Computation of Generalized Equivariant Cohomologies of Kac–Moody Flag Varieties

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Abstract. In 1998, Goresky, Kottwitz, and MacPherson showed that for certain projective varieties $X$ equipped with an algebraic action of a complex torus $T$, the equivariant cohomology ring $H^*_T(X)$ can be described by combinatorial data obtained from its orbit decomposition. In this paper, we generalize their theorem in three different ways. First, our group $G$ need not be a torus. Second, our space $X$ is an equivariant stratified space, along with some additional hypotheses on the attaching maps. Third, and most important, we allow for generalized equivariant cohomology theories $E^*_G$ instead of $H^*_T$. For these spaces, we give a combinatorial description of $E^*_G(X)$ as a subring of $\prod_i E^*_G(F_i)$, where the $F_i$ are certain invariant subspaces of $X$. Our main examples are the flag varieties $\mathcal{G}/P$ of Kac-Moody groups $\mathcal{G}$, with the action of the torus of $\mathcal{G}$. In this context, the $F_i$ are the $T$-fixed points and $E^*_G$ is a $T$-equivariant complex oriented cohomology theory, such as $H^*_T$, $K^*_T$ or $MU^*_T$. We detail several explicit examples.

1 Introduction and Background

The goal of this paper is to give a combinatorial description of certain generalized equivariant cohomologies of stratified spaces. The important examples to which our main theorems apply include $T$-equivariant cohomology, $K$-theory, and complex cobordism of Kac-Moody flag varieties. Although the examples that motivate us come from the theory of algebraic groups, our proofs rely heavily on techniques from algebraic topology. Indeed, we state the results of Sections 2 through 4 in the following context.

Let $G$ be a topological group and $E^*_G$ a $G$-equivariant cohomology theory (see [May96, Chapter XIII] for a definition) with a commutative cup product. Let $X$ be a stratified $G$-space such that successive quotients $X_i/X_{i-1}$ are homeomorphic to Thom spaces $Th(V_i)$ of $G$-orientable $G$-vector bundles $V_i \to F_i$. In this setting, and with the assumption that the Euler classes $e(V_i)$ are not zero divisors, we show that the restriction map

$$i^*: E^*_G(X) \to \prod_i E^*_G(F_i)$$

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is injective. Moreover, when $X$ and the $G$-action satisfy additional technical assumptions, we identify the image of $\iota^*$ as a subring of $\prod_i E^*_G(F_i)$ defined by explicit compatibility conditions involving divisibility by certain Euler classes. We also construct free $E^*_G$-module generators of $E^*_G(X)$.

Our theorems generalize known results in algebraic and symplectic geometry. When $X$ is a projective variety, $G$ a complex torus, and $E^*_G$ ordinary equivariant cohomology, then we recover a theorem of Goresky, Kottwitz and MacPherson [GKM98] that computes $H^*_T(X; \mathbb{C})$. They assume that $X$ has finitely many 0- and 1-dimensional $T$-orbits, and then consider the graph $\Gamma$ whose vertices are the fixed points $X^T$ and edges are the one-dimensional orbits. An edge $(v, w)$ in $\Gamma$ is decorated with the weight $\alpha_{(v, w)}$ of the $T$-action on the corresponding orbit. They provide a combinatorial description of $H^*_T(X)$ as a subring of $H^*_T(X^T)$ in terms of this graph. Each edge of $\Gamma$ gives a condition as follows. Let $x(v)$ denote the restriction of a class $x \in H^*_T(X)$ to $v \in X^T$. Then the condition reads

$$\alpha_{(v, w)} \mid x(v) - x(w). \quad (1.1)$$

We illustrate an example in Figure 1.1.

![Figure 1.1: This shows the graph $\Gamma$ for a flag variety $SL(3, \mathbb{C})/B$. The weight $\alpha_{(v, w)}$ is exactly the direction of the edge $(v, w)$, as explained in Section 5. There is a linear polynomial attached to each vertex, also depicted as a vector. The polynomials satisfy the compatibility conditions, so this does represent an equivariant cohomology class in $H^*_T(SL(3, \mathbb{C})/B)$.](image)

This article is organized as follows. In Section 2 we prove the injectivity of the map

$$\iota^*: E^*_G(X) \to \prod_i E^*_G(F_i).$$

Next, in Section 3, we identify the image of $\iota^*$, giving combinatorial conditions similar to those in (1.1). In Section 4, we give a description of module generators for $E^*_G(X)$. Finally, in Sections 5 and 6, we return to our motivating examples, which are homogeneous spaces $G/P$ for Kac-Moody groups $G$, equipped with the action of a torus $T$. For these spaces, our theory applies when $E^*_G$ is any complex oriented $T$-equivariant cohomology theory. We make explicit computations for three examples: a homogeneous space of $G_2$, the based loop space $\Omega SU(2)$, and a homogeneous space of $\widetilde{LSL}(3, \mathbb{C}) \times \mathbb{C}^*$.  

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Acknowledgments. The first version of this paper concerned ordinary $T$-equivariant cohomology and cell complexes with even dimensional cells. We thank the referee for pointing out that these results extend to arbitrary generalized cohomology theories and more general stratified spaces. (S)he also helped streamline our original proofs in Section 2, and in particular offered a proof of Theorem 2.3.

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2 The injectivity theorem for stratified spaces

Let $G$ be a topological group and $E_G^*$ a $G$-equivariant cohomology theory with commutative cup product. We consider stratified $G$-spaces

$$X = \bigcup_{i \geq 1} X_i, \quad X_1 \subseteq X_2 \subseteq X_3 \ldots$$

(2.1)

where the successive quotients $X_i/X_{i-1}$ are homeomorphic to the Thom spaces $Th(V_i)$ of some $G$-vector bundles $V_i \to F_i$. Moreover, we require that the above vector bundles be $E$-orientable (see [May96, p. 177]). In other words, $X$ is built by successively attaching disc bundles $D(V_i)$ via equivariant attaching maps $\varphi_i : S(V_i) \to X_{i-1}$. This should be compared to the way one builds CW complexes by successively attaching discs.

We recall that an $E$-orientation, or Thom class, of a $G$-vector bundle $V \to F$ is an element $u \in E_G^*(Th(V))$. For each closed subgroup $H < G$ and point $x \in F^H$, the restriction of $u$ to $V|_{G,x}$ is a generator of the free $E_H^*$-module $E_G^*((Th(V|_{G,x})) \simeq (E_H^*(D(V_x), S(V_x)))$. The Euler class $e(V)$ is the restriction of the Thom class $u$ to the base $F$ via the zero section map.

Remark 2.1 As with CW complexes, the stratification is often more naturally indexed by a poset $I$ rather than $\mathbb{N}$. In that case, one should replace the expression $X_i/X_{i-1}$ by $X_i/\bigcup_{j \leq i} X_j$. The poset $I$ is required to satisfy the condition that $\{i \in I : j < i\}$ is finite for all $i \in I$, which makes the inductive proofs work. In the proofs, we ignore this fact and pretend that $I = \mathbb{N}$. The only thing that we need is that for each $i \in I$, the subspace $X_i$ is obtained by a finite sequence of gluings, and that $X = \varinjlim X_i$.

Remark 2.2 In the examples in Sections 5 and 6, the group $G = T$ is a finite dimensional torus, the $T$-spaces $F_i$ are single points and the $V_i$ are complex $T$-representations. The stratification (2.1) expresses $X$ as a cell complex with even dimensional cells.

The main theorem of this section establishes the injectivity of the restriction map $E_G^*(X) \to E_G^*(\coprod F_i) \cong \prod E_G^*(F_i)$ when the Euler classes are not zero divisors.

Theorem 2.3 Let $X$ be a stratified $G$-space and let $E_G^*$ be a multiplicative cohomology theory as above. Assume that the Euler classes $e(V_i) \in E_G^*(F_i)$ of the vector bundles $V_i \to F_i$ are not zero.
divisors. Then the inclusion \( \prod F_i \hookrightarrow X \) induces an injection

\[
i^*: E^*_G(X) \to \prod_i E^*_G(F_i).
\]  

(2.2)

Moreover, let \( E^*_G(X) \) be given the induced filtration under the above inclusion. Then the associated graded \( E^*_G \)-module \( QE^*_G(X) \) is isomorphic to (the direct product of) the ideals generated by the Euler classes in the \( E^*_G(F_i) \). Explicitly,

\[
Q E^*_G(X) \cong \prod_i \epsilon(V_i) E^*_G(F_i).
\]  

(2.3)

Proof: We first prove the theorem when the stratification of \( X \) is finite. This is done by induction on the length of the stratification.

We first consider the assertion that (2.2) is injective. If the length of the stratification is 0, then \( X \) is empty, both sides of (2.2) are zero, and the result trivially holds. We now argue the inductive step. Assume that the stratification of \( X \) has length \( i \) (i.e. \( X = X_i \)) and consider the cofiber sequence

\[
X_{i-1} \to X_i \xrightarrow{p^*} Th(V_i).
\]  

(2.4)

It follows from the assumption on the Euler class that the long exact sequence in \( E \)-cohomology associated to (2.4) splits into short exact sequences

\[
0 \to E^*_G(Th(V_i)) \to E^*_G(X_i) \to E^*_G(X_{i-1}) \to 0.
\]  

(2.5)

To see this, we prove that \( p^* \) is an injection. Indeed, the composition

\[
E^*_G(F_i) \xrightarrow{m^*} E^*_G(Th(V_i)) \xrightarrow{p^*} E^*_G(X_i) \to E^*_G(F_i)
\]

is multiplication by the Euler class \( \epsilon(V_i) \), and is therefore injective. The first map is the Thom isomorphism (see [May96, Theorem 9.2]), so the middle map \( p^* \) must be injective.

Now consider the map of short exact sequences

\[
0 \to E^*_G(Th(V_i)) \to \prod_{j \leq i} E^*_G(F_j) \to E^*_G(F_i) \to 0
\]  

(2.6)

The left vertical map is injective by the assumption on \( \epsilon(V_i) \), with image \( \epsilon(V_i) E^*_G(F_i) \). The right vertical map is injective by induction. By the Five Lemma, the central map is also injective. This proves (2.2) when the filtration of \( X \) is finite.

We now prove (2.3). Again, the base case is trivial, since both sides of (2.3) are zero when the stratification has length zero. We now argue the inductive step. The associated graded \( Q E^*_G(X_i) \) is isomorphic to \( E^*_G(Th(V_i)) \oplus Q E^*_G(X_{i-1}) \). The image of \( Q E^*_G(X_{i-1}) \) under the rightmost vertical map in (2.6) is \( \prod_{j \leq i} \epsilon(V_j) E^*_G(F_j) \) by the induction hypothesis. So, the image of \( Q E^*_G(X_i) \) under the center vertical map is

\[
Q E^*_G(X_i) \cong \epsilon(V_i) E^*_G(F_i) \oplus \prod_{j \leq i} \epsilon(V_j) E^*_G(F_j) = \prod_{j \leq i} \epsilon(V_j) E^*_G(F_j),
\]
as claimed in (2.3).

For both statements (2.2) and (2.3), the general case \( X = \lim_{i} X_i \) follows directly from the finite case since

\[
E^*_G(X) = \lim_{i} E^*_G(X_i).
\]

Note that there is no Milnor \( \lim^1 \) term here because the maps \( E^*_G(X_i) \to E^*_G(X_{i-1}) \) are all surjective. \( \square \)

### 3 The combinatorial description of \( E^*_G(X) \)

We now identify the image of \( E^*_G(X) \) in \( \prod E^*_G(F_i) \); it is specified by simple combinatorial restrictions. This is the content of Theorem 3.1. In order to make this computation, we must make some additional assumptions on \( X \). We formalize our hypotheses on \( X \) below.

**Assumption 1** The space \( X \) is equipped with a \( G \)-invariant stratification

\[
X = \bigcup_{i \in I} X_i
\]

and each successive quotient \( X_i/X_{<i} \) is homeomorphic to the Thom space of a \( G \)-equivariant vector bundle \( \pi_i : V_i \to F_i \). Here \( X_{<i} \) denotes the subspace \( \bigcup_{j<i} X_j \subset X_i \).

**Assumption 2** The bundles \( V_i \to F_i \) are \( E \)-orientable and admit \( G \)-equivariant direct sum decompositions

\[
(\pi_i : V_i \to F_i) \cong \bigoplus_{j<i} (\pi_{ij} : V_{ij} \to F_i)
\]

into \( E \)-orientable vector bundles \( V_{ij} \). We allow the case \( V_{ij} = 0 \).

**Assumption 3** There exist \( G \)-equivariant maps \( f_{ij} : F_i \to F_j \) such that the attaching maps \( \varphi_i : S(V_i) \to X_{i-1} \), when restricted to \( S(V_{ij}) \), are given by

\[
\varphi_i|_{S(V_{ij})} = f_{ij} \circ \pi_{ij}.
\]

Here, we identify the \( F_j \) with their images in \( X_{i-1} \).

**Assumption 4** The Euler classes \( \epsilon(V_{ij}) \) are not zero divisors and are pairwise relatively prime in \( E^*_G(F_i) \). Namely, for any class \( x \in E^*_G(F_i) \), we have that

\[
(\forall j) \epsilon(V_{ij})|x \Leftrightarrow \epsilon(V_i)|x.
\]

With these assumptions, we may now formulate our main theorem.

**Theorem 3.1** Let \( X \) be a \( G \)-space satisfying Assumptions 1 through 4. Then the map

\[
i^* : E^*_G(X) \to \prod_i E^*_G(F_i)
\]

is injective with image

\[
R := \left\{ (x_i) \in \prod_i E^*_G(F_i) \mid \epsilon(V_{ij}) | x_i - f^*_{ij}(x_j) \text{ for all } j < i \right\}.
\]
When $V_{ij} = 0$ in the theorem above, the relation $\epsilon(V_{ij}) \mid x_i - f_{ij}^*(x_j)$ is vacuous because $\epsilon(0) = 1$. We introduce a decorated graph $\Gamma$ that carries all the information from $X$ necessary to compute the image $R$ of $E^*_G(X)$. Each edge of $\Gamma$ corresponds to a non-vacuous relation.

**Definition 3.2** The GKM graph $\Gamma$ associated to $X$ is the graph with one vertex $v_i$ for each subspace $F_i$ and an edge $(v_i, v_j)$ whenever $V_{ij}$ is non-zero. Each edge is labeled with the bundle $V_{ij}$ and the map $f_{ij} : F_i \to F_j$.

**Remark 3.3** In Sections 5 and 6, the description of $\Gamma$ simplifies greatly. In those examples, all the $F_i$ are single points, and the maps $f_{ij} : F_i \to F_j$ are the only possible ones. Moreover, the bundles $V_{ij}$ are all 1-dimensional complex $T$-representations. Hence $\Gamma$ is a graph with a character $\alpha \in \Lambda := \text{Hom}(T, S^1)$ attached to each edge.

**Remark 3.4** Theorem 3.1 generalizes many results found in the literature. We survey some of these results here.

A. Suppose that $X$ is a projective variety equipped with an algebraic action of a complex torus, with finitely many 0- and 1-dimensional orbits. Let $E^*_G$ be ordinary $T$-equivariant cohomology. In this setting, Theorem 3.1 is precisely the result of Goresky, Kottwitz, and MacPherson [GKM98].

B. Theorem 3.1 recovers the main theorem of [GH04] when $X$ is a compact Hamiltonian $T$-space with possibly non-isolated fixed points, and generalizes this result to equivariant $K$-theory.

C. When $E^*_G$ is $T$-equivariant $K$-theory with complex coefficients and $X$ is a GKM manifold, then Theorem 3.1 is identical to [KR03, Corollary A.5].

D. If $X$ is a Kac-Moody flag variety and $E^*_G$ is $T$-equivariant $K$-theory, then Theorem 3.1 is closely related to a result of Kostant-Kumar [KK87]. Indeed, their Theorem 3.13 identifies $K^*_T(\mathcal{G}/\mathcal{B})$ with the subring of elements of $\prod_W K^*_T$ that are mapped to $K^*_T$ by certain operators, which include the divided difference operators

$$\left( \delta_w - \delta_{w r_n} \right) \frac{1}{1 - e^\alpha}$$

for all $w \in W$ and reflections $r_n$. These are exactly the same conditions as in (3.1). Their Corollary 3.20 determines $K^*_T(\mathcal{G}/\mathcal{P})$ in a similar fashion.

Before proving Theorem 3.1, we give a Lemma which computes $E^*_G(X)$ when the stratification of $X$ has length 2.

**Lemma 3.5** Let $Y = F_1 \cup_\varphi D(V)$ be obtained by gluing the sphere bundle of $\pi : V \to F_2$ onto $F_1$, where $\varphi = f \circ \pi$ for a map $f : F_2 \to F_1$. Assume that $\epsilon(V)$ is not a zero divisor. Then the images of the restriction maps $i^* : E^*_G(Y, F_1) \to E^*_G(F_2)$ and $j^* : E^*_G(Y) \to E^*_G(F_1) \oplus E^*_G(F_2)$ are

$$i^*(E^*_G(Y, F_1)) = \left\{ g \in E^*_G(F_2) \mid \epsilon(V) \mid g \right\} \quad (3.2)$$

and

$$j^*(E^*_G(Y)) = \left\{ (g_1, g_2) \in E^*_G(F_1) \oplus E^*_G(F_2) \mid \epsilon(V) \mid g_2 - f^*(g_1) \right\}, \quad (3.3)$$

respectively.
**Proof:** Clearly \( E^*_G(Y, F_1) \cong E^*_G(\text{Th}(V)) \cong E^*_G(F_2) \) via the Thom isomorphism. The map

\[
E^*_G(F_2) \cong E^*_G(\text{Th}(V)) \xrightarrow{\pi^*} E^*_G(F_2)
\]

is multiplication by \( \epsilon(V) \), so \( \text{Im}(\pi^*) = \epsilon(V)E^*_G(F_2) \) as claimed in (3.2).

The space \( Y \) retracts onto \( F_1 \) via the map \( f \circ \pi \), so the long exact sequence associated to the pair \( (Y, F_1) \) splits. Now consider the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & E^*_G(Y, F_1) \\
\downarrow{\pi^*} & & \downarrow{\pi^*} \\
0 & \longrightarrow & E^*_G(F_2)
\end{array}
\]

Both rows split, and we get \( \text{Im}(j^*) = E^*_G(F_1) \oplus \text{Im}(\pi^*) \), where \( E^*_G(F_1) \) is mapped via the diagonal inclusion \((1, f^*) : E^*_G(F_1) \rightarrow E^*_G(F_1) \oplus E^*_G(F_2) \). It is now straightforward to check that \( \{(g_1, f^*(g_1))\} \oplus \{(0, g_2) : \epsilon(V) \mid g_2 \} \) is the same group as described in (3.3).

We now have the technical tool to prove our main theorem.

**Proof of Theorem 3.1:** The map \( \pi^* \) is injective by Theorem 2.3, so we must show that its image \( \text{Im}(\pi^*) \) equals the ring \( R \) of (3.1).

We first show that \( \text{Im}(\pi^*) \subseteq R \). Let \( Y_{ij} \) be the subspace of \( X \) given by

\[
Y_{ij} := F_j \cup_{f_j \circ \sigma_{ij}} D(V_{ij}).
\]

Consider a class \( x \in E^*_G(X) \), and let \( x_i \) denote its restriction to \( F_i \). Since \( (x_j, x_i) \) is the image of \( x|_{Y_{ij}} \in E^*_G(Y_{ij}) \) under the restriction map \( E^*_G(Y_{ij}) \rightarrow E^*_G(F_j) \oplus E^*_G(F_i) \), we know by Lemma 3.5 that

\[
\epsilon(V_{ij}) \mid x_i = f^*_j(x_j).
\]

The conditions (3.4) characterize \( R \), so we conclude \( (x_i) \subseteq R \).

We now have a map \( E^*_G(X) \rightarrow R \) and want to show that it is surjective. Following Remark 2.1, we are using \( I = \mathbb{N} \). We argue by induction on the length of the stratification. If the length is zero, then \( X = \emptyset \) and there is nothing to show. We now assume that \( X = X_i \) and that surjectivity holds for

\[
E^*_G(X_j) \rightarrow R_j := \left\{ (x_k) \in \prod_{k \leq j} E^*_G(F_k) \mid \epsilon(V_{k\ell}) \mid x_k = f^*_k(x_{\ell}) \text{ for all } \ell < k \right\}
\]

for all \( j < i \).

Let \( r_i : R_i \rightarrow R_{i-1} \) be the restriction map. By Assumption 4, its kernel can be written

\[
\ker(r_i) = \left\{ (x_j) \in \prod_{j \leq i} E^*_G(F_j) \mid \begin{array}{c}
x_j = 0 \text{ for } j < i \\
\epsilon(V_{ij}) \mid x_i \text{ for all } j < i
\end{array} \right\} \cong \epsilon(V_i)E^*_G(F_i), \quad (3.5)
\]

We now consider the following commutative diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & E^*_G(X_i, X_{i-1}) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \ker(r_i)
\end{array} \quad \begin{array}{ccc}
E^*_G(X_i) & \longrightarrow & E^*_G(X_{i-1}) \\
\downarrow & & \downarrow \\
\end{array} \quad \begin{array}{ccc}
R_i & \longrightarrow & R_{i-1}. \\
\downarrow{r_i} & & \\
\end{array}
\]
The top sequence comes from the long exact sequence of the pair, which splits into short exact sequences as shown in the proof of Theorem 2.3. By the induction hypothesis, we know that the right vertical arrow is an isomorphism. By comparing (3.2) and (3.5), the left vertical arrow is also an isomorphism. It is now an easy diagram chase to verify that \( r_i \) is surjective and that \( E^*_G(X_i) \cong R_i \).

Finally, we note that
\[
E^*_G(X) = \varprojlim E^*_G(X_i) = \varinjlim R_i = R,
\]
completing the proof. \qed

4 Module generators

The second part of Theorem 2.3 gives us a lot of information about the structure of \( E^*_G(X) \) as an \( E^*_G \)-module. When the spaces \( F_i \) consist of isolated fixed points, we can say more. With this assumption, (2.3) tells us that as an \( E^*_G \)-module, \( E^*_G(X) \) is (non-canonically) a product of principal ideals of \( E^*_G \):
\[
E^*_G(X) \cong \prod_{v \in F} \epsilon(V_v) E^*_G,
\]
where \( F = \bigcup F_i \) and \( V_v \) is the fiber over \( v \). Moreover, given a collection of classes \( x_v \in E^*_G(X) \), one for each \( v \in F \), it is very easy to check whether they form a set of free generators\(^4\) for \( E^*_G(X) \).

We write \( v < w \) when \( v \in F_i, w \in F_j \) and \( i < j \). We write \( v \leq w \) if \( v < w \) or \( v = w \). Let \( x_v(w) \) denote \( x_v \mid w \). We then have:

**Proposition 4.1** Suppose \( X \) satisfies Assumptions 1-4 and that the spaces \( F_i \) consist of isolated fixed points. Let \( x_v \in E^*_G(X) \) be classes satisfying
\[
\begin{align*}
x_v(w) &= 0 \text{ for } w \not< v; \\
x_v(v) &= \text{a generator of the ideal } \epsilon(V_v) E^*_G.
\end{align*}
\]
Then \( \{x_v\} \) is a set of free topological \( E^*_G \)-module generators. \qed

It might happen that a space \( X \) with \( G \)-action satisfies the Assumptions 1-4 for some cohomology theory \( E^*_G \), but that Assumption 4 fails for some closely related cohomology theory \( \widetilde{E}^*_G \). For example, this can happen when \( \widetilde{E}^*_G \) is non-equivariant \( E \)-cohomology \( E^*(X) := E^*_G(X \times G) \), or when \( E^*_G = H^*_G(\{-; \mathbb{Z}\}) \) and \( \widetilde{E}^*_G = H^*_G(\{-; \mathbb{Z}/2\}) \). In that case we have:

**Proposition 4.2** Suppose \( X \) satisfies Assumptions 1-4 for the cohomology theory \( E^*_G \), and that the \( F_i \) consist of isolated fixed points. Let \( \tilde{E}^*_G \) be a module cohomology theory over the ring cohomology theory \( E^*_G \). Then one can recover \( \tilde{E}^*_G(X) \) by tensoring
\[
\tilde{E}^*_G(X) = E^*_G(X) \widehat{\otimes}_{E^*_G} \tilde{E}^*_G.
\]
Here \( E^*_G(X) \) is viewed as a topological \( E^*_G \)-module and \( \widehat{\otimes} \) denotes the completed tensor product.

In particular, if \( E^*_G \) is an \( E^*_G \)-algebra and \( x_v \in E^*_G(X) \) satisfy (4.1), then \( x_v \otimes 1 \) are free \( \tilde{E}^*_G \)-module generators of \( E^*_G(X) \).

\(^4\)Here \( E^*_G(X) \) should be viewed as a topological \( E^*_G \)-module, and the word ‘generator’ should be interpreted in the topological sense.
**Proof:** We argue by induction on the length of the stratification. Without loss of generality, we may assume the $F_i$ are single points. The short exact sequence (2.5) consists of free $E_n^*$-modules. Therefore, the functor $- \otimes_{E_n^*} E_n^*$ preserves exactness, and we get the following commutative diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & E_n^*(Th(V_i)) \otimes_{E_n^*} E_n^* & \longrightarrow & E_n^*(X_i) \otimes_{E_n^*} E_n^* & \longrightarrow & E_n^*(X_{i-1}) \otimes_{E_n^*} E_n^* & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \alpha & & \downarrow \beta & & \\
E_n^*(Th(V_i)) & \longrightarrow & E_n^*(X_i) & \longrightarrow & E_n^*(X_{i-1}).
\end{array}
$$

The right vertical arrow is an isomorphism by induction. The left vertical arrow is an isomorphism since

$$E_n^*(Th(V_i)) \otimes_{E_n^*} E_n^* \cong E_n^*(F_i) \otimes_{E_n^*} E_n^* \cong \tilde{E}_n^*(Th(V_i)),$$

where the first and last isomorphisms are the equivariant suspension isomorphisms.

A diagram chase shows that $\beta$ is surjective, so the bottom long exact sequence splits and the map $\alpha$ is injective. We deduce by the Five Lemma that the middle vertical map is also an isomorphism, as desired.

Finally, if the filtration is infinite, we have

$$\tilde{E}_n^*(X) = \lim\limits_{\rightarrow} \tilde{E}_n^*(X_i) = \lim\limits_{\rightarrow} \left( E_n^*(X_i) \otimes_{E_n^*} E_n^* \right) = \left( \lim\limits_{\rightarrow} E_n^*(X_i) \right) \otimes_{E_n^*} E_n^* = E_n^*(X) \otimes_{E_n^*} E_n^*.$$

Assume now that $X$ is a CW complex with $G$-invariant cells\(^5\), that the filtration (2.1) is the usual filtration by skeleta (indexed by $\mathbb{N}$), and that $E_n^*(X) = H_n^*(X) := H^*(X \times G EG)$ is ordinary equivariant cohomology. In this case, we can give a canonical set of free generators for $H_n^*(X)$. As before, we let $F = \cup F_i$, where $F_i$ is now the set of the centers of the $i$-dimensional cells. We write $|v| = i$ whenever $v \in F_i$ and recall the notation $x_v(w)$ for $x_v|_w$.

**Proposition 4.3** Let $X$ be a CW complex as above. Then there is a unique set $\{x_v\}_{v \in F}$ of free generators for the $H_n^*$-module $H_n^*(X)$ satisfying the conditions:

1. each $x_v$ is homogeneous of degree $|v|$;
2. if $|w| \leq |v|$, $w \neq v$, then $x_v(w) = 0 \in H_n^*$; and
3. the element $x_v(v)$ is the equivariant Euler class $e(V_v) := e(V_v \times G EG \rightarrow BG) \in H_n^*$, where $V_v$ is the cell of $X$ with center $v$.

**Proof:** We first construct the classes $x_v$. Assume by induction that we have classes $x'_w$ in $H_n^*(X_{i-1})$ for $|w| < i$. To extend these to $H_n^*(X_i)$, consider the short exact sequence

$$
\begin{array}{cccccc}
0 & \longrightarrow & H_n^*(X_i, X_{i-1}) & \longrightarrow & H_n^*(X_i) & \longrightarrow & H_n^*(X_{i-1}) & \longrightarrow & 0
\end{array}
$$

and note that

$$H_n^*(X_i, X_{i-1}) \cong H_n^*(\bigvee_{|d| = i} Th(V_v)) \cong \prod_{|v| = i} H_n^*(Th(V_v)).$$

\(^5\)Careful: we don’t mean that $X$ is a $G$-CW complex.
The spaces $Th(V_i)$ are $G$-spheres, so each $H^*_G(Th(V_i))$ has a canonical generator $u_v$. The restriction of $u_v$ to the center $v$ of $V_i$ is the equivariant Euler class $e(V_i)$. The classes $x'_w$ of $H^*_G(X_{i-1})$ have a unique lift $x_w$ to $H^*_G(X_i)$ because $H^*_G(X_i, X_{i-1})$ is zero for all $k < i$. It is straightforward to check that these lifts, along with the images $x_v$ of the chosen generators $u_v$ of $H^*_G(X_i, X_{i-1})$, satisfy the above conditions and generate $H^*_G(X_i)$. We take a limit over $i$ to obtain the generators $x_v \in H^*_G(X)$. We show that Conditions 1, 2 and 3 characterize the generators $x_v$. Let $\{x_v\}$ be another set of generators satisfying the same conditions. Write them as $x_v = \sum_w a_{vw} x_w$. By Condition 2, we have $a_{vv} = 0$ whenever $|w| < |v|$ and $w \neq v$. By Condition 3, $a_{vv} = 1$. Finally, $a_{vw} = 0$ when $|w| > |v|$, because otherwise $x_v$ would not be homogeneous. □

**Remark 4.4** Suppose $X$ is a manifold with a $G$-invariant Morse function $f$ and a CW decomposition constructed from the Morse flow. Then the above construction is the same as the following: given a fixed point $v$, consider the flow-up manifold $\Sigma_v$ of codimension $|v|$. By Poincaré duality, it represents a cohomology class $x_v$. It is straightforward to see that the $x_v$ satisfy Conditions 1, 2 and 3 of Proposition 4.3.

**Remark 4.5** There are other situations when it is possible to find canonical module generators. For example, such generators exist when $X$ is a complex algebraic variety or a symplectic manifold, and $E^*_G$ is equivariant $K$-theory. The algebraic construction involves resolving the structure sheaf of the “flow-up” varieties $\Sigma_v$. See [BFM79] for details. The symplectic construction can be found in [GK03].

We illustrate these generators for some examples in Section 6.

## 5 Kac-Moody flag varieties

We now turn our attention to the main examples that motivate the results in this paper. These are homogeneous spaces $G/P$ for a (not necessarily symmetrizable) Kac-Moody group $G$, defined over $\mathbb{C}$, with $P$ a parabolic subgroup. Specific examples of such homogeneous spaces include finite dimensional Grassmannians, flag manifolds, and based loop spaces $\Omega K$ of compact simply connected Lie groups $K$.

We first take a moment to explicitly describe $\Omega K$ as a homogeneous space $G/P$. Let $LK$ be the group of polynomial loops

$$LK := \{\gamma : S^1 \to K\},$$

where the group structure is given by pointwise multiplication. By polynomial, we mean that the loop is the restriction $S^1 = \{z \in \mathbb{C} : |z| = 1\} \to K$ of an algebraic map $\mathbb{C}^* \to K\mathbb{C}$. The space of based polynomial loops is defined by

$$\Omega K := \{\sigma \in LK \mid \sigma(1) = 1 \in K\}.$$

The group $LK$ acts transitively on $\Omega K$ by

$$(\gamma \cdot \sigma)(z) = \gamma(z)\sigma(z)\gamma(1)^{-1}. \quad (5.1)$$

The stabilizer of the constant identity loop is exactly $K$, the subgroup of constant loops. Thus $\Omega K \cong LK/K$.

Now let $G$ be the affine Kac-Moody group $G = \widehat{LK}_\mathbb{C} \times \mathbb{C}^*$. Here, $LK_\mathbb{C}$ is the group of algebraic maps $\mathbb{C}^* \to K\mathbb{C}$, $\widehat{LK}_\mathbb{C}$ is the universal central extension of $LK_\mathbb{C}$, and the $\mathbb{C}^*$ acts on $LK_\mathbb{C}$ by rotating
the loop. The parabolic $\mathcal{P}$ is $K_c \times \mathbb{C}^*$, where $L^+ K_c$ is the subgroup of $LK_c$ consisting of maps $\mathbb{C}^* \to K_c$ that extend to maps $\mathbb{C} \to K_c$. It is shown in [PS86, 8.3] that $\Omega K$ can be identified as a homogeneous space $\mathcal{G}/\mathcal{P}$. We briefly sketch this argument. The group $LK$ acts on $\mathcal{G}/\mathcal{P}$ by left multiplication, and the stabilizer of the identity is $\mathcal{P} \cap LK$. This intersection is the set of polynomial maps $\mathbb{C}^* \to K_c$ which extend over 0, and which send $S^1$ to $K$. Thus, a loop $\gamma$ in $\mathcal{P} \cap LK$ satisfies the condition $\gamma(z) = \theta(\gamma(1/z))$, where $\theta$ is the Cartan involution on $K_c$. Therefore, since $\gamma$ extends over zero, by setting $\gamma(\infty) = \theta(\gamma(0))$, it also extends over $\infty$. But then $\gamma$ is an algebraic map from $\mathbb{P}^1$ to $K_c$, and is therefore constant, since $K_c$ is affine. Hence $\mathcal{P} \cap LK = K$.

**Remark 5.1** We have only considered the space of polynomial loops in $K$. However, our results still apply to other spaces of loops, such as smooth loops, 1/2-Sobolev loops, etc. Indeed, the polynomial loops are dense in these other spaces of loops [PS86, 3.5.3, [Mit87]. By Palais' theorem [Pal66, Theorem 12], these dense inclusions are weak homotopy equivalences. The inclusions of $T'$-fixed point sets for $T'$ a closed subgroup of $T$ are also equivalences. So the various forms of $\Omega K$ are actually equivariantly weakly homotopy equivalent.

Let us return to the general case. Let $T_\mathcal{G}$ be the maximal torus of $\mathcal{G}$. The center $Z(\mathcal{G})$ acts trivially on $X = \mathcal{G}/\mathcal{P}$, so the quotient group $T := T_\mathcal{G}/Z(\mathcal{G})$ acts on $X$. We need to check that this space $X$ with this $T$-action satisfy Assumptions 1–4 that are the hypotheses of Theorem 3.1. It is known (see for example [BD94, KP83, KK87, Mit87]) that $\mathcal{G}/\mathcal{P}$ admits a $T$-invariant CW decomposition

$$\mathcal{G}/\mathcal{P} = \bigsqcup_{[\bar{w}] \in W_\mathcal{G}/W_P} \mathcal{B} \bar{w} \mathcal{P}/\mathcal{P}, \quad (5.2)$$

where $W_\mathcal{G}$ and $W_\mathcal{P}$ are the Weyl groups of $\mathcal{G}$ and of $\mathcal{P}$ (the semisimple part of $\mathcal{P}$ respectively), and $\bar{w}$ is a representative of $w$ in $\mathcal{G}$. This is the filtration of Assumption 1. Each cell is homeomorphic to a $T$-representation and has a single $T$-fixed point $w := \bar{w} \mathcal{P}/\mathcal{P}$ at its center. These cells are the $V_i$ and the fixed points are the $F_i$. The $T$-representation $V_i$ is isomorphic to the tangent space

$$T_{\bar{w}} \mathcal{B} \bar{w} = T_{\bar{w}} \mathcal{B} \bar{w} \mathcal{P}/\mathcal{P} = b/b \cap \bar{w} \mathcal{P} \bar{w}^{-1} = b/b \cap w \cdot p.$$

This tangent space decomposes into 1-dimensional representations, corresponding to the roots contained in $b$ but not in $w \cdot p$. These subspaces are the $V_{ij}$ of Assumption 2.

We now check Assumption 3. Since the $F_i$ are points, we only need to show that the attaching map $\varphi_i : S(V_i) \to X_{i-1}$ maps each $S(V_{ij})$ onto the point $F_i$. In other words, we need to show that the closure of $V_{ij}$ is a 2-sphere with north and south poles $F_i$ and $F_j$. Pick a root $\alpha$ in $b$ but not in $w \cdot p$. Let $\epsilon_\alpha$, $\epsilon_{-\alpha}$ be the standard root vectors for $\alpha$, $-\alpha$. Let $SL(2, \mathbb{C})_\alpha$ be the subgroup of $\mathcal{G}$ with Lie algebra spanned by $\epsilon_\alpha$, $\epsilon_{-\alpha}$ and $[\epsilon_\alpha, \epsilon_{-\alpha}]$, and let $\mathcal{B}_\alpha$ be the Borel of $SL(2, \mathbb{C})_\alpha$ with Lie algebra spanned by $\epsilon_\alpha$ and $[\epsilon_\alpha, \epsilon_{-\alpha}]$. Let $\tilde{r}_\alpha := \exp(\pi(i\epsilon_\alpha - \epsilon_{-\alpha})/2)$ represent the element $r_\alpha$ of the Weyl group which is reflection along $\alpha$. Let $F_i$ be the point $w$ and $F_j$ the point $r_\alpha w$. The $\alpha$-eigenspace in the cell $\mathcal{B}w$ is $\mathcal{B}_\alpha w = V_{ij} \cong \mathbb{C}$. Its closure is $SL(2, \mathbb{C})_\alpha w \cong \mathbb{P}^1$, and the point at infinity is given by $\tilde{r}_\alpha w \mathcal{P}/\mathcal{P} = r_\alpha w = F_j$, as desired.

Finally, we need to check Assumption 4. To do this, we must show that for the roots contained in $b$ but not in $w \cdot p$, the corresponding Euler classes are pairwise relatively prime. This is true for a large class of $T$-equivariant complex oriented cohomology theories including $H^*_T(-; \mathbb{Z})$, $K^*_T$ and $MU^*_T$.

**Lemma 5.2** Let $E^*_T$ be $H^*_T(-; \mathbb{Z})$, $K^*_T$ or $MU^*_T$. Let $\alpha_i$ be any finite set of non-zero characters such that no two are collinear. Moreover, if $E^*_T = H^*_T(-; \mathbb{Z})$, assume that no prime $p$ divides two of the $\alpha_i$. Then the corresponding Euler classes $e(\alpha_i)$ are pairwise relatively prime in $E^*_T$. 11
Proof: The equivariant cohomology ring $H_T^*$ is the symmetric algebra \(^6\) $Sym^*(\Lambda)$ on the weight lattice of $T$. This is a unique factorization domain, and the Euler classes $e(\alpha_i) = a_i$ decompose into an integer times a primitive character. The result follows immediately in this case.

The equivariant $K$-theory ring $K_T^*$ is the group ring $\mathbb{Z}[\Lambda]$ generated by symbols $e^\alpha$. For each $\alpha$ in our set of characters, let $\alpha$ be the primitive character in that direction, so $\alpha = n\alpha$. The Euler classes $e(\alpha_i) = 1 - e^{\alpha_i}$ factorize as a product of cyclotomic polynomials

$$1 - e^{\alpha_i} = \prod_{d|\alpha_i} \Phi_d(e^{\alpha_i}).$$

The factors $\Phi_d(e^{\alpha_i})$ are all distinct, so the result follows.

To prove the result about complex cobordism, we argue by induction on the number of characters in our set. The base case is trivial. Assume by induction that the result holds for $n$ characters and that we are given a set $\alpha, \beta_1, \ldots, \beta_n$ of $n + 1$ characters satisfying the hypotheses of the lemma. Let $x$ be a class in $MU_T^*$ which is divisible by each of the Euler classes of the above characters. By induction, $x$ is divisible by the product $\prod_i e(\beta_i)$, so there exists a class $b$ such that $b \cdot \prod_i e(\beta_i) = x$.

We now consider the short exact sequence [Sin01, Theorem 1.2]

$$0 \longrightarrow MU_T^* \xrightarrow{-e(\alpha)} MU_T^* \xrightarrow{res} MU^*_{\text{Ker}(\alpha)} \longrightarrow 0.$$ 

Since $x$ is divisible by $e(\alpha)$,

$$res(b) \cdot \prod_i res(\beta_i) = res(x) = 0.$$ 

By assumption, the restrictions $\beta_i|_{\text{Ker}(\alpha)}$ are non-torsion in the group of characters of $\text{Ker}(\alpha)$. So by a result of Sinha [Sin01, Theorem 5.1] their Euler classes $e(\beta_i|_{\text{Ker}(\alpha)}) = res(e(\beta_i))$ are not zero divisors. We conclude that $res(b) = 0$. Hence $b$ is a multiple of $e(\alpha)$, completing the proof. \qed

Remark 5.3 It is shown in [CGK02] that any complex oriented $T$-equivariant cohomology theory $E_T^*$ is an algebra over $MU_T^*$. Combining this with Proposition 4.2 and Lemma 5.2, we may use our main Theorem 3.1 to compute $E_T^*(\mathcal{G}/\mathcal{P}) = MU_T^*(\mathcal{G}/\mathcal{P}) \hat{\otimes}_{MU_T^*} E_T^*$. 

We conclude this section with an explanation of how to obtain the pictures that we draw in Section 6. The GKM graph associated to $\mathcal{G}/\mathcal{P}$ has vertices $W_\mathcal{G}/W_\mathcal{P}$, with an edge connecting $[w]$ and $[r_w w]$ for all reflections $r_w$ in $W_\mathcal{G}$. The weight label on such an edge is $\alpha$. It turns out that it is possible to embed this GKM graph in $t^*$, the dual of the Lie algebra of $T$. Under this embedding, the weight $\alpha_{ij}$ is then the primitive element of $\Lambda \subset t^*$ in the direction of the corresponding edge. To produce this embedding, we pick a point in $t^*_\mathcal{G}$ whose $W_\mathcal{G}$-stabilizer is exactly $W_\mathcal{P}$, take its $W_\mathcal{G}$-orbit, and draw an edge connecting any two vertices related by a reflection in $W_\mathcal{G}$. This graph sits in a fixed level of $t^*_\mathcal{G}$ (this is only relevant when $\mathcal{G}$ is of affine type) and can therefore be thought of as sitting in $t^*$.

These ideas are borrowed from the theory of moment maps in symplectic geometry. In that context, $X$ is a symplectic manifold with $T$-action and admits a moment map $\mu : X \to t^*$. Consider the set $X^{(1)}$ of points with stabilizer of codimension at most 1. The GKM graph is the image of $X^{(1)}$ under the moment map $\mu$. In our situation, $X^{(1)}$ corresponds exactly to the union of the $V_{ij}$. Figure 5.1 shows the image of the moment map for the example $\Omega SU(2)$.

\(^6\)This is true if one restricts the $RO(T)$-grading of [May98] to the more familiar $\mathbb{Z}$-grading. Otherwise, one has various periodicities with respect to all zero-dimensional virtual $T$-representations.
6 Examples

6.1 A homogeneous space for $G_2$

The complex Lie group $G_2$ contains two conjugacy classes of maximal parabolic subgroups. They correspond to the two simple roots of $G_2$. We consider the case $X = G_2/P$ and its natural torus action, where $P = P_{\text{long}}$ is the parabolic generated by the Borel subgroup and the exponential of the negative long simple root. Equivalently, $X$ is the quotient of the compact group $G_2$ by a subgroup isomorphic to $U(2)$. The GKM graph is a complete graph on 6 vertices and is embedded in $t^* \cong \mathbb{R}^2$ as a regular hexagon.

We now compute explicitly module generators $x_v$ of $E^*_T (X)$ for a large class of cohomology theories $E^*_T$, following Section 4. We will represent them by their restrictions $x_v(w) := x_v|_w$ to the various $T$-fixed points $w \in F$. In this example, all the $x_v(w)$ happen to be Euler classes of complex $T$-representations. This allows us to use the following convenient notation to represent the classes $x_v$. On every vertex $w$ of $\Gamma$ we draw a bouquet of arrows $\beta_j \in \Lambda$. By this, we mean that the class $x_v(w) \in E^*_T(\{w\})$ is the Euler class

$$x_v(w) = \epsilon(\bigoplus_j \beta_j) = \prod_j \epsilon(\beta_j).$$

The vertices with no arrows coming out of them carry the class 0. Using these conventions, we draw the six module generators $1, x, y, z, s, t$ of $E^*_T(G_2/P)$ in Figure 6.1.

Recall that Assumptions 1-4 are satisfied for the cohomology theories $H^*_T(-; \mathbb{Z})$, $K^*_T$ and $MU^*_T$, as shown in Section 5. To check that the elements shown in Figure 6.1 are module generators, we need to check two things. First, we notice that the conditions (4.1) are satisfied. Second, we need to verify that the elements $x, y, z, s, t$ satisfy the criteria (3.1) for being elements of $E^*_T(X)$.

To check (3.1), note that $\epsilon(\alpha) \in E^*_T$ divides $\epsilon(\beta) - \epsilon(\gamma)$ whenever $\beta - \gamma$ is a multiple of $\alpha$ in $\Lambda$. This is a trivial fact when $E^*_T$ is ordinary $T$-equivariant cohomology or $T$-equivariant $K$-theory, and is a consequence of the theory of equivariant formal group laws when $E^*_T$ is an arbitrary $T$-equivariant complex oriented cohomology theory [CGK00, p. 374]. Similarly $\epsilon(\alpha)$ divides a difference of products $\prod \epsilon(\beta_j) - \prod \epsilon(\gamma_j)$ if the $\beta_j - \gamma_j$ are all multiples of $\alpha$. Now, for each of the classes in Figure 6.1, and for each edge $(v, w)$ of $\Gamma$ with direction $\alpha$, we note that the two bouquets of arrows $\{\beta_j\}$ at $v$ and $\{\gamma_j\}$ at $w$ can be ordered in such a way that the differences $\beta_j - \gamma_j$ are each in the direction of $\alpha$. So we have checked (3.1) and hence by Theorem 3.1, the classes in Figure 6.1 are elements of $E^*_T(X)$. Thus, by Proposition 4.1, they are free module generators.
Figure 6.1: The module generators for $E_8^*(G_2/P)$. We include the lattice $\Lambda$ in the first diagram.

Even though the module generators look very similar in all cohomology theories, the ring structures are different. We compute the ordinary $T$-equivariant cohomology and $K$-theory of $X = G_2/P$ to exhibit this phenomenon.

For cohomology theories $E^*_T$ such as $H^*_T(-;\mathbb{Z}/2)$, $H^*(-;\mathbb{Z})$, $K^*$, or $MU^*$ for which Assumption 4 fails, we still have a good understanding of $E^*_T(X)$ by Proposition 4.2. We exploit this to compute $H^*(X;\mathbb{Z})$ from $H^*_T(X;\mathbb{Z})$ and $K^*(X)$ from $K^*_T(X)$ below.

For the computation of $H^*_T(X;\mathbb{Z})$, it is convenient to let $a := \epsilon(\rightarrow), b := \epsilon(\backslash) \in H^*_T$ be the Euler classes of the characters $\rightarrow, \backslash \in \Lambda$. One then has $H^*_T = \mathbb{Z}[a, b]$. Using the embedding (2.2) $H^*_T(X;\mathbb{Z}) \hookrightarrow \prod_F H^*_F$, we compute:

\[
\begin{align*}
  x(x + a) &= y, \\
  x(x + a)(x + b) &= 2z, \\
  x(x + a)(x + b)(x + 2a + b) &= 2s, \quad \text{and} \\
  x(x + a)(x + b)(x + 2a + b)(x + 2b + a) &= 2t.
\end{align*}
\]

To get the non-equivariant cohomology $H^*(X;\mathbb{Z})$, it suffices by Proposition 4.2 to set $a = b = 0$:

\[
x^2 = y, \quad x^3 = 2z, \quad x^4 = 2s, \quad x^5 = 2t, \quad x^6 = 0.
\]  \tag{6.1}

In $K$-theory, it is more convenient to let $a, b \in K_T^0$ be the characters $\rightarrow$ and $\backslash \in \Lambda$ themselves (not their Euler classes). We then have $K_T^0 = \mathbb{Z}[a, a^{-1}, b, b^{-1}]$, and all other $K$-groups are either zero or isomorphic to $K^0$. We use the convention that the Euler class of a line bundle $L$ is $1 - L$.  

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We can now compute:

\[ x(ax + 1 - a) = y, \]
\[ x(ax + 1 - a)(bx + 1 - b) = (1 + a^{-1})z - a^{-1}s, \]
\[ x(ax + 1 - a)(bx + 1 - b)(a^2bx + 1 - a^2b) = (1 + b^{-1})s - b^{-1}t, \text{ and} \]
\[ x(ax + 1 - a)(bx + 1 - b)(a^2bx + 1 - a^2b)(ab^2x + 1 - ab^2) = (1 + a^{-1}b^{-1})t. \]

To get the non-equivariant \( K \)-theory, we set \( a = b = 1 \) according to Proposition 4.2:

\[ x^2 = y, \quad x^3 = 2z - s, \quad x^4 = 2s - t, \quad x^5 = 2t, \quad x^6 = 0. \tag{6.2} \]

We note that, as expected, the cohomology ring (6.1) of \( G_2/P \) is the associated graded of the \( K \)-theory ring (6.2).

### 6.2 Loops in \( SU(2) \)

We now compute explicitly the ring structure of \( H^*_\Gamma (\Omega SU(2); \mathbb{Z}) \) using the GKM graph \( \Gamma \subset t^* \) and the module generators \( x_v \) as constructed in Section 4. In this example, as in the previous one, all the restrictions \( x_v(w) \) at fixed points are elementary tensors in \( H^*_\Gamma (\{w\}) \cong \text{Sym}^*(\Lambda) \). So as before, we will represent the classes \( x_v \) by drawing on every vertex \( w \) a bouquet of arrows \( \beta_j \in \Lambda \) such that \( x_v(w) = \prod \beta_j \). The vertices with no arrows coming out of them carry the class 0.

The first few module generators are illustrated in Figure 6.2. We call \( x \) the generator of degree 2, and express the others in terms of it. The arrows in the expressions denote elements in \( H^*_\Gamma = \Lambda \).

![Figure 6.2: The degree 2,4,6, and 8 generators for \( H^*_\Gamma (\Omega SU(2); \mathbb{Z}) \).](image)

The map \( H^*_\Gamma (\Omega SU(2); \mathbb{Z}) \to H^*(\Omega SU(2); \mathbb{Z}) \) is simply the map that sends the arrows to zero. So we recover the well-known fact that the ordinary cohomology \( H^*(\Omega SU(2); \mathbb{Z}) \) is a divided powers algebra on a class in degree 2.

Note that the classes in Figure 6.2 are not generators for \( K \)-theory. Indeed, the conditions (3.1) are only satisfied when the classes in Figure 6.2 are interpreted in cohomology, but not when they are interpreted in \( K \)-theory.
To compute the generators of $K_T^*(\Omega SU(2))$, we introduce the following notation. Let

$$p_k(\lambda_1, \ldots, \lambda_n) := (1 - \lambda_1) \cdots (1 - \lambda_n) \cdot \sum_{0 \leq |\alpha| < k} \lambda^\alpha,$$

where $\lambda^\alpha = \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$. The first such polynomial $p_k(\lambda_1, \ldots, \lambda_n)$ is exactly the Euler class $\epsilon(\bigoplus \lambda_i)$ that appeared in Section 6.1. The other ones are slightly more complicated. To best draw our $K$-theory classes, we introduce a pictorial notation for $p_k(\lambda_1, \ldots, \lambda_n)$, for $\lambda_i \in \Lambda$. We will represent them by a bouquet $\{\lambda_i\}$ of arrows, and a small number $k$ at the vertex. We illustrate our generators using this notation in Figure 6.3. To check that these elements are indeed the generators of $K_T^*(\Omega SU(2))$, we need to check (4.1), which is immediate, and that they satisfy the GKM conditions (3.1). These latter turn out to be quite hard to check.

![Diagram](image_url)

**Figure 6.3:** The first few module-generators of $K_T^*(\Omega SU(2))$. (The class $x_0 = 1$ is omitted.)

Let $a := \longrightarrow$ and $q := 1 \in K_T^0$. Let us also identify the vertex set $F$ of $\Gamma$ with $\mathbb{Z}$ by taking the horizontal coordinate. The class $x_i$ drawn in Figure 6.3 is given by

$$x_i(m) = \begin{cases} p_{m-k}(a^{-1}q^{m-k}, a^{-1}q^{m-k+1}, \ldots, a^{-1}q^{m+\ell}) & \text{if } m > k \\ 0 & \text{if } -\ell \leq m \leq k \\ p_{m+\ell}(aq^{m-\ell}, aq^{m-\ell+1}, \ldots, aq^{m+k}) & \text{if } m < -\ell, \end{cases}$$

where $\ell = |i| - 1$ and $k = \frac{|i|}{2} - \frac{1}{2}$. Given an edge $(m, n) \in \Gamma$, we must check the condition given in (3.1), namely that the Euler class $1 - aq^{m+n}$ divides the difference

$$x_i(m) - x_i(n).$$

This involves several different cases. However, the problem has a few symmetries that allow us to reduce the cases to the following three.

If $m$ is between $-\ell$ and $k$ then $x_i(n)$ has either $(1 - aq^{m+n})$ or $(1 - a^{-1}q^{-m-n})$ as a factor and we are done.

If both $m$ and $n$ are bigger than $k$, then we must check that $1 - aq^{m+n}$ divides

$$p_{m-k}(a^{-1}q^{m-k}, \ldots, a^{-1}q^{m+\ell}) - p_{n-k}(a^{-1}q^{n-k}, \ldots, a^{-1}q^{n+\ell}).$$

(6.3)
This is equivalent to checking that (6.3) evaluates to 0 after setting \(a^{-1} = q^{m+n}\). So we are reduced to checking that

\[ p_{m-k}(q^{n-k}, \ldots, q^{n+\ell}) = p_{n-k}(q^{m-k}, \ldots, q^{m+\ell}). \]

The above formula is invariant under adding the same constant to the indices \(m, n\) and \(k\), and subtracting it from \(\ell\). So by letting \(k = 0\), we must prove the equivalent formula

\[ p_m(q^n, \ldots, q^{n+\ell}) = p_n(q^m, \ldots, q^{m+\ell}). \]

(6.4)

This is the content of Lemma 6.1.

Finally, if \(m > k\) and \(n < -\ell\) then we are reduced to checking that

\[ p_{m-k}(q^{n-k}, \ldots, q^{n+\ell}) = p_{n-\ell}(q^{m-\ell}, \ldots, q^{m+k}). \]

By replacing \(q\) with \(q^{-1}\), reversing the order of the arguments in the polynomial \(p\), and a couple changes of indices, this also reduces to Lemma 6.1.

**Lemma 6.1** The expression

\[ a_{m\ell n} := p_m(q^n, q^{n+1}, \ldots, q^{n+\ell}) \]

is symmetric in \(m\) and \(n\).

**Proof:** Let \((\cdot)_q\) denote the quantum binomial coefficient

\[ \binom{a}{b}_q = \frac{a!_q}{b!_q (a-b)!_q}, \]

where \(a!_q\) is the \(q\)-factorial\(^7\) \(a!_q = (1 - q)(1 - q^2) \ldots (1 - q^a)\). We can then rewrite the expression \(a_{m\ell n}\) as

\[ a_{m\ell n} = (1 - q^n) \cdots (1 - q^{n+\ell}) \cdot \sum_{i=0}^{m-1} q^i n^{\ell + i}_q. \]

(6.5)

See for example [And76, §3.3] for more detail. In particular, (6.5) is a truncated version of Equation (3.3.7) in [And76].

Now recall from [Zei93] that a “difference form”

\[ \omega = f(i, j) \delta i + g(i, j) \delta j \]

has “exterior difference”

\[ d\omega = [f(i, j + 1) - f(i, j)] \delta j \delta i + [g(i + 1, j) - g(i, j)] \delta i \delta j, \]

where \(\delta i\) and \(\delta j\) are anti-commuting symbols. Such a difference form can be viewed as a cellular 1-cochain on the standard square tiling of \(\mathbb{R}^2\), the exterior difference being the usual cellular coboundary operator. Consider the difference form

\[ \omega = q^{ij} \frac{(i + j)!_q}{i!_q j!_q} \cdot (1 - q^i) \delta i + (1 - q^j) \delta j. \]

\(^7\)Some authors define the quantum factorial \(a!_q\) to be \((1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^a)\). This agrees with our expression up to a power of \(1 - q\).
It is an easy exercise to verify that $\omega$ is closed. Therefore, by the discrete Stokes' theorem [Zei93],

$$\int_{\partial L} \omega = 0,$$

where $L$ is the rectangle $[0, m] \times [0, n]$. One now checks that the above integral is zero on the sides \{0\} \times [0, n] and [0, m] \times \{0\}, and equals $a_{m\ell}$ and $-a_{m\ell}$ on the remaining two sides. \hfill $\square$

**Remark 6.2** We do not know whether the generators illustrated in Figure 6.3 are the same as those mentioned in Remark 4.5.

### 6.3 A homogeneous space of type $A_1^{(1)}$

For our last example, we let $\mathcal{G}$ be the affine group associated to the Cartan matrix

$$
\begin{bmatrix}
2 & -1 \\
-4 & 2
\end{bmatrix}.
$$

This group is $\overline{LSL(3, \mathbb{C})} \rtimes \mathbb{Z}/2\mathbb{Z}$, where the $\mathbb{Z}/2\mathbb{Z}$-action on $LSL(3, \mathbb{C})$ is given by precomposition with the antipodal map $z \mapsto -z$ on $\mathbb{C}^3$ and composition with the outer automorphism $A \mapsto (A^t)^{-1}$ of $SL(3, \mathbb{C})$.

We consider the homogeneous space $\mathcal{G}/\mathcal{P}$ where the parabolic $\mathcal{P}$ has Lie algebra generated by $\mathfrak{b}$ and the negative of the simple short root. The degree 2, 4, 6, and 8 module generators for $H_\ell^+ (\mathcal{G}/\mathcal{P}; \mathbb{Z})$ are illustrated in Figure 6.4. The denominator in the degree $n$-th module generator is given by $n!2^{|\alpha|^2}$. 

![Figure 6.4: The degree 2, 4, 6, and 8 generators for $H_\ell^+ (\mathcal{G}/\mathcal{P}; \mathbb{Z})$.](image)
References


