A Constructive Proposal
for an Operator Approach
to the Crossing Property

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Dedicated to Jacques Bros on the occasion of his 70th birthday

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Abstract

In trying to generalize constructive ideas underlying the bootstrap-formfactor program beyond the limitation of d=1+1 factorizing models, the main obstacle is the lack of an operator interpretation of the crossing property. Here we propose a hypothesis involving a "masterfield" whose connected formfactors define an auxiliary thermal QFT, in which the KMS cyclicity equation for the thermal expectation values of the auxiliary field is identical to the crossing property of the formfactors of the master field. The speculative nature of the idea is somewhat mitigated by its successfull check in factorizing models.

1 History of the crossing property

The so-called crossing property of the S-matrix and formfactors\(^1\) is a deep and important but at the same time incompletely understood structure in particle physics. As a result of its inexorable link with analyticity properties in the quantum field theoretic setting of scattering theory, crossing is not a symmetry in the standard sense (of Wigner) even though it is often referred to as "crossing symmetry".

In contrast to the underlying causality principles which are "off-shell", i.e. are formulated in terms of local observables or fields with unrestricted Fourier transforms, the crossing property is "on-shell", that is to say it refers to particle states which are described by wave functions on the forward mass hyperboloid \(p^2 = m^2\). Particle properties are intrinsic to a theory, whereas fields are

\(^1\)In the setting of this paper the S-matrix as a transition amplitude between incoming and outgoing scattering states is a special case of a (generalized) formfactor associated with the identity operator.
(point-like [1] or string-like[2]) “coordinatizations” of local algebras i.e. only local equivalence classes of fields or the local algebras generated by fields deserve to be called intrinsic⁵. The use of the word “intrinsic” in local quantum physics (LQP) is reminiscent of the use “invariant” (as opposite to coordinate-dependent) in geometry; in this analogy the coordinates in geometry correspond to the coordinatization of spacetime-indexed algebras by pointlike field generators. In the Lagrangian quantization approach to quantum field theory, as well as in the more intrinsic algebraic approach to local quantum physics, crossing plays no significant role. Only in formulations of particle physics which start with on-shell quantities and aim at the construction of spacetime-indexed local algebras (or local equivalence classes of fields), the crossing becomes an important structural tool.

Examples par excellence of pure on-shell approaches are the various attempts at S-matrix theories which aim at direct constructions of scattering data without the use of local fields and local observables. The motivation behind such attempts was already (before renormalization theory) clearly spelled out by Heisenberg [3]: by limiting oneself to particles and their mass-shells one avoids (integration over) fluctuation over arbitrarily small spacelike distances which are the cause of ultraviolet divergencies.

This idea is quite different and more conservative than attempts at improving short-distance properties by introducing non-local interactions in a field theoretic framework (for a review of non-local attempts see [4]) which generally causes grave problems with the causality properties underlying particle physics. The main purpose of approaches using scattering concepts (“on-shell”) is to avoid such inherently singular objects as pointlike fields in calculational steps which is a reasonable aim independent of whether pointlike fields exist or not. Heisenberg’s S-matrix proposal can be seen as the first attempt in this direction. It incorporated unitarity, Poincaré invariance and certain analytic properties, but run into problems with the implementation of cluster factorization properties for the multiparticle scattering.

There exists a more recent scheme of “direct particle interaction” which solved this cluster factorization problem for the multi-particle representations of the Poincaré group in the presence of interactions by an iterative construction [5]. It is helpful at this point to recall that in multiparticles Schrödinger quantum mechanics the step from n to n+1 particles by simply adding the two-particle interactions of the new particle with the n previous ones manifestly complies (for sufficiently short range interactions) with the cluster factorizability of the unitary representors of the 10-parametric Galilei-group. But this infinite “Russian matrushka” picture of particle physics through increasing the particle number by adding the new (short range) interactions and in turn finding the old n-particle physics inside the n+1 particle physics by translating one particle to infinity runs into series problems in the relativistic context. In mathematical

⁵The individuality of classical fields is lost in QFT where e.g. a meson field is any local (relative local with respect to the local observables of the theory) covariant object with a nonvanishing matrix element between the vacuum and a one-meson state (“interpolating field”).
terms there exists a mismatch between the adding on of interactions associated with addition of particles and the cluster factorizability property i.e. the tensor factorization of the representation into the representation of the clusters (subgroup of particles) upon partitioning the total particle configuration into clusters and spatially translating them infinitely far from each other. The inductive adding on of interactions is conveniently done in terms of the invariant mass operator in the way of the Bakamjian-Thomas formalism [5]. For the two-particle system this construction automatically clusters but for 3 particles one runs into a problem: although the 3-particle S-matrix\(^3\) clusters [6], the representation of the Poincaré group fails to cluster (the Hamiltonian and the L-boosts are not asymptotically additive) and it was clear that by adding a fourth particle that even the 4-particle S-matrix would fail to cluster. The solution to this obstruction was later found [5]; it consisted in modifying the 3-particle system by adding a connected 3-particle interaction in such a way that the 3-particle S-matrix does not change. This is done by a so-called “scattering equivalence” [7] i.e. a unitary transformation which changes the (Bakamjian-Thomas) 3-particle representation in such a way that the 3-particle S-matrix is maintained.

It turns out that this process (of adding on interactions to the mass operator and then re-enforcing clustering by a scattering equivalence) works iteratively [5] and yields an \(n\)-particle interacting representation of the Poincaré group which then lead to Möller operators and an S-matrix which cluster. There is a prize to pay namely the use of scattering equivalences introduces an additional amount of nonlocality but it secures the “macro-locality” expressed by the (rapid in case of short range interactions) fall-off properties of the connected parts of the representation of the Poincaré group and the S-matrix. Different from the mass superselection rule in Galilei invariant quantum mechanics, there is no selection rule from Poincaré symmetry in this Coester-Polyzou (C-P) formalism which requires the absence of particle creation processes [5]. This poses the interesting question whether by coupling channels with an increasing number of particles one can approximate field theoretic models by mathematically controllable “direct particle interaction” situations.

Since the early 1950s, in the aftermath of renormalization theory, the relation between particles and fields received significant clarification through the derivation of time dependent scattering theory. It also became clear that Heisenberg’s proposal had to be amended by adding the crossing property i.e. a prescription of how to analytically continue particle momenta on the complex mass shell in order to relate matrix elements of local operators (or the identity in case of the S-matrix) between incoming ket and outgoing bra states. In physical terms it allows to relate matrix elements of operators between incoming ket- and outgoing bra-states to the vacuum polarization matrix elements where the ket-state is the vacuum vector. Whereas Heisenberg’s requirements on a relativistic S-matrix can be implemented in a C-P particle scheme, the implementation of crossing

\(^3\)The possibility of two-particle bound states entering as incoming particles requires the use of the framework of rearrangement collisions in which the space of (noninteracting) fragments is distinguished from the (Heisenberg) space on which the interacting Poincaré group is represented [6].
points to the presence of vacuum polarization for which QFT with its microcausality is the natural arena. At this point it should be clear to the reader why we highlighted the little known “direct particle interaction” theory; if one wants to shed some light on the mysterious crossing symmetry, it is helpful to contrast it with theories of relativistic particle scattering in which this property is absent.

The LSZ time-dependent scattering theory and the associated reduction formalism relates such a matrix element\(^4\) (generalized formfactor) in a natural way to one in which an incoming particle becomes “crossed” into an antiparticle on the backward real mass shell; it is at this point where analytic continuation enters. The important remark here is that the use of particle states requires the restriction of the analytic continuation to the complex mass shell (“on-shell”). If one were to allow interpolations via “off-shell” field states, the derivation of the crossing would be much easier since it would follow from off-shell spectral representations of the Jost-Lehmann-Dyson kind or perturbatively from Feynman diagrams and time-ordered functions. In this paper crossing will only be used in the for the more restrictive on-shell analytic continuation as it is needed for formfactors.

A rigorous on-shell derivation for two-particle scattering amplitude has been given by Bros\(^5\), Epstein and Glaser [8]. The S-matrix is the formfactor of the identity operator i.e. the identity operator sandwiched between incoming ket and outgoing bra multi-particle states. In that case there is no operator which can absorb a difference between total incoming and total outgoing momenta and hence the crossing of particle from a two-particle in state into an out state has to be accompanied by a reverse crossing of one of the outgoing particles into an incoming one. In this way the crossed situation is the analytic continuation of a physical scattering process. This simultaneous crossing of two particles in the elastic scattering amplitude gave origin to the name “crossing” and was the subject of rigorous analytic investigations [8]. A derivation of crossing in the setting of QFT for general multi-particle scattering configurations and for formfactors as one needs it for many applications (see later) does not exist. It is not clear to me whether the present state of QFT would permit to go significantly beyond the old and still impressive results quoted before.

The crossing property became the cornerstone of the so-called bootstrap S-matrix program and several ad hoc representations of analytic scattering amplitudes were invented (Mandelstam Regge...) in order to incorporate crossing in a more manageable form. An interesting early historical chance to approach QFT from a different direction by using on-shell global objects without short distance singularities was wasted when the S-matrix bootstrap approach ended in a verbal cleansing rage against QFT\(^6\).

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\(^4\)More precisely the crossing property involves the connected part of (generalized) formfactors (see next section).

\(^5\)For this reason the dedication of this particular work to Jacques Bros on the occasion of his 70\(^{th}\) birthday is particularly appropriate.

\(^6\)One glance at the old conference proceedings and review articles of the Chew S-matrix school reveals what is meant here. The ideological fervor is hard to understand, in particular
Some of the ideas were later used by Veneziano [9] in the construction of the dual model. But there is an essential difference in the way crossing was implemented. Whereas the field theoretic crossing involves a finite number of particles with the scattering continuum participating in an essential way, the dual model crossing does achieve this property already without a continuum by using instead a discrete infinite “particle tower” with ever increasing masses (the origin of what is called “stringyness”). This tower structure was later interpreted in terms of the particle excitations of a relativistic string. It is important to note that Veneziano’s successful experiment with properties of Gamma functions did satisfy the phenomenological desires in those days (in this sense it was more than a mathematical invention) which was calling for a one-particle “saturation” (suggested by the phenomenological use of Regge poles) of the crossing property in the setting of Mandelstam’s representation of the 2-particle scattering amplitude. It is important to mention these phenomenological ideas which separate the dual model from the field-theoretic crossing structure in order to counteract the impression that the dual model is the legacy of a QFT related S-matrix bootstrap approach.

There is some irony in the fact that Chew and his followers, who tried to find a philosophic basis why their S-matrix bootstrap should be considered a theory of everything (TOE), did not succeed in these attempts7, whereas Veneziano, who had no such aims, laid the seeds of string theory. Contrary to the original phenomenological intentions of the dual model, its string theoretical re-interpretation injected the aspects of a TOE (this time including gravity) and as a result the development took a quite speculative and ideological direction. Despite its phenomenological beginnings (and its later paradigmatic changes into an ultraviolet-finite candidate for quantum gravity), the physical content of string theory and its relation to the principles of particle physics became increasingly mysterious. Its endurance and attraction up to the present cannot be separated from the fact that it led to significant mathematical enrichments far beyond what could have been expected from its humble phenomenological origins.

The main reason why the old bootstrap approach ended in the dustbin of history (besides suffering the not unmerited consequences for its dismissive view of QFT even at the time of the discovery of the physical relevance of gauge theories beyond electromagnetic interactions) was its lack of success in taking the crossing property (the cornerstone of the bootstrap approach) out of its purely analytic setting and convert it into an operational tool, as well as its lack of mathematical appeal.

The issue of crossing within the setting of QFT and its use in model constructions will form the main issue of this paper. The reader who expects an in-depth study of the crossing property in the setting of QFT may however be disappointed by the content of this article; what we will offer instead is a highly speculative idea about the existence of an auxiliary non-local thermal the-

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1The S-matrix bootstrap returned many years later as a valuable tool (but not a TOE) of the “form factor program” in the limited context of d=1+1 factorising models of QFT[10].
ory which describes the formfactors of certain distinguished quantum fields (the "Masterfields") of the original QFT. We hope that since the crossing property ever since its discovery in the early 50s (despite numerous attempts to clarify its role in particle physics) has remained the most mysterious and at the same time most enigmatic among all incompletely understood properties of local quantum physics, a speculative attempts as the present one may generate new interest in this age old problem. Even though the support for our operational hypothesis about crossing comes from d=1+1 factorizing models\(^6\), the new “masterfield” proposal is not a priori limited by those algebraic restrictions [19] (see next section) which characterize the Zamolodchikov-Faddeev [11] algebra structure of factorizing models.

The motivation underlying this work is two-fold. On the one hand it is important to understand the conceptual position of crossing within the setting of algebraic QFT. On the other hand one would like to re-investigate the feasibility of the old “bootstrap dream” for an ultraviolet divergence-free formulation of particle physics in a wider context in which correlation functions of pointlike fields are avoided in favor of formfactors which do not suffer from short distance problems since there are no coalescing localization points. Algebraic QFT succeeds to account for the underlying causality and locality principles without using inherently singular pointlike field objects, but it has not yet come up with a calculational scheme which avoids the use of these delicate objects in computations. This paper also intends to give an account about how far one is still away from this goal.

Repetitions of results and ideas in a slightly changed context are intentional in a discursive and history rooted presentation.

2 The bootstrap-formfactor program in d=1+1 factorizing QFT

As already mentioned in the introduction, a more modest, but in its own right extremely successful version of the S-matrix bootstrap with strong field theoretic roots emerged in the second half of the 70s from some quasiclassical findings [12]. These seemingly exact quasiclassical observations on the very special two-dimensional "Sine-Gordon" models of QFT required an explanation beyond quasiclassical approximations [13], and from this in turn emerged a general constructive program of a bootstrap-factorizable construction of so-called d=1+1 factorizable models [10],[14],[15]. This new nonperturbative scheme of constructing a particular class of field theories produced a steady flux of new models and continues to be an important innovative area of research.

In contrast to what was expected from the ambitious original bootstrap approach, this program uses the S-matrix bootstrap idea in the limited context of

\(^6\)The reason for our preference of the name factorizing to integrable QFT is that the direct definition of factorizability in the quantum setting is conceptually much simpler than the classical notion of integrability which via quantization leads to quantum integrability.
an S-matrix Ansatz in which S factorize into 2-particle elastic components $S^{(2)}$. The advantage of this class of models is that the classification and calculation of
factorizing S-matrices [16] can be treated as a separate problem from the associated QFT. The S-matrix bootstrap is the first step in a bootstrap-formfactor
program where the second step consists in calculating formfactors of fields and
on-shell matrixelements of more general operators. In general one does not
expect such a separation, rather the construction of the S-matrix (which may
be considered as the special formfactor of the identity operator between in-out
multi-particle states) takes place simultaneously with that of formfactors.

It is interesting to note that the calculated formfactors of those factorizing
models which possess continuous coupling parameters are analytic functions
with a finite radius of analyticity around zero coupling strength. For the cor-
relation function on the other hand one does not expect expansibility into a
power series since their perturbative structure is indistinguishable from that of
other strictly renormalizable models and there exist general arguments against
the convergence of perturbative series. This raises the interesting question of
whether such a dichotomy between perturbatively converging on-shell objects
versus nonconverging (at best asymptotic) series for off-shell correlation func-
tions may apply in general. It would be quite startling if formfactors in renor-
malizable field theories have improved perturbative convergence properties which
are not shared by correlation functions.

The main motivating idea in favor of an on-shell approach, namely the total
avoidance of ultraviolet divergences, is brilliantly vindicated in the setting of
different models. The pointlike fields, which in the present state of develop-
ment are only known via their multi-particle formfactors, have an interesting
vacuum structure; despite their lack of real particle creation through scattering
processes they nevertheless possess the full vacuum polarization structure
which one expects in an interacting QFT and which in turn is the prerequisite
for the appearance of interaction-caused anomalous short distance dimensions.
In this respect factorizing models are more interesting than the (nonfactorizing)
polynomial interactions in $d=1+1$ which were the subject of existence proofs in
“constructive QFT” [17].

The experience one had with crossing indicates that contrary to the cluster
property, crossing is (similar to TCP symmetry) the characteristic imprint which
relativistic causality leaves in on-shell restrictions of QFT. In general it is not
possible to implement crossing in a relativistic particle setting as the C-P scheme
of direct particle interactions. Factorizing models seem to be an exception in
the sense that the S-matrix looks like that of a one-dimensional relativistic
quantum mechanics as the result of absence of particle creation via scattering.
An indication of a conceptual difference is that the C-P approach the interaction
is encoded into the generators of the connected part of the Poincaré group,
whereas in QFT this information resides in the discrete reflection symmetries
which involve time reversal interaction (see also section 4).

The fact that the bootstrap-formfactor approach does not need special pre-
scriptions (which one finds in the first systematic presentation [15]) but follows
from general principles of QFT becomes particularly transparent if the construc-
tion is placed into the Tomita-Takesaki modular theory of operator algebras as adapted to the local quantum physics setting (also referred to as the method of modular localization) [18][19][20] which will be briefly sketched below and (more technical aspects) in section 4.

This setting also highlights the “existence problem of QFT” in a new and promising fashion [21][22]. Here we remind the reader that even after almost 8 decades after its discovery and despite impressive perturbative and asymptotic (in the short distance regime) numerical successes, the description of interacting particles by covariant fields in 4-dimensional Minkowski spacetime remains still part of mathematical uncharted territory. This is the source of permanent discomfort unknown in other areas of theoretical physics.

The algebraic basis of the bootstrap-formfactor program for the special family of d=1+1 factorizable theories is the validity of a momentum space Zamolodchikov-Faddeev algebra [11]. The operators of this algebra are close to free fields in the on-shell sense of their Fourier transforms (see (6) in section 4), but they are non-local in the specific sense of being localizable precisely in wedge algebras [18][20]. From this follows that the wedge algebras generated via their Fourier transforms by on-shell Z-F field operators possess vacuum-polarization-free generators (PFGs). These are on-shell operators; the only property which distinguishes them from standard creation/annihilation operators is the fact that the product of two such operators in opposite orders is different by rapidity-dependent structure functions (matrix-valued in case of multi-component creation/annihilation operators). In fact the existence of “tempered” (Fourier transformable) wedge localized PFGs which implies the absence of real particle creation through scattering processes [19] turns out to be the prerequisite for the working of the present bootstrap-formfactor program (which is a calculational scheme in which one uses only formfactors and no short-distance singular correlation functions). According to an old structural theorem which uses analytic properties of a field theoretic S-matrix [23] virtual particle creation without real particle creation is only possible in d=1+1 theories. According to an argument by Karowski (private communication) even direct 3- or higher- particle elastic processes are ruled out by formfactor crossing, so that PFGs describing interacting d=1+1 QFT are synonymous with Z-F algebra structures.

The bootstrap-formfactor construction for d=1+1 factorizing models contains a valuable information about the uniqueness of the inverse scattering problem. As a result of the insensitivity of the S-matrix against local changes, the unicity of the inverse problem is defined to mean that the S-matrix determines the local equivalence classes of fields (rather than individual fields). This unique relation of a crossing symmetric S-matrix with a local equivalence class of fields is indeed what the bootstrap-formfactor program shows [18] (it is reminiscent but

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9 An operator which is localizable in a certain causally closed spacetime region is automatically localized in any larger region but not necessarily in a smaller region. The unspecific terminology “non-local” in the literature is used for any non pointlike localized field.

10 See a recent review [24] in which the minimal formfactor contributions, which are a joint property of the local equivalence class, have been separated from the polynomial contribu-
much tighter than the scattering equivalences mentioned in the introduction).

The crucial property which links scattering data with off-shell operators spaces is the crossing property. It relates the multiparticle component of vectors obtained by one-time application of a field to the vacuum with the (connected) formfactors of those fields. In other words, the crossing serves to construct from the vacuum-polarization components of wedge-localized operators their general (connected) matrix elements in the multi-particle basis between incoming ket- and outgoing bra- states. If a given S-matrix would lead to two different wedge algebras then the modular subspace i.e. the space which the Tomita-Takesaki modular theory associates with this algebra applied to the vacuum must already have been different. But this is impossible since the position of this subspace is determined by the Tomita operator which in turn is fixed in terms of the S-matrix [25]. This unicity property which results from this general crossing argument (without using the factorizability of the model) is based on a formulation of the crossing property\textsuperscript{11} which is stronger than what had been derived in the work of Bros and Epstein and Glaser [8] (see below). This is one of the few cases where crossing enters in an important way, without crossing it is not possible to establish uniqueness of the inverse scattering problem [26]. Nothing is known about conditions on crossing symmetric S-matrices which lead the existence of an associated QFT.

There remains the question of whether there exists a stronger constructive on-shell approach in terms of formfactors which is not a-priori limited by the existence of tempered PFGs and which could work also in $d>1+1$. It is clear that any approach which uses in its computations only formfactors and in this way (by avoiding correlation functions of pointlike field coordinates) will be free of ultraviolet problems.

From the previous remarks one expects that such an approach requires a much better operational understanding of crossing which has remained the most mysterious property of particle physics. Time and again has shown that analyticity properties on their own may be sufficient to derive certain sum rules, but are insufficient as the a of calculations and model constructions.

In the remainder of this paper an operational setting of crossing is presented which in principle does not suffer from the above limitations of temperate PFGs, although one is presently only able to test it by explicit model examples in the $d=1+1$ factorizing setting. It is based on the working hypothesis that each quantum field theory possesses a distinguished field called a “Masterfield” whose connected parts of its formfactors defines a global (i.e. no local substructure) quantum field theory in the on-shell momentum space variables in a thermal state at the KMS Hawking temperature in such a way that the cyclic KMS property (the thermal aspect of modular theory) is identical with the cyclic crossing property. By construction this theory obeys momentum space cluster decomposition properties in the rapidity variables. The simplicity

\textsuperscript{11}Without the crossing property hypothesis it is presently no possible to exclude the existence of several local field systems (or local nets) with one S-matrix [26].
of d=1+1 factorizing models finds its expression in the fact that the auxiliary operator, whose KMS correlation functions are identified with the connected formfactors of the masterfield, is an exponential of a bilinear expression in free creation/annihilation operators. There is a good chance that this structure is characteristic for factorizing models.

The subsequent content of the paper is organized as follows. The next section recalls the formal aspects of the crossing relations within the LSZ scattering theory. The fourth section reviews the modular approach to the problem of factorizing theories with special emphasis on the wedge-localized PFGs associated the Zamolodchikov-Faddeev algebra. These generators of the wedge algebra are then used in the computation of double cone intersections which exhibit the full vacuum polarization structure. The crossing property is an inseparable part of this formalism. In section 5 the working hypothesis of the existence of a “masterfield” is formulated. The validity of this hypothesis is checked in two models. These model illustrations make contact with some recent observations about clustering properties of formfactors in the momentum space rapidity variables [24] and with recent “free field representations” of formfactors [27].

3 Crossing recipe from LSZ scattering theory

Crossing has been first observed in Feynman perturbation before a formal derivation in the setting of LSZ scattering theory was given. Its formal aspects are easily obtained from the LSZ asymptotic convergence ($A^\#$ stands for $A$ or $A^*$)

$$\lim_{t \to \pm \infty} A^\#(f_t)\Phi = A^\#(f)_{in, out} \Phi$$

$$A(f_t) = \int f_t(x) U(x) A U^*(x) d^4 x, \quad A \in \mathcal{A}(O)$$

$$f_t(x) = \frac{1}{(2\pi)^4} \int e^{i[p \cdot x - \omega(p)]} f(p)d^4 x, \quad \omega(p) = \sqrt{p^2 + m^2}$$

which can be derived on a dense set of states. This is known to lead to the well-known LSZ reduction formulas (for a rigorous derivation see [30] [31]) which in terms of connected matrix elements (leaving out the test function smearing) read

$$\langle \text{out} | q_1, q_2, \ldots, q_m | F | p_{\alpha_1}, \ldots, p_{\alpha_2}, p_1 \rangle_{\text{conn}} =$$

$$-i \int \langle \text{out} | q_2, \ldots, q_m | K_T F A^*(y) | p_1, p_{\alpha_1} \rangle_{\text{conn}} q_1^4 y e^{-i q_1 \cdot y}$$

$$= -i \int \langle \text{out} | q_1, q_2, \ldots, q_m | K_T F A(y) | p_2, \ldots, p_{\alpha_2} \rangle_{\text{conn}} d^4 y e^{i q_1 \cdot y}$$

Here the time-ordering $T$ involving the original operator $F \in \mathcal{A}(O)$ and the interpolating Heisenberg field$^{12}$ $A(x) \equiv U(x) A U^*(x)$ or $A^*(x)$ (with $A \in \mathcal{A}(O)$)

$^{12}$The operators may be bounded or unbounded (with the well-known Wightman domain properties). The asymptotes are always local linear combinations of (necessarily unbounded) creation/annihilation operators.
for some $\mathcal{O}$) appears in the reduction of a particle from the bra- or ket state. For the definition of the time ordering between a fixed finitely localized operator $F$ and a field with variable localization $y$ we may use $TF A(y) = \theta(y) F A(y) + \theta(-y) A(y) F$, however as we place the momenta on-shell, the definition of time ordering for $y$ near $loc F$ is fortunately irrelevant. Each such reduction is accompanied by another disconnected contribution in which the creation operator of an outgoing particle $a_{\mu \nu} (q_1)$ changes to an incoming annihilation $a_{\mu \nu} (q_1)$ acting on the incoming configuration and there is a corresponding contraction in the reduction of a particle from the incoming state vector. These terms (which contain formfactors with two particle less in the bra- and ket-vektors) have been omitted since they do not contribute to generic nonoverlapping momentum contributions and to the analytic continuations (and hence do not enter the connected part). Under the assumption that there is an analytic path from $p \rightarrow -p$ (or $\theta \rightarrow \theta - i \pi , p_+ \rightarrow -p_+$ in the wedge adapted rapidity parametrization). The comparison between the two expressions gives the desired crossing property: a particle of momentum $p$ in the incoming ket state within the formfactor is crossed into an outgoing bra antiparticle at the analytically continued momentum $-p$ (here denoted as $-\tilde{p}$) and the connected and the connected formfactor remains invariant.

In order to obtain that required analytic path on the complex mass-shell of the $2 \rightarrow 2$ scattering amplitude it is convenient to pass from time ordering to retardation

$$TF A(y) = RFA(y) + \{ F, A(y) \} \quad (3)$$

The unordered (anticommutator) term does not have the pole structure on which the Klein-Gordon operator $K_y$ can have a nontrivial on-shell action and therefore drops out. The application of the JLD spectral representation puts the $p$-dependence into the denominator of the integrand of an integral representation from where the construction of the analytic path for formfactors proceeds in an analog fashion to the derivation of crossing for the S-matrix. Whereas it is fairly easy to find an off-shell analytic path, the construction of an on-shell path which remains in the complex mass shell is a significantly more difficult matter. The LSZ reduction formalism is suggestive of crossing but far to weak for securing the existence of paths on the complex mass shell which link real forward and backward mass shells.

The simplifications of the LSZ formalism resulting from factorizability of models can be found in an appendix of [33].

The result of the comparison between the reduction 2 applied to outgoing and incoming configurations may be written in the following suggestive way (for spinless particles).

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13 These on-shell reduction formulas remain valid if one uses as interpolating point like fields $A(x)$ the translates of bounded localized operators [31].

14 For the 2-particle S-matrix (which is the formfactor of the identity) one has to cross one incoming particle simultaneously with an outgoing particle which in the Feynman graph notation amounted to the crossing of two particle lines (and is the origin of the word “crossing”).
$$\begin{align*}
\text{out} \langle p_1, p_2, \ldots, p_r | F | q_k, q_{k-1}, \ldots, q_2, q_1 \rangle^{\text{in}} &= \\
\text{a.c.} \quad \text{out} \langle q_c, p_1, p_2, \ldots, p_r | F | q_k, q_{k-1}, \ldots, q_2 \rangle^{\text{in}} + \text{c.t} \\
\Rightarrow \text{out} \langle -q_1, p_1, p_2, \ldots, p_r | F | q_k, q_{k-1}, \ldots, q_2 \rangle^{\text{in}} + \text{c.t}. 
\end{align*}$$

where the contraction terms c.t. involve momentum space $\delta$-functions (which are part of the LSZ reduction theory) and the last line denotes a shorthand notation for the analytic continuation to the real negative mass shell. Instead of crossing from incoming ket to outgoing bras one may also cross in the reverse direction from bras to kets. The important physical role of the crossing property is to relate the vacuum polarization components of an operator to the connected part of the transition it causes between in and out scattering states via iterated crossing

$$\text{out} \langle p_1, p_2, \ldots, p_n | F | \Omega \rangle^\text{iteration} \text{ out} \langle p_k, p_{k+1}, \ldots, p_n | F | -\vec{p}_{k-1}, \ldots, -\vec{p}_2, -\vec{p}_1 \rangle^{\text{conn}} \tag{5}$$

Note that the vacuum polarization components are already connected. It is very important to realize that the simplicity of the crossing property occurs only for the connected part of the matrix elements; in order to write down the relation for the full matrix elements one must keep track of all the momentum space contraction terms in the iterative application of the LSZ formalism. It is the connected part which is described by one analytic “master function” whose different boundary values correspond to the connected part of the different matrix elements. This already indicates that one should expect problems if one wants to understand crossing as an operational property in the original theory of operators. Indeed the first naive attempts to relate crossing to the cyclicity property of thermal expectation values in KMS states failed\(^\text{15}\).

4 The Zamolodchikov-Faddeev algebra and its relation to modular localization

In this section we recall some details about how the modular localization formalism leads to a profound understanding of the bootstrap-formfactor construction which was already described from a historical perspective in the introduction. It turns out that the best insight into the importance of the wedge region is obtained if one first looks at the modular localization formalism in the simpler case of the net of algebras associated to free fields by starting from the modular structure of the Wigner representation theory of unitary irreducible positive energy representations of the Poincaré group. The mathematical implementation

\(^{15}\text{The structural similarity between the cyclicity of the crossing with the KMS-property has misled many authors (including the present author [35]) into claims that crossing has a KMS interpretation in the setting of wedge-localized algebras of the original theory. These conjectures (including “proofs” [34]) are incorrect.}\)
of this idea was known for a long time: the CCR-(Weyl)- or CAR functor applied to modular localized real subspaces of the complex Wigner representation space of integer or halfinteger spin representations maps these objects directly into local von Neumann algebras [37]. However it was not known at that time how to explicitly compute those subspaces by intrinsic (avoiding pointlike field coordinatizations) modular methods. The guiding idea was found only quite late, although its basic ingredience, namely the geometric nature of the wedge localization spaces, could have been known since the 1975 work of Bisognano and Wichmann [38] on the geometric aspects of the modular objects associated with wedge algebras.

There were two independent settings in which constructions based on modular localization emerged as computational tools. On the one hand it was shown by the present author that the prescriptions of the d=1+1 bootstrap-formfactor program (the Smirnov postulates and the Zamolodchikov-Faddeev algebraization) permitted a natural and profound explanation in the setting of modular localization in analogy to the modular localization construction in the absence of interactions [35][18].

On the other hand Brunetti Guido and Longo [39] systematically constructed the family of wedge-localized real subspaces for all positive energy Wigner representations by combining purely group-theoretical ideas with aspects of modular theory. Although spaces corresponding to causally complete subwedge regions (e.g. spacelike cones, double cones) have no geometric modular characterization, they can easily be constructed by intersecting wedge localized spaces. The work of these authors culminated in a surprising theorem stating that positive energy representations of the Poincare group always have nontrivial spacelike cone localized subspaces (for arbitrary small opening angles) and that the wedge like localization can be re-obtained by additivity from small spacelike cone localized subspaces which approximate the given region from the inside.

In case of (half)integer spin representations one can show that the compact spacelike cone intersections are nontrivial by constructing a dense set of double cone localized wave functions. However there were two classes of representations which did not allow compact localization (i.e. the double cone localization spaces were trivial) namely the d=1+2 representations with "anyonic" spin (≠ (half)integer) and the famous Wigner family of zero mass infinite helicity representations. Both cases lead to nontrivial spacelike cone localized subspaces and hence it was suspected that these representations may be associated with string-localized field theories [2][4]. This turned out to be true; but only in the infinite spin case the string-localized fields are genuine free fields whereas in the anyonic case the string localized fields applied to the vacuum are always accompanied by vacuum polarization which throws them off mass shell [40].

It has been my Leitmotiv for a number of years [35] that the spirit behind Wigner's physical representation theoretical approach suitably generalized to include interactions should lead to a framework in which constructions are following the intrinsic logic of local quantum physics without any need to refer to the quantization of classical Lagrangian structures reference to Lagrangian quantization. In the absence of interaction this was achieved by combining the
functorial relation (via the CCR/CAR functor) between subspaces and subalgebras. The presence of interactions renders this functorial relation useless, but does not affect the validity of the modular localization formalism. More specifically, the close relation between particles and fields (or local algebras) is lost, but there are still PFG operators associated with wedge algebras whose one-time application to the vacuum creates the one-particle states of the theory without any admixture of vacuum polarization contributions\textsuperscript{16}. On the other hand the knowledge of a wedge algebra (i.e. its position inside the global algebra in Fock space $\mathcal{A}(W) \subset B(\mathcal{H}_{Fock})$) suffices to construct the net of tighter localized algebras by intersecting wedge algebras. Unlike in the standard approach, the interacting fields and their correlation functions appear not the formulation of the problem as the carriers of the Lagrangian quantization but rather at the end as the universal generators of algebras and their modular properties after the algebraic intersections have been carried out.

It is comforting to know that besides the global definition of the presence of interactions in terms of a nontrivial ($S \neq 1$) S-matrix, the modular approach leads to an intrinsic local characterization in terms of an ubiquitous presence of vacuum polarization clouds in the attempt to create one-particle states by applying operators from (causally closed) subwedge regions to the vacuum. One would expect that the interaction-caused vacuum polarization clouds possess a joint core (independent of the individual operators) which only depends on the causally complete region and on the particular model and that a better future understanding will permit a classification of interactions in terms of characteristic properties of vacuum polarization clouds.

Before we specialize to the case of $d=1+1$ factorizing models for which the inverse scattering problem has a unique solution, it is interesting to isolate the property which is responsible for this unicity and explore the possibility of a general unicity argument without factorizability. This crucial property is the crossing property for formfactors which allows to pass from the vacuum polarization components of a localized operator to its matrix elements between arbitrary multi-particle states. The main nontrivial step in which the above formulated property of formfactor crossing enters in an essential way is the uniqueness of the inverse scattering problem in QFT, which for a crossing symmetric situation with $S=1$ amounts to the equivalence class of the free field as the only field system with the trivial $S$-matrix. For the $S$-matrix of the $d=1+1$ factorizing models (whose $S$-matrix is given in terms of a combinatorial formula in terms of the elastic two-particle scattering) the unicity is implicit in the construction of a unique system of formfactors (without being forced to invoke an additional selection principle) of an associated composite field system.

There are two observations about factorizing models which turned out to be a perfect match. On the one hand it has been noted \textsuperscript{18} that the Fourier transforms of the Zamolodchikov-Faddeev algebra creation and annihilation operators in the case of absence of boundstates\textsuperscript{17} are generators of wedge-localized

\textsuperscript{16}The proof breaks down in the presence of zero mass caused infrared problems [30].

\textsuperscript{17}A situation which in case of factorizing models with variable coupling (as e.g. the Sine-
Weyl algebras. In the simplest case we have (in the absence of bound state poles of \( S(\theta) \))

\[
\phi(x) = \frac{1}{\sqrt{2\pi}} \int e^{i p(x) x} Z(\theta) + \text{h.c.} \ d\theta
\]

\[
Z(\theta) Z^*(\theta') = S^{(1)}(\theta - \theta') Z^*(\theta') Z(\theta) + \delta(\theta - \theta')
\]

\[
Z(\theta) Z(\theta') = S^{(2)}(\theta' - \theta) Z(\theta') Z(\theta)
\]

Here \( p(\theta) = m(\hbar \theta, \hbar \theta) \), is the rapidity parametrizations of the d=1+1 mass-shell and \( x = r(h \chi, h \chi) \) parametrizes the right hand wedge in Minkowski spacetime; \( S(\theta) \) is the physical S-matrix of elastic two particle scattering and \( S(z) \) its analytic continuation. It fulfills unitarity \( S^2(z) = S(\bar{z}) (-z) \) and crossing [16]. The \( Z \) operators are deformations of standard creation/annihilation operators; the definition of \( Z \) on a standard Fock space vector created by an ordered product \( (\theta_1 > \theta_2 > \ldots > \theta_n) \) of incoming \( a^*(p) \) reads

\[
Z^*(\theta) a^*(\theta_1) a^*(\theta_2) \ldots a^*(\theta_n) \Omega = \prod_{i=1}^{k} \left[ S(\theta - \theta_i) a^*(\theta_1) a^*(\theta_2) \ldots a^*(\theta_i) a^*(\theta_n) \right] \Omega
\]

where \( \theta < \theta_i \ i = 1..k \), \( \theta > \theta_i \ i = k+1..n \). The \( Z-F \) commutation relations follow from this definition. In the presence of bound states it is better to work with the (appropriately modified) state vector definition than with the \( Z-F \) algebra. The general Zamolodchikov-Faddeev algebra is a matrix generalization of this structure. As a result of the rapidity dependent commutation relations of the \( Z^i \)'s, the field is Poincaré-covariant but not local in the sense of the standard use of this terminology[18]. It is however wedge-localized in the sense that the generating family of operator for the righthand wedge \( W \) algebra \( \{ A(f), \text{supp} f \subset W \} \) commute with the TCP transformed algebra \( \{ J A(g) J, \text{supp} g \subset W \} \) which is the left wedge algebra [20]

\[
[A(f), J A(g) J] = 0
\]

\[
J = J_0 S_{sc\theta}
\]

where \( J_0 \) is the TCP symmetry of the free field theory associated with \( a^*(\theta) \) and \( S_{sc\theta} \) is the factorizing \( S \)-matrix which has the form

\[
S_{sc\theta} a^*(\theta_1) a^*(\theta_2) \ldots a^*(\theta_n) \Omega = \prod_{i \neq j} S^{(2)}(\theta_i - \theta_j) a^*(\theta_1) a^*(\theta_2) \ldots a^*(\theta_n) \Omega
\]

if, as before, we identify the \( a^*(\theta) \) with the incoming creation/annihilation operators. It is then possible to give a rigorous proof [20] that the Weyl-like

Gordon theory) can always be obtained by choosing a sufficiently small coupling.
algebra generated by exponential unitaries is really wedge-localized and fulfills
the Bisognano-Wichmann property
\[
\mathcal{A}(W) = \text{alg} \left\{ e^{iA(t)} | \text{supp} f \subset W \right\}
\]
\[
\mathcal{A}(W)' = J \mathcal{A}(W) J = \mathcal{A}(W')
\]
where the dash on operator algebras is the standard notation for their von
Neumann commutant and the dash on spacetime regions stands for the causal
complement. The operator TCP operator $J$ is the (antiunitary) angular part of
the polar decomposition of Tomita’s algebraically defined unbounded antilinear
$S$-operator with the following characterization
\[
SA\Omega = A\Omega, \ A \in \mathcal{A}(W)
\]
\[
S = J \Delta^\frac{1}{2}, \ \Delta = U(A(-2\pi t))
\]
with $A(\chi)$ being the Lorentz boost at the rapidity $\chi$. At this point the setup
looks like relativistic quantum mechanics since the $A(f)$ (similar to genuine free
fields if applied to the vacuum) do not generate vacuum polarization clouds.
The advantage of the algebraic modular localization setting is that vacuum
polarization is generated by algebraic intersections which is in agreement with
the intrinsic definition of the notion of interaction presented in terms of PFGs
in the previous section
\[
\mathcal{A}(D) \equiv \mathcal{A}(W) \cap \mathcal{A}(W'_o) = \mathcal{A}(W) \cap \mathcal{A}(W'_o)'
\]
\[
D = W \cap W'_o
\]
This is the operator algebra associated with a double cone $D$ (which is chosen
symmetric around the origin by intersecting suitably translated wedges and their
causal complements). The problem of computing intersected von Neumann
algebras algebras is difficult since there are no known general computational
techniques.

The problem becomes more amenable if one considers instead of operators
their formfactors i.e. their matrix elements between incoming ket and outgoing
bra state vectors. In the spirit of the old LSZ formalism one can then make an
Ansatz in form of a power series in $Z(\theta)$ and $Z^*(\theta) \equiv Z(\theta - i\pi)$ (corresponding
to the power series in the incoming free field in LSZ theory)
\[
A = \sum \frac{1}{n!} \int_C \cdots \int_C a_n(\theta_1, \ldots, \theta_n) : Z(\theta_1) \cdots Z(\theta_n) :
\]
Each integration path $C$ extends over the upper and lower part of the rim of
the $(0, -i\pi)$ strip in the complex $\theta$-plane. The strip-analyticity of the coefficient
functions $a_n$ expresses the wedge-localization of $A$. It is easy to see that these
coefficients are identical to the vacuum polarization form factors of $A$
\[
\langle \Omega | A \Omega \rangle_{p_1, \ldots, p_n}^{in} = a_n(\theta_1, \ldots, \theta_n)
\]
whereas the crossing of some of the particles into the left hand bra state (see the previous section) leads to the connected part of the formfactors

\[
\langle p_1, \ldots, p_l | A | p_0, \ldots, p_{n+1} \rangle_{\text{conn}}^{\text{out}} = a_n (\delta_1 + i \pi, \ldots, \delta_l + i \pi, \theta_{l+1}, \ldots, \theta_n) \quad (15)
\]

Hence the crossing property of formfactors are encoded into the notation of the operator formalism (13) in that there is only one analytic function \(a_n\) which describes the different possibilities of placing \(\theta\) on the upper or lower rim of \(C\). This is analogous to the LSZ expansion formulas of the interacting Heisenberg fields in terms of free fields in which the \(n^{\text{th}}\) term was also given in terms of different boundary values of one and the same retarded function.

In terms of the formfactors the relative commutant (12) results from restricting the series (13) by requiring that the \(\Lambda\)'s commute with the generators of the shifted algebra \(\mathcal{A}(W_c)\)

\[
[A, U(a)A(f)U(a)^*] = 0 \quad (16)
\]

Using the Z-F algebra relation this leads to a recursive equation for the coefficient function which relates the \(a_n\) coefficients whose \(n\) differs by 2. This is nothing else than the famous “kinematical pole condition” (the only restriction in the absence of bound states) first introduced as one of the construction recipes by Smirnov [15]. This together with the Paley-Wiener Schwartz analytic properties and the crossing property (which links the crossed formfactor to the analytic continuation between the two rims of the \(\theta\)-strip \(\mathbb{R} + i(0, \pi)\)) characterizes the space of formfactors associated with the algebra \(\mathcal{A}(D)\).

The multiplicative structure (i.e. the realization that the formfactors define not only spaces but are even algebras) is outside of mathematical control as long as one is unable to control the convergence of the infinite sums; in this respect the situation is at first sight not better than that of the old expansion formulas of Glaser, Lehmann, Symanzik and Zimmermann. The linear space of formfactors can be parametrized in terms of a covariant basis which corresponds to the formfactors of a basis of “would be” composite fields. It turns out that the dependence on the individual composite field in the Borchers class of relatively local fields can be encoded into a polynomial factor [33] after splitting of a common factor which is the same for all fields in the same class. This tells us that if we knew that those operator subalgebras characterized by the vanishing of the relative commutant (16) are nontrivial, then the associated quantum field theory exists and we have a nonperturbative formalism to compute formfactors of pointlike fields or of more general operators in \(\mathcal{A}(D)\). Since the formalism avoids correlation functions of pointlike fields, it is free of ultraviolet problems (and a fortiori does not require renormalizations of infinities).

There has been extensive work on the calculation of formfactors of composite fields. Similar to Wick polynomials there is a basis of (composite) fields in the same superselection sector. The formfactors of this family contain one factor which is common to all of them (the so-called minimal formfactor); this factor is associated with the core of the vacuum polarization cloud which is common to all states created with operators from the same spacetime region and which is
characteristic for the model. The remaining vacuum cloud factor is a polynomial and carries the information about the individual fields; this is analogous to the different Wick polynomials of free fields, polarization individual fields\(^1\). The polynomial factors actually complicate the calculation of correlation functions as convergent series over formfactors. In fact, apart from two-point functions in very special cases, this program has had no success despite serious attempts over many years.

The suggestion from the modular approach would be to look at formfactors of bounded operators in \(A(D)\) which would have better asymptotic behavior for large rapidities than pointlike fields. The best strategy for securing the existence of factorizable models would be to show the nontriviality of the relative commutant algebra \(A(D)\) obeying (16) by the powerful methods of operator algebras. Recently Buchholz and Lechner [21] proposed a very elegant criterion for the nontriviality of \(A(D)\) in terms of an operator algebraic property of the wedge algebra \(A(W)\). Lechner tested this criterion in the case of the Ising field theory [22] and there seems to be a well-founded hope that the already impressive calculational results of the bootstrap-formfactor program for factorizing models will be backed up by a structural argument of the existence of their local algebras without having to control the convergence of infinite sums over formfactors. The latter problem has been attempted by many authors but without success apart from some numerical estimates.

5 Going beyond factorizable models, the hypothesis of a Masterfield

For factorizable models, the crossing relation of the analytic coefficient functions in the series representation (13) is a consequence of the algebraic properties of the \(Z\)'s. Since there are no \(Z\)-\(F\) operators for models with non-factorizing \(S\)-matrices, one must look for a more general operational formulation of crossing. To obtain an idea in what direction to look for, let us first recall the precise conceptual position of factoring models within the general setting of massive models to which scattering theory applies.

As was mentioned in section 2, PFG with generating properties for wedge-localized algebras only exist for \(d=1+1\) theories with \(S\)-matrices which factorize into 2-particle contributions \(S^{(2)}\). This is a very peculiar situation in which cluster separability does not distinguish between the two contributions in \(S^{(2)} = 1 + T^{(2)}\).

So the crucial question is how can one get an operational formulation of crossing in formfactors\(^2\) beyond such special situations? We already dismissed the idea of interpreting crossing as KMS property in the same theory as in-

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1. This factor is different for bounded operators where one obtains a decrease for large momenta which may help in the control of the convergence in (13).

2. We always mean the connected part of the formfactors which is what one gets by starting with the outgoing components of the one field (or operator from a local algebra) state and crossing from outgoing bras to incoming kets.
correct. The only alternative idea which maintains a KMS interpretation of
crossing would consist in declaring simply the formfactors of an operator \( A \) to
be correlation functions in a KMS state at the Hawking-Unruh temperature \( 2\pi \)
of (nonlocal) operators \( R^{(A)} \) in rapidity momentum space (the auxiliary \( R \)'s
will be referred to as “Rindler operators”)

\[
\langle \theta_1, \ldots, \theta_n | A | \Omega \rangle \overset{\beta}{=} \left( R^{(A)}(\theta_1) \ldots R^{(A)}(\theta_n) \right)
\]  

(17)

But this idea only works if we find special operators \( A \) in the original theory
whose formfactors define a system of positive correlation functions so that the
GNS reconstruction would lead to an auxiliary operator field theory. Since
positivity can hardly be checked directly\(^\text{21}\), let us first look for a property which
is much easier to check namely the cluster separation property. It is known
that this property holds also in global operator algebras (i.e. without a local net
substructure) as long as the operator algebra happens to be a von Neumann
factor in which case it is related to the property of asymptotic abelieness [36][29].
In many factorizable models one was able to identify such fields with rapidity
space clustering [41] and in no model one was able to exclude this possibility.
We will formulate the following slightly stronger hypothesis about the existence
of a quantum field which we call a ”masterfield”

**Definition 1** A masterfield \( M(x) \) associated to a QFT is a distinguished scalar
Boson field within the Barchers class of locally equivalent fields whose connected
formfactors defines a thermal auxiliary “Rindler QFT” at the Hawking temperature
\( \beta = 2\pi \) in terms of a nonlocal field \( R(\theta, p_\perp) \) in the sense of the above
formula (17) with \( A \) being the masterfield \( M(x) \) at \( x=0 \).

The KMS relation in \( \theta \) reads

\[
\left( R^{(M)}(\theta_1, p_{1\perp}) \ldots R^{(M)}(\theta_{n-1}, p_{n-1\perp}) R^{(M)}(\theta_n, p_{n\perp}) \right)_{\beta=2\pi} = \left( R^{(M)}(\theta_n - 2\pi i, p_{n\perp}) R^{(M)}(\theta_1, p_{1\perp}) \ldots R^{(M)}(\theta_{n-1}, p_{n-1\perp}) \right)_{\beta=2\pi} \\
= (R^{(M)}(\theta_n - 2\pi i, p_{n\perp}) \Omega_{\beta=2\pi}) R^{(M)}(\theta_1, p_{1\perp}) \ldots R^{(M)}(\theta_{n-1}, p_{n-1\perp}) \Omega_{\beta=2\pi} = [J R^{(M)}(\theta_n - \pi i, p_{n\perp}) \Omega_{\beta=2\pi}) R^{(M)}(\theta_1, p_{1\perp}) \ldots R^{(M)}(\theta_{n-1}, p_{n-1\perp}) \Omega_{\beta=2\pi} \\
= (J R^{(M)}(\theta_n - \pi i, p_{n\perp}) \Omega_{\beta=2\pi}) R^{(M)}(\theta_1, p_{1\perp}) R^{(M)}(\theta_1, p_{1\perp}) \ldots R^{(M)}(\theta_{n-1}, p_{n-1\perp}) \Omega_{\beta=2\pi} \\
\]

where in the last two lines we used the more convenient state vector notation
for the thermal expectation values and in the last line we used modular theory
in order to convert \( \Delta^\beta \) into \( J \). The identification of this expression with the
crossing property of the formfactor of \( M(0) \)

\[
\langle 0 | M(0) | p_1, \ldots, p_n \rangle = (\Omega_n | M(0) | p_1, \ldots, p_{n-1}) \\
= (J R^{(M)}(\theta_n - \pi i, p_{n\perp}) \Omega_{\beta=2\pi}) R^{(M)}(\theta_1, p_{1\perp}) \ldots R^{(M)}(\theta_{n-1}, p_{n-1\perp}) \Omega_{\beta=2\pi} \\
\]

\(^\text{21}\)I am not aware of nontrivial model whose existence was established by just looking at
correlation without referring to a Hilbert space and (auxiliary) operators.
requires the action of the auxiliary $J$ as $J R^{(M)}(\theta_n - \pi i, p_{n\perp}) \Omega_{\beta=\pm \pi} = R^{(M)}(\theta_n - 2\pi i, -p_{n\perp})^* \Omega_{\beta=\pm \pi}$.

In $d=1+1$ the interpretation of crossing in terms of KMS of an auxiliary theory simplifies since there is no transverse momentum $p_{\perp}$.

It is important to notice that the auxiliary field theory associated with the formfactors of the master field is not subject to the restriction of wedge-localized PFGs which led to factorizable models. In fact being a global (i.e. without a local net structure) KMS theory, the concept of particles and in particular the concept of PFG becomes meaningless.

Let us first look at the rather trivial illustration of a free master field namely

$$M(x) \equiv e^{\gamma A(x)}, \quad A(x) = \text{free field}$$

$$\langle M(0)|p(\theta_1, p_{1\perp})...p(\theta_n, p_{n\perp})\rangle = e^{\gamma}$$

where the positive constant $\gamma$ is related to the vacuum-one particle normalization of $A$. Clearly among all compositions of the free field which lead to $\theta$-independent connected formfactors the only case with the correct combinatorics complying with clustering is the above exponential field. The auxiliary algebra of $R^{(M)}$ is the trivial abelian algebra which permits states for every KMS temperature. The free field case is the only model in which the formfactors of the master field define an abelian auxiliary theory; a nontrivial $S$-matrix prevents abelianness.

The master field hypothesis remains nontrivial even in the setting of factorizable models. In the following we use two quite different models to illustrate its working. We first recall some formalism of KMS states on free fields.

For bosonic quasifree KMS states at the KMS temperature $\beta$ one obtains

$$\langle c(q)c(q')\rangle_\beta = e^{\beta q} \left\{ \langle c(q)c(q')\rangle_\beta - [c(q), c(q')] \right\}$$

$$\gamma \langle c(q)c(q')\rangle_\beta = \frac{e^{\beta q}}{e^{\beta q} - 1} \delta(q + q')$$

For the first illustration we take the Sinh-Gordon theory. The field which leads to formfactors which have the cluster factorization property in the rapidity variable is again an exponential $M(x) = e^{\varphi}$ operator in terms of the basic Sinh-Gordon field $\varphi$ [41]. They have the following structure

$$(\Omega | M(0)|| p(\theta_1) ... p(\theta_n)) = K_n(\theta) \prod_{i<j} F(\theta_{ij})$$

$$K_n(\theta) = \sum_{l_i=0,1} ... \sum_{l_n=0,1} (-1)^{l_1} \prod_{i<j} (1 + (l_i - l_j) \frac{i \sin \pi \nu}{\sinh \theta_{ij}}) \prod_k C e^{i \varphi / 2 (-1)^{l_k}}$$

where the coupling strength $\beta$ and $\nu$ are related by $\frac{1}{\nu} = \frac{8 \pi}{\beta}$. The product factor involving the 2-particle formfactor $F$ has the combinatorics of an exponential which is bilinear in $c(q)$ free Boson operators. This suggests to start from the
complex exponential

\[ C(\theta) = e^{i\alpha(\theta)} \]

\[ a(\theta) = \int dq u(q) c(q)e^{i\alpha(\theta - i\pi/2)} \]

and look for a Rindler operator \( R^{22} \) as a Hermitian combination of the form

\[ R(\theta) = N \left\{ e^{i\alpha(\theta - i\pi/2)} + h.a. \right\} \]

\[ C(\theta)C(\theta') = S(\theta - \theta')C(\theta') \]

\[ S(\theta) = \exp \int_0^{\infty} dq \hat{f}(q) \sinh \frac{\theta}{i\pi} \]

\[ \langle C(\theta - i\pi/2)C(\theta' - i\pi/2) \rangle_{2\pi} = \exp \int_0^{\infty} dq \hat{f}(q) \frac{1 - \hat{c} \hat{a}(\pi + i(\theta - \theta'))}{2\sinh \pi q} \]

\[ \equiv F_{\text{min}}(\theta - \theta') \]

The function \( F_{\text{min}}(\theta) \) is the so-called minimal 2-particle formfactor of the model i.e. the unique function which obeys \( F(\theta) = S(\theta)F(-\theta) \) and is holomorphic in the closed strip. In the last line in (24) we used the fact that the KMS state at the inverse temperature \( 2\pi \) fixes the quasi-free state on the Rindler creation/annihilation operator algebra which in turn determines the thermal expectations of the \( C \)-operators.

The Sinh-Gordon S-matrix

\[ S_{sh}(\theta) = \frac{\sinh \frac{\theta}{2}}{\sinh \frac{\theta}{2} + ik}, \quad \kappa = \frac{\pi \beta^2}{8\pi + \beta^2} = \pi B \leq 1 \]  

(25)

fixes the quasifree commutation relation of the Rindler operators \( R(\theta) \) with

\[ f(q) = \frac{2\sinh \frac{\pi}{2} \sinh \frac{\pi}{2} \sinh \frac{\pi}{2}}{q \cosh \frac{\pi}{2}} = \frac{2\sinh \frac{\pi}{2} \sinh \frac{\pi}{2} (2 - B)}{q \cosh \frac{\pi}{2}} \]

(26)

The n-point function

\[ \langle C(\theta_1 - i\pi/2) \ldots C(\theta_n - i\pi/2) \rangle_{2\pi} \sim \prod_{i<k} F_{\text{min}}(\theta_{ik}) \]

(27)

fulfills the commutation relation of the \( R \)-algebra (which is identical to that of the \( C \)-algebra as well as the KMS condition. Our interest lies in the Hermitian field operator \( R \). For convenience we have adjusted our notation to the resulting combinatorics for the thermal \( Z \)-expectation which are sums of terms with different \( C_l(\theta) := C(\theta - il\pi/2) \), \( l = \pm \)

\[ \langle C_{l_1} \ldots C_{l_n} \rangle_{2\pi} = \prod_{i<k} F_{\text{min}}(\theta_{ik}) \left\{ 1 - (l_k - l_i) \frac{\text{isink}}{\sin k} \right\}, \quad l_i = \pm 1 \]

(28)

\[ \langle R_{\theta_1} \ldots R_{\theta_1} \rangle_{2\pi} \sim \sum_{C_{l_1} \ldots C_{l_n}} \langle C_{l_1} \ldots C_{l_n} \rangle_{2\pi} \exp i\pi a(l_1 + l_2 + \ldots l_n) \]

\[ \text{We use the letter } Z \text{ for the particle physics representation of the Zamolodchikov algebra (the Minkowski spacetime operators which are related to the FFG wedge generators) whereas } R \text{ is used for the thermal Rindler representation.} \]
where \( \alpha \) depends on the numerical prefactors.

With this construction of an auxiliary global Rindler QFT for the formfactors of the masterfield we have reproduced a curious observation by Lukyanov [27] which is known in the literature on factorizing models as “free field representations” (for a recent account see also [28]). Besides the fact that some of our physical concepts and mathematical formalism is somewhat different from that of Lukyanov, the actual calculation is practically identical. The thermal states from a Rindler localization setting turn out to be KMS states which lack a natural thermodynamic limit interpretation in terms of tracial Gibbs states in the heat bath setting. Unique KMS states on operator algebras lead to von Neumann factors which in turn fulfill a weak cluster property [29] and it was the cyclicity of crossing together with the mysterious cluster properties in the rapidity variables (about which I learned first from the work of Babujian and Karowski) which led me to this problem of an operator interpretation of the crossing property.

Since fields whose formfactors cluster have been found in many similar factorizing models of Toda type \( ^{23} \), one would expect that the idea of an auxiliary Rindler theory works in all of them. Moreover it would be tempting to conjecture that the simplifying feature of factorizing models consists in the auxiliary formfactor theory being bilinear exponential in \( e^q \) creation/annihilation operators. This conjecture draws also support from a recent observation by Babujian and Karowski who observed that a suitably generalized form of clustering for also holds in statistics changing \( Z_n \)-models [42] of which the lowest one is the Ising field theory. In that case a combination of disorder/order field formfactors leads to clustering [24].

In the following we briefly show that in the Ising model a “twisted” KMS state at the temperature \( \beta = \pi \) does the job \( ^{24} \). The twisting consists in changing the KMS formula by a -sign.

\[
\langle c(q) c(q') \rangle_\beta = - e^{\beta q} \left\{ \langle c(q) c(q') \rangle_\beta - [c(q), c(q')] \right\}
\]

\[
\langle c(q) c(q') \rangle_\beta = \frac{e^{\beta q}}{1 + e^{\beta q}} e(q) \delta(q + q')
\]

\[
\langle c(q) c(q') \rangle_\pi = \frac{e^{\pi q}}{2 \cosh \frac{\pi q}{2}} e(q) \delta(q + q')
\]

where the third line contains the KMS two-point function at \( \beta = \pi \) which we are going to use together with the following definition of \( a(\theta) \)

\[
a(\theta) = \int c(q) dq
\]

\[
\langle a(\theta) a(\theta') \rangle_{\beta = \pi} = \int_{0}^{\infty} \frac{\sinh q(i\theta - i\theta' + \frac{\pi}{2})}{\cosh \frac{\pi q}{2}} \frac{dq}{q} = \ln \tanh \frac{\theta - \theta'}{2}
\]

\(^{23}\)In [42] it was shown that distinguished fields with clustering formfactors exist for all \( A_{n-1} \) affine Toda field theories of which the Sinh-Gordon is the lowest member.

\(^{24}\)The masterfield hypothesis should not be considered as an axiom but rather as a suggestion which may still require flexibility in adapting to concrete situations.
which finally leads to the well-known disorder/order Ising formfactors which is given by a combinatorial expression in the two-particle formfactor of the disorder operator (which correspond to an even number of particles)
\[
\left\langle e^{\theta(x)}e^{\theta(x')}\right\rangle_\tau = \tanh \frac{\theta - \theta'}{2}
\]

(31)

As the Sine-Gordon model is the simplest representative of the class of \(A_{n-1}\) affine Toda models [42], the Ising field theory is the beginning of the family of \(Z_n\) models. These models are more difficult as a consequence of their preferred \(Z_n\) braid group statistics and a candidate for a masterfield is not immediately visible. The suggestion from the Ising case would be that a suitable combination of all disorder/order operators would be a candidate for a field which fulfills some generalized clustering (i.e. adjusted to the exotic statistics).

Outside factorizing models one can only hope for a perturbative treatment in which the formfactors of the masterfield must be determined recursively together with the S-matrix which controls the algebraic properties. To find some guidance, it may be helpful to investigate whether the masterfield formfactors of factorizing models can be constructed in such a perturbative manner.

6 Concluding remarks

The present masterfield hypothesis leaves one with the problem of what to expect in non-factorizing models and in particular in higher spacetime dimensions. In that case one cannot expect a non-perturbative description in terms of KMS states on nonlocal exponential expressions in terms of free creation/annihilation operators but one could hope for a finite perturbative approach. Different from the old bootstrap, the masterfield hypothesis has converted the elusive crossing property into the KMS property well-known in operator algebras. Such an approach must reproduce the on-shell results of renormalized Lagrangian perturbation theory\(^{35}\). The fact that we had to distinguish a masterfield in order to be able to formulate an operational hypothesis about crossing does not mean that there is no KMS operator interpretation of crossing properties formfactors for other fields. If the formfactors of other fields do allow an auxiliary KMS description, it will be conceptually more complicated.

It should be mentioned that the masterfield hypothesis is only one of several ideas which aim at making some constructive headway beyond Lagrangian quantization methods. Another potentially more important idea which is not directly related to particles and scattering theory consists in studying QFT through its appropriately formulated (in the setting of algebraic QFT) holographic lightfront projection [43]. As a global object the algebra associated with the upper lightfront horizon of a wedge is equal to the wedge algebra. Since the symmetry group of the lightfront algebra is a 7-parametric subgroup of the Poincaré group

\[^{35}\text{From the formfactors of the masterfield one can in particular extract the S-matrix for certain scattering processes.}\]
one only needs to know the action of the unitaries which represent the remaining
3 Poincaré transformations in order to generate the full net from the holographi-
cal lightfront projection of one fixed wedge algebra. The holographic lightfront
projections turn out to be described by a chiral conformal theories with a rather
simple transverse extension [43][44]. These are algebras about which one knows
an impressive amount. Whereas this together with the Poincaré group action
may not be sufficient in order to start explicit calculations of intersections of
wedge algebras, it could well be a good starting point for a Buchholz-Lechner
type of argument which secures the nontriviality of intersections and in this way
insures the existence of a QFT model.

Since the invention of new problems has done little for the progress of particle
physics, a return to the deep unsolved QFT problems of the past, which certainly
includes the crossing property, combined with the new modular localization
ideas may be our best investment into the future of particle physics.

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