Set Coverings and Invertibility of Functional Galois Connections

Marianne Akian
Stéphane Gaubert
Vassili Kolokoltsov

Set coverings and invertibility of Functional Galois Connections

Marianne Akian, Stéphane Gaubert, and Vassili Kolokoltsov

ABSTRACT. We consider equations of the form $Bf = g$, where $B$ is a Galois connection between lattices of functions. This includes the case where $B$ is the Fenchel transform, or more generally a Moreau conjugacy. We characterize the existence and uniqueness of a solution $f$ in terms of generalized subdifferentials, which extends K. Zimmermann's covering theorem for max-plus linear equations, and give various illustrations.

1. Introduction

We call functional Galois connection a (dual) Galois connection between a sublattice $\mathcal{F}$ of $\mathbb{R}^Y$ and a sublattice $\mathcal{G}$ of $\mathbb{R}^X$, where $X,Y$ are two sets and $\mathbb{R} = \mathbb{R} \cup \{+\infty\}$, see Section 2 for definitions. An important example of functional Galois connection is the Fenchel transform, and more generally, the Moreau conjugacy [19] associated to a kernel $b : X \times Y \to \mathbb{R}$, 

\begin{equation}
B : \mathbb{R}^Y \to \mathbb{R}^X, \quad Bf(x) = \sup\{b(x, y) - f(y) \mid y \in Y\},
\end{equation}

with the convention $(-\infty) + (+\infty) = (+\infty) + (-\infty) = -\infty$. Moreau conjugacies are instrumental in nonconvex duality, see [25, Ch. 11, § E], [27]. Max-plus linear operators with kernel, which are of the form $f \mapsto B(-f)$, arise in deterministic optimal control and asymptotics, and have been widely studied, see in particular [10, 18, 4, 17, 1, 14]. Other examples of functional Galois connections include dualities for quasi-convex functions (see for instance [27, 29, 28]).

We consider here general Galois connections between the set $\mathcal{F}$ of lower semi-continuous functions from a Hausdorff topological space $Y$ to $\mathbb{R}$ and $\mathcal{G} = \mathbb{R}^X$. As shown in Section 2, any functional Galois connection $B : \mathcal{F} \to \mathcal{G}$ has the form:

\[ Bf(x) = \sup\{b(x, y, f(y)) \mid y \in Y\}, \]

where $b : X \times Y \times \mathbb{R} \to \mathbb{R}$ is such that $b(x, \cdot, a) \in \mathcal{F}$, for all $x \in X$ and $a \in \mathbb{R}$, and $b(x, y, \cdot)$ is nonincreasing, right continuous and send $+\infty$ to $-\infty$, for all $x \in X$ and $y \in Y$.

\textit{Date:} July 26, 2003.  
This work has been partially supported by the Erwin Schroedinger International Institute of mathematical Physics (ESI).
Given a map \( g \in \mathcal{G} \) and a functional Galois connection \( B : \mathcal{F} \to \mathcal{G} \), we consider the problem:

\[
(\mathcal{P}) : \text{ Find } f \in \mathcal{F} \text{ such that } Bf = g .
\]

In particular, we look for effective conditions on \( g \) for the solution \( f \) to exist and be unique. When \( X, Y \) are finite sets, \( \mathcal{F} = \mathbb{R}^Y \) and \( \mathcal{G} = \mathbb{R}^X \), and \( B \) is as in (1) with \( b(x, y) \in \mathbb{R} \), the existence and uniqueness of the solutions of \( (\mathcal{P}) \) was characterized by K. Zimmermann [30] (see [7]), who gave conditions involving coverings of \( X \) by sets. This is one of the basic results in max-plus linear algebra, and it has been instrumental in the understanding of the geometry of images of max-plus linear operators. This result has also been the source of several important developments, including the characterization by Butkovič [7, 8] of locally injective ("strongly regular") max-plus linear maps in terms of optimal assignment problems.

We extend here K. Zimmermann’s theorem to the case of functional Galois connections: the existence (Section 3) and uniqueness (Section 4) of the solutions of \( (\mathcal{P}) \) are characterized in terms of generalized subdifferentials of \( g \). As in the work of K. Zimmermann, when \( X \) and \( Y \) are finite, our results yield an algorithm to check existence or uniqueness (Section 5). We also illustrate our results on various Moreau conjugacies (Section 6). In the special case where \( B \) is the Fenchel transform, our results show (see Section 6.1) that essentially smooth convex functions have a unique preimage by the Fenchel transform, a fact which is the essence of the classical Gärtner-Ellis theorem (see e.g. [11, Th. 2.3.6,(c)] for a general presentation, the use of the uniqueness of the preimage of the Fenchel transform is made explicit in [21, Th. 4.1 (c)], [15, Th. 4.7 and 5.3] and [22, Lemmas 3.2 and 3.5]). We also consider Moreau conjugacies with kernels of the form \( b(x, y) = \varphi \|x - y\|^p \), with \( 0 < p \leq 1 \), whose images are the spaces of Hölder continuous functions (see Section 6.3).

Problem \( (\mathcal{P}) \) arises when looking for the rate function in large deviations: further applications of our results are given in [3], where we use our characterizations of the uniqueness in Problem \( (\mathcal{P}) \) to give a new proof, as well as generalizations, of Gärtner-Ellis theorem. Let us also mention that Problem \( (\mathcal{P}) \) arises in the characterization of dual solutions of the Monge-Kantorovich mass transfer problem (see e.g. [23, §3.3]).

The results of the present paper were announced in [2].

2. Representation of Functional Galois Connections

Many basic results of convex analysis are specializations of general properties of Galois connections in lattices, that we next recall (the proofs can be found in [5, Chapter V, Section 8], [12], [6, chapter 1, Section 2], [4, Section 4.4.2], and [13, Chapter 0, Section 3]).

Let \( (\mathcal{F}, \leq_{\mathcal{F}}) \) and \( (\mathcal{G}, \leq_{\mathcal{G}}) \) be two partially ordered sets, and let \( B : \mathcal{F} \to \mathcal{G} \) and \( C : \mathcal{G} \to \mathcal{F} \). We say that \( B \) is antitone if \( f \leq_{\mathcal{F}} f' \implies Bf' \leq_{\mathcal{G}} Bf \). The pair \((B, C)\) is a dual Galois connection between \( \mathcal{F} \) and \( \mathcal{G} \) if it satisfies one of the following equivalent conditions:

\[
\begin{align*}
(2a) & \quad I_{\mathcal{F}} \geq_{\mathcal{F}} CB, \quad I_{\mathcal{G}} \geq_{\mathcal{G}} BC, \quad \text{and } B, C \text{ are antitone maps,} \\
(2b) & \quad (g \geq_{\mathcal{G}} Bf \iff f \geq_{\mathcal{F}} Cg) \quad \forall f \in \mathcal{F}, g \in \mathcal{G}, \\
(2c) & \quad Cg = \min_{\mathcal{F}} \{f \mid g \geq_{\mathcal{G}} Bf\} \quad \forall g \in \mathcal{G}, \\
(2d) & \quad Bf = \min_{\mathcal{G}} \{g \mid f \geq_{\mathcal{F}} Cg\} \quad \forall f \in \mathcal{F},
\end{align*}
\]
where \( \min_{\mathcal{F}} \) and \( \min_{\mathcal{G}} \) denote the minimal elements for the orders \( \preceq_{\mathcal{F}} \) and \( \preceq_{\mathcal{G}} \), respectively, and where \( I_{\mathcal{G}} \) denotes the identity on a set \( \mathcal{A} \).

It follows from (2c) that for any \( B \), there is at most one map \( C \) such that \((B, C)\) is a dual Galois connection. We denote this \( C \) by \( B^* \). It also follows from (2c) that for all \( g \in \mathcal{G} \) and \( h \in \mathcal{F} \),

\[
g = Bh \Rightarrow B^* g = \min_{\mathcal{F}} \{ f \mid g = B f \} .
\]

The two inequalities in (2a) imply that

\[
BB^* B = B \quad \text{and} \quad B^* BB^* = B^* .
\]

From (3) or (4), one deduces

\[
B f = g \text{ has a solution } f \in \mathcal{F} \iff BB^* g = g .
\]

If \((B, C)\) is a dual Galois connection, so does \((C, B)\), by symmetry. Hence, \((B^*)^* = B\). If \( B \) yields a dual Galois connection, then

\[
B (\inf_{\mathcal{F}} F) = \sup_{\mathcal{G}} \{ B f \mid f \in F \}
\]

for any subset \( F \) of \( \mathcal{F} \) such that the infimum of \( F \) exists. In particular, if \( \mathcal{F} \) has a maximum element, \( \top_{\mathcal{F}} \), we get by specializing (6) to \( F = \emptyset \) that \( \mathcal{F} \) has a minimum element, \( \bot_{\mathcal{G}} \), and

\[
B(\top_{\mathcal{F}}) = \bot_{\mathcal{G}} .
\]

Moreover, if \( \mathcal{F} \) is a complete ordered set, i.e., if any subset of \( \mathcal{F} \) has a greatest lower bound, property (6) characterizes the maps \( B \) that yield a dual Galois connection.

Ordinary (non dual) Galois connections are defined by reversing the order of \( \mathcal{F} \) and \( \mathcal{G} \) in (2). One also finds in the literature the names of residuated maps \( B \) and dually residuated maps \( C \), which are defined by reversing the order of \( \mathcal{F} \), but not the order of \( \mathcal{G} \), in (2). All these notions are equivalent.

We call lattice of functions a sublattice \( \mathcal{F} \) of \( S^\vee \), where \((S, \leq)\) is a lattice, \( Y \) is a set, and \( S^\vee \) is equipped with the product ordering (that we still denote by \( \leq \)). When \( \mathcal{F} \subset S^\vee \) and \( \mathcal{G} \subset T^\vee \) are lattices of functions, we say that \((B, B^*)\) is a functional (dual) Galois connection.

When \( S \) has a maximum element \( \top_S \), we define the Dirac function at point \( y \in Y \) with value \( s \in S \):

\[
\delta'_y \in S^\vee : \quad \delta'_y (y') = \begin{cases} s & \text{if } y' = y, \\ \top_S & \text{otherwise}. \end{cases}
\]

**Theorem 2.1.** Let \( S, T \) be two lattices that have a maximum element, let \( X, Y \) be arbitrary nonempty sets and let \( \mathcal{F} \subset S^\vee \) (resp. \( \mathcal{G} \subset T^\vee \)) be a lattice of functions containing all the Dirac functions of \( S^\vee \) (resp. \( T^\vee \)). Then, \((B, B^*)\) is a dual Galois connection between \( \mathcal{F} \) and \( \mathcal{G} \) if and only if, there exists two maps \( b : X \times Y \times S \to T \) and \( b^* : X \times Y \times T \to S \) such that: for all \((x, y) \in X \times Y\), \((b(x, y, \cdot), b^*(x, y, \cdot))\) is a dual Galois connection between \( S \) and \( T \); for all \((x, t) \in X \times T\), \( b^*(x, \cdot, t) \in \mathcal{F} \); and for all \((y, s) \in Y \times S\), \( b(\cdot, y, s) \in \mathcal{G} \); and

\[
\begin{align*}
B f &= \sup_{\mathcal{G}} \{ b(\cdot, y, f(y)) \mid y \in Y \}, \quad \forall f \in \mathcal{F}, \\
B^* g &= \sup_{\mathcal{F}} \{ b^*(x, \cdot, g(x)) \mid x \in X \}, \quad \forall g \in \mathcal{G}.
\end{align*}
\]
When \( Y \) is a \( T_1 \) topological space and \( S \) has a maximum element, Theorem 2.1 can be applied to the set \( \mathcal{F} = \text{Is}(Y, S) \) of lower semicontinuous maps \( Y \to S \) (we say that a map \( f : Y \to S \) is lower semicontinuous, or l.s.c., if for all \( s \in S \), the sublevel set \( \{ y \in Y \mid f(y) \leq s \} \) is closed). In this case, \( \sup_{\mathcal{F}} = \sup \) since the sup of l.s.c. maps is l.s.c.

Kolokoltsov already proved [16] (see also [17, Theorem 1.4]) a “Riesz representation theorem” similar to Theorem 2.1, for a continuous map \( B \) between (non-complete) lattices of continuous functions \( \mathcal{F} \) and \( \mathcal{G} \), assuming that \( B \) preserves finite sups. Singer [27, Th. 7.3] also proved Theorem 2.1 when \( \mathcal{F} = S^Y, \mathcal{G} = T^X \) and \( S \) and \( T \) are complete lattices (Th. 7.3 of [27] is stated in the case where \( S \) and \( T \) are included in \( \mathcal{F} \), but it is remarked in [27, page 419] that this result is valid for general complete lattices \( S \) and \( T \).

**Remark 2.2.** If \( S, T, \mathcal{F}, \mathcal{G} \) are as in Theorem 2.1, \( \mathcal{F} \) has a maximal element, namely, the constant function \( y \mapsto T_S \) (which necessarily belongs to \( \mathcal{F} \) because it is equal to the Dirac function \( \delta_s^y \) for any \( y \in Y \)). Then, if \( (B, B^*) \) is a dual Galois connection between \( \mathcal{F} \) and \( \mathcal{G} \), the remark before (7) shows that \( \mathcal{G} \) has a minimum element. Moreover, by Theorem 2.1, the existence of a dual Galois connection between \( \mathcal{F} \) and \( \mathcal{G} \) implies the existence of dual Galois connection between \( S \) and \( T \), hence, by (7), \( T \) has a minimum element, \( \bot_T \), and since \( \mathcal{G} \) contains Dirac functions, the minimum element of \( \mathcal{G} \) is necessarily the constant function \( x \mapsto \bot_T \).

Symmetrically, \( S \) (resp. \( \mathcal{F} \)) has a minimum element, \( \bot_S \) (resp. the constant function \( y \mapsto \bot_S \)).

**Proof of Theorem 2.1.** Let us first assume that \( (B, B^*) \) is a dual Galois connection, and define

\[
\begin{align*}
b(\cdot, y, s) &= B\delta_y^s \in \mathcal{G}, & b^*(x, \cdot, t) &= B^*\delta_x^t \in \mathcal{F}.
\end{align*}
\]

We have

\[
\begin{align*}
b(x, y, s) \leq t & \iff B\delta_y^s(x) \leq t \\
& \iff B\delta_y^s \leq \delta_x^s \\
& \iff B^*\delta_x^s \leq \delta_y^s \quad (\text{by } (2b)) \\
& \iff b^*(x, y, t) \leq s,
\end{align*}
\]

which shows, by (2b) again, that \( (b(x, y, \cdot), b^*(x, y, \cdot)) \) is a dual Galois connection between \( S \) and \( T \).

Using (6) and \( f = \inf_{\mathcal{F}} \{ \delta_y^s \mid y \in Y \} \), which holds for all \( f \in \mathcal{F} \), we get (8a). The representation (8b) is obtained by symmetry.

Conversely, let us assume that \( (B, B^*) \) are defined by (8) where \( b \) and \( b^* \) satisfy the conditions of the theorem. (This means in particular that the \( \sup_{\mathcal{G}} \) and \( \sup_{\mathcal{F}} \) in (8) exist.) Then, for all \( f \in \mathcal{F}, g \in \mathcal{G} \),

\[
Bf \leq g \iff b(\cdot, y, f(y)) \leq g, \forall y \in Y
\]

\[
\iff b(x, y, f(y)) \leq g(x), \forall (x, y) \in X \times Y
\]

(since \( \leq \) is the product ordering on \( \mathcal{G} \))

\[
\iff b^*(x, y, g(x)) \leq f(y), \forall (x, y) \in X \times Y \quad (\text{by } (2b))
\]

\[
\iff B^*g \leq f,
\]

which, by (2b) again, shows that \( (B, B^*) \) is a dual Galois connection. \( \square \)
Proposition 2.3. If \((b, b^*)\) are as in Theorem 2.1, where \(Y\) is a \(T_1\) topological space and \(\mathcal{F} = \text{lsc}(Y, S)\), then, for all \((x, s) \in X \times S\), the map \(b(x, \cdot, s) : Y \to T\) is l.s.c.

Proof. By Theorem 2.1, \(b^*(x, \cdot, t) \in \mathcal{F} = \text{lsc}(Y, S)\), for all \((x, t) \in X \times T\). The equivalence (2b) shows that \(b^*(x, \cdot, t)\) is l.s.c. for all \(t \in T\), if, and only if, \(b(x, \cdot, s)\) is l.s.c. for all \(s \in S\). □

By symmetry, when \(X\) is a \(T_1\) topological space, and \(\mathcal{B} = \text{lsc}(X, T)\), the map \(b^*(\cdot, y, t)\) is l.s.c. for all \((y, t) \in Y \times T\).

We say that \(\mathcal{F}\) is a lattice of subsets if there exists a set \(Y\) such that \(\mathcal{F} \subset \mathcal{P}(Y)\), the set of all subsets of \(Y\), and \(\mathcal{F}\) is a lattice for the \(\subset\) ordering. Taking for \(S\) the complete lattice of Booleans \(\{\{0, 1\}, \leq\}\), one can identify \(\mathcal{F}\) to a lattice of functions included in \(S^Y\), by using the lattice isomorphism: \(F \in \mathcal{P}(Y) \mapsto 1_F\), where \(1_F(y) = 1\) if \(y \in F\) and \(1_F(y) = 0\) otherwise. Hence, specializing Theorem 2.1 to the case where \(S = T = \mathbb{R}\) equal the lattice of Booleans, we get:

Corollary 2.4. Let \(X, Y\) be arbitrary nonempty sets and let \(\mathcal{F} \subset \mathcal{P}(Y)\) (resp. \(\mathcal{G} \subset \mathcal{P}(X)\)) be a lattice of subsets containing the empty set and all singletons of \(Y\) (resp. \(X\)). Then, \((B, B^*)\) is a Galois connection between \(\mathcal{F}\) and \(\mathcal{G}\) if, and only if, there exists a set \(\mathcal{B} \subset X \times Y\) such that: for all \(x \in X\), 
\[ \mathcal{B}_x = \{ y \in Y \mid (x, y) \in \mathcal{B} \} \in \mathcal{F}; \]
for all \(y \in Y\), 
\[ \mathcal{B}^y = \{ x \in X \mid (x, y) \in \mathcal{B} \} \in \mathcal{G}; \]
and
\[(9a) \quad B F = \inf_{\mathcal{F}} \{ \mathcal{B}^y \mid y \in F \}, \quad \forall F \in \mathcal{F}\]
\[(9b) \quad B^* G = \inf_{\mathcal{G}} \{ \mathcal{B}_x \mid x \in G \}, \quad \forall G \in \mathcal{G}\].

If \(\mathcal{G}\) (resp. \(\mathcal{F}\)) is a complete sublattice of \(\mathcal{P}(X)\) (resp. \(\mathcal{P}(Y)\)), then \(\inf_{\mathcal{G}}\) (resp. \(\inf_{\mathcal{F}}\)) is the intersection operation.

Remark 2.5. Many classical Galois connections are of the form (9) (but \(\mathcal{F}\) and \(\mathcal{G}\) need not contain the singletons and the empty set). For instance, if \(Y\) is an extension of a field \(K\), if \(\mathcal{F}\) is the set of intermediate fields \(F\): \(K \subset F \subset Y\), if \(X\) is the group of automorphisms of \(Y\) fixing every element of \(K\), and if \(\mathcal{G}\) is the set of subgroups \(G\) of \(X\), we obtain the original Galois connection by setting
\[ \mathcal{B} = \{ (g, y) \in X \times Y \mid g(y) = y \}. \]

In the sequel, we shall only consider the case when \(S = T = \mathbb{R}\). In this case, the property that \((b(x, y, \cdot), b^*(x, y, \cdot))\) is a dual Galois connection can be made explicit:

Lemma 2.6. A map \(h : \mathbb{R} \to \mathbb{R}\) yields a dual Galois connection if, and only if, \(h\) is nonincreasing, right-continuous, and \(h(\pm \infty) = \mp \infty\). □

Example 2.7. When \(b \in \mathbb{R}\), the map \(h : \mathbb{R} \to \mathbb{R}, \lambda \mapsto b - \lambda\), with the convention \((- \infty) + (+ \infty) = (+ \infty) + (- \infty) = - \infty\), yields a dual Galois connection (by Lemma 2.6). Moreover \(h^* = h\).

Example 2.8. Let \(S = T = \mathbb{R}\), and \(X, Y, \mathcal{F}, \mathcal{G}\) be as in Theorem 2.1. Assume in addition that \(\mathcal{F}\) and \(\mathcal{G}\) are stable by the addition of a constant (again with the convention: \((- \infty) + (+ \infty) = (+ \infty) + (- \infty) = - \infty\)). Let \(\tilde{b} : X \times Y \to \mathbb{R}\) be a map, and let \(B : \mathcal{F} \to \mathcal{G}\) and \(B^* : \mathcal{G} \to \mathcal{F}\) be defined by (8) with
\[ b(x, y, \alpha) = b^*(x, y, \alpha) = \tilde{b}(x, y) - \alpha \quad \forall x \in X, \ y \in Y, \ \alpha \in \mathbb{R}. \]
(with the same convention as above). Theorem 2.1 shows that \((B, B^*)\) is a dual Galois connection if, and only if, \(b^*(x, \cdot) \in \mathcal{F}\) for all \(x \in X\), and \(b(\cdot, y) \in \mathcal{G}\) for all \(y \in Y\). This result can be applied, in particular, when \(\mathcal{F} = \text{isc}(Y, \mathbb{R})\) and \(\mathcal{G} = \mathbb{R}^X\). Such functional Galois connections are called Moreau conjugacies. In particular, taking two topological vector spaces in duality \(X\) and \(Y\), and \(b : X \times Y \to \mathbb{R}\), \((x, y) \mapsto \langle x, y \rangle\), we obtain the classical Legendre-Fenchel transform \(f \mapsto Bf = f^*\).

**Remark 2.9.** The set \(\mathbb{R}\) can be equipped with the semiring structure of \(\mathbb{R}^{\text{max}}\), in which the addition is \((a, b) \mapsto \max(a, b)\) and the multiplication is \((a, b) \mapsto a + b\), with the convention \(-\infty + (+\infty) = (+\infty) = (-\infty) = -\infty\). Then, if \(\mathcal{Z}\) is a set, \(\mathbb{R}^\mathcal{Z}\) can be equipped with two different \(\mathbb{R}^{\text{max}}\)-semimodule structures. The *natural semimodule*, denoted \(\mathbb{R}^{\text{max}}_{\mathcal{Z}}\), is obtained by taking the addition \((f, f') \mapsto f + f'\), with \((f + f')(z) = \max(f(z), f'(z))\) for all \(z \in \mathcal{Z}\), and the action \((\lambda, f) \mapsto \lambda f\) with \((\lambda f)(z) = \lambda f(z)\) for all \(z \in \mathcal{Z}\), again with the convention \((-\infty) + (+\infty) = (+\infty) = (-\infty) = -\infty\). The *opposite semimodule*, denoted \(\mathbb{R}^{\text{max}}_{\mathcal{Z}}^\text{op}\), is obtained by taking the addition \((f, f') \mapsto f - f'\), with \((f - f')(z) = \min(f(z), f'(z))\) for all \(z \in \mathcal{Z}\), and the action \((\lambda, f) \mapsto \lambda' f\) with \((\lambda' f)(z) = -\lambda f(z)\) for all \(z \in \mathcal{Z}\), with the dual convention \((-\infty) + (+\infty) = (+\infty) + (-\infty) = +\infty\) (see [9]). Then, the Moreau conjugacies, that is the functional Galois connections of Example 2.8 are \(\mathbb{R}^{\text{max}}\)-linear from \(\mathbb{R}^{\text{max}}_{\mathcal{Z}}\) to \(\mathbb{R}^{\text{max}}_{\mathcal{Z}}^\text{op}\).

### 3. Existence of Solutions of \(Bf = g\)

**3.1. Statement of the existence result.** In the following, we take \(S = T = \mathbb{R}\), we assume that \(X\) and \(Y\) are Hausdorff topological spaces, and take \(\mathcal{F} = \text{isc}(Y, \mathbb{R})\), \(\mathcal{G} = \mathbb{R}^X\), together with \(B, B^*, b, b^*\) as in Theorem 2.1. The case when \(\mathcal{F} = \mathbb{R}^Y\) and \(X, Y\) are arbitrary sets can be obtained by taking discrete topologies on \(X\) and \(Y\). Since, by Theorem 2.1, \((b(x, y, \cdot), b^*(x, y, \cdot))\) is a dual Galois connection, Lemma 2.6 shows that \(b(x, y, \cdot)\) and \(b^*(x, y, \cdot)\) are nonincreasing and right-continuous maps from \(\mathbb{R}\) to itself, which take the value \(-\infty\) at \(+\infty\).

We shall assume that there is a subset \(\mathcal{S} \subset X \times Y\) such that the following assumptions hold: (A1) \(\mathcal{S}_x = \{y \in Y \mid (x, y) \in \mathcal{S}\} \neq \emptyset\) for all \(x \in X\); (A2) \(\mathcal{S}_y = \{x \in X \mid (x, y) \in \mathcal{S}\} \neq \emptyset\) for all \(y \in Y\); (A3) \(b(x, y, \cdot)\) is a bijection \(\mathbb{R} \to \mathbb{R}\) for all \((x, y) \in \mathcal{S}\), (A4) \(b(x, y, \cdot) \equiv -\infty\) for \((x, y) \in X \times Y \setminus \mathcal{S}\).

When \(B\) is the Moreau conjugacy given by (8,10), Assumptions (A1–A4) are satisfied if, and only if, \(\tilde{b}(x, y) \in \mathbb{R} \cup \{-\infty\}\) for all \((x, y) \in X \times Y\), and \(\mathcal{C} := \{(x, y) \in X \times Y \mid \tilde{b}(x, y) \in \mathbb{R}\}\) satisfies (A1,A2), that is, for all \(x \in X\) and \(y \in Y\), \(\tilde{b}(x, \cdot)\) and \(\tilde{b}(\cdot, y)\) are not identically \(-\infty\). These assumptions are fulfilled in particular for the kernel of the Fenchel transform.

Rather than Problem (P), we will consider the more general problem:

\[(P'):\quad \text{Find } f \in \mathcal{F} \text{ such that } Bf \leq g, \quad Bf(x) = g(x) \quad \forall x \in X',\]

where \(g \in \mathcal{G}\) and \(X' \subset X\) are given.

To state our results, we need some definitions and notations. When \(B\) is the Fenchel transform, these notations and definitions will correspond to those defined classically in convex analysis. First, for any map \(g\) from a topological space \(Z\) to
In $\mathbb{R}$, we define the lower domain, upper domain, domain, and inner domain:

(11a) $\text{ldom}(g) = \{ z \in Z \mid g(z) < +\infty \}$,
(11b) $\text{udom}(g) = \{ z \in Z \mid g(z) > -\infty \}$,
(11c) $\text{dom}(g) = \text{ldom}(g) \cap \text{udom}(g)$,
(11d) $\text{idom}(g) = \{ z \in \text{dom}(g) \mid \limsup_{z'\to z} g(z') < +\infty \}$.

The set $\text{idom}(g) \subseteq \text{dom}(g)$ is an open subset of $\text{udom}(g)$. When $X$ is endowed with the discrete topology, $\text{idom}(g) = \text{dom}(g)$. We say that $g$ is proper if $g(z) \neq -\infty$ for all $z \in Z$ and if there exist $z \in Z$ such that $g(z) \neq +\infty$, which means that $\text{udom}(g) = Z$ and $\text{dom}(g) \neq \emptyset$.

Next, given $f \in \mathcal{F}, y \in Y, g \in \mathcal{G}$ and $x \in X$, we define the subdifferentials:

(12a) $\partial f(y) = \{ x \in X \mid (x, y) \in \mathcal{S}, \ b(x, y, f(y)) \leq b(x, y, f(y')) \forall y' \in Y \}$,
(12b) $\partial^* g(x) = \{ y \in Y \mid (x, y) \in \mathcal{S}, \ b^*(x', y, g(x')) \leq b^*(x, y, g(x)) \forall x' \in X \}$.

Then,

(13a) $\partial f(y) = \{ x \in X \mid (x, y) \in \mathcal{S} \text{ and } B f(x) = b(x, y, f(y)) \}$,
(13b) $\partial^* g(x) = \{ y \in Y \mid (x, y) \in \mathcal{S} \text{ and } B^* g(y) = b^*(x, y, g(x)) \}$.

and when $b(x, y, a) = (x, y) - a$ is the kernel of the Legendre-Fenchel transform, we recover the classical definition of subdifferentials.

**Definition 3.1.** When $F$ is a map from a set $Z$ to the set $\mathcal{P}(W)$ of all subsets of some set $W$, we denote by $F^{-1}$ the map from $W$ to $\mathcal{P}(Z)$ given by $F^{-1}(w) = \{ z \in Z \mid w \in F(z) \}$, and we define the domain of $F$: $\text{dom}(F) := \{ z \in Z \mid F(z) \neq \emptyset \} = \bigcup_{w \in W} F^{-1}(w)$. If $Z' \subseteq Z$ and $W' \subseteq W$, we say that $\{ F(z) \}_{z \in Z'}$ is a covering of $W'$ if $\bigcup_{z \in Z'} F(z) \supseteq W'$.

When $F, Z, Z', W$ are as in Definition 3.1, the family $\{ F^{-1}(w) \}_{w \in W}$ is a covering of $Z'$ if, and only if, $Z' \subseteq \text{dom}(F)$. By (13),

(14a) $(\partial f)^{-1}(x) = \arg \max_{y \in \mathcal{S}_x} b(x, y, f(y))$
(14b) $(\partial^* g)^{-1}(y) = \arg \max_{x \in \mathcal{S}_y} b^*(x, y, g(x))$.

**Definition 3.2.** We say that $b$ is continuous in the second variable if for all $x \in X$ and $a \in \mathbb{R}, b(x, \cdot, a)$ is continuous. We say that $b$ is coercive if for all $x \in X$, all neighborhoods $V$ of $x$ in $X$, and all $a \in \mathbb{R}$, the function

(15) $y \in Y \mapsto b_{x, V}^a(y) = \sup_{z \in Z} b(z, y, b(x, y, a))$, 

has relatively compact finite sublevel sets, which means that $\{ y \in Y \mid b_{x, V}^a(y) \leq \beta \}$ is relatively compact for all $\beta \in \mathbb{R}$.

The continuity of $b$ in the second variable holds trivially when $Y$ is discrete (and in particular when $Y$ is finite). The coercivity of $b$ holds trivially, and independently of the topology on $X$, when $Y$ is compact (and in particular when $Y$ is finite). If $X = Y = \mathbb{R}^n$ and $b(x, y, a) = (x, y) - a$, then $b$ is continuous in the second variable, and for all neighborhood $V$ of $x$, and all $a \in \mathbb{R}, b_{x, V}^a(y) \geq \| y \| + a$, for some $\varepsilon > 0$. 

so that $b$ is coercive. Similarly, if $b(x, y, a) = a\|x - y\|^2 - a$, where $a \in \mathbb{R} \setminus \{0\}$ and $\| \cdot \|$ is the Euclidean norm, then $b$ is continuous in the second variable, and for all neighborhood $V$ of $x$, and all $a \in \mathbb{R}$, $b_{a, V}(y) \geq \varepsilon \|y - x\| - 1 + a$, for some $\varepsilon > 0$, so that $b$ is coercive.

We also denote by $\mathcal{F}_c$ the set of all $f \in \mathcal{F}$ such that $y \mapsto b(x, y, f(y))$ has relatively compact finite superlevel sets, which means that, for all $x \in X$ and $\beta \in \mathbb{R}$, the set $\{ y \in Y \mid b(x, y, f(y)) \geq \beta \}$ is relatively compact. When $Y$ is compact, $\mathcal{F}_c = \mathcal{F}$.

**Theorem 3.3.** Let $X' \subset X$, and $g \in \mathcal{G}$. Consider the following statements:

(16) Problem $(P')$ has a solution,

(17) $X' \subset \text{dom}(\partial^* g)$

(18) $\{(\partial^* g)^{-1}(y)\}_{y \in \mathcal{Y}}$ is a covering of $X'$,

(19) $\{(\partial^* g)^{-1}(y)\}_{y \in \text{dom}(B^* g)}$ is a covering of $X' \cap \text{dom}(g)$,

(20) $\{(\partial^* g)^{-1}(y)\}_{y \in \text{dom}(B^* g)}$ is a covering of $X' \cap \text{dom}(g)$.

We have: $(17) \Rightarrow (18) \Rightarrow (19) \Rightarrow (16, 20)$. The implication $(16) \Rightarrow (19)$ holds if we assume that $b$ is continuous in the second variable, that either $Y$ is discrete or $B^* g(y) > -\infty$ for all $y \in Y$, and that either (i) $B^* g \in \mathcal{F}_c$ or (ii) $b$ is coercive and $X' \subset \text{dom}(g) \cup g^{-1}(\infty)$. In particular, it holds when $Y$ is finite. The implication $(20) \Rightarrow (19)$ holds in case (ii). Finally, $(16-20)$ are true when $g = +\infty$ or $g = -\infty$.

**Corollary 3.4.** Consider $g \in \mathcal{G}$. Assume that $Y$ is finite. Then, $Bf = g$ has a solution $f \in \mathcal{F}$ if, and only if, $\{(\partial^* g)^{-1}(y)\}_{y \in \mathcal{Y}}$ is a covering of $X$.

The condition in Theorem 3.3 that $B^* g(y) > -\infty$ for all $y \in Y$, is fulfilled in very general situations: if $g(x) < +\infty$ for all $x \in X$, in particular if $\text{dom}(g) = \text{dom}(g)$ as in Corollary 3.5 below; or if $\mathcal{F} = X \times Y$ and $g \not= +\infty$ (this is the case when $B$ is the Legendre-Fenchel transform, and $g$ is proper).

**Corollary 3.5.** Consider $g \in \mathcal{G}$ such that $\text{dom}(g) = \text{dom}(g)$. Assume that $b$ is continuous in the second variable and coercive. Then, $Bf = g$ has a solution $f \in \mathcal{F}$ if, and only if, $\{(\partial^* g)^{-1}(y)\}_{y \in \mathcal{Y}}$ is a covering of $X$.

When $B$ is the Fenchel transform over $\mathbb{R}^n$, the implication $(16) \Rightarrow (20)$ of Theorem 3.3 says in particular that any l.s.c. proper convex function $g$ admits subdifferentials in $\text{dom}(g)$, and that these subdifferentials belong to the domain of $g^* = B^* g$, a well known result since for any l.s.c. convex function $g$ on $\mathbb{R}^n$, $\text{dom}(g)$ is the interior of $\text{dom}(g)$. Corollary 3.5 shows that if $g$ is finite and locally bounded from above everywhere, then $g$ is l.s.c and convex if, and only if, it has nonempty subdifferentials everywhere.

We also get:

**Corollary 3.6.** Let $g \in \mathcal{G}$. Assume that $B^* g \in \mathcal{F}_c$, that $b$ is continuous in the second variable, and that $B^* g(y) > -\infty$ for all $y \in Y$. Then, $Bf = g$ has a solution $f \in \mathcal{F}$ if, and only if, $\{(\partial^* g)^{-1}(y)\}_{y \in \mathcal{Y}}$ is a covering of $X$.

In that case, $g(x) < +\infty$ for all $x \in X$.

**Example 3.7.** The following counter-example shows that Assumption (A3) is useful. Consider $Y = \{ y_1 \}$, $\mathcal{F} = \mathbb{R}^\mathbb{N}$, $X = \{ x_1, x_2 \}$, $\mathcal{G} = \mathbb{R}^X$, and the Moreau
conjugacy (8.10) with \( \tilde{b}(x_1, y_1) = 0, \tilde{b}(x_2, y_1) = +\infty \). We have
\[
Bf = \begin{pmatrix} Bf(x_1) \\ Bf(x_2) \end{pmatrix} = \begin{pmatrix} -f(y_1) \\ +\infty - f(y_1) \end{pmatrix} \quad \forall f \in \mathcal{F}
\]
and
\[
B^* g(y_1) = \max(-g(x_1), (+\infty) - g(x_2)) \quad \forall g \in \mathcal{G}.
\]
Let
\[
g = \begin{pmatrix} g(x_1) \\ g(x_2) \end{pmatrix} = \begin{pmatrix} 0 \\ +\infty \end{pmatrix}.
\]
Then, \( Bf = g \) has a solution, namely, \( f(y_1) = 0 \). However, taking any \( \mathcal{F} \subset X \times Y \) in (14b), we get
\[
(\delta^* g)^{-1}(y_1) \subset \operatorname{arg \ max}_{x \in [x_1, x_2]} \tilde{b}(x, y_1) - g(x) = \{ x_1 \}
\]
which does not cover \( X = \{ x_1, x_2 \} \). Therefore, the implication (16) \( \Rightarrow \) (18) of Theorem 3.3 does not extend to the case of kernels \( b \) which take the \( +\infty \) value, even when \( Y \) is finite: these kernels do not satisfy Assumption (A3).

Let now
\[
g = \begin{pmatrix} g(x_1) \\ g(x_2) \end{pmatrix} = \begin{pmatrix} -\infty \\ 0 \end{pmatrix}.
\]
Then, \( Bf = g \) has no solution, but taking \( \mathcal{F} = X \times Y \) in (14b), we get
\[
(\delta^* g)^{-1}(y_1) = \operatorname{arg \ max}_{x \in [x_1, x_2]} b^*(x, y_1, g(x)) = X
\]
Therefore, the implication (18) \( \Rightarrow \) (16) of Theorem 3.3 does not extend to the case of kernels \( b \) which take the \( +\infty \) value for some \( (x, y) \in \mathcal{F} \).

### 3.2. Additional properties of \( B \), and proof Theorem 3.3.

In this section, we state several lemmas and prove successively the different assertions of Theorem 3.3. We first show some properties of the kernels \( b \) and \( b^* \).

By Theorem 2.1, we know that \((b(x, y, \cdot), b^*(x, y, \cdot))\) is a dual Galois connection between \( \overline{\mathcal{R}} \) and \( \mathcal{R} \). The following result, which uses Assumptions (A3) and (A4), shows that \((b(x, y, \cdot), b^*(x, y, \cdot))\) is almost a (non dual) Galois connection:

**Lemma 3.8.** For all \((x, y) \in \mathcal{S}, b(x, y, \cdot) \) is a decreasing homeomorphism of \( \overline{\mathcal{R}} \) with inverse \( b^*(x, y, \cdot) \).

For all \((x, y) \in X \times Y \), we have
\[
(21a) \quad (b(x, y, \beta) \geq \alpha \text{ and } \alpha, \beta > -\infty) \iff (b^*(x, y, \alpha) \geq \beta \text{ and } \alpha, \beta > -\infty).
\]
Moreover,
\[
(21b) \quad (b(x, y, \beta) \geq \alpha \text{ and } \alpha > -\infty) \Rightarrow b^*(x, y, \alpha) \geq \beta.
\]

**Proof.** When \((x, y) \in \mathcal{S}, b(x, y, \cdot) \) which is bijective and nonincreasing is trivially an homeomorphism of \( \overline{\mathcal{R}}, \) and by \((3), b^*(x, y, \cdot) \) is the inverse of \( b(x, y, \cdot). \) When the left-hand side of \((21b) \) is satisfied, we have \((x, y) \in \mathcal{S}, \) so \( b^*(x, y, \cdot) \) is the inverse of \( b(x, y, \cdot), \) which shows \((21b) \). Together with the symmetric implication, this shows \((21a) \).

We readily get from the first assertion of Lemma 3.8 and (13):

**Proposition 3.9.**
\[
(22) \quad g \in \mathcal{G} \text{ and } g = BB^* g \iff (\delta^* g)^{-1} = \delta B^* g.
\]
Remark 3.10. When $B$ is the Legendre-Fenchel transform, a function is in the image of $B$, or of $B^*$, if, and only if, it is convex, l.s.c., and proper, or identically $+\infty$, or identically $-\infty$. Then, (22) gives the classical inversion property of subdifferentials, $(\partial g)^{-1} = \partial^* g$, which holds for all convex l.s.c proper functions $g$.

Proof of (18)⇒(16) of Theorem 3.3. If $\{(\partial^* g)^{-1}(y)\}_{y \in Y}$ is a covering of $X'$, then $\partial^* g(x) \neq \emptyset$ for all $x \in X$. Let us prove that $f = B^* g$ satisfies (P'). Since by (2a), $BB^* g \leq g$, it is enough to prove that
\[(23) \quad \partial^* g(x) \neq \emptyset \Rightarrow BB^* g(x) \geq g(x).
\]
If $y \in \partial^* g(x)$, then, by (13b), $b^*(x,y,g(x)) = B^* g(y)$ and $(x,y) \in \mathcal{F}$, which yields $BB^* g(x) \geq b(x,y,B^* g(y)) = b(x,y,b^*(x,y,g(x)))$. By Lemma 3.8, $b(x,y,b^*(x,y,g(x))) = g(x)$, hence $BB^* g(x) \geq g(x)$, and (23) is shown. □

To pursue the proof of Theorem 3.3, we next show some properties of subdifferentials which generalize proposition 3.9.

Lemma 3.11. Consider $f \in \mathcal{F}$ and $g \in \mathcal{G}$ such that $Bf \leq g$, and let $E = \{x \in X \mid Bf(x) = g(x)\}$. Then,
\[(24a) \quad f \geq B^* g,
\]
\[(24b) \quad BB^* g(x) = g(x), \text{ for all } x \in E,
\]
and for all $y \in Y$,
\[(24c) \quad \partial f(y) \cap E = \begin{cases} \partial B^* g(y) \cap E &= (\partial^* g)^{-1}(y) \quad \text{if } f(y) = B^* g(y), \\ \emptyset & \text{otherwise.} \end{cases}
\]

Proof. By (2b), $Bf \leq g$ implies $f \geq B^* g$, which gives (24a). Since $B$ is antitone, applying $B$ to $f \geq B^* g$, we get $Bf \leq BB^* g$. By (2a), $BB^* g \leq g$, hence $Bf \leq BB^* g \leq g$, which implies (24b).

Let $x \in \partial f(y) \cap E$. Then, by (13a), $(x,y) \in \mathcal{F}$ and $Bf(x) = b(x,y,f(y))$. Using (24a) and the definition of $E$ and (A3), we get that $f(y) = b^*(x,y,g(x))$. Using (24a) and the definition of $B^*$, we obtain $B^* g(y) \leq f(y) = b^*(x,y,g(x)) \leq B^* g(y)$, which yields the equality. It follows that $f(y) = B^* g(y)$ and, using (13b), $y \in \partial^* g(x)$. This shows that $\partial f(y) \cap E \subset (\partial^* g)^{-1}(y)$ for all $y \in Y$, and that $\partial f(y) \cap E = \emptyset$ when $f(y) \neq B^* g(y)$. To prove the converse inclusion in (24c), let $y \in Y$ such that $f(y) = B^* g(y)$. Let $x \in (\partial^* g)^{-1}(y)$, then, by (13b), $(x,y) \in \mathcal{F}$ and $f(y) = B^* g(y) = b^*(x,y,g(x))$. It follows that $g(x) = b(x,y,f(y)) \leq Bf(x) \leq g(x)$, which yields the equality. Hence, $x \in E$ and, by (13a), $x \in \partial f(y)$. This shows that $\partial f(y) \cap E = (\partial^* g)^{-1}(y)$ when $f(y) = B^* g(y)$. Replacing $f$ by $B^* g$, we obtain that $BB^* g(y) \cap \{x \in X \mid BB^* g(x) = g(x)\} = (\partial^* g)^{-1}(y)$ for all $y \in Y$. Taking the intersection with $E$, using (24b) and using $(\partial^* g)^{-1}(y) = \partial f(y) \cap E \subset E$, we obtain $BB^* g(y) \cap E = (\partial^* g)^{-1}(y) \cap E = (\partial^* g)^{-1}(y)$ when $f(y) = B^* g(y)$. □

Lemma 3.12. Consider $f \in \mathcal{F}$ and $g \in \mathcal{G}$ such that $Bf \leq g$, and let $E = \{x \in X \mid Bf(x) = g(x)\}$. We have
\[(25a) \quad \bigcup_{y \in \partial f^{-1}(\infty)} \partial f(y) \cap E = \bigcup_{y \in \partial^* g^{-1}(\infty)} (\partial^* g)^{-1}(y) = g^{-1}(\infty),
\]
\[(25b) \quad \bigcup_{y \in \text{dom}(f)} \partial f(y) \cap E \subset \bigcup_{y \in \text{dom}(B^* g)} (\partial^* g)^{-1}(y) \subset \text{dom}(g),
\]
\[(25c) \quad \bigcup_{y \in \partial f^{-1}(-\infty)} \partial f(y) \cap E \subset \bigcup_{y \in \partial^* g^{-1}(-\infty)} (\partial^* g)^{-1}(y) \subset g^{-1}(\infty).
\]
Proof. The inclusion of the left term into the center term in (25a), (25b) and (25c) follows readily from (24c). If \( x \in (\partial^g)^{-1}(y) \) with \( y \in Y \), then, by (13b), \((x, y) \in \mathcal{S} \) and \( B^g(y) = b^*(x, y, g(x)) \). Using that \( b^*(x, y, \cdot) \) is a decreasing homeomorphism (Lemma 3.8), we get the inclusion of the center term into the right term in (25a), (25b) and (25c), which finishes the proof of (25b) and (25c). It remains to prove the inclusion of the right term into the left term in (25a). Consider \( x \in X \) such that \( g(x) = -\infty \). Since \( Bf \leq g \), we get that \( x \in E \). Moreover, since \( \mathcal{S}_x \neq \emptyset \), there is an \( y \in Y \) such that \( b(x, y, \cdot) \) is bijective. This implies that \( f(y) \geq B^g(y) \geq b^*(x, y, g(x)) = b^*(x, y, -\infty) = +\infty \), hence \( f(y) = +\infty \). Moreover, \( Bf(x) = g(x) = -\infty = b(x, y, +\infty) = b(x, y, f(y)) \), which shows, by (13a), that \( x \in \partial f(y) \). It follows that \( x \in \partial f(y) \cap E \) with \( f(y) = +\infty \), which finishes the proof of the inclusion of the right term into the left term in (25a). \( \square \)

Proof of (17) \( \Leftrightarrow \) (18) \( \Leftrightarrow \) (19) \( \Rightarrow \) (20) and of (20) \( \Rightarrow \) (19), in Theorem 3.3. The equivalence (17) \( \Leftrightarrow \) (18) holds trivially by the definition of \( \text{dom}(\partial^g) \) and of a covering. By (25a), we get \( \bigcup_{y \in \text{dom}(\partial^g)} (\partial^g)^{-1}(y) = \bigcup_{y \in \text{dom}(\partial^g)} (\partial^g)^{-1}(y) = X - \text{udom}(g) \), and since \( (B^g)^{-1}(+\infty) = Y \setminus \text{idom}(B^g) \), we deduce (18) \( \Rightarrow \) (19). By (25c), \( \bigcup_{y \in \text{dom}(\partial^g)} (\partial^g)^{-1}(y) \subset \bigcup_{y \in \text{dom}(\partial^g)} (\partial^g)^{-1}(y) \), hence we always have (19) \( \Rightarrow \) (20). Since \( \text{dom}(B^g) \subset \text{ldom}(B^g) \) and \( X' \cap \text{dom}(g) = X' \cap \text{dom}(g) \), we get (20) \( \Rightarrow \) (19) in case (ii). \( \square \)

Conditions (16–20) of Theorem 3.3 are trivial in the following degenerate cases:

**Proposition 3.13.** We have

\[(26a) \quad g \equiv +\infty \quad \Leftrightarrow \quad B^g \equiv -\infty , \]
\[(26b) \quad g \equiv -\infty \quad \Rightarrow \quad B^g \equiv +\infty . \]

In both cases, \( BB^g = g \) and \( \{(\partial^g)^{-1}(y)\}_{y \in Y} \) is a covering of \( X \). Moreover, if \( B^g \equiv +\infty \) and \( BB^g = g \), then \( g \equiv -\infty \).

Proof. The implication \( \Rightarrow \) in (26a) follows from (7). By symmetry, \( f \equiv +\infty \) implies \( Bf \equiv -\infty \). Taking \( f = B^g \), we get

\[(27) \quad B^g \equiv +\infty \quad \Rightarrow \quad BB^g \equiv -\infty , \]

which implies the last assertion of the lemma. If \( g \equiv -\infty \), then for all \( y \in Y \), taking \( x \in \mathcal{S}_y \), we get \( B^g(y) \geq b^*(x, y, g(x)) = +\infty \), which shows (26b). By symmetry, if \( f \equiv -\infty \), then \( Bf \equiv +\infty \). Applying this property to \( f = B^g \), we get that \( B^g \equiv -\infty \) implies \( BB^g \equiv +\infty \), and since \( g \geq BB^g \), \( g \equiv +\infty \), which shows the implication \( \Leftarrow \) in (26a), together with \( BB^g = g \). When \( g \equiv -\infty \), combining (26b) and (27), we also get \( BB^g = g \). Moreover, since \( \text{udom}(g) = \emptyset \), (19) is trivial with \( X' = X \), and by the equivalence (18) \( \Leftrightarrow \) (19), which has already been proved, we get that \( \{(\partial^g)^{-1}(y)\}_{y \in Y} \) is a covering of \( X \). It remains to show that the same covering property holds when \( g \equiv +\infty \). For all \( x \in X \) and \( y \in Y \), we have \( B^g(y) = +\infty \equiv b^*(x, y, g(x)) \). Taking \( y \in \mathcal{S}_x \), we get \( y \in \partial^g(x) \) by (13b), which shows that \( \bigcup_{y \in Y} (\partial^g)^{-1}(y) = X \) in this case, too. \( \square \)

We next mention some direct consequences of the definition of continuous in the second variable and coercive kernels.

**Lemma 3.14.** The kernel \( b \) is continuous in the second variable if, and only if, for all \( x \in X \) and \( a \in \mathbb{R} \), \( b(x, \cdot, a) \) is upper semicontinuous (u.s.c.). In that case, \( b(x, \cdot, a) \) is continuous, for all \( x \in X \) and \( a \in \mathbb{R} \cup \{+\infty\} \).
Proof. Proposition 2.3 shows that \( b(x,\cdot, a) \) is l.s.c. for all \( x \in X \) and \( a \in \mathbb{R} \). Hence, for all \( x \in X \) and \( a \in [0,\infty) \), \( b(x,\cdot, a) \) is u.s.c. if, and only if, it is continuous. Moreover, since, by (7), \( b(x,\cdot, +\infty) \equiv -\infty \), \( b(x,\cdot, +\infty) \) is always (trivially) continuous.

Note that the continuity assumption does not require that \( b(x,\cdot, -\infty) \) is continuous or u.s.c. (indeed, in the special case when \( b(x,\cdot, a) = b(x,\cdot) - a \), we have \( b(x,\cdot, -\infty) = b(x,\cdot) + \infty \), which need not be u.s.c if \( b(x,\cdot) \) is continuous and takes the \(-\infty\) value). The next lemma shows that assuming \( b \) or \( b^* \) to be u.s.c. (or equivalently continuous) is the same:

**Lemma 3.15.** Let \( x \in X \). Then, \( b(x,\cdot, \beta) \) is u.s.c. for all \( \beta \in \mathbb{R} \) if, and only if, \( b^*(x,\cdot, a) \) is u.s.c. for all \( a \in \mathbb{R} \).

**Proof.** We already observed in the proof of Lemma 3.14 that \( b(x,\cdot, +\infty) \) and \( b^*(x,\cdot, +\infty) \) are u.s.c, so it is enough to show that

\[
\begin{align*}
\text{(28)} && (b(x,\cdot, \beta) \text{ is u.s.c. } \forall \beta \in \mathbb{R} \cup \{+\infty\}) \iff (b^*(x,\cdot, a) \text{ is u.s.c. } \forall a \in \mathbb{R} \cup \{+\infty\})
\end{align*}
\]

The left hand side of (28) is equivalent to

\[
\begin{align*}
\text{(29)} && \{y \in Y \mid b(x, y, \beta) \geq a\} \text{ is closed } \forall a, \beta \in \mathbb{R} \cup \{+\infty\}
\end{align*}
\]

Applying (21a), we get that (29) is equivalent to

\[
\begin{align*}
\text{(30)} && \{y \in Y \mid b^*(x, y, a) \geq \beta\} \text{ is closed } \forall a, \beta \in \mathbb{R} \cup \{+\infty\},
\end{align*}
\]

which is exactly the upper semicontinuity of all the maps \( b^*(x,\cdot, a) \), with \( a \in \mathbb{R} \cup \{+\infty\} \). □

**Lemma 3.16.** If \( f \) is continuous in the second variable, then for all maps \( f \in \mathcal{F} \) such that \( f(y) > -\infty \) for all \( y \in Y \), the map \( y \mapsto b(x, y, f(y)) \) is u.s.c. for all \( x \in X \).

**Proof.** We have to show that \( \{y \in Y \mid b(x, y, f(y)) \geq \beta\} \) is closed, for all \( \beta \in \mathbb{R} \cup \{+\infty\} \). Since \( f(y) > -\infty \), for all \( y \in Y \), (21a) yields \( \{y \in Y \mid b(x, y, f(y)) \geq \beta\} = \{y \in Y \mid b^*(x, y, \beta) \geq f(y)\} \), which is closed since \( f \) is l.s.c. and \( b^*(x,\cdot, \beta) \) is u.s.c. (by Lemma 3.14). □

The following observation shows that we could have replaced "relatively compact" by "compact" in the definition of coercivity.

**Proposition 3.17.** If \( b \) is continuous in the second variable and coercive, then, for all \( a, \beta \in \mathbb{R} \), for all \( x \in X \) and neighborhoods \( V \) of \( x \), \( \{y \in Y \mid b^*_a V(y) \leq \beta\} \) is compact.

**Proof.** Since \( b^*_a V \) is given by the sup in (15), we have \( \{y \in Y \mid b^*_a V(y) \leq \beta\} = \cap_{z \in Y} Y_z \), where \( Y_z = \{y \in Y \mid b(z, y, b^*(x, y, a)) \leq \beta\} \). By (2b), \( Y_z \) is the set \( \{y \in Y \mid b^*(z, y, a) \leq b^*(x, y, a)\} \), which is closed because \( b^*(z,\cdot, \beta) \) is l.s.c. (by Theorem 2.1), and \( b^*(x,\cdot, a) \) is u.s.c. for \( a \in \mathbb{R} \) (by Lemma 3.15 and the continuity of \( b \) in the second variable). Therefore, \( \cap_{z \in Y} Y_z \), which is closed and relatively compact, is compact. □

The proof of (16)⇒(19) in Theorem 3.3 relies on the following result:
THEOREM 3.18. Let $f \in \mathcal{F}$. Assume that $b$ is continuous in the second variable, and that either $Y$ is discrete or $f(y) > -\infty$ for all $y \in Y$. (i) If $f \in \mathcal{F}_c$, then $\{\partial f(y)\}_{y \in \text{dom}(f)}$ is a covering of $\text{udom}(Bf)$. (ii) If $b$ is coercive, then $\{\partial f(y)\}_{y \in \text{dom}(f)}$ is a covering of $\text{dom}(Bf)$.

PROOF. We set $g = Bf$. We gather the proofs of Assertions (i) and (ii) by setting $X' = \text{udom}(g)$ when $f \in \mathcal{F}_c$, and $X' = \text{dom}(g)$ when $b$ is coercive. We thus need to prove that $X' \subseteq \bigcup_{y \in \text{dom}(f)} \partial f(y)$. Since, by (25a), $\bigcup_{y \in Y \setminus \text{dom}(f)} \partial f(y) = X' \setminus \text{udom}(g)$, and since $X' \subseteq \text{udom}(g)$, it is sufficient to prove that $X' \subseteq \bigcup_{y \in Y} \partial f(y)$.

We will prove:

$$x \in X' \implies \exists y \in Y, \ b(x, y, f(y)) = \sup_{y' \in Y} b(x, y', f(y')).$$ 

Indeed, if (31) holds, then for all $x \in X'$, there exists $y \in Y$ such that $b(x, y, f(y)) \geq Bf(x) = g(x)$ and since $X' \subseteq \text{udom}(g)$, $g(x) \neq -\infty$. Hence $b(x, y, f(y)) \neq -\infty$, whence $(x, y) \in \mathcal{F}$, which implies with $b(x, y, f(y)) = Bf(x)$ that $x \in \partial f(y)$. This shows that $X' \subseteq \bigcup_{y \in Y} \partial f(y)$.

To prove (31), it suffices to show that

$$\forall x \in X', \forall \alpha \in \mathbb{R}, \ L_\alpha(x) = \{y \in Y \mid b(x, y, f(y)) \geq \alpha \}$$

is compact.

Let us first prove that the sets $L_\alpha(x)$ are closed for all $x \in X$ and $\alpha \in \mathbb{R}$. When $Y$ is discrete, this is trivial. Otherwise, by the assumptions of the theorem, $f(y) > -\infty$, for all $y \in Y$, and $b$ is continuous in the second variable, therefore, by Lemma 3.16, $y \mapsto b(x, y, f(y))$ is an u.s.c. map for all $x \in X$. This implies again that the sets $L_\alpha(x)$ are closed for all $x \in X$ and $\alpha \in \mathbb{R}$.

It remains to show that the sets $L_\alpha(x)$ are relatively compact for all $x \in X'$ and $\alpha \in \mathbb{R}$. By definition of $\mathcal{F}_c$, this holds trivially for any $X' \subseteq X$, when $f \in \mathcal{F}_c$. It remains to consider the case where $b$ is coercive and $X' = \text{dom}(g)$. Let $x \in \text{dom}(g)$ and $\alpha \in \mathbb{R}$. There exists $\beta \in \mathbb{R}$ such that $\lim_{x' \to x} g(x') < \beta$, so there exists a neighborhood $V$ of $x$ in $X$ such that $\sup_{x' \in V} g(x') \leq \beta$. Then, by (21b):

$$b(x, y, f(y)) \geq \alpha \implies b^*(x, y, \alpha) \geq f(y)$$

and since

$$f(y) \geq B^*g(y) \geq b^*(z, y, g(z)) \geq b^*(z, y, \beta) \quad \forall z \in V,$$

we obtain:

$$b(x, y, f(y)) \geq \alpha \implies \forall z \in V, \ b^*(x, y, \alpha) \geq b^*(z, y, \beta)$$

$$\implies \forall z \in V, \ \beta \geq b(z, y, b^*(x, y, \alpha)),$$

which shows that $L_\alpha(x) \subseteq \{y \in Y \mid b_{z, y}^*(x) \leq \beta\}$. By the coercivity of $b$, the latter set is relatively compact, and thus $L_\alpha(x)$ is also relatively compact. This finishes the proof of (32) in all cases. $\square$

Proof of (16) $\Rightarrow$ (19) in Theorem 3.3. If $(\mathcal{P}')$ has a solution $f \in \mathcal{F}$, then, by Lemma 3.11, $f = B^*g$ is also a solution of $(\mathcal{P}')$. Fix $f = B^*g$. By Lemma 3.12, $(\bigcup_{y \in \text{dom}(f)} \partial f(y)) \cap X' \subset \bigcup_{y \in \text{dom}(B^*g)} (\partial^* g)^{-1}(y)$. Hence, applying Theorem 3.18 to $f$ in Case (i), we deduce that $\text{udom}(Bf) \cap X' \subset \bigcup_{y \in \text{dom}(B^*g)} (\partial^* g)^{-1}(y)$. Since it is clear that $X' \cap \text{udom}(g) \subseteq X' \cap \text{udom}(Bf)$, we get (19) in Case (i) of Theorem 3.3 without any condition on $X'$. Similarly, applying Theorem 3.18 to $f$ in Case (ii), we obtain that $\text{dom}(Bf) \cap X' \subset \bigcup_{y \in \text{dom}(f)} \partial f(y) \cap X' \subset \bigcup_{y \in \text{dom}(B^*g)} (\partial^* g)^{-1}(y)$.
Moreover, it is easy to show that $X' \subset \text{dom}(g) \cup g^{-1}(-\infty)$ implies $X' \cap \text{dom}(Bf) \subset X' \cap \text{dom}(Bf)$, hence (19) follows in Case (ii) of Theorem 3.3.

Proof of Corollary 3.6. The assumptions of the corollary imply that of the case (i) of Theorem 3.3. Hence the equivalence (16)$\Leftrightarrow$(18) when $X'=X$ yields the first assertion of the corollary. Assume that $\{(\partial^x g)^{-1}(y)\}_{y \in Y}$ is a covering of $X$. Then, for all $x \in X$, there exists $y \in \partial^x g(x)$. By (13b), $(x, y) \in \mathcal{F}$ and $B^x g(y) = b^*(x, y, g(x))$. Since $B^x g(y) > -\infty$ for all $y \in Y$, and $b^*(x, y, \cdot)$ is a decreasing homeomorphism of $\mathbb{R}$ (Lemma 3.8), we deduce that $g(x) < +\infty$, for all $x \in X$.

4. Uniqueness of Solutions of $Bf = g$

4.1. Statement of the uniqueness results. To give an uniqueness result, we need some additional definitions.

Definition 4.1. Let $F$ be a map from a set $Z$ to the set $\mathcal{P}(W)$ of all subsets of some set $W$, and let $Z' \subset Z$ and $W' \subset W$ be such that $\{F(z)\}_{z \in Z'}$ is a covering of $W'$. An element $y \in Z'$ is said algebraically essential with respect to the covering $\{F(z)\}_{z \in Z'}$ if there exists $w \in W'$ such that $w \not\in \cup_{z \in Z' \setminus \{y\}} F(z)$. When $Z$ is a topological space, an element $y \in Z'$ is said topologically essential with respect to the covering $\{F(z)\}_{z \in Z'}$ of $W'$ if for all open neighborhoods $U$ of $y$ in $Z'$, there exists $w \in W'$ such that $w \not\in \cup_{z \in Z' \setminus \{y\}} F(z)$. The covering of $W'$ by $\{F(z)\}_{z \in Z'}$ is algebraically (resp. topologically) minimal if all elements of $Z'$ are algebraically (resp. topologically) essential.

Algebraic minimality implies topological minimality. Both notions coincide if $Z$ is a discrete topological space.

Definition 4.2. Let $f \in \mathcal{F}$ and $X' \subset X$. We say that $y \in Y$ is an exposed point of $f$ relative to $X'$ if there exists $x \in X'$ such that $(x, y) \in \mathcal{F}$ and

$$b(x, y', f(y')) < b(x, y, f(y)) \quad \forall y' \in Y \setminus \{y\}.$$

When $B$ is the Fenchel transform and $X' = X$, this notion coincides with the definition given in [11, Def. 2.3.3] of an exposed point of $f$. It is equivalent to the property that $(p, f(y))$ is an exposed point of the epigraph of $f$ [24, Sections 18 and 25]. We readily get from Definitions 3.1 and 4.2:

Lemma 4.3. Let $f \in \mathcal{F}$ and let $X' \subset \cup_{y \in Y} \partial f(y)$. An element $y \in Y$ is an exposed point of $f$ relative to $X'$ if, and only if, $y$ is algebraically essential with respect to the covering $\{\partial f(z)\}_{z \in Y}$ of $X'$.

Definition 4.4. We say that a map $h : X \to Y$ is quasi-continuous if for all open sets $G$ of $Y$, the set $h^{-1}(G)$ is semi-open, which means that $h^{-1}(G)$ is included in the closure of its interior.

One can see for instance [26] for definitions and properties of quasi-continuous functions or multi-applications. If $h : X \to \mathbb{R}$ is l.s.c., then $h$ is quasi-continuous if, and only if, $h = \text{lsc(usc)}(h)$, where lsc (resp. usc) means the l.s.c. (resp. u.s.c.) hull. The notion of quasi-continuous function, and the properties of l.s.c. or u.s.c. quasi-continuous functions have also been studied in [26].
Definition 4.5. We say that $B$ is regular if for all $f \in \mathcal{F}$, $Bf$ is l.s.c. on $X$ and quasi-continuous on its domain, which means that the restriction of $Bf$ to its domain is quasi-continuous for the induced topology.

The notion of regularity for $B^*$ is defined in the symmetric way. When $X$ (resp. $Y$) is endowed with the discrete topology, $B$ (resp. $B^*$) is always regular. When $\mathcal{F} = X \times Y$ and \{b(c, y, a)\}_{y \in Y, a \in \mathbb{R}} is an equicontinuous family of functions, then $Bf$ is continuous on $X$ for any $f \in \mathcal{F}$, so $B$ is regular. The Fenchel transform on $\mathbb{R}^n$ is regular (see Lemma 6.1 below).

Theorem 4.6. Let $X' \subset X$, and $g \in \mathcal{G}$. Assume that $\{(\delta^*g)^{-1}(y)\}_{y \in \text{idom}(B^*g)}$ is a covering of $X' \cap \text{udom}(g)$, and denote by $Z_\alpha$ (resp. $Z_\beta$) the set of algebraically (resp. topologically) essential elements with respect to this covering. Assume that $b$ is continuous in the second variable, that either $Y$ is discrete or $B^*g(y) > -\infty$ for all $y \in Y$, and that either (i) $B^*g \in \mathcal{F}_c$ or (ii) $b$ is coercive together with $X' \subset \text{idom}(g) \cup g^{-1}(-\infty)$. Then, Problem $(P')$ has a solution and any solution $f$ of $(P')$ satisfies

$$f \geq B^*g, \quad f(y) = B^*g(y) \quad \text{for all } y \in Z_\alpha.$$  

If, in addition, $B^*g$ is quasi-continuous on its domain, and int($Z_\beta$) denotes the interior of $Z_\beta$, relatively to idom($B^*g$), then any solution $f$ of $(P')$ satisfies also

$$f(y) = B^*g(y) \quad \text{for all } y \in \text{int}(Z_\beta).$$  

Theorem 4.7. Let $X' \subset X$, and $g \in \mathcal{G}$. Consider the following statements:

(35) Problem $(P')$ has a unique solution,

(36) $\{(\delta^*g)^{-1}(y)\}_{y \in \text{idom}(B^*g)}$ is a topologically minimal covering of $X' \cap \text{udom}(g)$.

We have (35)$\Rightarrow$(36). Assume that $b$ is continuous in the second variable, that either $Y$ is discrete or $B^*g(y) > -\infty$ for all $y \in Y$, and that either (i) $B^*g \in \mathcal{F}_c$ or (ii) $b$ is coercive together with $X' \subset \text{idom}(g) \cup g^{-1}(-\infty)$. Then, the implication (35)$\Rightarrow$(36) holds. The equivalence (35)$\Leftrightarrow$(36) holds if we assume in addition that $B^*g$ is quasi-continuous on its domain. In particular, it holds when $Y$ is finite.

The topological minimality in (36) is a relaxation of algebraic minimality, which is a generalized differentiability condition. Indeed, if $\{(\delta^*g)^{-1}(y)\}_{y \in \text{idom}(B^*g)}$ is a covering of $X' \cap \text{udom}(g)$, this covering is algebraically minimal if, and only if, for all $y \in \text{idom}(B^*g)$, there is an $x \in X' \cap \text{udom}(g)$ such that $\delta^*g(x) = \{y\}$. This is in particular fulfilled when $\{(\delta^*g)(x)\}_{x \in X' \cap \text{udom}(g)}$ is a covering of Idom($B^*g$), and for all $x \in X' \cap \text{udom}(g)$, $\delta^*g(x)$ is a singleton, a condition which, in the case where $B$ is the Fenchel transform, means that $g$ is differentiable at $x$.

Corollary 4.8. Consider $g \in \mathcal{G}$. Assume that $Y$ is finite. Then, the equation $Bf = g$ has a unique solution $f \in \mathcal{F}$, if, and only if, $\{(\delta^*g)^{-1}(y)\}_{y \in \text{idom}(B^*g)}$ is a topologically minimal covering of $\text{udom}(g)$.

Corollary 4.9. Let $g \in \mathcal{G}$. Assume that $b$ is continuous in the second variable, that $B^*g(y) > -\infty$ for all $y \in Y$, that $B^*g \in \mathcal{F}_c$, and that $B^*g$ is quasi-continuous on its domain. Then, the equation $Bf = g$ has a unique solution $f \in \mathcal{F}$, if, and only if, $\{(\delta^*g)^{-1}(y)\}_{y \in \text{idom}(B^*g)}$ is a topologically minimal covering of $\text{udom}(g)$.
Since (35) implies that Problem (P) has at most one solution, Theorem 4.7 yields a sufficient uniqueness condition for the solution of Problem (P). However, for Problem (P), the necessary uniqueness condition implied by Theorem 4.7 only holds when \( B^s g \in \mathcal{F}^* \), or when \( X = \text{dom}(g) \cup g^{-1}(-\infty) \), or when \( X = \text{dom}(\partial^s g) \).

To give a more specific uniqueness result, we shall use the following condition:

there exists a basis \( \mathcal{B} \) of neighborhoods such that

\[
(\mathcal{C}) : \quad \forall U \in \mathcal{B}, \, \exists \varepsilon > 0, \, \forall x \in X, \quad \sup_{y \in U \cap X_0, \, a \in \mathbb{R}} (b(x, y, a) - b(x, y, a + \varepsilon)) < +\infty.
\]

Condition (\( \mathcal{C} \)) is satisfied in particular when \( \{ b(x, y, \cdot) \}_x \subseteq Y \) is a family of \( \beta \)-Hölder continuous functions (for \( 0 < \beta \leq 1 \)), uniformly in \( y \in U \), for all small enough open sets \( U \), or if \( \{ b(x, y, \cdot) \}_x \subseteq Y \) is an equicontinuous family, for all small enough open sets \( U \). In particular, condition (\( \mathcal{C} \)) is satisfied when \( b(x, y, a) = b(x, y) - a \) or when \( b(x, y, a) = -\left| (x, y) \right| + 1 \). 

**Theorem 4.10.** Let \( g \in \mathcal{F} \). Then, the existence and uniqueness of a solution of Problem (P) implies (36), when any of the following assumptions holds:

1. \( X' = \text{dom}(\partial^s g) \), and \( Y \) is discrete.
2. \( X' = \text{dom}(\partial^s g) \), \( Y \) is locally compact, \( b \) is continuous in the second variable, and \( B^s g(y) > -\infty \) for all \( y \in Y \).
3. \( X' = \text{dom}(g) \), \( b \) is continuous in the second variable and coercive, \( B^s g(y) > -\infty \) for all \( y \in Y \), \( B \) is regular, \( \text{dom}(g) \) is included in the closure of \( \text{dom}(g) \), and either \( Y \) is locally compact or condition (\( \mathcal{C} \)) holds.

4.2. **Proofs of the uniqueness results.** Let us first state a general property of quasi-continuous functions.

**Lemma 4.11.** Let \( f \in \text{lsc}(Y, \mathbb{R}) \), and let \( h : Y \to \mathbb{R} \) be a quasi-continuous map. Then, the set \( V = \{ y \in Y \mid h(y) < f(y) \} \) is semi-open. In particular, if \( V \) is non-empty, \( V \) has a non-empty interior.

**Proof.** It suffices to consider the case when \( V \) is non-empty. Let \( z \in V \). There exists \( a \in \mathbb{R} \) such that \( h(z) < a < f(z) \). Consider \( V_1 = \{ y \in Y \mid h(y) < a \} \), \( U_1 \) the interior of \( V_1 \), and \( U_2 = \{ y \in Y \mid a < f(y) \} \). We have \( z \in V_1 \cap U_2 \subset V \). Since \( h \) is quasi-continuous, \( V_1 \) is semi-open, hence \( V_1 \) is included in the closure of \( U_1 \), that we denote by \( \overline{U_1} \). Since \( f \) is l.s.c., \( U_2 \) is open. We have \( z \in V_1 \cap U_2 \subset U_1 \cap U_2 \subset U_1 \cap U_2 \), and since \( U_1 \cap U_2 \) is open and included in \( V \), we get that \( z \) belongs to the closure of the interior of \( V \).

**Proof of Theorem 4.6.** Note first that since, when \( B f \leq g \), the equation \( B f = g \) on \( X' \) is equivalent to the equation \( B f = g \) on \( X' \cap \text{dom}(g) \), we can assume without restriction of generality that \( X' \subset \text{dom}(g) \), in the proofs of Theorems 4.6 and 4.7.

Let \( Y' = \text{idom}(B^s g) \), assume that \( \{ (\partial^s g)^{-1}(y) \}_{y \in Y'} \) is a covering of \( X' \), and denote by \( Z_0 \) (resp. \( Z'_0 \)) the set of algebraically (resp. topologically) essential elements with respect to this covering. Since (19) holds, it follows from Theorem 3.5, that (\( \mathcal{P}^* \)) has a solution \( f \in \mathcal{F} \), and by Lemma 3.11, \( B^s g \) is necessarily another solution and \( B^s g \leq f \).

Let \( f \in \mathcal{F} \) be a solution of (\( \mathcal{P}^* \)) and denote by \( F = \{ y \in Y' \mid B^s g(y) = f(y) \} \) and by \( V \) its complementary in \( Y' \). Since, \( B^s g \leq f \), \( V = \{ y \in Y' \mid B^s g(y) < f(y) \} \). We claim that

\[
(37) \quad \forall x \in X', \exists y \in F \text{ such that } x \in (\partial^s g)^{-1}(y).
\]
If (37) is proved, then the following holds
\begin{equation}
Z_c \subseteq F \text{ and } Z_t \subseteq \overline{F},
\end{equation}
where \( \overline{F} \) denotes the closure of \( F \), relatively to \( Y' \). Indeed, let us first consider \( z \in Z_c \). Since \( z \) is algebraically essential with respect to the covering \( \{(\delta^* g)^{-1}(y)\} \) of \( X' \), there exists \( x \in X' \) such that
\begin{equation}
x \in (\delta^* g)^{-1}(z) \setminus \bigcup_{y \in Y' \setminus \{z\}} (\delta^* g)^{-1}(y).
\end{equation}
Moreover, by (37), there exists \( y \in F \) such that \( x \in (\delta^* g)^{-1}(y) \). Using (39), this implies that \( y = z \) and \( z \in F \), which shows \( Z_c \subseteq F \). Let us now consider \( z \in Z_t \). Since \( z \) is topologically essential with respect to the covering \( \{(\delta^* g)^{-1}(y)\} \) of \( X' \), for all open neighborhoods \( U \) of \( z \) in \( Y' \), there exists \( x_U \in X' \) such that
\begin{equation}
x_U \in \bigcup_{y \in Y'} (\delta^* g)^{-1}(y) \setminus \bigcup_{y \in Y' \setminus \{y_U\}} (\delta^* g)^{-1}(y_U).
\end{equation}
Moreover, by (37), there exists \( y_U \in F \) such that \( x_U \in (\delta^* g)^{-1}(y_U) \). Using (40), this implies that \( y_U \in U \). We have thus proved that for all open neighborhood \( U \) of \( z \) in \( Y' \), there exists \( y_U \in U \cap F \), which means that \( z \in \overline{F} \), and shows \( Z_t \subseteq \overline{F} \).

Now from (38), we get (33). When \( Y \) is discrete, \( Z_c = Z_t \) and thus (34) holds trivially by (33). Otherwise, \( Y' = \text{Idom}(B^* g) = \text{Idom}(B^* g) \) and if \( B^* g \) is quasi-continuous on its domain, then, since \( f \) is l.s.c. on \( Y' \), and a fortiori on \( Y' \), we obtain, by Lemma 4.11, that \( V \) is semi-open in \( Y' \). It follows that its complement in \( Y' \), \( F \), contains the interior of its closure \( \overline{F} \), relatively to \( Y' \). Using (38), this yields int(\( Z_t \)) \subseteq F, which means (34).

Let us prove (37). When \( f \) is a solution of \((P')\), \( X' \subset \text{Idom}(g) \) implies \( X' \subset \text{Idom}(B f) \), \( X' \subset \text{Idom}(g) \) implies \( X' \subset \text{Idom}(B f) \), and \( B^* g \in \mathscr{F} \) if \( f \in \mathscr{F} \) (since \( f \geq B^* g \)). Hence, Theorem 3.18 shows that \( X' \subset \bigcup_{y \in X'} \partial f(y) \). Since \( B^* g \leq f \) implies that \( \text{Idom}(f) \subset \text{Idom}(B^* g) = Y' \), we get that \( X' \subset \bigcup_{y \in Y'} \partial f(y) \).

Hence, for all \( x \in X' \), there exists \( y \in Y' \) such that \( x \in \partial f(y) \). Since then \( \partial f(y) \) an \( X' \neq \emptyset \), Lemma 3.11 shows that \( f(y) = B^* g(y) \) and \( \partial f(y) \subset \{\delta^* g \} \) \( \} \). Hence, \( y \in F \) and \( x \in (\delta^* g)^{-1}(y) \) which shows (37).

We now prove the different assertions of Theorem 4.7.

**Proof of (35,19)⇒(36) in Theorem 4.7.** We assume, as in the above proof, that \( X' \subset \text{Idom}(g) \). Set \( Y' = \text{Idom}(B^* g) \). Assume that (35) and (19) hold, that is \((P')\) has a unique solution and \( \{(\delta^* g)^{-1}(y)\} \) is a covering of \( X' \). Assume by contradiction that this covering is not topologically minimal, i.e., that there exists an open set \( U \) of \( Y \) such that \( U \cap Y' \neq \emptyset \), and such that for all \( x \in X' \), there exists \( y \in Y' \setminus U \) such that \( x \in (\delta^* g)^{-1}(y) \), which means, by (13b), that \( (x, y) \in \mathscr{F} \) and \( B^* g(y) = B^*(x, y, g(x)) \). Then, \( g(x) = \sup_{y \in Y' \setminus U} b(x, y, B^* g(y)) \) and since \( g \geq B^* g \) we get:
\begin{equation}
g(x) = \sup_{y \in Y' \setminus U} b(x, y, B^* g(y)) \quad \forall x \in X'.
\end{equation}
To contradict the uniqueness of the solution of \((P')\), it suffices to build a map \( f \in \mathscr{F} \) such that \( f \neq B^* g \) and
\begin{equation}
f \geq B^* g, \quad f = B^* g \text{ on } Y \setminus U.
\end{equation}
Indeed, for any function \( f \) satisfying (42), we have \( B f \leq g \), and, by (41), \( B f \geq g \) on \( X' \), hence \( f \) is a solution of \((P')\). The function \( f = B^* g \) satisfies trivially \( f \in \mathscr{F} \) and (42). Defining \( f \) by \( f = B^* g \) on \( Y \setminus U \) and \( f = +\infty \) on \( U \), we obtain that
\( f \in \mathcal{F} \), \( f \) satisfies (42) and since \( Y' \cap U \neq \emptyset \), \( f \neq B^*g \), which concludes the proof. \( \square \)

**Proof of (35)\( \Rightarrow \) (36) in Theorem 4.7.** Assume that \( b \) is continuous in the second variable, that either \( Y \) is discrete or \( B^*g(y) > -\infty \) for all \( y \in Y \), and that either (i) \( B^*g \in \mathcal{F} \), or (ii) \( b \) is coercive and \( X' \subseteq \text{Idom}(g) \cup g^{-1}(-\infty) \). If (35) holds, it follows that \((P')\) has a solution, hence, by Theorem 3.3, (19) holds. From the implication (35,19)\( \Rightarrow \) (36) that we already proved, we obtain (35)\( \Rightarrow \) (36). \( \square \)

**Proof of (36)\( \Rightarrow \) (35) in Theorem 4.7.** The assumptions for this implication imply that of Theorem 4.6, together with the property that any element of \( \text{Idom}(B^*g) \) is topological essential for the covering \( \{ (\partial^*g)^{-1}(y) \}_{y \in \text{Idom}(B^*g)} \) of \( X' \cap \text{udom}(g) \). This means that \( Z_0 = \text{Idom}(B^*g) \) in Theorem 4.6, thus \( \text{int}(Z_0) = Z_0 = \text{Idom}(B^*g) \). Hence, the conclusions of Theorem 4.6 imply that \((P')\) has a solution and that any solution \( f \) of \((P')\) satisfies \( f \geq B^*g \) and \( f(y) = B^*g(y) \) for all \( y \in \text{Idom}(B^*g) \). Since \( f \geq B^*g \), then \( f(y) = B^*g(y) = +\infty \) for all \( y \in Y \setminus \text{Idom}(B^*g) \). Therefore, \( f = B^*g \).

We now prove Theorem 4.10.

**Proof of Theorem 4.10 in Cases (1) and (2).** Assume that \( Bf = g \) has a unique solution \( f \in \mathcal{F} \) and that Condition (1) or Condition (2) holds. In particular, \( BB^*g = g \). Note first that in Case (1), all the properties of Case (2) hold but the condition \( B^*g(y) > -\infty \) for all \( y \in Y \), which is replaced by the condition that \( Y \) is discrete. Set \( Y' = \text{Idom}(B^*g) \) and \( X' = \text{dom}(\partial^*g) \). By Theorem 3.3, \( \{ (\partial^*g)^{-1}(y) \}_{y \in Y'} \) is a covering of \( X' \cap \text{udom}(g) \). We shall show that this covering is minimal. Arguing by contradiction, and using the same arguments as in the proof of the implication (35,19)\( \Rightarrow \) (36) in Theorem 4.7, we obtain that there exists an open set \( U \) of \( Y \) such that \( U \cap Y' \neq \emptyset \) and (41) holds. Possibly after replacing \( U \) by an open subset, we can assume that \( U \) is relatively compact, that is its closure is compact. Taking \( f = B^*g \) on \( Y \setminus U \) and \( f = +\infty \) on \( U \), we deduce, as in the proof of (35,19)\( \Rightarrow \) (36), that \( f \neq B^*g \), \( Bf \leq g \) and \( Bf = g \) on \( X' \) (we obtain \( Bf \leq g \) and \( Bf = g \) on \( X' \cap \text{udom}(g) \), and since \( Bf \leq g \Rightarrow Bf = g \) on \( g^{-1}(-\infty) \), we get \( Bf = g \) on \( X' \)), hence

\[ S \overset{\text{def}}{=} \{ x \in X \mid Bf(x) \neq g(x) \} = \{ x \in X \mid Bf(x) < g(x) \} \subset X \setminus X', \]

(43)

It remains to check that \( S = \emptyset \), in order to contradict the uniqueness of the solution \( f \) of \( Bf = g \). Assume by contradiction that \( S \neq \emptyset \). If \( x \in S \), it follows from (42), that \( \sup_{y \in Y \setminus U} b(x, y, B^*g(y)) \leq Bf(x) < g(x) \). Since \( BB^*g = g \), we get

\[ g(x) = \sup_{y \in U} b(x, y, B^*g(y)). \]

Since \( U \) has a compact closure \( \overline{U} \), and the function \( y \mapsto b(x, y, B^*g(y)) \) is u.s.c (trivially in Case (1) and by Lemma 3.16 in Case (2)),

(44)

\[ \exists y \in \overline{U} \text{ such that } g(x) = b(x, y, B^*g(y)). \]

Since \( g(x) > -\infty \), we get that \( (x, y) \in \mathcal{F} \) and, by (13a) and Proposition 3.9, \( x \in \partial B^*g(y) = (\partial^*g)^{-1}(y) \), which implies that \( x \in \text{dom}(\partial^*g) = X' \). By (43), we get a contradiction. \( \square \)
Proof of Theorem 4.10 in Case (3). Assume that \( Bf = g \) has a unique solution \( f \in \mathcal{S} \) and that Condition (3) holds. In particular, \( BB^*g = g \). Set \( Y' = \text{Idom}(B^*g) = \text{dom}(B^*g) \) and \( X' = \text{Idom}(g) \). Theorem 3.3 shows that \( \{ (B^*g)^{-1}(y) \}_{y \in Y'} \) is a covering of \( X' \cap \text{udom}(g) = \text{dom}(g) \). We shall show that this covering is minimal.

Arguing by contradiction, and using the same arguments as before, there exists an open set \( U \) of \( Y \) such that \( U \cap Y' \neq \emptyset \) and (41) holds. For any given basis of open neighborhoods in \( Y, \mathcal{B} \), possibly after replacing \( U \) by an open subset, we can assume that \( U \in \mathcal{B} \) and \( U \cap Y' \neq \emptyset \). We shall take either \( \mathcal{B} \) as in condition (3), or \( \mathcal{B} \) as the basis of relatively compact open sets.

Fix \( \varepsilon > 0 \) and consider the l.s.c. finite function \( w : Y \to [0,1] \) given by \( w(y) = 0 \) for \( y \in Y \setminus U \) and \( w(y) = \varepsilon \) for \( y \in U \). Taking \( f = B^*g + w \), we get that \( f \) is l.s.c. \( f \) satisfies (42), and \( f \neq B^*g \) since \( f(y) = B^*g(y) + \varepsilon > B^*g(y) \) for \( y \in \text{dom}(B^*g) \cap U = Y' \cap U \neq \emptyset \). As in the proof of the (35,19) ⇒ (36), we deduce that \( Bf \leq g \), \( Bf = g \) on \( X' \), hence (43) holds. As in the above proof, it remains to show that \( S = \emptyset \).

Assume by contradiction that \( S \neq \emptyset \). We first prove that

\[
\begin{align}
(45a) & \quad \text{Idom}(g) \subset \text{Idom}(Bf), \\
(45b) & \quad \text{udom}(g) \subset \text{udom}(Bf).
\end{align}
\]

Indeed, (45a) follows from \( Bf \leq g \). Let \( x \in \text{udom}(g) \), hence \( g(x) > -\infty \). Since \( g(x) = \sup_{y \in Y} b(x, y, B^*g(y)) \), there exists \( y \in Y \) such that \( b(x, y, B^*g(y)) > -\infty \), hence \( (x, y) \in \mathcal{S} \) and \( B^*g(y) < +\infty \), then \( f(y) \leq B^*g(y) + \varepsilon < +\infty \) and \( Bf(x) \geq b(x, y, f(y)) > -\infty \), which concludes the proof of (45b).

Since \( S \subset \text{Idom}(Bf) \cap \text{udom}(g) \), we deduce from (45b), that \( S \subset \text{dom}(Bf) \). Since \( B \) is regular, \( Bf \) and \( g \) are l.s.c. on \( X \) and quasi-continuous on their domain. Hence, by Lemma 4.11, \( S \) is semi-open relatively to \( \text{dom}(Bf) \). Since \( S \neq \emptyset \), \( S \) has a nonempty interior relatively to \( \text{dom}(Bf) \). This means that there exists an open set \( V \) of \( X \) such that

\[
(46) \quad \emptyset \neq V \cap \text{dom}(Bf) \subset S.
\]

By (46) and (45), we get

\[
(47) \quad V \cap \text{dom}(g) \subset S.
\]

If we know that \( V \cap \text{Idom}(g) \neq \emptyset \), then since we assumed that \( \text{Idom}(g) \) is dense in \( \text{dom}(g) \), we get \( V \cap \text{Idom}(g) \neq \emptyset \), so by (47), \( S \cap \text{Idom}(g) \neq \emptyset \), i.e. \( S \cap X' \neq \emptyset \), which contradicts (43). It remains to show that \( V \cap \text{dom}(g) \neq \emptyset \).

If \( Y \) is locally compact, (44) holds as before for all \( x \in S \). Since \( B^*g(y) > -\infty \) for all \( y \in Y \), we deduce that \( g(x) < +\infty \), hence \( S \subset \text{Idom}(g) \). Since we also have \( S \subset \text{udom}(g) \), we get \( S \subset \text{dom}(g) \) and by (46) and (45), \( V \cap \text{dom}(g) = V \cap \text{dom}(Bf) \neq \emptyset \).
Otherwise, \( b \) satisfies Condition (C), and if \( U \in \mathcal{B} \) and \( \varepsilon > 0 \) is chosen as in (C), we get that for all \( x \in X \),
\[
g(x) \geq B f(x) = \sup_{y \in Y, \, B^s g(y) < +\infty} b(x, y, B^s g(y) + w(y))
\]
\[
\geq \sup_{y \in \mathcal{X}, \, B^s g(y) < +\infty} b(x, y, B^s g(y))
\]
\[
+ \inf_{y \in \mathcal{X}, \, B^s g(y) < +\infty} \left( b(x, y, B^s g(y) + w(y)) - b(x, y, B^s g(y)) \right)
\]
\[
\geq g(x) + \inf_{y \in \mathcal{X} \cap U, \, \alpha \in \mathbb{R}} \left( b(x, y, \alpha + \varepsilon) - b(x, y, \alpha) \right).
\]

By (C), we obtain \( \text{dom}(g) = \text{dom}(B f) \), which shows, by (46), that \( V \cap \text{dom}(g) \neq \emptyset \). \( \square \)

5. Algorithmic issues and geometrical interpretation

When \( X \) and \( Y \) are finite, and when the kernels \( b \) and \( b^s \) are given in some effective way, Corollaries 3.4 and 4.8 yield an algorithm à la K. Zimmermann to solve the equation \( B f = g \) and to decide the uniqueness of its solution. Let us explain the algorithm by taking \( X = \{x_1, x_2\} \) and \( Y = \{y_1, y_2, y_3\} \). Let \( B : \mathbb{R}^3 \to \mathbb{R}^X \) denote the functional Galois connection whose kernel \( b \) is given by the following table:

\[
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix}
= \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\begin{pmatrix}
-\lambda & 4 - 3\lambda & 2 - \lambda \\
-\text{sgn}(\lambda)\lambda^2 & 3 - \lambda & -\lambda
\end{pmatrix},
\]

which means for instance that \( b(x_1, y_2, \lambda) = 4 - 3\lambda \). (We denote by \( \text{sgn}(\lambda) \in \{0, \pm 1\} \) the sign of a scalar \( \lambda \).) Assumptions (A1–A4) are clearly satisfied with \( \mathcal{X} = X \times Y \).

Then, the kernel \( b^s \) of \( B^s \) is given by the following table:

\[
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix}
= \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\begin{pmatrix}
-\lambda & -\text{sgn}(\lambda)\sqrt{|\lambda|} \\
(4 - \lambda)/3 & 3 - \lambda & -\lambda
\end{pmatrix}.
\]

Let \( g \in \mathbb{R}^X \) denote the map given by the table

\[
g : \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= \begin{pmatrix}
8 \\
6
\end{pmatrix},
\]

and let us compute \( B^s g \):

\[
B^s g : \begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix}
= \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\begin{pmatrix}
-8 & 8 - \sqrt{6} \\
(4 - 8)/3 & 3 - 6 \\
2 - 8 & -6
\end{pmatrix} = \begin{pmatrix}
-\sqrt{6} \\
-4/3 \\
-6
\end{pmatrix}
\]

where we underlined the terms which determine the maximum, and where \( \lor \) denotes the sup law. By (14b), the sets \( (\partial^s g)^{-1}(y_j) \) can be read directly from (49) by
picking, for each row \( y_j \), the \( x_i \) variables corresponding to the underlined terms:

\[
(\delta^* g)^{-1}(y_1) = \{x_2\},
(\delta^* g)^{-1}(y_2) = \{x_1\},
(\delta^* g)^{-1}(y_3) = \{x_1, x_2\}.
\]

Since the union of these subsets is equal to \( X = \{x_1, x_2\} \), it follows from Corollary 3.4 that \( f = B^* g \) is a solution of \( B f = g \). It follows from Corollary 4.8 that this solution is not unique, because the covering \( \{(\delta^* g)^{-1}(y_j)\}_{1 \leq j \leq 3} \) of \( X \) is not minimal: for instance, \( \{(\delta^* g)^{-1}(y_3)\} \) is a subcovering of \( X \), which reflects the fact that setting \( f(y_1) = f(y_2) = +\infty \) and \( f(y_3) = -6 \) yields another solution of \( B f = g \).

More generally, a minimal covering of a set of cardinality \( n \) must consist of at most \( n \) sets, which implies that when \( X \) and \( Y \) are finite, the number of elements of \( Y \), i.e., the number of “scalar unknowns”, must not exceed the number of elements of \( X \), i.e., the number of “scalar equations”, for the solution of \( B f = g \) to be unique.

To show a uniqueness case, consider the restriction \( B_{1,2} : \mathbb{F}^{|y_1, y_2|} \rightarrow \mathbb{F}^X \), which is obtained by specializing \( B \) to those \( f \) such that \( f(y_3) = +\infty \). Then, the covering \( \{(\delta^* g)^{-1}(y_j)\}_{1 \leq j \leq 2} \) of \( X \) is minimal, which shows that setting \( f(y_1) = -\sqrt{6} \), \( f(y_2) = -4/3 \) yields the only solution of \( B_{1,2} f = g \).

To illustrate the case where \( B f = g \) has no solution, consider:

\[
g' : x_1 \begin{pmatrix} 3 \\ -3 \end{pmatrix}.
\]

We get

\[
B^* g' : \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \begin{pmatrix} 1 \\ 4 - 3/3 \\ 2 - 3 \end{pmatrix} \begin{pmatrix} -3 \\ \sqrt{3} \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} (4 - 3)/3 \\ 3 + 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 3 \end{pmatrix}.
\]

We see from Corollary 3.4 that \( B f = g' \) has no solution, because \( \bigcup_{1 \leq j \leq 3} (\delta^* g')^{-1}(y_j) = \{x_2\} \) is not a covering of \( X \).

Finally, let us interpret these computations in geometric terms. For each \( 1 \leq j \leq 3 \), denote by \( B_j \) the restriction of \( B : \mathbb{F}^{|y_j|} \rightarrow \mathbb{F}^X \), which is obtained by specializing \( B \) to those \( f \) such that \( f(y_k) = +\infty \) for \( k \neq j \), so that the corresponding kernels \( b_j \) are given by:

\[
b_1 : x_1 \begin{pmatrix} -\lambda \\ -\text{sgn}(\lambda)\lambda^2 \end{pmatrix}, \quad b_2 : x_1 \begin{pmatrix} 4 - 3\lambda \\ 3 - \lambda \end{pmatrix}, \quad b_3 : x_1 \begin{pmatrix} 2 - \lambda \\ -\lambda \end{pmatrix}.
\]

The (set of finite points of the) image of the operator \( B_1 \) is the curve \( \lambda \rightarrow \text{sgn}(\lambda)\lambda^2 \) which is depicted on Figure 1. The image of \( B_2 \) (resp. \( B_3 \)) is the line with slope \( 1/3 \) (resp. \( 1 \)) on the figure. The image of \( B \) can be computed readily from the images of \( B_j \); since \( B f = B_1 f(y_1) \cup B_2 f(y_2) \cup B_3 f(y_3) \), the image of \( B \) is the sup-subsemilattice of \( \mathbb{F}^X \) generated by these curve and two lines, which corresponds to the gray region on Figure 1.

Now, for each Galois connection \( B \), observe that \( BB^* g \) is the maximal element of the image of \( B \) which is below \( g \). Thus, \( P = BB^* \) is a nonlinear projector on the
image of $B$, and for $j = 1, 2, 3$, consider the nonlinear projector $P_j = B_j B_j^*$ on the image of $B_j$. By definition of Galois connections, $P_{j,g}(x) = b(x, y_j, B^* g(y_j))$, thus

$$P = \sup_{1 \leq j \leq 3} P_j.$$ 

The element $g$, and its image by the projectors $P_j$, are shown on the figure (and can be computed directly from the figure). For each $1 \leq j \leq 3$, the set $(\partial^* g)^{-1}(y_j)$ represents the subset of elements $x_i$ of $\{x_1, x_2\}$ such that $(P_{j,g})(x_i) = g(x_i)$. Thus, the covering condition $X \subset \bigcup_{1 \leq j \leq 3}(\partial^* g)^{-1}(y_j)$ is nothing but a combinatorial rephrasing of $g = \sup_{1 \leq j \leq 3} P_j g$.

6. Some Moreau Conjugacies examples

We give now some applications of the results of Sections 3 and 4 to the case of Moreau conjugacies $B$ and $B^*$ given by (8,10), with finite kernels, that is such that $\bar{b} : X \times Y \to \mathbb{R}$. Since $\mathcal{I} = X \times Y$, $B^* g(y) > -\infty$ for all $y \in Y$, and $g \in \mathcal{I}$ such that $g \not= +\infty$.

6.1. The Fenchel transform. Let us consider the case where $X = Y = \mathbb{R}^n$ and $B = B^*$ is the Fenchel transform, that is $B$ and $\bar{b}$ are given by (8,10), with $\bar{b}(x, y) = \langle x, y \rangle$. We have already shown in Section 3.1 that $\bar{b}$ is continuous in the second variable and coercive. We also have:

**Lemma 6.1.** The Fenchel transform on $\mathbb{R}^n$ is regular.

**Proof.** We need to show that for any function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $g = f^*$ is l.s.c. on $\mathbb{R}^n$ and is quasi-continuous on its domain dom$(g)$. We know that $g$ is either $\equiv +\infty$, or $\equiv -\infty$ or a l.s.c. proper convex function. Hence, it is l.s.c. and in the first two cases, the domain is empty. In the last case, since $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is l.s.c. on $\mathbb{R}^n$, and a fortiori on dom$(g)$, it is sufficient to prove that $g = \text{lsc}(\text{usc}(g))$ where lsc and usc envelopes are applied to the restrictions to dom$(g)$. Moreover, since $g$ is l.s.c., we get $g \leq \text{lsc}(\text{usc}(g))$, hence it is sufficient to prove that $g \geq \text{lsc}(\text{usc}(g))$. The following properties of l.s.c. proper convex functions can be found in [24]: $g$ is continuous in the relative interior of dom$(g)$, that we denote by ridom$(g)$ [recall that the relative interior of a convex set is the interior of the set for the topology of the affine hull of the set], and for any affine line $L$, the restriction of $g$ to $L$ is
continuous on its domain \( \text{dom}(g) \cap L \) and \( \text{ridom}(g) \cap L \) is dense in \( \text{dom}(g) \cap L \). From this, we get that \( \text{usc}(g) \leq g \) on \( \text{ridom}(g) \). Let us fix \( x_0 \in \text{ridom}(g) \). For all \( x \in \text{dom}(g) \), take the affine line \( L \) containing \( x_0 \) and \( x \). Since there exists a sequence \( (x_n)_{n \in \mathbb{N}} \) in \( \text{ridom}(g) \cap L \) converging to \( x \) and since \( g \) is continuous on \( \text{dom}(g) \cap L \), we obtain that \( g(x) = \lim_{n \to \infty} g(x_n) = \lim_{n \to \infty} \text{usc}(g)(x_n) \geq \text{usc}(\text{usc}(g))(x) \), which finishes the proof.

These properties allow us to apply Theorems 3.3, 4.6 and 4.7. In particular using Theorems 3.3 and 4.6, we get

**Proposition 6.2.** Let \( g \) be a l.s.c. proper convex function on \( \mathbb{R}^n \). Then, \( \{(\partial g)^{-1}(y)\}_{y \in \text{dom}(g')} \) is a covering of \( \text{dom}(g) \). Let \( Z_\circ \) (resp. \( Z_t \)) be the set of algebraically (resp. topologically) essential elements with respect to this covering, and let \( \text{int}(Z_t) \) denotes the interior of \( Z_t \), relatively to \( \text{dom}(g') \). Then,

\[
(f^* \leq g \text{ and } f^*(x) = g(x) \text{ for all } x \in \text{dom}(g)) \Rightarrow (g^* \leq f \text{ and } f(y) = g^*(y) \text{ for all } y \in Z_\circ \cup \text{int}(Z_t))
\]

Since \( Y \) is locally compact, and for any l.s.c. proper convex function on \( \mathbb{R}^n \) such that \( \text{dom}(g) \) has a nonempty interior \( \text{dom}(g) \), \( \text{dom}(g) \) is included in the closure of \( \text{dom}(g) \) [24, Th. 6.3], we can also apply Theorem 4.10. We deduce:

**Proposition 6.3.** Let \( g \) be a l.s.c. proper convex function on \( \mathbb{R}^n \) such that \( \text{dom}(g) \neq \emptyset \). The following statements are equivalent:

1. \( (f^* \leq g \text{ and } f^*(x) = g(x) \text{ for all } x \in \text{dom}(g)) \Rightarrow f = g^* ;
2. \( \{(\partial g)^{-1}(y)\}_{y \in \text{dom}(g')} \) is a topologically minimal covering of \( \text{dom}(g) \);
3. \( f^* = g \Rightarrow f = g^* \).

The following classical notion is intermediate between algebraic and topological minimality in (51): a l.s.c. proper convex function \( g \) on \( \mathbb{R}^n \) is **essentially smooth** if \( \text{dom}(g) \neq \emptyset \), \( g \) is differentiable in \( \text{dom}(g) \), and the norm of the differential of \( g \) at \( x \) tends to infinity, when \( x \) goes to the boundary of \( \text{dom}(g) \), see [24, §26]. A l.s.c. proper convex function \( f \) on \( \mathbb{R}^n \) is **essentially strictly convex** if the restriction of \( f \) to any affine line (or segment) in \( \text{dom}(f) \) is strictly convex. A l.s.c. proper convex function \( g \) is essentially smooth if, and only if, its conjugate \( g^* \) is essentially strictly convex [24, Theorem 26.3]. The following result, which can be compared with [24, Corollary 26.4.1], is a corollary of Proposition 6.3. It is the underlying argument of G"artner-Ellis theorem and it is explicitly used in [21, Th. 4.1 (c)], in [15, Th. 4.7 and 5.3] and in [22, Lemmas 3.2 and 3.5].

**Corollary 6.4.** Let \( g \) be an essentially smooth l.s.c. proper convex function on \( \mathbb{R}^n \). If \( f \) is a l.s.c. function such that \( f^* \leq g \) and \( f^*(x) = g(x) \) for all \( x \in \text{dom}(g) \), then \( f = g^* \). In particular, \( g \) has a unique preimage by the Fenchel transform.

**Proof.** First, since \( f^* \leq g \) implies \( f \geq g^* \), so \( f = g^* \) outside \( \text{dom}(g^*) \), one can replace \( Y \) by the affine hull of \( \text{dom}(g^*) \), so that \( \text{dom}(g^*) \) is the relative interior of \( \text{dom}(g^*) \) (and is thus nonempty). The conditions on the differentials of \( g \) imply that \( \partial g(x) \) is a singleton when \( x \in \text{dom}(g) \) and is empty elsewhere (see [24, Theorem 26.1]). Hence, applying Theorem 3.18 to \( g \), we get that for all \( y \in \text{dom}(g^*) \) there exists \( x \in \text{dom}(g) = \text{dom}(g^*) \) such that \( y \in \partial g(x) \). Since \( \partial g(x) \neq \emptyset \), we get that \( x \in \text{dom}(g) \), and \( \{y\} = \partial g(x) \), hence any \( y \in \text{dom}(g^*) \).
cannot be removed in the covering of idom(\(y\)) by \(\{(\partial g)^{-1}(y)\}_{y \in \text{dom}(g)}\). Moreover, since dom(\(g^*\)) is included in the closure of idom(\(g^*\)), any open set \(U\) of dom(\(g^*\)) contains a point \(y \in \text{idom}(g^*)\), so \(U\) cannot be removed in the covering of idom(\(y\)) by \(\{(\partial g)^{-1}(y)\}_{y \in \text{dom}(g)}\). This shows that the covering of idom(\(g\)) by \(\{(\partial g)^{-1}(y)\}_{y \in \text{dom}(g)}\) is topologically minimal. The implication (51)\(\Rightarrow\)(50) in Proposition 6.3 yields the result of the corollary.

\[\]

**Example 6.5.** The following function \(g\) satisfies (51) and thus (50), but is not essentially smooth: consider \(X = Y = \mathbb{R}^2\), \(g = f^*\) where \(f: \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}\), with \(f(y) = y_1^2 + y_2^2\) if \(|y| < 1\) and \(f(y) = +\infty\) elsewhere. Indeed, since \(f\) is l.s.c. and convex, \(f = g^*\). Since \(f\) is not strictly convex on \(y = 0\), \(g\) is not essentially smooth. If \(f\) is essentially strictly convex in a neighborhood of \(y \in \text{dom}(f)\), the point \(y\) cannot be removed in the covering of dom(\(\partial g\)) by \(\{(\partial g)^{-1}(y)\}_{y \in \text{dom}(f)}\).

Then, since idom(\(y\)) = dom(\(\partial g\)) = dom(\(y\)) = \(\mathbb{R}^2\) and the last of strict convexity of \(f\) occurs only on a line, any open set of dom(\(f\)) intersects the “part” of idom(\(f\)) where \(f\) is essentially strictly convex. This implies that the covering of idom(\(g\)) = \(\mathbb{R}^2\) by \(\{(\partial g)^{-1}(y)\}_{y \in \text{dom}(f)}\) is topologically minimal.

### 6.2. Quadratic kernels

Let us consider the case where \(Y = X = \mathbb{R}^n\) and \(b\) is given by (10) with \(b(x, y) = b_a(x, y) := (x, y) - \frac{1}{2\alpha}\|y\|^2\), where \(\|\cdot\|\) is the Euclidean norm and \(a \in \mathbb{R}\) is some constant. Denoting \(B_a\) and \(B_a^*\) the corresponding Galois connections given by (8), we get that \(B_a f = (f + \frac{1}{2\alpha}\|\cdot\|^2)^*\) and \(B_a^* g = -\frac{1}{2\alpha}\|\cdot\|^2 + g^*\), hence the properties of \(B_a\) can be deduced from that of the Fenchel transform. In particular:

**Corollary 6.6.** Let \(g: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}\) be an essentially smooth l.s.c. proper convex function. If \(f\) is a l.s.c. function such that \(B_a f \leq g\) and \(B_a f(x) = g(x)\) for all \(x \in \text{idom}(g)\), then \(f = B_a^* g\).

Such kernels are useful, for instance, if we want to identify a function \(f\) which is semiconvex but not convex. Indeed, if \(f\) is semiconvex, there exists \(a \in \mathbb{R}\) such that \(f + \frac{1}{2\alpha}\|\cdot\|^2\) is convex and by increasing \(a\), one can obtain that \(f + \frac{1}{2\alpha}\|\cdot\|^2\) is essentially strictly convex. Hence, \(g = B_a f\) satisfies the assumptions of Corollary 6.6. Note that by standard results of convex analysis, we know that when \(f\) is l.s.c. proper and convex, \(B_a f\) is the inf-convolution of \(f^*\) and \((\frac{1}{2\alpha}\|\cdot\|^2)^*\) is the Moreau-Yoshida regularisation of \(f^*\), which explain why \(B_a f\) is essentially smooth.

The same kind of result can be obtained when replacing the kernel \(b_a\) by: \(b_a^*(x, y) = -\frac{1}{2\alpha}\|x - y\|^2\) with \(a \neq 0\).

### 6.3. \(\omega\)-Lipschitz continuous maps

Let \(E\) be a Hausdorff topological vector space and \(\omega: E \to \mathbb{R}^+\) be a continuous subadditive map:

\[
\omega(x + y) \leq \omega(x) + \omega(y)
\]

for all \(x, y \in E\)

such that \(\omega(-x) = \omega(x)\) for all \(x \in E\), and \(\omega(x) = 0 \iff x = 0\). We say that a function \(f: E \to \mathbb{R}\) is \(\omega\)-Lipschitz continuous if:

\[
|f(y) - f(x)| \leq \omega(y - x)
\]

for all \(x, y \in E\),

and we denote by \(\text{Lip}_{\omega}(E)\) the set of \(\omega\)-Lipschitz continuous functions \(f: E \to \mathbb{R}\).

If \(E\) is a normed vector space with norm \(\|\cdot\|\), then \(\omega(x) = a\|x\|^p\) satisfies the above properties for all \(a > 0\) and \(p \in (0, 1]\), and in that case \(\text{Lip}_{\omega}(E)\) is the set of Hölder
continuous functions $f : E \to \mathbb{R}$ with exponent $p$ and multiplicative factor less or equal to $a$.

Take $Y = X = E$ and consider the kernel $b$ given by (10) with $\bar{b}(x, y) = b_\omega(x, y) := -\omega(y - x)$. We denote by $B_\omega$ and $B_\omega^*$ the corresponding Galois connections given by (8). We have $B_\omega = B_\omega^*$. The kernel $b$ is continuous (in the second variable), but $b$ is in general not coercive since when $\omega = a|x|$ and $V$ is the ball of center $x$ and radius $\varepsilon$, $b_\omega^*(y) = \sup_{x \in V} \alpha(||y - x||)$, then $b_\omega^*(y) = a\varepsilon + a$. We have:

**Proposition 6.7.** Let $g \in \mathcal{F}$. Then, $g = B_\omega B_\omega^* g$ if, and only if, either $g \equiv +\infty$, or $g \equiv -\infty$, or $g \in \text{Lip}_\omega(E)$. In that case, we have $B_\omega^* g = -g$.

**Proof.** Let $f \in \mathcal{F}$. If $B_\omega f \not\equiv +\infty$ and $B_\omega f \not\equiv -\infty$, then $f(y) > -\infty$ for all $y \in Y$ and there exists $y \in Y$ such that $f(y) < +\infty$. Moreover, by the subadditivity of $\omega$, $b_\omega(\cdot, y) - a$ is a $\omega$-Lipschitz continuous function for all $y \in E$ and $a \in \mathbb{R}$. Hence, $B_\omega f$ which is a nonempty supremum of $\omega$-Lipschitz continuous functions, is also $\omega$-Lipschitz continuous, which shows the “only if” part of the proposition.

Conversely, if $g \equiv +\infty$ or $g \equiv -\infty$, then $B_\omega^* g = -g$ and $g = B_\omega B_\omega^* g$. Let $g \in \text{Lip}_\omega(E)$. Since $g(y) - g(x) \leq \omega(y - x)$ for all $x, y \in E$, we deduce:

$$B_\omega^* g(y) = \sup_{x \in E} -\omega(y - x) - g(x) \leq -g(y).$$

Moreover, taking $x = y$ in the supremum, we get that $B_\omega^* g(y) \geq -g(y)$, hence $B_\omega^* g = -g$. Since $B_\omega = B_\omega^*$ and since $-g$ is also in $\text{Lip}_\omega(E)$, it follows, by application of the same argument, that $B_\omega (-g) = g$, hence $B_\omega B_\omega^* g = B_\omega (-g) = g$, which shows the “if” part of the proposition. \(\square\)

Proposition 6.7 implies that $B_\omega = B_\omega^*$ is regular, since any map of the form $B_\omega f$ is continuous. Also, $\text{dom}(B_\omega f) = \text{dom}(B_\omega f)$ is equal to $E$ or $\emptyset$ for all $f \in \mathcal{F}$. We also have:

**Lemma 6.8.** A map $f \in \mathcal{F}$ is in $\mathcal{F}_\omega$ if, and only if, $f + \omega$ has relatively compact finite sublevel sets, which means that $\{ y \in E \mid f(y) + \omega(y) \leq \beta \}$ is relatively compact, for all $\beta \in \mathbb{R}$.

**Proof.** By definition, $f \in \mathcal{F}$ if, and only if, $b_\omega(x, \cdot) - f$ has relatively compact finite superlevel sets. Since $b_\omega(0, \cdot) = -\omega$, $f \in \mathcal{F}$ implies that $f + \omega$ has relatively compact finite sublevel sets. Conversely, if $f + \omega$ has relatively compact finite sublevel sets, then, by the subadditivity of $\omega$, $\{ y \in E \mid b_\omega(x, y) - f(y) \geq \beta \} \subseteq \{ y \in E \mid f(y) + \omega(y) \leq \omega(x) - \beta \}$ is relatively compact for all $\beta \in \mathbb{R}$ and $x \in E$. \(\square\)

The condition of Lemma 6.8 holds in particular when $E = \mathbb{R}^n$, $\omega = a\| \cdot \|$ for some norm on $E$ and $f$ is Lipschitz continuous with a Lipschitz constant $b < a$. We can apply Corollaries 3.6 and 4.9, and Theorem 4.6 to any map $g$ such that $f = B_\omega^* g$ satisfies the condition of Lemma 6.8. Using Proposition 6.7, we obtain:

**Corollary 6.9.** Let $g \in \text{Lip}_\omega(E)$ be such that $g - \omega$ has relatively compact finite superlevel sets, then $\{(B_\omega^* g)^{-1}(y)\}_{y \in E}$ is a covering of $E$. If

$$|g(y) - g(x)| < \omega(y - x) \quad \text{for all } x, y \in E \text{ such that } x \neq y,$$

then $(f \in \mathcal{F}$ and $B_\omega f = g)$ $\implies f = -g.$
We denote by $B$ and $B^*$ the corresponding Galois connections given by (5). When $p \leq 1$, and $x = (x', x''') \in E \times (0, +\infty)$, $B f(x) = B_w f(x')$ where $B_w$ is given as in Section 6.3 with $w = x''||$. We denote by $b$ and $b^*$ the corresponding Galois connections given by (6). When $E = \mathbb{R}^n$, $\| \cdot \|$ is the Euclidean norm, $p = 2$, and $x = (x', x''') \in E \times (0, +\infty)$, $B f(x) = x''||x'||^2 + B_{2x''} f(x''x')$ where $B_{2x''}$ is given as in Section 6.2, and the results of this latter section show that $B$ is injective on the set of semiconvex functions with exponent $p$. When $p = 1$, $B$ is used in [26] to define a (semi-)distance in the set of quasi-continuous functions from $E$ to $\mathbb{R}$. We first prove some preliminary results.

Lemma 6.10. Let $f \in \mathcal{F}$. Then, either $B f \equiv +\infty$, or $B f \equiv -\infty$, or there exists $a \geq 0$ such that

\begin{equation}
E \times [a, +\infty) = \text{dom}(B f) \subset \text{dom}(B f) \subset E \times [a, +\infty).
\end{equation}

More precisely, for all $x = (x', x''')$, $z = (x', z'') \in X$ such that $x'' > z''$, we have

\begin{equation}
B f(x) \leq B f(z) + K(x, z)
\end{equation}

with $K(x, z) := \begin{cases} \left( (z'')^{-1} - (x'')^{-1} \right) \frac{1}{1-p} \|x' - z'|^p & \text{when } p > 1 \\ \|z''| x' - z'|^p & \text{when } p \leq 1 \end{cases}$

Proof. Assume that $B f \not\equiv +\infty$, and $B f \not\equiv -\infty$. Then, there exists $y \in Y$ such that $f(y) < +\infty$, which implies that $B f(x) > -\infty$ for all $x \in X$, hence $\text{dom}(B f) \neq \emptyset$. Assume first that (58) is proved. Let $a = \inf \{x'' \in (0, +\infty) \mid \exists x' \in E \text{ such that } B f(x', x'') < +\infty \}$. 


Then, \( \text{dom}(Bf) \subset E \times [a, +\infty) \), and since \( \text{dom}(Bf) \) is open and included in \( \text{dom}(Bf) \), we get that \( \text{dom}(Bf) \subset E \times (a, +\infty) \). Conversely, let \( x_0 = (x'_0, x''_0) \in E \times (a, +\infty) \). By definition of \( a \), there exists \( z = (z', z'') \in E \times (a, x''_0) \) such that \( Bf(z) < +\infty \). Let \( \varepsilon = \frac{x''_0 - a}{2} > 0 \) and consider the neighborhood \( V \) of \( x_0 \) given by \( V = \{ (x', x'') \in X : \|x' - x'_0\| \leq \varepsilon \text{ and } \|x'' - x''_0\| \leq \varepsilon \} \). Then, \( x'' - z'' \geq \varepsilon \) and \( \|x' - z'\| \leq \|x'_0 - z'\| + \varepsilon \) for all \( (x', x'') \in V \). Using (58), we obtain:

\[
\sup_{x \in V} Bf(x) \leq Bf(z) + \sup_{x \in V} K(x, z)
\]

\[
\leq Bf(z) + \left( (\frac{x''_0 - a}{2})^{1-p} - (z'' + \varepsilon)^{1-p} \right) (\|x'_0 - z'\| + \varepsilon)^p \quad \text{when } p > 1
\]

\[
(z''(\|x'_0 - z'\| + \varepsilon)^p) \quad \text{when } p \leq 1
\]

\[
< +\infty
\]

hence \( x_0 \in \text{dom}(Bf) \), which finishes the proof of (57).

Let us now prove (58). Let \( x = (x', x'') \), \( z = (z', z'') \in X \) be such that \( x'' > z'' \). Using the definition of \( Bf \), we deduce that \( Bf(x) \leq Bf(z) + K_0(x, z) \) with \( K_0(x, z) \) satisfying:

\[
K_0(x, z) = \sup_{y \in E} (-x''_0 \|y - x'\|^p + z'' \|y - z'\|^p)
\]

\[
\leq \sup_{y \in E} (-x''_0 \|y - x'\|^p + z'' (\|y - x'\| + \|x' - z'\|)^p)
\]

\[
\leq \sup_{\rho \geq 0} (-x''_0 \rho^p + z'' (\rho + \|x' - z'\|)^p).
\]

Computing the supremum in (59), we obtain that \( K_0(x, z) \leq K(x, z) \) with \( K \) given by (58b), which shows (58).

\[\square\]

**Proposition 6.11.** Let \( f \in \mathcal{F} \). Then, \( f = B^aBf \) if and only if, \( f \equiv -\infty \), or there exists \( a > 0 \) such that \( f + a\| \cdot \|_p \) is lower bounded. Moreover, \( \text{dom}(Bf) \neq \emptyset \iff f = B^aBf \).

**Proof.** If \( f = B^aBf \) and \( f \equiv -\infty \), then there exists \( x \in X \) such that \( Bf(x) < +\infty \). By Lemma 6.10, either \( Bf \equiv -\infty \), or (57) holds. In the first case, \( f \equiv +\infty \), thus \( f + a\| \cdot \|_p \) is lower bounded, for all \( a > 0 \). In the second case, there exists \( a > 0 \) such that \( \text{dom}(Bf) = E \times (a, +\infty) \). Taking \( a' > a \), we get that \( Bf(0, a') < +\infty \), which means that \( f + a'\| \cdot \|_p \) is lower bounded.

Conversely, if \( f \equiv -\infty \) or \( f \equiv +\infty \), then \( f = B^aBf \). Assume that \( f + a\| \cdot \|_p \) is lower bounded for some \( a > 0 \) and that \( f \equiv +\infty \). Then, \( Bf(0, a) < +\infty \) and \( Bf(x) > -\infty \) for all \( x \in X \), which shows that \( \text{dom}(Bf) \neq \emptyset \). It remains to prove the last assertion of the proposition.

Let \( f \in \mathcal{F} \) be such that \( \text{dom}(Bf) \neq \emptyset \). Then, (57) holds. Let \( y \in E \). If \( f(y) = -\infty \), then \( f(y) \leq B^aBf(y) \). Otherwise, let \( a < f(y) \). Since \( f \) is l.s.c., there exists \( \varepsilon > 0 \) such that \( a \leq f(\zeta) \) for all \( \zeta \in E \) such that \( \|z - \zeta\| \leq \varepsilon \). Hence, \( a \leq c\|z - y\|^p + f(z) \) for all \( z \in E \) such that \( \|z - y\| \leq \varepsilon \) and all \( c > 0 \). When \( \|z - y\| \geq \varepsilon \) and \( c > a' > a \), we have

\[
c\|z - y\|^p + f(z) \geq (c - a')\varepsilon^p - Bf(y, c')
\]

Since, by (57) \( Bf(y, c') < +\infty \) for all \( a' > a \), there exists \( c > c' > a \) such that the right hand side of (60) is greater or equal to \( a \). For such a constant \( c \), we get that \( c\|z - y\|^p + f(z) \geq a \) for all \( z \in E \), hence \( Bf(y, c) \leq a \). Since \( B^aBf(y) \geq -Bf(y, c) \) for all \( c > 0 \), we deduce that \( B^aBf(y) \geq a \). Taking the limit when \( a \)
goes to \( f(y) \), we get \( B^* B f(y) \geq f(y) \). Since \( B^* B f \leq f \) holds for all \( f \in \mathcal{F} \), we get the equality. \( \square \)

In order to deduce covering properties as in Section 4, we need to show some properties of \( b \) and \( B \). First, it is clear that \( b \) is continuous (in the second variable).

**Lemma 6.12.** When \( E = \mathbb{R}^n \), the kernel \( b \) is coercive.

**Proof.** Let \( x \in \mathbb{R}^n \), \( \alpha \in \mathbb{R} \) and \( V \) be a neighborhood of \( x \). The map \( b^{\alpha}_{x,v} \) defined in (15) satisfies:

\[
b^{\alpha}_{x,v}(y) = \sup_{z \in v} \tilde{b}(z, y) - \tilde{b}(x, y) + \alpha
\]

Let \( \varepsilon > 0 \) be such that \( V \supset \{ (z', z'') \in X \mid \|z' - x'\| \leq \varepsilon, \|z'' - x''\| \leq \varepsilon \} \). We get that \( b^{\alpha}_{x,v}(y) \geq \varepsilon \|y - x'\| - \varepsilon \|y - x''\| + \alpha \). Hence \( b^{\alpha}_{x,v} \) has bounded sublevel sets. When \( E = \mathbb{R}^n \), this implies that \( b^{\alpha}_{x,v} \) has relatively compact sublevel sets, hence \( b \) is coercive. \( \square \)

**Lemma 6.13.** \( B \) is regular.

**Proof.** Let \( f \in \mathcal{F} \) and \( g = B f \). Then, \( g \) is l.s.c. as the supremum of continuous maps. To show that \( g \) is quasi-continuous on its domain it is thus sufficient to prove that \( g = \text{lsc(usc}(g)) \) where \( \text{lsc} \) and \( \text{usc} \) envelopes are applied to the restrictions to \( \text{dom}(g) \). Moreover, since \( g \) is l.s.c., we get \( g \leq \text{lsc(usc}(g)) \), hence it is sufficient to prove that \( g \geq \text{lsc(usc}(g)) \). This is true if \( g \equiv +\infty \) or \( g \equiv -\infty \). Otherwise, (57) and (58) hold and \( \text{udom}(g) = X \). In particular, since \( x \mapsto K(x, z) \) is continuous on the open set \( E \times (z''', +\infty) \) for all fixed \( z = (z', z'') \in X \), and since \( g(x) \leq g(z) + K(x, z) \) for all \( x \in E \times (z''', +\infty) \), we get that

\[
\text{usc}(g)(x) \leq g(z) + K(x, z) \quad \text{for all } x \in E \times (z''', +\infty) .
\]

Since \( \text{udom}(g) \) is open (in \( \text{udom}(g) = X \)), for all \( x \in (x', x'') \in \text{dom}(g) \), there exists \( \varepsilon > 0 \) such that \( z = (z', x'' - \varepsilon) \in \text{dom}(g) \). Hence, using (61), we get that

\[
\text{usc}(g)(x) \leq g(z) + K(x, z) = g(z) .
\]

It follows that

\[
\text{usc}(g)(x) \leq \lim_{\varepsilon \to 0+} g(x', x'' - \varepsilon) .
\]

Moreover, it is clear that for all fixed \( x' \in E \), \( x'' \in (0, +\infty) \mapsto g(x', x'') \) is a nonincreasing l.s.c. proper convex map. Since \( (0, +\infty) \) is one dimensional, this implies in particular that this map is continuous on its domain (see [24]). Therefore, it follows from (62) that \( \text{usc}(g)(x) \leq g(x) \) for all \( x \in \text{dom}(g) \). This shows that \( g \) is continuous in the interior of its domain. Now, by (57), if \( x = (x', x'') \in \text{dom}(g) \), \( (x', x'' + \varepsilon) \in \text{dom}(g) \) for all \( \varepsilon > 0 \), and since \( x'' \in (0, +\infty) \mapsto g(x', x'') \) is continuous on its domain, we get

\[
\text{lsc}(\text{usc}(g))(x) \leq \liminf_{\varepsilon \to 0+} \text{usc}(g)(x', x'' + \varepsilon) = \liminf_{\varepsilon \to 0+} g(x', x'' + \varepsilon) = g(x)
\]

which finishes the proof of \( \text{lsc}(\text{usc}(g)) \leq g \). \( \square \)

Since \( b \) is continuous, and coercive when \( E = \mathbb{R}^n \), \( B \) is regular, Condition (C) holds, and, by Lemma 6.10, \( \text{dom}(B f) \) is included in the closure of \( \text{dom}(B f) \), Theorem 4.10 in Case (3) and Theorem 4.7 yield:
Corollary 6.14. Assume that $E = \mathbb{R}^n$. Let $g \in \mathcal{F}$ such that $\text{dom}(g) \neq \emptyset$ and $g = B^* g$. Then, $(f \in \mathcal{F}$ and $B f = g) \implies f = B^* g$ and $(\{ (\partial^g)^{-1}(y) \} \in \text{dom}(B^* g))$ is a topological minimal covering of $\text{dom}(g)$. Moreover, if $B^* g$ is quasi-continuous on its domain, then $(B f \leq g$ and $B f = g$ on $\text{dom}(g)) \implies f = B^* g$.

References
To appear in Linear Algebra and Appl.
16. V. Kolokoltsov, On linear, additive, and homogeneous operators, (1992), Appeared in [18, p. 87–102].

Marianne Akian, INRIA, Domaine de Voluceau, B.P. 105, 78153 Le Chesnay Cedex, France.
E-mail address: Marianne.Akian@inria.fr

Stéphane Gaubert, INRIA, Domaine de Voluceau, B.P. 105, 78153 Le Chesnay Cedex, France.
E-mail address: Stephane.Gaubert@inria.fr

Vassili Kolokoltsov, Dep. of Computing and Mathematics, Nottingham Trent University, Burton Street, Nottingham, NG1 4BU, UK, and Institute for Information Transmission Problems of Russian Academy of Science, Moscow, Russia.
E-mail address: vk@maths.ntu.ac.uk