Positive Energy Unitary Irreducible Representations 
of the Superalgebras $\text{osp}(1-2n,R)$

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Positive Energy Unitary Irreducible Representations of the

Superalgebras $osp(1|2n, \mathbb{R})$

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We give the classification of the positive energy (lowest weight) unitary irreducible representations of the superalgebras $osp(1|2n, \mathbb{R})$.

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I. INTRODUCTION

Recently, superconformal field theories in various dimensions are attracting more interest, in particular, due to their duality to AdS supergravities, cf. [1-58] and references therein. Until recently only those for $D \leq 6$ were studied since in these cases the relevant superconformal algebras satisfy [59] the Haag-Lopuszanski-Sohnius theorem [60]. Thus, such classification was known only for the $D = 4$ superconformal algebras $su(2, 2/N)$ [61] (for $N = 1$), [62-65] (for arbitrary $N$). More recently, the classification for $D = 3$ (for even $N$), $D = 5$, and $D = 6$ (for $N = 1, 2$) was given in [66] (some results are conjectural), and then the $D = 6$ case (for arbitrary $N$) was finalized in [67].

On the other hand the applications in string theory require the knowledge of the UIRs of the conformal superalgebras for $D > 6$. Most prominent role play the superalgebras

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$osp(1|2n)$, cf. their applications in, e.g., [68–77]. Initially, the superalgebra $osp(1|32)$ was put forward for $D = 10$ [68]. Later it was realized that $osp(1|2n)$ would fit any dimension, though they are minimal only for $D = 3, 9, 10, 11$ (for $n = 2, 16, 16, 32$, resp.) [74]. In all cases we need to find first the UIRs of $osp(1|2n, R)$. This can be done for general $n$. Thus, in this paper we treat the UIRs of $osp(1|2n, R)$ only while the implications for conformal supersymmetry for $D = 9, 10, 11$ shall be treated in a follow-up paper.

II. REPRESENTATIONS OF THE SUPERALGEBRAS $osp(1|2n)$ AND $osp(1|2n, R)$

A. The setting

Our basic references for Lie superalgebras are [78, 79]. The conformal superalgebras in $D = 9, 10, 11$ are $\mathcal{G} = osp(1|2n, R)$, $n = 16, 16, 32$, resp., cf. [68, 74]. The even subalgebra of $osp(1|2n, R)$ is the algebra $sp(2n, R)$ with maximal compact subalgebra $\mathcal{K} = u(n) \cong su(n) \oplus u(1)$. The algebra $sp(2n, R)$ contains the conformal algebra $\mathcal{C} = so(D, 2)$, while $\mathcal{K}$ contains the maximal compact subalgebra $so(D) \oplus so(2)$ of $\mathcal{C}$, $so(2)$ being identified with the $u(1)$ factor of $\mathcal{K}$.

We label the relevant representations of $\mathcal{G}$ by the signature:

$$\chi = [d; a_1, \ldots, a_{n-1}]$$

where $d$ is the conformal weight, and $a_1, \ldots, a_{n-1}$ are non-negative integers which are Dynkin labels of the finite-dimensional UIRs of the subalgebra $su(n)$ (the simple part of $\mathcal{K}$).

Our aim is to classify the UIRs of $\mathcal{G}$ following the methods used for the $D = 4, 6$ conformal superalgebras, cf. [62–65],[67], resp. The main tool is an adaptation of the Shapovalov form on the Verma modules $V^\chi$ over the complexification $\mathcal{G}^c = osp(1|2n)$ of $\mathcal{G}$.

B. Verma modules

To introduce Verma modules we use the standard triangular decomposition:

$$\mathcal{G}^c = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^-$$

(2)
where $\mathcal{G}^+$, $\mathcal{G}^-$, resp., are the subalgebras corresponding to the positive, negative, roots, resp., and $\mathcal{H}$ denotes the Cartan subalgebra.

We consider lowest weight Verma modules, so that $V^\Lambda \cong U(\mathcal{G}^+) \otimes v_0$, where $U(\mathcal{G}^+)$ is the universal enveloping algebra of $\mathcal{G}^+$, and $v_0$ is a lowest weight vector $v_0$ such that:

$$Z\ v_0 = 0, \quad Z \in \mathcal{G}^-$$

$$H\ v_0 = \Lambda(H)\ v_0, \quad H \in \mathcal{H}. \quad (3)$$

Further, for simplicity we omit the sign $\otimes$, i.e., we write $p\ v_0 \in V^\Lambda$ with $p \in U(\mathcal{G}^+)$. The lowest weight $\Lambda$ is characterized by its values on the simple roots of the superalgebra. In the next subsection we describe the root system.

C. Root systems

We recall some facts about $\mathcal{G}^\theta = osp(1|2n)$ (denoted $B(0,n)$ in [78]). Their root systems are given in terms of $\delta_1, \ldots, \delta_n$, $(\delta_i, \delta_j) = \delta_{ij}$, $i, j = 1, \ldots, n$. The even and odd roots systems are [78]:

$$\Delta_\text{even} = \{ \pm \delta_i \pm \delta_j, 1 \leq i < j \leq n, \pm 2\delta_i, 1 \leq i \leq n \}, \quad \Delta_\text{odd} = \{ \pm \delta_i, 1 \leq i \leq n \} \quad (4)$$

(we remind that the signs $\pm$ are not correlated). We shall use the following distinguished simple root system [78]:

$$\Pi = \{ \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n \}, \quad (5)$$

or introducing standard notation for the simple roots:

$$\Pi = \{ \alpha_1, \ldots, \alpha_n \}, \quad (6)$$

$$\alpha_j = \delta_j - \delta_{j+1}, \quad j = 1, \ldots, n - 1, \quad \alpha_n = \delta_n.$$

The root $\alpha_n = \delta_n$ is odd, the other simple roots are even. The Dynkin diagram is:

$$\begin{array}{c}
\circ \\
1 \\
\cdots \\
\circ \\
n-1 \\
\Rightarrow \\
n
\end{array} \quad (7)$$

The black dot is used to signify that the simple odd root is not nilpotent, otherwise a gray dot would be used [78]. In fact, the superalgebras $B(0,n) = osp(1|2n)$ have no nilpotent generators unlike all other types of basic classical Lie superalgebras [78].
The corresponding to \( \Pi \) positive root system is:

\[
\Delta_0^+ = \{ \delta_i \pm \delta_j \mid 1 \leq i < j \leq n \}, \quad \Delta_1^+ = \{ \delta_i \mid 1 \leq i \leq n \} \quad (8)
\]

We record how the elementary functionals are expressed through the simple roots:

\[
\delta_k = \alpha_k + \cdots + \alpha_n . \quad (9)
\]

The even root system \( \Delta_0^+ \) is the root system of the rank \( n \) complex simple Lie algebra \( sp(2n) \), with \( \Delta_0^+ \) being its positive roots. The simple roots are:

\[
\Pi_0 = \{ \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, 2\delta_n \} = \{ \alpha_1^0, \ldots, \alpha_n^0 \} , \quad (10)
\]

\[
\alpha_j^0 = \delta_j - \delta_{j+1} , \quad j = 1, \ldots, n-1 , \quad \alpha_n^0 = 2\delta_n .
\]

The Dynkin diagram is:

\[
\circ_1 \cdots \circ_{n-1} \leftrightarrow \circ_n \quad (11)
\]

The superalgebra \( \mathcal{G} = osp(1|2n, \mathbb{R}) \) is a split real form of \( osp(1|2n) \) and has the same root system.

**D. Lowest weight through the signature**

Since we use a Dynkin labelling, we have the following relation with the signature \( \chi \) from (1):

\[
(\Lambda, \alpha_k^\vee) = -a_k , \quad k < n ,
\]

\[
\tilde{d} , \quad k = n ,
\]

\[
(\Lambda, \alpha_k^\vee) = -a_k , \quad k < n ,
\]

\[
\tilde{d} , \quad k = n ,
\]

\[
(\Lambda, \alpha_k^\vee) = -a_k , \quad k < n ,
\]

\[
\tilde{d} , \quad k = n ,
\]

where \( \alpha_k^\vee \equiv 2\alpha_k/(\alpha_k, \alpha_k) \), and \( \tilde{d} \) differs from the conformal weight \( d \) as explained below. The minus signs in the first row are related to the fact that we work with lowest weight Verma modules (instead of the highest weight modules used in [79]) and to Verma module reducibility w.r.t. the roots \( \alpha_k \) (this is explained in detail in [64]). The value of \( \tilde{d} \) is a matter of normalization so as to correspond to some known cases. Thus, our choice is:

\[
\tilde{d} = 2d + a_1 + \cdots + a_{n-1} .
\]

\[
(\Lambda, \alpha_k^\vee) = -a_k , \quad k < n ,
\]

\[
\tilde{d} , \quad k = n ,
\]
Having in hand the values of \( \Lambda \) on the basis we can recover them for any element of \( \mathcal{H}^* \). In particular, for the values on the elementary functionals we have using (9), (12) and (13):

\[
(\Lambda, \delta_j) = d + \frac{1}{2}(a_1 + \cdots + a_{j-1} - a_j - \cdots - a_{n-1}) .
\]

(14)

Using (12) and (13) one can write easily \( \Lambda = \Lambda(\chi) \) as a linear combination of the simple roots or of the elementary functionals \( \delta_j \), but this is not necessary in what follows. We shall need only \( (\Lambda, \beta') \) for all positive roots \( \beta \) and from (14) we have:

\[
(\Lambda, (\delta_i - \delta_j)') = (\Lambda, \delta_i - \delta_j) = -a_i - \cdots - a_{j-1}
\]

\[
(\Lambda, (\delta_i + \delta_j)') = (\Lambda, \delta_i + \delta_j) = 2d + a_1 + \cdots + a_{i-1} - a_j - \cdots - a_{n-1}
\]

\[
(\Lambda, \delta_i') = (\Lambda, 2\delta_i) = 2d + a_1 + \cdots + a_{i-1} - a_i - \cdots - a_{n-1}
\]

\[
(\Lambda, (2\delta_i)') = (\Lambda, \delta_i) = d + \frac{1}{2}(a_1 + \cdots + a_{i-1} - a_i - \cdots - a_{n-1})
\]

(15)

E. Reducibility of Verma modules

Having established the relation between \( \chi \) and \( \Lambda \) we turn our attention to the question of reducibility. A Verma module \( V^\Lambda \) is reducible w.r.t. the positive root \( \beta \) iff the following holds [79]:

\[
(\rho - \Lambda, \beta') = m_\beta , \quad \beta \in \Delta^+ , \quad m_\beta \in \mathbb{N} ,
\]

(16)

where \( \rho \in \mathcal{H}^* \) is the very important in representation theory element given by the difference of the half-sums \( \rho_\delta, \rho_\pi \) of the even, odd, resp., positive roots (cf. (8):

\[
\rho \triangleq \rho_\delta - \rho_\pi = (n - \frac{1}{2})\delta_1 + (n - \frac{3}{2})\delta_2 + \cdots + \frac{3}{2}\delta_{n-1} + \frac{1}{2}\delta_n ,
\]

(17)

\[
\rho_\delta = n\delta_1 + (n - 1)\delta_2 + \cdots + 2\delta_{n-1} + \delta_n ,
\]

\[
\rho_\pi = \frac{1}{2}(\delta_1 + \cdots + \delta_n) .
\]

To make (16) explicit we need first the values of \( \rho \) on the positive odd roots:

\[
(\rho, \delta_i) = n - i + \frac{1}{2} .
\]

(18)
Then for \((\rho, \beta^\vee)\) we have:

\[
\begin{align*}
(\rho, (\delta_i - \delta_j)^\vee) &= j - i , \\
(\rho, (\delta_i + \delta_j)^\vee) &= 2n - i - j + 1 , \\
(\rho, \delta_i^\vee) &= 2n - 2i + 1 , \\
(\rho, (2\delta_i)^\vee) &= n - i + \frac{1}{2} .
\end{align*}
\] (19)

Naturally, the value of \(\rho\) on the simple roots is 1: \((\rho, \alpha_i^\vee) = 1, \ i = 1, \ldots, n.\)

Consecutively we find that the Verma module \(V^{\Lambda(\alpha)}\) is reducible if one of the following relations holds (following the order of (15) and (19):

\[
\begin{align*}
IN \ni m_{ij}^- &= j - i + a_i + \cdots + a_{j-1} , \quad (20a) \\
IN \ni m_{ij}^+ &= 2n - i - j + 1 + a_i + \cdots + a_{n-1} - a_i - \cdots - a_{i-1} - 2d , \quad (20b) \\
IN \ni m_i &= 2n - 2i + 1 + a_i + \cdots + a_{n-1} - a_i + \cdots - a_{i-1} - 2d , \quad (20c) \\
IN \ni m_{ii}^- &= n - i + \frac{1}{2}(1 + a_i + \cdots + a_{n-1} - a_i + \cdots - a_{i-1}) - d . \quad (20d)
\end{align*}
\]

Note that \(m_i = 2m_{ii}\), thus, whenever (20d) is fulfilled also (20c) is fulfilled.

If a condition from (20) is fulfilled then \(V^\Lambda\) contains a submodule which is a Verma module \(V^{\Lambda'}\) with shifted weight given by the pair \(m, \beta\): \(\Lambda' = \Lambda + m\beta\). The embedding of \(V^{\Lambda'}\) in \(V^\Lambda\) is provided by mapping the lowest weight vector \(v'_0\) of \(V^{\Lambda'}\) to the singular vector \(v^{m,\beta}_s\) in \(V^\Lambda\) which is completely determined by the conditions:

\[
\begin{align*}
X v^{m,\beta}_s &= 0 , \quad X \in \mathcal{G}^- , \\
H v^{m,\beta}_s &= \Lambda'(H) v_0 , \quad H \in \mathcal{H} , \quad \Lambda' = \Lambda + m\beta .
\end{align*}
\] (21)

Explicitly, \(v^{m,\beta}_s\) is given by a polynomial in the positive root generators:

\[
v^{m,\beta}_s = P^{m,\beta} v_0 , \quad P^{m,\beta} \in U(\mathcal{G}^+) .
\] (22)

Thus, the submodule of \(V^\Lambda\) which is isomorphic to \(V^{\Lambda'}\) is given by \(U(\mathcal{G}^+) P^{m,\beta} v_0\).

Here we should note that we may eliminate the reducibilities and embeddings related to the roots \(2\delta_i\). Indeed, let (20d) hold, then the corresponding singular vector \(v^{m_{ii},2\delta_i}_s\) has the properties prescribed by (21) with \(\Lambda' = \Lambda + m_{ii} 2\delta_i\). But as we mentioned above in this situation also (20c) holds and the corresponding singular vector \(v^{m_i,\delta_i}_s\) has the properties prescribed by (21) with \(\Lambda'' = \Lambda + m_i \delta_i\). But due to the fact that \(m_i = 2m_{ii}\) it is clear
that $\Lambda'' = \Lambda'$, which means that the singular vectors $v^{m,2\delta_i}_s$ and $v^{m,\delta_i}_s$ coincide (up to nonzero multiplicative constant). On the other hand if (20c) holds with $m_i$ being an odd number, then (20d) does not hold (since $m''_i = m_i/2$ is not integer).

Further, we notice that all reducibility conditions in (20a) are fulfilled. In particular, for the simple roots from those condition (20a) is fulfilled with $\beta \rightarrow \alpha_i = \delta_i - \delta_{i+1}, i = 1, \ldots, n-1$ and $m^-_i = m^-_{i+1} = 1 + a_i$. The corresponding submodules $I^\Lambda_i = U(G^+) v^i_0$, where $\Lambda_i = \Lambda + m^-_i \alpha_i$ and $v^i_0 = (X^+_i)^{1+a_i} v_0$, where $X^+_i$ are the root vectors of the these simple roots. These submodules generate an invariant submodule which we denote by $I^\Lambda_i$. Since these submodules are nontrivial for all our signatures instead of $V^\Lambda$, we shall consider the factor-modules:

$$F^\Lambda = V^\Lambda / I^\Lambda_i.$$  

(23)

We shall denote the lowest weight vector of $F^\Lambda$ by $|\Lambda\rangle$ and the singular vectors above become null conditions in $F^\Lambda$:

$$(X^+_i)^{1+a_i} |\Lambda\rangle = 0, \quad i = 1, \ldots, n-1.$$  

(24)

If the Verma module $V^\Lambda$ is not reducible w.r.t. the other roots, i.e., (20b,c,d) are not fulfilled, then $F^\Lambda$ is irreducible and is isomorphic to the irrep $I^\Lambda$ with this weight.

Other situations shall be discussed below in the context of unitarity.

**F. Realization of $osp(1|2n)$ and $osp(1|2n,\bar{R})$**

The superalgebras $osp(m|2n) = osp(m|2n)_{\bar{5}} + osp(m|2n)_{\bar{1}}$ are defined as follows [78]:

$$osp(m|2n)_s = \{ X \in gl(m/2n; \mathcal{F})_s : X W + \ i^s W \ i^s X = 0 \}, \quad s = \bar{5}, \bar{1},$$

where $W$ is a matrix of order $m + 2n$:

$$W = \begin{pmatrix} iI_m & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{pmatrix}$$
The even part $osp(m|2n)_5$ consists of matrices $X$ such that:

$$X = \begin{pmatrix}
S & 0 & 0 \\
0 & B & C \\
0 & D & \bar{t}B
\end{pmatrix}$$

\(\bar{t}S = -S, \quad \bar{t}C = C, \quad \bar{t}D = D.\)

In our case $m = 1$ and $S = 0$, the Cartan subalgebra $\mathcal{H}$ consists of diagonal matrices $H$ such that:

$$H = \begin{pmatrix}
0 & 0 & 0 \\
0 & B & 0 \\
0 & 0 & -B
\end{pmatrix}$$

We take the following basis for the Cartan subalgebra:

$$H_i = \begin{pmatrix}
0 & 0 & 0 \\
0 & B_i & 0 \\
0 & 0 & -B_i
\end{pmatrix}, \quad i < n,$n, \quad i = n,$

$$H_n = \begin{pmatrix}
0 & I_n & 0 \\
0 & 0 & 0 \\
0 & 0 & -I_n
\end{pmatrix}$$

where

$$B_i = \text{diag}(0, \ldots, 0, 1, -1, 0, \ldots, 0)$$

the first non-zero entry being on the $i$-th place. This basis shall be used also for the real form $osp(1|2n, \mathbb{R})$ and is chosen to be consistent with the fact that the even subalgebra $sp(2n, \mathbb{R})$ of the latter has as maximal noncompact subalgebra the algebra $sl(n, \mathbb{R}) \oplus \mathbb{R}$.

Via the Weyl unitary trick this related to the structure of $sp(2n, \mathbb{R})$ as a hermitean symmetric space with maximal compact subalgebra $u(n) \cong su(n) \oplus u(1)$.

The root vectors of the roots $\delta_i - \delta_j, \ (i \neq j), \ \delta_i + \delta_j, \ (i \leq j), \ -(\delta_i + \delta_j), \ (i \leq j)$, respectively, are denoted $X_{ij}, \ X_{ij}^+, \ X_{ij}^-$, respectively. The latter are given by matrices of the type (25) with $S = 0$, given (up to multiplicative normalization) by $B = E_{ij}, \ C = E_{ij} + E_{ji}, \ D = E_{ij} + E_{ji}$, respectively, where $E_{ij}$ is $n \times n$ matrix which has only one non-zero entry equal to 1 on the intersection of the $i$-th row and $j$-th column. Explicitly,
(including some choice of normalization), this is:

\[
X_{ij} = \begin{pmatrix}
0 & 0 & 0 \\
0 & E_{ij} & 0 \\
0 & 0 & -E_{ji}
\end{pmatrix}, \quad i \neq j,
\]

\[
X_{ij}^+ = \begin{pmatrix}
0 & 0 & 0 \\
0 & -E_{ij} - E_{ji} & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad i < j, \quad X_{ii}^+ = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -E_{ii} \\
0 & 0 & 0
\end{pmatrix}
\]

\[
X_{ij}^- = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & E_{ij} + E_{ji} & 0
\end{pmatrix}, \quad i < j, \quad X_{ii}^- = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & E_{ii}
\end{pmatrix}
\]

The odd part \(osp(m|2n)_\Omega\) consists of matrices \(X\) such that:

\[
X = \begin{pmatrix}
0 & \xi & -\eta \\
\tau\xi & 0 & 0 \\
\tau\eta & 0 & 0
\end{pmatrix}
\]

The root vectors \(Y_i^+, \ Y_i^-\), of the roots \(\delta_i, -\delta_i\) correspond to \(\eta, \xi\), resp., with only non-zero \(i\)-th entry. Explicitly this is:

\[
Y_i^+ = \begin{pmatrix}
0 & 0 & -E_{1i} \\
E_{i1} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
Y_i^- = \begin{pmatrix}
0 & E_{1i} & 0 \\
0 & 0 & 0 \\
E_{i1} & 0 & 0
\end{pmatrix}
\]

In the calculations we need all commutators of the kind \([X_\beta, X_{-\beta}] = H_\beta, \ \beta \in \Delta_0^+\).
Explicitly, we have:

\[
[X_{ij}, X_{ji}] = H_{ij} = H_i + H_{i+1} + \cdots + H_{j-1}, \quad 1 \leq i < j \leq n ,
\]

\[
[Y_i^+, Y_i^-]_+ = H'_i \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_{ii} & 0 \\ 0 & 0 & -E_{ii} \end{pmatrix}, \quad 1 \leq i \leq n ,
\]

\[
[X_{ij}^+, X_{ij}^-] = H''_{ij} = -H'_i - H'_j, \quad 1 \leq i < j \leq n ,
\]

\[
[X_{ii}^+, X_{ii}^-] = -H'_i, \quad 1 \leq i \leq n .
\]

The minus sign in (29d) is consistent with the relations:

\[
\frac{1}{2}[Y_i^+, Y_i^-]_+ = (Y_i^+)^2 = X_{ii}^+ .
\]

We note also the following relations:

\[
[Y_i^+, Y_j^-]_+ = X_{ij}, \quad i \neq j ,
\]

\[
[Y_i^+, Y_j^+]_+ = X_{ij}^+, \quad i \neq j ,
\]

\[
H_n = H'_1 + \cdots + H'_n.
\]

We shall use also the abstract defining relations of \(osp(1|2n)\) through the Chevalley basis. Let \(\hat{H}_i, i = 1, \ldots, n\), be the basis of the Cartan subalgebra \(\mathcal{H}\) associated with the simple roots, and \(X_i^+, i = 1, \ldots, n\), be the simple root vectors (the Chevalley generators). The connection with the basis above is:

\[
\hat{H}_i = H_i , \quad i < n , \quad \hat{H}_n = H'_n ,
\]

\[
X_i^+ = X_{i,i+1}^+ , \quad i < n , \quad X_n^+ = Y_n^+ .
\]

Let \(A = (a_{ij})\) be the Cartan matrix [78]:

\[
A = (a_{ij}) = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
0 & 0 & 0 & \cdots & 0 & -2 & 2
\end{pmatrix}
\]
We shall also use the decomposition: \( A = A^dA^s \), where \( A^d = \text{diag} (1, \ldots, 1, 2) \), and \( A^s \) is a symmetric matrix:

\[
A^s = (a^s_{ij}) = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 \\
\end{pmatrix}
\tag{34}
\]

Then the defining relations of \( \mathfrak{osp}(1|2n) \) are:

\[
[\hat{H}_i, \hat{H}_j] = 0, \quad [\hat{H}_i, X^\pm_j] = \pm a^\pm_{ij} X^\pm_j, \\
[X^+_i, X^-_j] = \delta_{ij} \hat{H}_i, \\
(\text{Ad} X^\pm_j)^{n_{jk}}(X^\pm_k) = 0, \quad j \neq k, \quad n_{jk} = 1 - a_{jk},
\tag{35}
\]

where in (35) one uses the supercommutator:

\[
(\text{Ad} X^\pm_j)(X^\pm_k) = [X^\pm_j, X^\pm_k] \equiv X^\pm_j X^\pm_k - (-1)^{\deg X^\pm_j \deg X^\pm_k} X^\pm_k X^\pm_j.
\tag{36}
\]

G. Shapovalov form and unitarity

The Shapovalov form is a bilinear \( \mathfrak{g} \)-valued form on \( U(\mathcal{G}^+) \) \([80]\), which we extend in the obvious way to Verma modules, cf. e.g., \([65]\). We need also the involutive antiautomorphism \( \omega \) of \( U(\mathcal{G}) \) which will provide the real form we are interested in. Since this is the split real form \( \mathfrak{osp}(1|2n, \mathbb{R}) \) we use:

\[
\omega(X_\beta) = X_{-\beta}, \quad \omega(H) = H, \tag{37}
\]

where \( X_\beta \) is the root vector corresponding to the root \( \beta, H \in \mathcal{H} \).

Thus, an adaptation of the Shapovalov form suitable for our purposes is defined as follows:

\[
( u, u' ) = ( p v_0, p' v_0 ) \equiv ( v_0, \omega(p) p' v_0 ) = ( \omega(p') p v_0, v_0 ) \tag{38}
\]

\[
u = p v_0, \quad u' = p' v_0, \quad p, p' \in U(\mathcal{G}^+), \quad u, u' \in \mathcal{V}^\Lambda,
\]
supplemented by the normalization condition \( (v_0, v_0) = 1 \). The norms squared of the states would be denoted by:
\[
\| u \|^2 \equiv (u, u).
\] (39)

Now we need to introduce a PBW basis of \( U(G^+) \). We use the so-called normal ordering, namely, if we have the relation:
\[
\beta = \beta' + \beta'', \quad \beta, \beta', \beta'' \in \Delta^+
\]
then the corresponding root vectors are ordered in the PBW basis as follows:
\[
\ldots (X_{\beta'}^+)^{k'} \ldots (X_{\beta}^+)^{k} \ldots (X_{\beta''}^+)^{k''} \ldots, \quad k, k', k'' \in \mathbb{Z}_+.
\] (40)

We have also to take into account the relation (30) between the root vectors corresponding to the roots \( \delta_i \) and \( 2\delta_i \). Because of this relation and consistently with (40) the generators \( X_{\delta_i}^+, i = 1, \ldots, n \), are not present in the PBW basis. On the other hand the PBW basis of the even subalgebra of \( U(G^+) \) would differ from the above only in the fact that the powers of \( X_i^+ \), \( i = 1, \ldots, n \), are only even representing powers of the even generators \( X_{\delta_i}^+, i = 1, \ldots, n \).

### III. Unitarity

#### A. Calculation of some norms

In this subsection we show how to use the form (38) to calculate the norms of the states. We shall use the isomorphism between the Cartan subalgebra \( \mathcal{H} \) and its dual dual \( \mathcal{H}^* \). This is given by the correspondence: to every element \( \beta \in \mathcal{H}^* \) there is unique element \( H_\beta \in \mathcal{H} \), so that:
\[
\mu(H_\beta) = (\mu, \beta^\vee),
\] (41)
for every \( \mu \in \mathcal{H}^*, \mu \neq 0 \). Applying this to the positive roots we have: to \( \beta = \delta_i - \delta_j, \delta_i, \delta_i + \delta_j \), resp., correspond: \( H_\beta = H_{ij}, H_i^i, H_i^i + H_j^j \), resp.

We give now explicitly the norms of the one-particle states introducing also notation for
future use:

\[
x_{ij}^- \equiv \| X_{ij} v_0 \|^2 = ( X_{ij} v_0, X_{ij} v_0 ) = \\
= ( v_0, X_{ji} X_{ij} v_0 ) = ( v_0, (X_{ij} X_{ji} - H_{ij}) v_0 ) = \\
= -\Lambda(H_{ij}) = - (\Lambda(\delta_i - \delta_j)) = a_i + \cdots + a_{j-1}, \quad i < j,
\]
\[
x_{ij}^+ \equiv \| X_{ij}^+ v_0 \|^2 = ( X_{ij}^+ v_0, X_{ij}^+ v_0 ) = \\
= ( v_0, X_{ij}^+ X_{ij}^+ v_0 ) = ( v_0, (X_{ij}^+ X_{ji} - H_{ij}^l) v_0 ) = -\Lambda(H_{ij}^l) = \\
= \Lambda(H_{ij}^l + H_{ij}^l) = (\Lambda(\delta_i + \delta_j)) = 2d + a_1 + \cdots + a_{i-1} - a_i - \cdots - a_{n-1},
\]
\[
x_i \equiv \| X_i^+ v_0 \|^2 = ( X_i^+ v_0, X_i^+ v_0 ) = \\
= ( v_0, X_i^- X_i^+ v_0 ) = ( v_0, (X_i^+ X_i^- + H_i^l) v_0 ) = \Lambda(H_i^l) = \\
= (\Lambda(\delta_i)) = 2d + a_1 + \cdots + a_{i-1} - a_i - \cdots - a_{n-1}.
\]

Positivity of all these norms gives the following necessary conditions for unitarity:

\[
a_i \geq 0, \quad i = 1, \ldots, n - 1, \\
d \geq \frac{1}{2}(a_1 + \cdots + a_{n-1}).
\]

In fact, the boundary values are possible due to factoring out of the corresponding null states when passing from the Verma module to the unitary irreducible factor module.

Further, we shall discuss only norms which involve the conformal weight since the others are related to unitarity of the irrep restricted to the maximal simple compact subalgebra \( su(n) \). The norms that we are going to consider can be written in terms of factors \( d - \ldots \), and the leading term in \( d \) has a positive coefficient. Thus, for \( d \) large enough all norms will be positive. When \( d \) is decreasing there is a critical point at which one (or more) norm(s) will become zero. This critical point (called the 'first reduction point' in [61]) can be read off from the reducibility conditions, since at that point the Verma module is reducible (and it is the corresponding submodule that has zero norm states).

The maximal \( d \) coming from the different possibilities in (20b) are obtained for \( m_{ij}^\pm = 1 \) and they are, denoting also the corresponding root:

\[
d_{ij} \equiv n + \frac{1}{2}(a_j + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1} - i - j),
\]

the corresponding root being \( \delta_i + \delta_j \). The maximal \( d \) coming from the different possibilities
in \((20c,d)\), resp., are obtained for \(m_i = 1, m_{ii} = 1\), resp., and they are:

\[
d_i \equiv n - i + \frac{1}{2}(a_i + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1}) ,
\]
\[
d_{ii} = d_i - \frac{1}{2},
\]

the corresponding roots being \(\delta_i, 2\delta_j\), resp. These are some orderings between these maximal reduction points:

\[
d_1 > d_2 > \cdots > d_n ,
\]
\[
d_{i,i+1} > d_{i,i+2} > \cdots > d_{in} ,
\]
\[
d_{1,j} > d_{2,j} > \cdots > d_{j-1,j} ,
\]
\[
d_i > d_{j,k} > d_\ell , \quad i \leq j < k \leq \ell .
\]

Obviously the first reduction point is:

\[
d_1 = n - 1 + \frac{1}{2}(a_1 + \cdots + a_{n-1}) .
\]

B. Main result

**Theorem:** All positive energy unitary irreducible representations of the superalgebras \(osp(1\vert 2n, \mathbb{H})\) characterized by the signature \(\chi\) in (1) are obtained for real \(d\) and are given in the following list:

\[
d \geq d_1 = n - 1 + \frac{1}{2}(a_1 + \cdots + a_{n-1}) , \quad \text{no restrictions on } a_j,
\]
\[
d = d_{12} = n - 2 + \frac{1}{2}(a_2 + \cdots + a_{n-1} + 1) , \quad a_1 = 0 ,
\]
\[
\ldots
\]
\[
d = d_{j-1,j} = n - j + \frac{1}{2}(a_j + \cdots + a_{n-1} + 1) , \quad a_1 = \ldots = a_{j-1} = 0 ,
\]
\[
\ldots
\]
\[
d = d_{n-1,n} = \frac{1}{2} , \quad a_1 = \ldots = a_{n-1} = 0 .
\]

**Proof:**

**Necessity:**

We give examples of states with negative norm in the excluded intervals \(d < d_1\), cf. [81].

** Sufficiency:**
The statement of the Theorem for $d > d_i$ is clear form the general considerations above. For $d = d_i$, we have the first zero norm state which is naturally given by the corresponding singular vector $v^{1, \delta_i} = \mathcal{P}^{1, \delta_i} v_0$. In fact, all states of the embedded submodule $V^{\Lambda+\delta_i}$ built on $v^{1, \delta_i}$ have zero norms. Due to the above singular vector we have the following additional null condition in $F^\Lambda$:

$$\mathcal{P}^{1, \delta_i} [\Lambda] = 0.$$ (49)

The above conditions factorizes the submodule built on $v^{1, \delta_i}$. There are no other vectors with zero norm at $d = d_i$ since by a general result [79], the elementary embeddings between Verma modules are one-dimensional. Thus, $F^\Lambda$ is the UIR $I_\Lambda = F^\Lambda$.

Further we consider the remaining discrete points of unitarity for $d < d_i$, i.e., $d = d_{i;i+1}$, $i = 1, \ldots, n-1$. The corresponding roots are $\delta_i + \delta_{i+1} = \alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_n$. The corresponding singular vectors $v^{1, \delta_i + \delta_{i+1}} = \mathcal{P}^{1, \delta_i + \delta_{i+1}} v_0$.

Now, fix $i$, where $i \in \{1, \ldots, n-1\}$. All states of the embedded submodule $V^{\Lambda+\delta_i+\delta_{i+1}}$ built on $v^{1, \delta_i+\delta_{i+1}}$ have zero norms for $d = d_{i;i+1}$. Due to the above singular vector we have the following additional null condition in $F^\Lambda$:

$$\mathcal{P}^{1, \delta_i+\delta_{i+1}} [\Lambda] = 0, \quad d = d_{i;i+1}. \tag{50}$$

At this point the states built on the vector $v^{1, \delta_i}$ and on the vectors $v^{1, \delta_k + \delta_{k+1}}$ for $k < i$ (all of these are not singular vectors at $d = d_{i;i+1}$) have negative norm except when $a_1 = \cdots = a_i = 0$. For this statement we may use the explicit form of these vectors. This explicit form is the same as the singular vectors of the same weight for the Lie algebra $B_n = \text{so}(2n+1)$. For $v^{1, \delta_i}$ this can be read off from [82] (in fact, there it is for the more general situtation of the quantum group $U_q(B_n)$):

$$v^{1, \delta_i} = \sum_{k_1 = 0}^1 \cdots \sum_{k_{n-1} = 0}^1 b_{k_1 \ldots k_{n-1}} (X_1^+)^{1-k_1} \cdots (X_{n-1}^+)^{1-k_{n-1}} \times X_n^+ (X_{n-1}^+)^{k_{n-1}} \cdots (X_1^+)^{k_1} v_0 \equiv \mathcal{P}^{1, \delta_i} v_0, \tag{51a}$$

$$b_{k_1 \ldots k_{n-1}} = (-1)^{k_1 + \cdots + k_{n-1}} \frac{1 + a_1}{1 + a_1 - k_1} \cdots \frac{n-1 + a_1 + \cdots + a_{n-1} - k_{n-1}}{n-1 + a_1 + \cdots + a_{n-1} - k_{n-1}} = \tag{51b}$$

$$\frac{(\rho - \Lambda)(H^1)}{(\rho - \Lambda)(H^1) - k_1} \cdots \frac{(\rho - \Lambda)(H^{n-1})}{(\rho - \Lambda)(H^{n-1}) - k_{n-1}} \tag{51c}$$

$$= \frac{1 + a_1}{n - 1 + a_1 + \cdots + a_{n-1} - k_{n-1}}$$

$$= (-1)^{k_1 + \cdots + k_{n-1}} (a_1 + k) \frac{2 + a_1 + a_2}{2 + a_1 + a_2 - k_2} \cdots \frac{n - 1 + a_1 + \cdots + a_{n-1} - k_{n-1}}{n - 1 + a_1 + \cdots + a_{n-1} - k_{n-1}} \tag{51d}$$
where \( H^s = \hat{H}_1 + \hat{H}_2 + \cdots + \hat{H}_s \), (cf. f-lae (13) from [82] with \( q = 1, t = n - 1, m = 1, \lambda \rightarrow -\Lambda \) (the last change due to the fact that in [82] are considered highest weight modules)); in (51c) we have inserted our signatures:

\[
(\rho - \Lambda)(H^s) = (\rho - \Lambda, (\alpha_1 + \cdots + \alpha_s)') = (\rho - \Lambda, \delta_1 - \delta_{s+1}) = m_{1,s+1} = s + a_1 + \cdots + a_s
\]

and in (51d) we have made the choice of constant \( b_0 = a_1 \) in order to make the expression valid also for \( a_1 = 0 \). It is easy to see that for \( a_1 = 0 \) the vector \( v^{1,\delta_1} \) is not independent, but is a descendant of the singular vector \( v^1_s = X^+_1 v_0 : \)

\[
v^{1,\delta_1} = \sum_{k_2=0}^1 \cdots \sum_{k_{n-1}=0}^1 b_{k_2,\ldots,k_{n-1}} (X^+_2)^{1-k_2} \cdots (X^+_n)^{1-k_{n-1}} \times \\
x X^+_n (X^+_n)^{k_{n-1}} \cdots (X^+_2)^{k_2} X^+_1 v_0.
\]  

(52)

Thus, \( v^{1,\delta_1} \) is not present in \( F^A \) for any \( d \) and \( a_1 = 0 \) since the null condition (49) follows from case \( i = 1 \) of the null conditions (24). Analogously, if \( i > 1 \) and fixing now \( k < i \), the vector \( v^{1,\delta_k+\delta_{k+1}} \) has negative norm at \( d = d_{i,i+1} \) except if \( a_{k+1} = 0 \), when it is not independent, but is a descendant of the singular vector \( v^{k+1}_s = X^+_k v_0 \), and hence is not present in \( F^A \). (This will be given more explicitly in [81].) Thus, for \( d = d_{i,i+1} \) together with \( a_1 = \cdots = a_i = 0 \), the condition (50) factorizing the submodule built on \( v^{1,\delta_i+\delta_{i+1}} \), is the only condition - in addition to (24) - needed to obtain the UIR \( L_{\Lambda} = F^A \) at \( d = d_{i,i+1}, i = 1, \ldots, n - 1 \).

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