Dequantisation: Direct and Semi–Direct Sums of Idempotent Semimodules

Edouard Wagneur


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Edouard Wagneur
École Polytechnique de Montréal
and
GERAD HEC 3000, chemin de la Cte-Sainte-Catherine Montréal (Québec) Canada, H3T 2A7

e-mail: Edouard.Wagneur@gerad.ca

Abstract
In classical module theory, a module over a principal ideal domain splits into the direct sum of a free module and a torsion module. This decomposition does not hold in general for a semimodule over a semiring. We give here a necessary and sufficient condition for an idempotent semimodule to be the direct sum of two subsemimodules. Similar to the classical theory, where the decomposition of modules is derived from that of abelian groups, we first deal with idempotent abelian monoids, or, equivalently, semilattices. We introduce the semi-direct sum of semilattices, extend it to idempotent semimodules, and then use the concepts of torsion, Boolean, and semi-Boolean semimodules, to show how a general semimodule over an idempotent semiring splits into a semi-direct sum of a free, a Baerian, a semi-Boolean and a torsion semimodule. This provides a constructive insight into the existence of a left inverse to the inclusion map of a subsemimodule. Also, our decomposition suggests how the classification of finite dimensional semimodules may be simplified.

Keywords: Max-algebra, torsion semimodule, direct and semidirect sum

1 Introduction
Consider an additively idempotent semifield \((P, \lor, \cdot)\). We write \(0\) for the neutral element w.r. to \(\lor\), and \(1\) for the neutral element w.r. to \(\cdot\), which we assume to be distributive over \(\lor\). Usually, the \(\cdot\) will be omitted. \(P\) is ordered by \(a \leq b \iff a \lor b = b\). We assume that \(P\) is totally ordered, that \((P \setminus \{0\}, \cdot)\) is an abelian group - hence \((P \setminus \{0\}, \cdot, \leq)\) is a \(\ell\)-group - and is complete with least element \(0\).

A semimodule over an idempotent semifield is defined in a similar way as a module over a ring. More details can be found in e.g. [1], [6], [7], [8], [10], and [14]. We also write \(\lor\) for the internal composition law of \(M\), and \(0\) for its neutral element. The external multiplication law is written multiplicatively, and \(\cdot\) will usually be omitted.
Let $X \subseteq M$. We write $M_X$ for the semimodule generated by $X$, i.e., $M_X$ is the set of finite linear combinations $\sum \lambda_i x_i$, $\lambda_i \in P$, $x_i \in X$, and $X$ is a set of generators of $M_X$.

We say that $X$ is independent if, for any $x \in X$, we have $x \notin M_X \setminus \{x\}$.

When $X$ is finite, then this condition guarantees the existence of a basis.

Accordingly, a set $X$ of independent elements is a basis of $M_X$. The basis of a semimodule is unique, up to a rescaling of the form $x_i \mapsto \lambda_i x_i$ ([11], [14]). The dimension of a semimodule $M$ with basis $X$ is the cardinality of $X$.

Note that our concept of independence corresponds to that of weak independence of [14].

Also, $M$ is partially ordered by “$\leq$”, where $x \leq y$ iff $x \lor y = y$. It follows that $(M, \leq)$ is a (sup) semilattice with least element $0$, the neutral element of $\lor$. For $X \subseteq M$, the set $X^+$ of finite sums $\sum x_i$, $x_i \in X$ is a subsemilattice of $M_X$. We restrict here to the case where $X$ is finite. Then the set of (sup)-irreducible elements of $X$ generates $M_X$, and a basis of $M_X$ may be extracted from this set by removing all but one of the elements of the form $\lambda x$, $\lambda \in P$ ($\lambda \neq 0$).

Let $M$, $N$ be two idempotent semimodules, and $\Phi : M \rightarrow N$ a morphism of semimodules. It is well-known that $\Phi$ is isotope, i.e. $\forall x_i, x_j \in M$, $x_i \leq x_j \Rightarrow \Phi(x_i) \leq \Phi(x_j)$.

For $n = 2$, the basis $X = \{x_1, x_2\}$, is a poset. Hence $(X, \leq)$ is either an antichain, or a chain. Since an isomorphism is isotope, the semimodules generated cannot be isomorphic.

Yet, there is another case, where $x_1 \leq x_2$, and $x_2 \leq \lambda x_1$, for some $\lambda \in P$, $\lambda > 1$. We may represent the generators by the columns of a matrix:

1. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
2. $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
3. $C = \begin{bmatrix} 1 & 1 \\ 1 & \lambda \end{bmatrix}$, $\lambda > 1$

(for $\lambda < 1$, the rescaling $x_2 \mapsto \lambda^{-1} x_2$ yields $x_1 < x_2 < \lambda^{-1} x_1$, with $\lambda^{-1} > 1$).

In the latter case, we say that $M_X$ is a torsion semimodule (cf definition in Section 2 below).

Note also that case 3 defines a one-parameter family of two-dimensional semimodules.

Clearly, case 1 defines the free semilattice with two generators, which is the direct sum $P \oplus P$. Neither of the other semimodules defined in cases 2-3 admit such a decomposition.

This leads to the following questions:

1. When is $M$ the direct sum of two semimodules $M_1 \oplus M_2$?
2. For the general case, is there a weaker decomposition, say, $M = M_1 \oplus M_2$ such that $M_1$ and $M_2$ have different characteristics, (e.g. $M_2$ concentrates all the torsion, or $M_1$ is free, a.s.o.) and if so, how can such a decomposition be characterized?

In [5], Cohen et al, using residuation theory, prove a necessary and sufficient condition (in terms of transversality) for the direct sum of two sub-semimodules of $P^n$ to yield the free semimodule. In [16], this condition was stated in terms of the existence of a left inverse to the inclusion map. These conditions are clearly equivalent, since the exactness condition $\text{Im}\Phi = \text{Ker}\Phi$ (and the existence of a left inverse) in [16], is a transversality condition.

This condition is quite general, but very difficult to verify in practice. It will be given here a more geometric flavor (cf Theorem 8).

The aim of this paper is to give a satisfactory answer to the questions raised above by intro-
ducing the concept of semidirect sum, and to provide a simple proof of the decomposition
Theorem.
In Section 2 below, we briefly recall the definition of semi-Boolean and torsion semimodules
introduced in [15]. Then, in Section 3 - and just as the classical decomposition theorem for
modules follows that of abelian groups - we first introduce the semidirect sum of idempotent
monoids (i.e., semilattices). This key issue lies on the existence of a left inverse to the
inclusion map of a submonoid (subsemilattice). This definition is extended to semimodules.
Then in Section 4, we prove a general statement for the decomposition of a semimodule
into four subsemimodules, which are free, Boolean, semi-Boolean, and torsion semimodules
respectively.
We also state the necessary and sufficient conditions for a decomposition to be a direct
sum. The semidirect sum decomposition is not unique in general, as suggested by some
examples in Section 5. A short discussion then concludes the paper. Throughout this
paper, the reader may think of \((P, \vee, \cdot)\) as \(\mathbb{R}_{\text{max}} = (\mathbb{R} \cup \{-\infty\}, \{\infty\}, \text{max}, +)\) (with
max written as \(\vee\), and usual addition written multiplicatively), and with the convention
that \((+\infty) \vee (-\infty) = -\infty\), or as \(\mathbb{R}_{\text{min}} = (\mathbb{R} \cup \{+\infty\}, \{-\infty\}, \text{min}, +)\) (with min written
as \(\vee\), and with the opposite convention w.r. to the \(\vee\) composition of \(+\infty\) with \(-\infty\).

2 Semi-Boolean and torsion semimodules

Let \(X\) be the basis of a semimodule \(M\). The relation \(x \prec y \iff \exists \lambda \in P\) such that
\(x \leq \lambda y\) defines a quasi-order on \(M\). Therefore, it also induces a quasi-order on \(X\) and
\(X^+\).

It is easy to see that \(x \sim y \iff x \prec y\) and \(y \prec x\) is a congruence relation with respect
to the semilattice structure of \(M\). An equivalence class containing more than one element
will often be called a torsion cycle. For every \(x \in M\), let \(c_x\) stand for the equivalence class
of \(x\). Then \(\prec\) induces an order relation on the set of equivalence classes (since the context
will usually limit the risks of ambiguity, we will also write \(\leq\) for this order relation), and
\(\{c_x \mid x \in X\}\) generates the semilattice \(S(X^+)\) of \(mod \sim\) equivalence classes of \(X^+.\) This
is summarized in the following statement.

**Theorem 1** ([15])

For any two bases \(X, Y\), the quotient semilattices \(S(X^+)\) and \(S(Y^+)\) are isomorphic.

**Theorem 1** states that the semilattice \(S(X^+)\) is an intrinsic invariant of \(M\), therefore, we
call it the structural semilattice of \(M\). We also write \(S(M)\) for this semilattice.

Note that in case 3 above, \(S(M_C)\) reduces to a point, while \(X^+\) has two elements. This
motivates the following definition.

**Definition 2.1**

We say that a \(P\)-semimodule \(M\) is semi-Boolean if there is a basis \(X\) such that
$p: X^+ \to S(X^+)$ is an isomorphism of semilattices.

Given $X$, we construct the weighted oriented graph $(V(X^+), E(\prec), w_{ij})$, where

- $V(X^+)$ is the set of vertices of $X^+$,
- $(x_i, x_j) \in E(\prec) \iff x_i \prec x_j$, and
- $\forall (x_i, x_j) \in E(\prec), \ w_{ij} = \inf \{\lambda | x_i \leq \lambda x_j\}$.

Note that, $(x_i, x_j) \in E(\prec) \Rightarrow c_{x_i} \leq c_{x_j}$. In particular all the edges in a weighted oriented graph constructed with the vertices of $S(X^+)$ have weight 1, and there is a surjective morphism of weighted oriented graphs:

$p_G: (V(X^+), E(\prec), w_{ij}) \to (S(X^+), E(S(X^+)), w_{ij}(S(X^+)))$.

**Definition 2.2**

We say that $M$ is Boolean if there is a basis $X$ of $M$ such that $p_G$ is an isomorphism of weighted oriented graphs.

Thus a Boolean semimodule has a basis $X$ such that $\forall (x_i, x_j) \in E(\prec), \ w_{ij} = 1$. Clearly free $\Rightarrow$ Boolean $\Rightarrow$ semi-Boolean. The converse does not hold. (Example 2.3.2 below).
Examples 2.3
2.3.1. Let $x_1 = (1\ 0\ 0)'$, $x_2 = (1\ 1\ 0)'$, $x_3 = (\lambda\ 1\ 1)'$, $\lambda > 1$. Then $S(X^+)$ is isomorphic to $X^+$, and $M_X$ is semi-Boolean. However, $x_1 \leq \lambda^{-1}x_3$, hence $w_{13} = \lambda^{-1} < 1$ in $(\mathcal{V}(X^+), E(\prec), w_{ij})$. Thus $M_X$ is not Boolean.

2.3.2. Let $X$ stand for the set of columns of $C$ in 3 above, then $M_X$ is not semi-Boolean.

2.3.3 Let $A = \begin{bmatrix} 0 & 1 & \lambda \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $\lambda > 1$. We also write $M_A$ for $\text{Im}A$ (and $x_j =$ column $j$ of $A$). Clearly, $M_A$ is not semi-Boolean, since $x_1 \lor x_2 \sim x_3$.

2.3.4. $A = \begin{bmatrix} 0 & 1 & \lambda \\ 0 & 1 & \mu \\ 1 & 0 & 1 \end{bmatrix}$, with $\mu > \lambda > 1$; $M = M_A$.

Just as in 2.3.3, the structural semilattice $S(M)$ is (isomorphic to) $2^2 \setminus \{0\}$.

Remarks 2.4
2.4.1 Note that a torsion semimodule may contain elements which do not belong to a torsion cycle (e.g. $x_1$, $x_2$ in Examples 2.3.3, and 2.3.4). In Example 2.3.3, we have $p^{-1}(e_{x_3}) = \{x_1 \lor x_2, x_3\}$, while in Example 2.3.4, we have $p^{-1}(e_{x_3}) = \{x_1 \lor x_2, x_3, x_4\}$.

Definition 2.5
A semimodule which is not semi-Boolean is called a general semimodule.

Let $x, y \in X^+$ be such that $x \leq y \leq \lambda x$, where $\lambda = \inf\{\mu : x \leq \mu y\}$. Then $\lambda$ is the torsion coefficient of $x$ and $y$. It is equal to $\exp(\delta(x, y))$, where $\delta(x, y)$ is the Hilbert-Birkhoff pseudometric.\footnote{This pseudometric has been invented by Hilbert in citek:Hilbert03. Its application to positive linear operators has first been made by G. Birkhoff [3].}

Since $X, X^+$, and $S(X^+)$ play an essential role in the decomposition of a semimodule, we first investigate the existence problem for the direct and semidirect sum decomposition of a semilattice. Note that, for $X \cup \{0\}$, we have $(X \cup \{0\})^+ = X^+ \cup \{0\}$, and, unless otherwise stated (abusing the notation), we will always assume that $0 \in X^+$.

3 Direct and semidirect sum decomposition of semilattices
An idempotent semimodule is an idempotent commutative monoid with 0. Hence, it is a semimodule with 0. Also, if $P$ is complete, then it is not to see that every finitely generated semimodule over $P$ is complete. Therefore all semimodules considered in this section will be assumed to be complete. We first recall some well-known facts on semilattice (resp. semimodule) morphisms (see for example [2]).

Let $S, T$ be semilattices with 0, and $\varphi : S \rightarrow T$ a semilattice morphism. The kernel of $\varphi$ is the subset $\text{Ker} \varphi = \{k \in S \mid \varphi(k) = 0\}$. Then $\text{Ker} \varphi$ is an ideal of $S$. Also, a semilattice morphism determines a congruence relation $x \equiv y \Leftrightarrow \varphi(x) = \varphi(y)$. Conversely, given an ideal $K$ in a semilattice with 0, there is a semilattice epimorphism $\varphi$ of $S$ onto a semilattice $T$ with zero, such that $K$ is the kernel of $\varphi$. It is also well-known ([2]) that,
contrary to the situation in groups, the ideal does not in general uniquely determine the associated congruence relation. In other words:
- a semilattice morphism (of a semilattice with 0 into a semilattice with 0) uniquely determines a congruence relation and an ideal $K = \text{Ker } \varphi$,
- for a given ideal $I$ (in a semilattice with 0) there exists a semilattice epimorphism $\varphi$ onto a semilattice with 0 such that $\text{Ker } \varphi = I$. However, neither $\varphi$, nor the associated congruence relation are uniquely determined.

It follows in particular that the primitive concept is that of morphism. The congruence relation defined by $\varphi$ induces a short exact sequence $S \xrightarrow{\pi_{\varphi}} S/\varphi \rightarrow 0$, where $S/\varphi$ is the quotient semilattice defined by the set of $\varphi$-equivalence classes.

Also, recall that a sequence $A \xrightarrow{\Phi} B \xrightarrow{\psi} C$ is said to be exact if $\text{Im } \Phi = \text{Ker } \psi$. Here, the kernel of $S/\varphi \rightarrow 0$ is the whole of $S/\varphi$, which is $\text{Im } \pi_{\varphi}$, and the map $S \xrightarrow{\pi_{\varphi}} S/\varphi$ must be surjective.

Dually, if $S$ is a subsemilattice of $T$, then the inclusion map $i$ is injective, hence its kernel is necessarily 0, which can be summarized by a short exact sequence $0 \rightarrow S \xrightarrow{i} T$. However, a map with kernel 0 need not be injective (just take the map from the 3-chain $\{0, 1, 2\}$ to the 2-chain $\{0, 1\}$ which maps 0 to 0 and 1, 2 to 1).

Notwithstanding this remark, we will abuse the notation, and write $0 \rightarrow S \xrightarrow{\varphi} T$ to mean that $\varphi$ is injective, rather than just that its kernel is 0. In [16], we give a necessary and sufficient condition for the existence of a short exact sequence $0 \rightarrow S \xrightarrow{\varphi} T \xrightarrow{\psi} T/\psi \rightarrow 0$ in terms of the existence of a left inverse to $\varphi$. This condition will be given a more constructive flavor below.

Let $S$ be a subsemilattice with 0 of a semilattice (with 0) $T$. We have an injective map $0 \rightarrow S \xrightarrow{i} T$.

Recall that (cf [4]):
- A poset morphism $f: (E, \leq) \rightarrow (F, \prec)$ is said to be residuated if
  \[ \forall y \in F, \exists \bigvee_{E} \{ x \mid f(x) \prec y \}. \]
  In this case, the residuated map $f^\#: (F, \prec) \rightarrow (E, \leq)$ defined by $f^\#(y) = \bigvee_{E} \{ x \mid f(x) \prec y \}$, is a morphism of posets, is unique, and
  \[ f \circ f^\# \prec 1_T, \quad \text{and} \quad f^\# \circ f \geq 1_S, \]
- $f^\#$ is a generalized inverse of $f$, i.e. $f \circ f^\# \circ f = f$, and $f^\# \circ f \circ f^\# = f^\#$.

It follows in particular that $f$ injective $\iff f^\# \circ f = 1_S$ $\iff f^\#$ surjective.
- In the category of complete lattices with 0, the canonical injection $i: S \rightarrow T$ of a subsemilattice such that $0_S = 0_T$ is residuated.

Note that, even if $S, T$ are complete lattices with bottom, the residuated map of the canonical injection $i: S \rightarrow T$ of a sumsemilattice such that $i(0) = 0$, may fail to be a morphism of semilattices.
Example 3.1

Let $S = \{0, s_1 \}$, $s_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, s_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, s_1 \lor s_2, t_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, t_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, t_1 \lor t_2$.

$s_1 \lor t_2, s_2 \lor t_1$

$S_1 = \{0, s_1, s_2, s_1 \lor s_2\}, S_2 = \{0, t_1, t_2, t_1 \lor t_2\}$.

The canonical semilattice injections $i_j : S_j \rightarrow S$ with $i_j(0) = 0, j = 1, 2$ are residuated, and we have $i_2^\#(s_1) = 0, i = 1, 2$. Thus $i_2^\#(s_1) \lor i_2^\#(s_2) = 0 < i_2^\#(s_1 \lor s_2) = t_1$.

Similarly, $i_1^\#(t_1) = s_1, i_1^\#(t_2) = 0 \Rightarrow i_1^\#(t_1) \lor i_1^\#(t_2) = s_1 < i_1^\#(t_1 \lor t_2) = s_1 \lor s_2$.

We introduce the following definition.

Definition 3.2

Let $S_1, S_2$ be semilattices with 0, such that $S_1 \cap S_2 = \{0\}$. We say that $S$ is the semidirect sum of $S_1$ and $S_2$, written $S_1 \oplus S_2$, if the following conditions hold:

i) The semilattice monomorphisms $S_j \rightarrow S$ satisfy $i_j(0_{S_j}) = 0_S, j = 1, 2$.

ii) The residuate maps $S \rightarrow S$, $j = 1, 2$ are semilattice morphisms, and satisfy $i_1 \circ i_1^\# \lor i_2 \circ i_2^\# = 1_S$.

Remark 3.3

Assume $S = S_1 \oplus S_2$. Then: $i_k^\# \circ i_j = \begin{cases} 1_{S_j} & k = j, \\
0 & \text{otherwise}
\end{cases}$, since $i_1^\# \circ i_j = 1_{S_j}$, and

$\forall s_j \in S_j, i_k^\# \circ i_j(s_j) = 0 \Rightarrow \{s_k \in S_k | i_k(s_k) \leq i_j(s_j)\} = \{0\}$.

Hence, the semidirect sum is a generalization of the direct sum.

The following theorem gives a necessary and sufficient condition for the semidirect sum decomposition of a semilattice into two subsemilattices.

Theorem 2

A semilattice $S$ admits a decomposition into a semidirect sum of sub-semilattices $S_1 \oplus S_2$ iff

1. $\forall s \in S, \exists s_1 \in S_1, s_2 \in S_2, \text{such that } s = s_1 \lor s_2$.

2. For $j = 1, 2$, and $j \neq k (1 \leq j, k \leq 2), \forall s, t \in S, \text{we have:}$

$\{x \in S_j | t_j \circ i_j^\#(s) \lor t_j^\#(t) < t_j(x) \lor s \lor t\} = \emptyset$.

Proof of Theorem 2

In order to simplify the notation, we write $i_j^\# = \pi_j, j = 1, 2$. Clearly, conditions 1 and 2 are necessary. We prove they are also sufficient.

Since $S_1, S_2$ are subsemilattices of $S$, the injections $i_j, j = 1, 2$, are injective semilattice morphisms, and $0_S = i(0_{S_j}), j = 1, 2$. Hence i) of Definition 3.2 holds.

This condition implies that $\pi_j$ is a morphism of poset, and $\pi_j \circ i_j = 1_{S_j}$ for $j = 1, 2$.

It remains to show that $\pi_j$ is a morphism of semilattices for $j = 1, 2$, and that
\[i_1 \circ \pi_1 \lor i_2 \circ \pi_2 = 1_S.\]

For the morphism, it is enough to show that this holds for \(\pi_1\). Let \(s \lor t \in S\).

\[\blacklozenge\] If \(s = i_1(s_1), t = i_1(t_1), s_1,t_1 \in S_1\), then \(s \lor t = i_1(s_1) \lor i_1(t_1) = i_1(s_1 \lor t_1)\), and
\[\pi_1(s \lor t) = \pi_1 \circ i_1(s_1 \lor t_1) = s_1 \lor t_1 = \pi_1 \circ i_1(s_1) \lor \pi_1 \circ i_1(t_1) = \pi_1(s) \lor \pi_1(t).\]

\[\Diamond\] If \(s = i_2(s_2), t = i_2(t_2), s_2,t_2 \in S_2\), then \(\pi_1(s \lor t) = \sup\{s_o \in S_1 | i_1(s_o) \leq s \lor t\}\).
Clearly, \(\{x \in S_1 | i_1(x) \leq s\} \subseteq \{x \in S_1 | i_1(x) \leq s \lor t\}\), hence \(\pi_1(s) \leq \pi_1(s \lor t)\). Similarly,
\[\pi_1(t) \leq \pi_1(s \lor t).\]

Therefore \(\pi_1(s) \lor \pi_1(t) \leq \pi_1(s \lor t)\).

Let \(x = \pi_1(s \lor t)\). We have \(i_1(x) < s \lor t\) (since \(s \lor t \in \text{Im } i_2\) and \(\text{Im } i_1 \cap \text{Im } i_2 = \emptyset\)). But by 2, \(\pi_1(s) \lor \pi_1(t) < x\) is excluded, thus we must have \(\pi_1(s) \lor \pi_1(t) = x = \pi_1(s \lor t)\).

\[\blacklozenge\] If \(s = i_1(s_1), s_1 \in S_1\), and \(t = i_2(s_2), s_2 \in S_2\). We conclude readily \(\pi_1(s) \lor \pi_1(t) \leq x = \pi_1(s \lor t)\), as above. Since \(s \lor t \geq i_1(x) \in \text{Im } i_1\), while \(s \lor t \not\in \text{Im } i_1\), we have \(i_1(x) < s \lor t\).

Again, by 2, we must have \(\pi_1(s) \lor \pi_1(t) = x = \pi_1(s \lor t)\).

If follows that \(\pi_j = i_j^\#\) is a morphism of semilattices for \(j = 1,2\).

We show that \(i_1 \circ \pi_1 \lor i_2 \circ \pi_2 = 1_S\), since \(i_j \circ \pi_j \leq 1_S, j = 1,2\).

\[\blacklozenge\] Let \(s \in S\), then, by 1, \(s = i_1(s_1) \lor i_2(s_2), s_j \in S_j, j = 1,2\).

We have \((i_1 \circ \pi_1 \lor i_2 \circ \pi_2)(s) = i_1 \circ \pi_1(s) \lor i_2 \circ \pi_2(s) = [i_1 \circ \pi_1(i_1(s_1) \lor i_2(s_2))] \lor [i_2 \circ \pi_2(i_1(s_1) \lor i_2(s_2))] = [i_1 \circ \pi_1(i_1(s_1) \lor i_1(s_1) \lor i_2(s_2))] \lor [i_2 \circ \pi_2(i_1(s_1) \lor i_2(s_2))] = i_1(s_1) \lor i_2(s_2) \lor i_1 \circ \pi_1 \circ i_2 \circ \pi_2 \lor i_2 \circ \pi_2 \circ i_1 \circ i_2 \circ \pi_2 \lor i_1 \circ i_2 \circ \pi_2 \circ i_1 \circ i_2 \circ \pi_2 \geq s.\]

\[\square\]

**Remark 3.4**

Condition 2 of Theorem 2 is necessary and sufficient for the existence of a semilattice morphism \(\pi: T \to S\), which is a left inverse to the inclusion \(0 \rightarrow S \leftarrow\rightarrow T\). More precisely, we have the following Corollary.

**Corollary 3.5**

Let \(i\) stand for the inclusion \(0 \rightarrow S \leftarrow\rightarrow T\) of a semilattice \(S\) with \(0\) into a semilattice \(T\) with \(0\). Then the map \(\pi: T \to S, t \mapsto \sup\{s \in S | i(s) \leq t\}\) is a left inverse to \(i\) if and only if \(\forall t_1, t_2 \in T, \{s \in S | i(t_1) \vee i(t_2) < i(s) < t_1 \lor t_2\} = \emptyset\). \[\square\]

**Remark 3.6**

Example 3.1 shows that although any decompositional morphism is always available, it may not be satisfactory. Definition 3.2, and Theorem 2 (together with its Corollary 3.5) states how to choose the appropriate subsemilattices.
Theorem 3  (direct sum decomposion of semilattices)
Let $S_1, S_2$ be semilattices satisfying the conditions of Theorem 2. Then $S = S_1 \oplus S_2$ if and only if for $j = 1, 2$, and $k \neq j (1 \leq k \leq 2), \forall s_j \in S_j$, \{s_k \in S_k \mid s_k \leq s_j\} = \{0\}.

Proof
Clearly, \{s_k \in S_k \mid s_k \leq s_j\} = \{0\} \Rightarrow \pi_k \circ i_j(s_j) = 0 \quad (k \neq j, 1 \leq j, k \leq 2). \quad \Box

Theorem 4
Every lattice $S \neq 2$ decomposes into the semidirect sum of two semilattices:
$S = S_1 \oplus S_2$ such that $S_1$ is the largest free subsemilattice of $S$.

Proof
Let $X$ stand for the set of nonzero irreducible elements of $S$, and let $x \in X$, cover 0.
The semilattices $S_1$ generated by 0 and $x$, and $S_2$ generated by 0 and $X \setminus \{x\}$ satisfy the conditions of Theorem 2. Then $S_1$ is free.
Now just take $k$ to be the largest integer such that,
for $X_1 = \{x_1, \ldots x_k\} \subset X$ , and $S_1$, (resp$S_2$) generated by (0 and) $X_1$, (resp$X \setminus X_1$),
we have that $S_1$ is free, and condition 2 of theorem 2 holds. \quad \Box

Remark 3.7
Let $E_i = 2^n, i = 2, \ldots , n$. Then the antichain $A = 2^n$ may be written as the direct sum $\bigoplus_{i=1}^{n} E_i$. By contrast, a chain $0 < x_1 < \ldots , x_n$, may be written as $\bigoplus_{i=1}^{n} E_i$, with $i_k(0) = 0$, and $i_k(1) = x_k, k = 1, \ldots n$.

4  The decomposition theorem for idempotent semimodules

Just as we did for semilattices, we try to define a left inverse $\pi$ to the inclusion $i$ of a semimodule $M$ into a semimodule $N : 0 \rightarrow M \rightarrow N$. For every $y \in N$, either $y$ dominates some $x \in M$ (i.e $i(x) \leq y$) or not. In the first case, we set $\pi(y) = i_#(y) = \sup\{x \in M \mid i(x) \leq y\}$. In the second case, we set $\pi(y) = 0$.

We extend Definition 3.2 to semimodules as follows.

Definition 4.1
Let $M_1, M_2$ be subsemimodules of a semimodule $M$ such that $M_1 \cap M_2 = \{0\}$.
We say that $M$ is the semidirect sum of $M_1$ and $M_2$, written $M_1 \oplus M_2$ if the following conditions hold:
\begin{enumerate}
  \item[i)] The semimodules monomorphisms $M_j \rightarrow M$ satisfy $i_j(bzeroM_j) = 0_M, j = 1, 2$.
  \item[ii)] The residuate maps $M \rightarrow M_j x \mapsto i_j^#(x), j = 1, 2$ are semimodule morphisms satisfying: $i_1 \circ i_1^# \vee i_2 \circ i_2^# = 1_M$.
\end{enumerate}
Remark 4.2
Just as in Remark 3.2, we can show that our definition generalizes that of the direct sum of semimodules.

The following statement follows readily from Theorem 2, and the definition of $\pi_j : M \to M_j$, $j = 1, 2$.

Theorem 5
Let $M_1, M_2$ be finite dimensional subsemimodules of a finite dimensional semimodule $M$, such that $M_1 \cap M_2 = \{0\}$. Then $M$ splits into a semi-direct sum $M_1 \oplus M_2$ iff the following hold:
1. $\forall x \in M, \exists x_1 \in M_1, x_2 \in M_2$, such that $x = x_1 \lor x_2$.
2. For $j = 1, 2$, and $j \neq k$, $(1 \leq j \leq 2)$, $\forall x, y \in M$, we have:
   \[ \{ z \in M_j | i_j^j(x) \lor i_j^j(y) < i_j^j(z) < x \lor y \} = \emptyset. \]

Proof of Theorem 5
Since $M$ is a semilattice, by Theorem 2, it suffices to show that $i_j^j(\lambda x) = \lambda i_j^j(x)$, $j = 1, 2$.

It is enough to prove the case $j = 1$. For simplicity of the notation, we write $i_1^1 = \pi_1$.
Let $A_x = \{ x_1 \in M_1 | i_1^1(x_1) \leq x \}$. Clearly $\forall (\lambda, x) \in P \times M$, $x_1 \in A_x \Rightarrow \lambda x_1 \in A_{\lambda x}$, and for every upper bound $y$ of $A_x$, then $\lambda y$ is an upper bound of $A_{\lambda x}$. In particular, this holds for $y = \pi(x)$, therefore $\pi(\lambda x) \leq \lambda \pi(x)$. But then $\pi(x) = \pi(\lambda^{-1} \lambda x) \leq \lambda^{-1} \pi(\lambda x) \Rightarrow \lambda \pi(x) \leq \pi(\lambda x)$, which completes the proof. \qed

Remark 4.3
The decomposition of $M$ into a semidirect sum is not unique in general (cf Examples 5). Let $X$ be the (finite) basis of $M$, and $c \in S(M)$. The set $\pi^{-1}(c) \in X^+$ is called the fiber over $c$. The number of elements in this fiber depends on the choice of the basis $X$.

Definition 4.4
We say that $X$ is regular if:
\[ \forall c \in S(M) \text{ the cardinality of the fiber } \pi^{-1}(c) \subset X^+ \text{ is minimal} \]
(cf the discussion in Example 5.2 below).

Collecting Theorem 2 and Theorem 5, we can state the following, which may be used for the decomposition of general semimodules.

Proposition 4.5
Let $X$ be a regular basis of $M$. Let $X_o = X \cup \{0\}$, and let $X_1, X_2$ be such that:
\[ X_1 \cap X_2 = \{0\} \]
and write $M_j$ for $M_{X_j}$, $j = 1, 2$. Assume further that:
\[ \forall x, y \in M, x \sim y \Rightarrow \exists x_j, y_j \in M_j, z \in M \text{ s.t. } x = x_j \lor z, y = y_j \lor z \text{ with } x_j \sim y_j (1 \leq j \leq 2). \]
Then we have:
\[ M = M_1 \oplus M_2 \Rightarrow X_o^+ = X_1^+ \oplus X_2^+ \]

**Proof of Proposition 4.5**

In order to eliminate any risk of confusion, we use the following notation:
\( i_j^M: M_j \rightarrow M \) and \( \pi_j^M: M \rightarrow M_j \) (resp. \( i_j^X: X^+ \rightarrow X^+ \) and \( \pi_j^X: X^+ \rightarrow X_j^+ \)), for the inclusions and their natural projections (i.e. residuates), respectively.

Let \( \tilde{M}(x, y) = \{ z \in M \mid \exists j \text{ s.t. } i_j^M(x) < z < y \} \), and \( \tilde{X}^+(x, y) = \{ z \in X^+ \mid \exists j \text{ s.t. } i_j^X(x) < z < y \} \). Clearly \( \tilde{X}^+(x, y) \subset \tilde{M}(x, y) \). Hence \( \tilde{M}(x, y) = \emptyset \Rightarrow \tilde{X}^+(x, y) = \emptyset. \)

**Remarks 4.6**

4.6.1. Proposition 4.5 states a necessary condition for the subdirect (resp direct) sum decomposition of \( M \).

4.6.2. The regular basis condition is also necessary if we want the map \( \rho: M \rightarrow X^+ \) to split into \( \rho_i: M_i \rightarrow X_i^+ \), \( i = 1, 2 \). Indeed, let \( X \) and \( Y \) be two bases with \( X \) only a regular basis. Then while \( M_X = M_Y \), \( X^+ \) and \( Y^+ \) are not isomorphic (cf Example 5.3 below).

In our decomposition of torsion semimodules, we want to choose \( X_1 \), and \( X_2 \) such that \( M_1 = M_{X_1} \) is a maximal semi-Boolean semimodule, while \( M = M_1 \oplus M_2 \). The maximality of \( M_1 \) is to be understood in the sense that, for any other semidirect sum decomposition \( M = N_1 \oplus N_2 \), with \( N_1 \) semi-Boolean, we have \( \dim N_1 \leq \dim M_1 \).

Consider the map \( X^+ \overset{\rho}{\rightarrow} S(X^+) = S(M) \). For any \( x \in X \), we have \( x \in p^{-1}(c_x) \), and \( p^{-1}(c_x) \) is the set of elements \( (x) \) characterizes the fact that \( x \) belongs to a torsion cycle in \( M \). In this case, both \( x \) and \( p(x) \) will be called torsion elements.

Let \( X_o = \{ x \in X \mid p^{-1}(c_x) = \{ x \} \} \). Clearly the set \( X_1 \) of generators we want must belong to \( X_o \). It will fail to be equal to \( X_o \) in general, since the elements of \( X_o \) may generate torsion elements, without being torsion elements themselves (consider for instance the case where \( x_1, \ldots, x_4 \in X_o \) with \( x_1 \lor x_2 < x_3 \lor x_4 < (x_1 \lor x_2) \); then we have \( p^{-1} \circ p(x_i) = \{ x_i \} \), \( i = 1, \ldots, 4 \), while \( p^{-1} \circ p(x_1 \lor x_2) = \{ x_1 \lor x_2, x_3 \lor x_4 \} \).

For every \( Y \in \mathcal{P}(X_o) \), consider the semilattice \( Y^+ \) such that:
\[ \forall \ c \in S(Y^+) \), the fiber \( p^{-1}(c) \) contains at most one element of \( Y^+ \),
\[ \land \ M_Y \text{ and } M_{X \setminus Y} \text{ satisfy the conditions of Theorem 5.} \]

Condition \( \Diamond \) ensures that \( M_Y \) is semi-Boolean, and condition \( \heartsuit \) ensures that \( M_X \) splits into a semidirect sum of semimodules.

A semimodule \( M \) for which \( \Diamond \) hold for no \( c \in S(Y^+) \), is called a pure torsion semimodule.

The restricted map \( p_{Y^+} \) is an isomorphism of semilattices, and \( M_Y \) is a semi-Boolean semimodule. Let \( \mathcal{Y} \) stand for the set of all the \( Y \in \mathcal{P}(X_o) \) satisfying \( \Diamond \land \heartsuit \).

Note that \( \mathcal{Y} \) may be any subset of \( P(X_o) \). If \( \mathcal{Y} = \emptyset \), then \( M \) is a pure torsion semimodule
(cf Example 5.7 below). If \( Y = \mathcal{P}(X_o) \), then we can take \( X_1 = X_o \). For the general case, we want to take \( Y \) as large as possible. The relation \( Y^+ \ll Y'^{+} \iff |Y^+| \leq |Y'^{+}| \), where \( |Y^+| \) is the cardinal of \( Y^+ \), defines a quasorder on \( Y \). Then choose \( X_1 \) to be the set of generators in a maximal element in \((\mathcal{Y}, \ll)\). This choice is not unique in general (cf Examples 5.2, 5.4, 5.8 below). We have the following statement.

**Theorem 6**

Let \( M \) be an idempotent finite dimensional general semimodule such that \( \mathcal{Y} \neq \emptyset \). Then there are non trivial subsemimodules \( M_1, M_2 \) of \( M \), such that:

1) \( M_1 \) is a maximal semi-Boolean semimodule such that \( M = M_1 \oplus M_2 \).

2) \( M_2 \) is a pure torsion semimodule.

**Proof of Theorem 6**

Clearly \( \mathcal{Y} \neq \emptyset \implies X_1 \neq \emptyset \), and the fact that \( M \) is a general semimodule imply that \( X_2 = (X \setminus X_1) \neq \emptyset \). By Theorem 5 and the construction of \( \mathcal{Y} : M = M_1 \oplus M_2 \) holds. By the choice of \( X_1 \in \mathcal{Y} \), \( M_1 \) is maximal. \( \square \)

**Remark 4.7**

The conditions \( x \sim y \) and \( x \in M_1 \Rightarrow y \in M_1 \) in Proposition 4.5 require that \( \forall x \in X^+_1, p^{-1}(p(x)) \in X^+_1 \), while condition \( \text{\textdiamond} \), only requires that (using the same notation, i.e. \( Y = X_1 \)) the intersections \( p^{-1}(p(x)) \cap X^+_1 \) of the fibers with the semilattice \( X^+_1 \) are singletons. Thus Theorem 6 yields larger semi-Boolean semimodule components than Proposition 4.5., while Proposition 4.5. yields larger torsion semimodule components, in general (cf Examples 5.2-5.3).

Let \( X_o \) stand for the basis of a semi-Boolean semimodule \( M \), and let \( X_1 \subset X_o \) be the largest subset of \( X_o \) such that \( M_1 = M_{X_1} \) is a Boolean semimodule and the conditions of Theorem 5 are satisfied by \( M_2 = M_{X_o \setminus X_1} \). Then \( M = M_1 \oplus M_2 \), where \( M_1 \) is the largest Boolean subsemimodule of \( M \).

Now let \( X_2 \subset X_1 \) be the largest subset of \( X_1 \) such that \( M_2 = M_{X_2} \) is a free semimodule and the conditions of Theorem 5 are satisfied by \( M_2 \) and \( M_3 = M_{X_1 \setminus X_2} \). Then \( M_1 = M_2 \oplus M_3 \), where \( M_2 \) is the largest free subsemimodule of \( M \).

We have shown the following general decomposition theorem for semimodules.

**Theorem 7**

Every semimodule splits into a semidirect sum decomposition \( M = M_1 \oplus M_2 \oplus M_3 \oplus M_4 \), where \( M_1 \) is free, \( M'' = M_1 \oplus M_2 \) is Boolean, \( M' = M_1 \oplus M_2 \oplus M_3 \) is semi-Boolean, and \( M_4 \) pure torsion, with:

- \( M' \) is largest semi-Boolean subsemimodule of \( M \).
- \( M'' \) is the largest Boolean subsemimodule of \( M' \).
- \( M_1 \) is the largest free subsemimodule of \( M'' \). \( \square \)
We can now state the direct sum decomposition theorem for semimodules.

**Theorem 8**
Let $M_1$, $M_2$ be semimodules satisfying the conditions of Theorem 3. Then $M$ splits into the direct sum $M_1 \oplus M_2$ if and only if
\[
\forall x \in M_k, \{ y \in M_j \mid y \leq x \} = \{0\}, \quad 1 \leq j, k \leq 2, k \neq j.
\]

**Proof of Theorem 8**
This follows directly from Theorem 3, since if the conditions in the statement of the Theorem hold then
\[
\pi_k \circ \iota_j = 0, \quad 1 \leq j, k \leq 2, k \neq j,
\]
Conversely, if $M$ splits into a direct sum $M_1 \oplus M_2$, then, for $k = 1, 2$, we have
\[
\pi_j \circ \iota_k = 0 \implies [\iota_k(x) \leq \iota_j(y) \iff x = 0].
\]

**Remark 4.8**
There is a direct sum analog of Theorem 7. Its statement and proof are straightforward.

## 5 Examples
In these examples, the generators are given by the columns of the matrices, with $x_i$ given by column $i$ of the matrix.

### 5.1 Let $M$ be generated by the columns of $A = \begin{bmatrix} \mid & 0 & \mid \\
0 & \mid & \mid \\
\mid & \mid & \lambda \end{bmatrix}$, $\lambda > 1$. The torsion cycle $x_1 \vee x_2 \sim x_3$ contains one generator.
Following Proposition 4.5., we must have both $x_1 \vee x_2$ and $x_3$ in the same component. Hence, we will have $X_1 = \{x_1\}$ (or $X_1 = \{x_2\}$), and $X_2 = \{x_2, x_3\}$ (or $X_2 = \{x_1, x_3\}$), with $M = M_{X_1} \oplus M_{X_2}$.
Let $X^+$ stand for the semilattice generated by (0 and) the $x_i$'s. It is easy to see that, for $X_1 = \{x_1, x_2\}$, $X_2 = \{x_3\}$ the conditions of Theorem 2 hold; in particular \{ $x \in X_2 | \tau_2(x) \leq x \} = \{ x \in X_2 | 0 < \tau_2(x) < (1 1 1)^T \} = 0$. Hence $X^+ = X_1^+ \oplus X_2^+$.
However, in the semimodules $M_{X_1}$, $M_{X_2}$, we have \{ $x \in M_2 | \tau_2(x) \leq \tau_1(x) \} = \lambda^{-1} x_3$. Therefore, there is no decomposition of $M$ associated with the decomposition of the basis into $X_1$, $X_2$. This Example shows in particular that the converse of Proposition 4.5 does not hold.

### 5.2 Let $A = \begin{bmatrix} \mid & \mid & \mid \\
\mid & \mid & \mid \\
\mid & \mid & \mid \end{bmatrix}$, $\lambda > 1$. The torsion cycle $x_1 \sim x_2 \leq x_3$. There are three possible decompositions:
i) $X_1 = \{x_2, x_3\}$, $X_2 = \{x_1\}$.
ii) $X_2 = \{x_1, x_3\}$, $X_2 = \{x_2\}$.
iii) $X_1 = \{x_1, x_2\}$, $X_3 = \{x_3\}$.

For all three cases, we have $M = M_1 \oplus M_2$. In the cases i), ii) the torsion in $M$ comes from the semidirect sum composition of the two semimodules. In this composition, the torsion relation $x_1 < x_2 < \lambda x_1$ must be specified.

Such situations have to be taken into account when building up semimodules from simpler ones.

In case iii), $M_1$ is a torsion semimodule.

Note that the generators of $M_1$ are both dominated by the generator $x_3$ of the Boolean semimodule. Let $y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, then $Y = \{x_1, x_2, y\}$ is not a regular basis. Indeed, with this basis, we have two torsion cycles: $x_1 \sim x_2$, and $x_2 \lor y \sim y$. Then $p^{-1}(p(x_3))$ contains two elements. With $X = \{x_1, x_2, x_3\}$, we have a unique torsion cycle $x_1 \sim x_2$, and $p^{-1}(p(x_3)) = \{x_3\}$. Also $X^+$ is a chain, while $Y^+$ is not.

**5.3** Let $A = \begin{bmatrix} 1 & 0 & \lambda & \lambda \\ 0 & 1 & \mu & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $1 < \lambda < \mu$.

For $X_1 = \{x_1, x_2\}$, $X_2 = \{x_3, x_4\}$, just as in Example 5.1, we have a semidirect sum decomposition of the lattice $X^+ = X_1 \oplus X_2$. Also, in the semimodules $M_{X_1}$, $M_{X_2}$, we have $\{x \in M_2 | \pi_2(x_1) \lor \pi_2(x_2)\} < \pi_2(x) < x_1 \lor x_2 = \lambda^{-1} x_3$, and there is no decomposition of $M$ associated with the decomposition of the basis into $X_1, X_2$.

Note that here, the condition of Proposition 4.5. that every vector in a torsion cycle belongs to the same component holds, which was not the case in 5.1 above. This shows in particular that the condition that $X^+$ splits (Proposition 4.5) is not sufficient to guarantee that $M_X$ splits.

**5.4** Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & \lambda & \mu \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $1 < \lambda \leq \mu$. The torsion cycle $x_2 \sim x_3$ contains two generators. A maximal semi-Boolean semimodule obtains (Theorem 4) by taking $X_1 = \{x_1, x_2\}$, and $X_2 = \{x_3\}$, and the torsion in $M$ comes from the semidirect sum composition $M = M_1 \oplus M_2$. From the requirement that torsion cycles belong to the same summand, we will have $X_1 = \{x_1\}$, and $X_2 = \{x_2, x_3\}$. The generator $x_1$ of the Boolean semimodule is dominated by the torsion cycle $x_2 \sim x_3$.

**5.5** Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Let $X_1 = \{X_1, X_2\}$. Then we get a semidirect sum decomposition $M = M_{X_1} \oplus M_{X_2} \setminus X_1$, with $M_{X_1}$ a free (and $M_{X_2} \setminus X_1$ a Boolean) subsemimodule of $M$. On the other hand, let $Y = \{x_1, x_3\}$. Then we get a **direct sum** decomposition $M_Y \oplus M_{X_2} \setminus Y$ of two Boolean semimodules. Note that, since $x_1 \leq x_3$, and $x_2 \leq x_4$, then this (direct sum) decomposition is unique (i.e., $x_1, x_3$ [resp. $x_2, x_4$] have to belong to the
same summand).
5.6 Let \( A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & \lambda \\ 0 & 0 & 1 & \mu \end{bmatrix} \), \( 1 < \lambda < \mu \). We get a semi-direct sum decomposition \( M_1 \oplus M_2 \), of a semi-Boolean semimodule \( M_1 \) generated by \( X_1 = \{ x_1, x_2 \} \), and a torsion semimodule \( M_2 \) generated by \( X_2 = \{ x_3, x_4 \} \). Note that \( x_1 \lor x_3 = x_2 \lor x_3 \), hence this is not a direct sum.

\[
\begin{bmatrix}
\| & \lambda & \lambda & & & & \\
\| & \| & \xi & & & & \\
\| & 0 & 0 & \| & \| & & \\
0 & 0 & 0 & 0 & 0 & \| & \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7
\end{bmatrix}
\]

5.7 Let \( A = \begin{bmatrix} 1 & \lambda & \lambda & \mu & 0 & 0 \\ 1 & \lambda & \xi & \xi & 0 & 0 \\ 0 & 0 & \| & & \| & & \\
0 & 0 & 0 & \| & \| & \| & \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7
\end{bmatrix} \), \( 1 < \lambda < \xi < \mu < \nu \).

Clearly \( M \) is a pure torsion semimodule. Let \( M_1 = M_{\{x_1,x_2\}} \), \( M_2 = M_{\{x_3,x_4,x_5\}} \) \( M_3 = M_{\{x_6,x_7\}} \). Note the two direct sum compositions with summand \( M_3 \) in two alternative decompositions of \( M \):

\( M = (M_1 \oplus M_3) \oplus M_2 \), and

\( M = (M_1 \oplus M_2) \oplus M_3 \).

The semilattices \( X^+ \) has 15 vertices, we only represent \( S(M) \) below.

![Figure 1: The structural semilattice of Example 5.7](image)

All vectors in a torsion cycle belong to the same semimodule. Hence, by Proposition 4.5, the decomposition also applies to the semilattices \( X^+ \) and \( S(X^+) = S(M) \).

5.8 Let \( M \) be given by \( A = \begin{bmatrix} 1 & 1 & 0 & 1 & \xi \\ 1 & 0 & 1 & \lambda & \lambda \\ 0 & 1 & 1 & 1 & \mu \end{bmatrix} \), with \( 1 < \lambda < \xi < \mu \).

We have \( x_1 \lor x_2 = x_1 \lor x_3 = x_2 \lor x_3 \sim x_4 \sim x_5 \). Let \( X_1 = \{ x_1, x_2 \} \), \( X_2 = \{ x_3, x_4, x_5 \} \), then \( X^+ = X_1^\lor \oplus X_2^\lor \). However: \( \iota_2(\pi_2(x_1) \lor \pi_2(x_2)) = 0 < \iota_2(\lambda^{-1} x_4) < x_1 \lor x_2 \), and \( M \) does not split into a semidirect sum \( M_1 \oplus M_2 \). The situation would be similar for \( X_1 = \{ x_1, x_3 \} \), or \( X_1 = \{ x_2, x_3 \} \).

Let \( X_1 = \{ x_1, x_2, x_3 \} \), \( X_2 = X \setminus X_1 \). Clearly \( M = M_1 \oplus M_2 \).

![Figure 2: The representations of Example 5.7](image)
Figure 2: The semilattices $X^+$ and $S(M)$ of Example 5.8
6 Conclusions and further research

In this paper, we have shown that unlike as in the theory of modules, the direct sum decomposition is not possible in general for idempotent semimodules. We define the concept of semidirect sum for semilattices and semimodules, and prove a general decomposition theorem for finite dimensional idempotent semimodules into a semidirect sum of idempotent semimodules, which states that every semimodule may be decomposed into four subsemimodules: a free, a Boolean, a semi-Boolean, and a torsion semimodule.

Also, we prove a necessary and sufficient condition for the direct sum decomposition of semilattices. By contrast with the direct sum, which is a free sum, the semidirect sum is not free, in the sense that the knowledge of the two summands $M_1$, $M_2$ (or $S_1$, $S_2$ for semilattices) is not sufficient for the characterization of $M_1 \oplus M_2$ (resp. $(S_1 \oplus S_2)$. We also need to know something about the relations binding the elements of one of the summands with those of the other. However, since idempotent semimodules are very complicated in general, our decomposition allows for some simplification in the presentation of these structures.

Also, since a semilattice is an idempotent commutative monoid, the results of Section 3 give some insight into the decomposition of commutative idempotent monoids.

The direct and semidirect sum decompositions are illustrated with examples, and we show that semidirect sum decomposition is not unique in general. The decomposition of idempotent semimodules into four parts can also be used in order to get a simpler classification of the isomorphism classes of these structures. Indeed, although there are only 3 families of isomorphy classes in the 2-dimensional case, for the 3-dimensional case, this number jumps to near 60 (e.g. [12], [13]).

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