}^{+\infty} dz_2 \ldots \int_{-\infty}^{+\infty} f(z_n) \, dz_n
\]
d) \( D_{h_1} D_{h_2} = D_{h_1+h_2} \)

Proof. a) Convergence of the integral for \( \Re u > 0 \) is obvious. Let us show rapid decreasing of \( g := D_h f \) at \( z \to \infty \).

Let \( \Re h < 0 \). We represent (2.11) as a contour integral
\[
\frac{\Gamma(h+1)}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(u) \, du}{(-u + z)^{h+1}}
\]
Denote \( R := |z| \). Then we replace a part \( (-iR, +iR) \) of the contour of the integration \( \mathbb{R} \) by the semi-circle \( R \exp(i\varphi) \), where \( \varphi \in (-\pi, \pi) \). Since (2.10), all the 3 summands of the integral rapidly tend to 0 as \( R \) tend to \( \infty \).

If \( \Re h \geq 0 \), we integrate our expression by parts and obtain
\[
(-1)^k \frac{\Gamma(h-k+1)}{2\pi i} \int_{-\infty}^{+\infty} (z - u)^{h-k-1} f^{(k)}(u) \, du
\]
We choose \( k > \Re(h+1) \) and repeat the same consideration.
Also, \[ \frac{d}{dz} D_h f = -D_{h+1} f \]
and this implies rapid decreasing (2.10) of derivatives.

b) This is the Cauchy integral representation for derivatives.

c) First, we give a remark that formally is not necessary. Factor \( \Gamma(1 + h) \)
has a pole for \( h = -m \). Let us show that \( \int \) vanishes at this point. Indeed, we have the expression
\[ \int_{-i\infty}^{i\infty} (z - u)^{m-1} f(u) \, du \]
We replace a part \((-iR, iR)\) of the contour of the integration by the semi-circle \( R \exp(i\phi) \) as above and tend \( R \) to \( \infty \).

Now give a proof of c). Consider the operator of indefinite integration
\[ I f(z) = \int_{-i\infty}^z f(u) \, du \quad (2.13) \]
Changing the contour as above, we obtain \( I f \in \mathcal{K} \).

For \( f \in \mathcal{K} \) we have
\[ \Gamma(1 - s) \int_{-i\infty}^{i\infty} (z - u)^{s-1} f(u) \, du = \]
\[ = (-1)^m \Gamma(1 - s + m) \int_{-i\infty}^{i\infty} (z - u)^{s-m-1} (I^m f)(u) \, du = \]
Now we can substitute \( s = m \).

d) This is valid for \( \text{Re} \, h_1 = \text{Re} \, h_2 = 0 \) by the statement b) of following
Lemma 2.4. Then we consider the analytic continuation in \( h \).

**Lemma 2.4.** a) For \( s \in \mathbb{R} \), the operator \( D_{is} \) is a unitary operator in each \( \mathcal{H}_\mu \). Its kernel (in the sense of \( \mathcal{H}_\mu \)) is
\[ \frac{\Gamma(\mu + is)}{\Gamma(is)} (z + \overline{\nu})^{-\mu - is} \]
\[ b) \quad D_{is_1} D_{is_2} = D_{is_1 + is_2} \]

**Proof.** a) Consider the operator \( U_{is} \) in \( L^2_\mu (\mathbb{R}_+) \) given by
\[ U_{is} f(x) = f(x)^{i^s} \]
Let us evaluate the kernel of the operator
\[ \mathcal{L}_\mu U_{is} \mathcal{L}_\mu^{-1} : \mathcal{H}_\mu \to \mathcal{H}_\mu \]
By (2.6), we must evaluate the function \( \mathcal{L}_\mu U_{is} \mathcal{L}_\mu^{-1} \Xi_u \). We have
\[ \mathcal{L}_\mu^{-1} \Xi_u(x) = \exp(-\pi x), \]
\[ U_{is} \mathcal{L}_\mu^{-1} \Xi_u(x) = \exp(-\pi x)x^{is}; \]
\[ \mathcal{L}_\mu U_{is} \mathcal{L}_\mu^{-1} \Xi_u(z) = (z + \overline{\nu})^{-\mu - is} \Gamma(\mu + is)/\Gamma(is) \]
For $\mu = 1$ our operator coincides with the operator $D_{1,\nu}$ defined above. In fact, all the operators $L_\mu U_{1,\nu} L_\mu^{-1} : H_\mu \to H_\mu$ induce the same operator $D_{1,\nu}$ in $K$. Indeed, the operator $L_\mu^{-1} L_\nu$ is the operator of multiplication by $x^\mu - \nu$, and this operator commutes with $U_{1,\nu}$.

b) corresponds to the identity $x^{i\xi_1 + i\xi_2} = x^{i\xi_1} x^{i\xi_2}$ after the Laplace transform.

3. Operators in spaces of holomorphic functions

3.1. Some operators acting in $S(\mathbb{R}_+).$ We consider the following one-parameter groups of operators in the space $S(\mathbb{R}_+)$ (see 2.4).

\[ U_\alpha f(x) = f(x^\alpha), \quad \alpha > 0; \]  \hspace{1cm} (3.1)

\[ V_\alpha(x)f(x) = f(ax), \quad a > 0; \]  \hspace{1cm} (3.2)

\[ W_h f(x) = x^h f(x), \quad h \in \mathbb{C} \]  \hspace{1cm} (3.3)

The last group is a complex one-parameter group, i.e., a real two-parameter group. The infinitesimal generators of these groups are respectively

\[ E_1 f(x) = x \ln x \frac{d}{dx} f(x); \quad E_2 f(x) = x \frac{d}{dx} f(x); \quad E_3 f(x) = (\ln x) f(x) \]

They satisfy the commutation relations

\[ [E_1, E_2] = -E_2; \quad [E_1, E_3] = E_3; \quad [E_2, E_3] = 1 \]  \hspace{1cm} (3.4)

Thus we obtain a real 6-dimensional Lie algebra $\mathfrak{g}$ spanned by the operators

\[ E_1, \quad E_2, \quad E_3, \quad iE_3, \quad 1, \quad i \]

The algebra $\mathfrak{g}$ is solvable and it contains a two-dimensional center $\mathbb{R} \cdot 1 + \mathbb{R} \cdot i$.

Also $\mathfrak{g}$ is a real subalgebra (but not a real form) in 4-dimensional complex Lie algebra

\[ \mathbb{C} \cdot E_1 + \mathbb{C} \cdot E_2 + \mathbb{C} \cdot E_3 + \mathbb{C} \cdot 1 \]

Obviously,

\[ U_\alpha W_h U_\alpha^{-1} = W_{\alpha h}; \]  \hspace{1cm} (3.5)

\[ U_\alpha V_\alpha U_\alpha^{-1} = V_{\alpha / \alpha} \]  \hspace{1cm} (3.6)

Consider the group $G$ generated by the one-parameter groups (3.1)-(3.3). General element of this group is an operator of the form

\[ \lambda \cdot R(h, \alpha, a) \]

where $\lambda \in \mathbb{C}$ and

\[ R(h, \alpha, a)f(x) = x^h f(ax^\alpha) = W_h U_\alpha V_\alpha f(x) \]  \hspace{1cm} (3.7)
The product is given by
\[ R(g, \beta, b)R(h, \alpha, a) = b^h R(g + h\beta, \alpha \beta, ab^\alpha) \] (3.8)

We also can add the operator
\[ U_{-1} f(x) = f(1/x) \]
or equivalently we can allow \( a \in \mathbb{R} \setminus 0 \) in (3.1), (3.7), (3.8). Then we obtain a Lie group \( G^* \) of operators containing two connected components; the group \( G \) defined above is the connected component containing 1.

3.2. Operators in \( L^2_{\mu}(\mathbb{R}_+) \). Now, let us fix \( \mu > 0 \). Then the operators
\[ |a|^{1/2}a^{1/2} R((a - \mu)/2 + is, \alpha, a) \]
are unitary in \( L^2_{\mu}(\mathbb{R}_+) \). Such operators form a 4-dimensional solvable Lie group with an one-dimensional center; denote this group by \( G^*_\mu \).

Remark. All other operators \( R(h, \alpha, a) \) are unbounded in \( L^2_{\mu}(\mathbb{R}) \).

3.3. Operators in \( H_\mu \). Let us evaluate the kernel in \( H_\mu \) of the operator \( \mathcal{L}_\mu R(h, \alpha, a)\mathcal{L}_\mu^{-1} \) using 2.3.

We have
\[ \mathcal{L}_\mu^{-1} \Xi_u(x) = \exp(-x \alpha); \]
\[ R(h, \alpha, a)\mathcal{L}_\mu^{-1} \Xi_u(x) = x^h \exp(-x^\alpha); \]
\[ \mathcal{L}_\mu R(h, \alpha, a)\mathcal{L}_\mu^{-1} \Xi_u(z) = \frac{1}{\Gamma(\mu)} \int_0^\infty x^{\mu-1} \exp(-a x^\alpha) - z x \, dx = \]
\[ = z^{-\mu-1} \Pi_{\alpha, h+\mu}(a/z^\alpha) \] (3.9)

and we obtain (for \( \mu > 1 \)) the integral operator
\[ \tilde{R}(h, \alpha, a)F(z) = \frac{1}{\pi^2(\mu - 1)^{z^{-\mu-1}} \int_{\Pi_{\alpha, h+\mu}(a/z^\alpha)F(u)}} \] (3.10)

For \( \mu = 1 \) we understand (3.11) as (2.8), and for \( \mu < 1 \) as (2.9).

Formally, our algorithm of evaluating the kernel is valid only for bounded operators. Thus we proved the following theorem

Theorem 3.1. Let
\[ \text{Re } h = (a - \mu)/2 \]

Then an operator \( \tilde{R}(h, \alpha, a) \) defined by (3.11) is unitary in \( H_\mu \) up to a scalar factor. The product of two operators \( \tilde{R} \) is given by the formula
\[ \tilde{R}(g, \beta, b)\tilde{R}(h, \alpha, a) = b^h \tilde{R}(g + h\beta, \alpha \beta, ab^\alpha) \] (3.12)

These operators generate a 4-dimensional solvable Lie group isomorphic to the group \( G^*_\mu \) described in 3.2.
3.4. Operators in the space $\mathcal{K}$. The group $G^*$ defined in 3.1 acts in the space $\mathcal{S}(\mathbb{R}^+)$. Since the weighted Laplace transform $\mathcal{L}_\mu$ identifies $\mathcal{S}(\mathbb{R}^+)$ and $\mathcal{K}$, this group also acts in $\mathcal{K}$. For the subgroup $G^* \subset G^*$, the action was constructed in the previous subsection, but formula (3.11) for the kernel almost survives for a general element of $G^*$. We consider separately $a > 0$ and $a < 0$.

a) Let $a > 0$. We substitute $y = x^a$ to the expression (3.9) and obtain

$$\frac{1}{|a|} \int_{\delta}^{\infty} y^{(b+\mu)/\alpha-1} \exp(-a y^a - z y^{1/\alpha}) \, dy$$  

(3.13)

If $h$ satisfies the condition

$$(\text{Re } h + \mu)/\alpha > 0$$  

(3.14)

then our integral is convergent (otherwise, we have a non-integrable singularity at 0). The expression (3.13) is the Laplace transform of the function $y^{(b+\mu)/\alpha-1} \exp(-z y^{1/\alpha})$. Since this function is an element of $L^1$, its Laplace transform is a bounded function in $\Pi$. Hence, for $F \in \mathcal{K}$, the integral (3.11) is convergent and is holomorphic in $h$. Thus the formula (3.11) defines the $\mathcal{L}_\mu$-image of $R(h, a, a)$ for all triples $h, a, a$ satisfying (3.14).

Next, we write

$$R(h, a, a) = W_{-n} \circ R(h + n, a, a)$$  

(3.15)

For sufficiently large $n$, the $\mathcal{L}_\mu$-image of $R(h + n, a, a)$ is defined by formula (3.11); the $\mathcal{L}_\mu$-image of $W_{-n}$ is the iterated indefinite integration (2.12).

b) $a < 0$. We again transform the integral to the form (3.13). Now the integrand is smooth at 0; for convergence at infinity we are need in the condition $(\text{Re } h + \mu)/\alpha < 0$. Then we repeat (3.15)\footnote{This is not really necessary, since for $\text{Re } a > 0$ integral (3.13) is convergent. But for the case $a = 1$ it is pleasant to have an expression for $\text{Re } a = 0$.}

Thus we obtain the following theorem

**Theorem 3.2.** Fix $\mu > 0$. Let $h \in \mathbb{C}$, $a \in \mathbb{R} \setminus 0$, and $a > 0$. If $\text{Re } h + \mu > 0$, then we define the integral operator $\tilde{R}(h, a, a)$ in $\mathcal{K}$ by (3.11). Otherwise, we consider $n$ such that $\text{Re } h + n > 0$ and define

$$\tilde{R}(h, a, a) = (-1)^a I^n \circ \tilde{R}(h + n, a, a)$$

there $I$ is the operator of indefinite integration (2.13) Then all these operators are bounded in $\mathcal{K}$ and their product is given by (3.12). The group generated by $\tilde{R}(h, a, a)$ is isomorphic to the group $G^*$ defined in 3.1.

**Remark.** For different $\mu$ we obtain the same group of operators in $\mathcal{K}$; but identification of this group with $G^*$ depends on $\mu$.

3.5. Statements formulated in Introduction. Let $\mu = 1$. The operator $A_a$ given by (0.5) is the $L_1$-image of $U_a$, the fractional derivation $D_h$ is the $L_1$-image of $W_h$, and the operator $R_{a}$ is the $L_1$-image of $V_a$. Now (0.7), (0.9) follow from (3.5), (3.6).
The operator $F \mapsto z F$ is the $L$-image of $d/dx$, and this implies (0.8).

3.6. **Hankel type transforms.** The kernel of the operator $\hat{R}(h, -1, 1)$ is

$$K(z, u) = \text{const} \cdot \int_0^\infty x^{h+\beta-1} \exp(-\pi/x-zx) \, dx$$

This expression is a modified Bessel function of Macdonald, see [6]. The corresponding integral transform is similar to the Hankel transform.

3.7. **Another group of symmetries.** Now we consider the group of unitary operators in $L^2(\mathbb{R}^+_3)$ generated by

$$U^\alpha f(x) = a^{|\alpha|/2} f(x^\alpha x^{(a-1)/2});$$
$$V^\alpha f(x) = a^{|\alpha|/2} f(ax);$$
$$T_\beta (is) f(x) = \exp(is x^\beta) f(x), \quad s \in \mathbb{R}, \beta \in \mathbb{R}$$

This group is infinite dimensional since it contains all the operators having the form

$$f(x) \mapsto f(x) \exp(i \sum_{j=1}^N s_j x^{\beta_j})$$

Obviously, we have

$$U^\alpha T_\beta (is) U^{-1}_\alpha f(x) = T_{a^\beta}(is); \quad V^\alpha T_\beta (is) V^{-1}_\alpha = T_{a^\beta}$$

(3.16)

Consider the image of our group of operators under the standard Laplace transform $L$. The operators $U^\alpha, V^\alpha$ are contained in the group $G^a$ and their images $\hat{U}^\alpha, \hat{V}^\alpha$ are described above. The image of $T_\beta (is)$ is the convolution operator

$$\hat{T}_\beta (is) F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} M(z-u) F(u) \, du$$

where

$$M(z) = \int_0^\infty \exp(is x^\beta - zx) \, dx = \mathbb{L}_\beta,1(is^z/x^\beta)$$

In particular, we obtain the identity

$$\hat{U}^\alpha \hat{T}_\beta (is) \hat{U}^{-1}_\alpha f(x) = \hat{T}_{a^\beta} (is)$$

for our operators with $\mathbb{L}$-kernels.

4. **Operators in space of functions on half-line**

4.1. **Spaces of functions.** Consider the space $P$ consisting of $C^\infty$-functions on half-line $x \geq 0$ satisfying

a) $f^{(k)}(0) = 0$ for all $k \geq 0$

b) $\lim_{x \to +\infty} f(m)(x) \exp(-e x) = 0$ for all $e > 0, m \geq 0$. 

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Consider also the space $\mathcal{F}$, whose elements are functions holomorphic in the half-plane $\text{Re } z > 0$ satisfying the condition

for each $\varepsilon > 0$, $N > 0$ where exists $C$ such that

$$|f(z)| < C/|z|^{-N} \quad \text{for } \text{Re } z > \varepsilon$$

**Lemma 4.1.** The image of the space $\mathcal{P}$ under the Laplace transform is $\mathcal{F}$.

**Proof.** a) For $f \in \mathcal{P}$ denote $F = \mathcal{L}f$. Fix small $\delta > 0$.

$$F(p) = \int_0^\infty f(x)e^{-px} \, dx = \int_0^\infty \left[ f(x)e^{-\delta x} \right] e^{-(p-\delta)x} \, dx$$

Since $f(x)e^{-\delta x}$ is an element of the Schwartz space, for each $N$ we have an estimate

$$|F(p)| \leq C \text{Re } p - \delta^{-N}$$

for $\text{Re } p \geq \delta$. For $\text{Re } p > 2\delta$, we can write

$$|F(p)| \leq 2^N C |p|^{-N}$$

Thus $F \in \mathcal{F}$.

b) Let $F \in \mathcal{F}$. The inversion formula for $\mathcal{L}$ gives

$$f(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{px} F(p) \, dp = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} F(a + it) \, dt$$

Since $F(a + it)$ is an element of Schwarcz space $\mathcal{S}(\mathbb{R})$, the function

$$e^{-\sigma x} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} F(a + it) \, dt$$

is an element of $\mathcal{S}(\mathbb{R})$. By the Paley–Wiener theorem this function is supported by $\mathbb{R}^+$. Since $a > 0$ is arbitrary, we obtain we required statement.

**4.2. Fractional derivations.** Consider the Riemann–Liouville operators of fractional integration (see [20])

$$J_r f(x) = \frac{1}{\Gamma(r)} \int_0^x f(y) (x - y)^{r-1} \, dy \quad (4.1)$$

in the space $\mathcal{P}$. The integral is convergent for $r > 0$. For integer positive $r = n$ we have

$$J_r f(x) = \int_0^x dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_n-1} f(x_n) \, dx_n$$

For fixed $f$ and $x$, the function $J_r f(x)$ admits a holomorphic continuation to the whole plane $r \in \mathbb{C}$, and for integer negative $r = -n$ we have

$$J_{-n} f(x) = \frac{d^n}{dx^n} f(x)$$
We also have the identity
\[ J_r J_p = J_{r+p} \]
for all \( r, p \in \mathbb{C} \).

Laplace transform \( \mathcal{L} \) identifies the Riemann–Liouville fractional integrations \( J_r \) with the operators in \( \mathcal{F} \) given by
\[ F(z) \mapsto z^{-r} F(z) \]

4.3. **Operators.** Now we consider the semigroup \( \Gamma \) of operators in \( \mathcal{F} \) consisting of transformations
\[ \lambda Q(\theta, \alpha, a) \]
where \( \lambda \in \mathbb{C}^* \),
\[ Q(\theta, \alpha, a) \tilde{f}(z) = z^\theta \tilde{f}(a z^\alpha) \tag{4.2} \]
and the parameters satisfy the conditions
\[ 0 < \alpha < 1, \quad \arg a + a \pi/2 < \pi/2, \quad \arg a - a \pi/2 > -\pi/2, \quad \theta \in \mathbb{C} \tag{4.3} \]

**Remark.** The restrictions (4.4) mean that \( z \in \Pi \) implies \( a z^\alpha \in \Pi \).

Obviously, we have
\[ Q(\theta', \alpha', a') Q(\theta, \alpha, a) = (a')^\theta Q(\theta' + \theta \alpha', \alpha \alpha', a(a')^\alpha) \tag{4.4} \]
Thus \( \Gamma \) is a 7-dimensional semigroup with 2-dimensional center.

**Remark.** The semigroup \( \Gamma \) can be embedded to a 7-dimensional Lie group.

The parameters of this group are \( \lambda \in \mathbb{C}^* \), \( a \in \mathbb{C}^* \), \( \theta \in \mathbb{C} \), \( a > 0 \). The multiplication in this group is determined by the formula (4.4), where \( \alpha > 0 \) and \( \theta \), \( a \in \mathbb{C} \). But the corresponding operators (4.2) are not well-defined in the space \( \mathcal{F} \).

The \( \mathcal{L}^{-1} \)-image of the operators \( Q(\theta, \alpha, a) \) is given by the formula
\[ \tilde{Q}(\theta, \alpha, a) \tilde{f}(x) = \int_0^\infty N(x, y) \tilde{f}(y) \, dy \tag{4.5} \]
where the kernels \( N(x, y) \) are given by
\[ N(x, y) = \frac{1}{2\pi i} \int_{|z|=\infty} z^{\theta} \exp(-a z^\alpha y + a z x) \, dz \]
For \( \theta > -1 \), we transform this integral in the same way as in 1.5 and obtain
\[ N(x, y) = \frac{1}{2\pi i y^{\theta+1}} \left\{ e^{i(\theta+1)\pi/2 \Pi_{\alpha, \theta+1} (yax^{-\alpha} e^{i\pi\alpha/2})} - 
 - e^{-i(\theta+1)\pi/2 \Pi_{\alpha, \theta+1} (yax^{-\alpha} e^{-i\pi\alpha/2})} \right\} \]
For real \( a > 0 \), \( \theta > -1 \) we have an expression of the form (1.10).
If \( \theta < -1 \), we write
\[
z^{\theta} F(a z^\alpha) = z^{-n} z^{\theta+n} F(a z^\alpha)
\]
and
\[
\tilde{Q}(\theta, a, a) := J_n \circ \tilde{Q}(\theta + n, a, a)
\]
for sufficiently large \( n \).

**Theorem 4.1.** For \( a \), \( \alpha \) satisfying (4.3), the operators \( \tilde{Q}(\theta, a, a) \) defined by (4.5), (4.6) are bounded in \( \mathcal{P} \) and their product is given by (4.4).

Formulae (0.11)-(0.12) given in Introduction are particular cases of this statement.

**References**


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