On \((\text{min}, \text{max}, +)\)-Inequalities

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ABSTRACT. In this paper we concentrate on processes consisting of events on which constraints are put on the order in which the individual events can take place. We ask ourselves the question if the constraints allow the processes to happen at all, or do contain intrinsic bottlenecks. Further, if the processes can take place and are supposed to repeat themselves, we investigate whether or not a periodic behavior is possible. We distinguish between a periodic behavior with one overall period, and the case in which the more general cycle time vector has to be considered. We will use a (min, max, +) formulation and present our results in graph theoretic terms. Also we will also discuss some of the computational aspects of our approach.

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1. Introduction

In this paper we study processes consisting of events of which only the duration and the interdependency is given. The latter means on the one hand that we know of each event which other events have to be completed before it can take place, while on the other hand we know of each event which events can take place only after it has been completed. These requirements lead naturally to a description of the processes using operators like the maximum, the minimum and the addition. Given such requirements we investigate in this paper under which conditions the processes can happen at all. Furthermore, in case the processes are to be repeated, we study the question whether or not this repetition can be done by means of some sort of regular time table. We will also discuss some of the computational aspects of our approach, although it is not our intention to go for the fastest algorithms. We begin to illustrate our ideas by means of two examples.

1.1. Example 1. Many processes in every day life require a lot of planning and synchronization. Think for instance of the preparation of a meal with various dishes that have to be served at the same time. For the preparation of an individual dish one needs the availability of the ingredients at the right time. Its duration will furthermore depend on the labor involved, like the time it takes to boil water, clean the salad and so on. Crucial for a dish to be ready for being served are the time instants at which each of its ingredients is available. The earliest time at which the whole meal can be served is after the latest time instant at which all dishes are ready. In this process some dishes probably will be already available before the time of serving. Depending on the particulars of these dishes, one should prepare them as late as possible in order not to ruin their taste during the idle time between being ready and serving. Hence this idle time must be kept to a minimum. Thus the cook faces a decision process with at least the maximization operator (the maximum of the time instants when all dishes are ready) as well as the minimum operator (minimizing the idle times).
1.2. **Example 2.** Consider the following part of a railway network.

The part consists of three lines, called line I, II and III. Line I goes from station 1 to station 4 via station 3, line II goes from station 2 to station 3 and back, and line III goes from station 5 to station 3 and back. In station 3 the trains on lines I and II are supposed to wait for each other in order to give passengers the possibility to change over. Also in station 3 the trains on lines I and III are supposed to use the same platform and therefore can not be in station 3 at the same time. The first requirement implies that the \((k+1)\)-st departure time in station 3 of the train to station 4 is after the \(k\)-th arrival time in station 3 of the trains from both stations 1 and 2. However, the second requirement implies that the \((k+1)\)-st departure time in station 3 of the train to station 4 should be before the \(k\)-th arrival time in station 3 of the train from station 5. It is tacitly assumed here that the frequencies of the trains on all tracks is the same.

In order to make things a bit more specific we denote \(a_{ij}\) for the travel time of the train from station \(j\) to station \(i\). We denote \(\delta\) for the passenger-change-over time. Further, we denote \(x_i(k)\) for the \(k\)-th departure time of the train in station \(i\), where \(x_3(k)\) denotes the departure time at station 3 of the train to station 4. The departure time at station 3 of the train to station 5 will not play a role in our model. Observe that the \(k\)-th arrival time at station \(i\) of the train from station \(j\) can be expressed as \(x_j(k) + a_{ij}\). Then the fact that the \((k+1)\)-st departure time in station 3 of the train to station 4 is after the \(k\)-th arrival time in station 3 of the trains from stations 1 and 2 plus the passenger-change-over time \(\delta\) yields that for all non-negative integers \(k\)

\[
x_3(k+1) \geq \max(x_1(k) + a_{31} + \delta, x_2(k) + a_{32} + \delta).
\]

The requirement that the \((k+1)\)-st departure time in station 3 of the train to station 4 should be before the \(k\)-th arrival time in station 3 of the train from station 5 yields that for all non-negative integers \(k\)

\[
x_3(k+1) \leq x_5(k) + a_{35}.
\]

If the above should be part of a periodic behavior, then we have to look for initial departure times, say \(x_i^0\), \(i = 1, 2, 3, 4, 5\), and a period, say \(\lambda\), such that

\[
\max(x_1^0 + a_{31} + \delta, x_2^0 + a_{32} + \delta) \leq x_3^0 + \lambda \quad \text{and} \quad x_3^0 + \lambda \leq x_5^0 + a_{35}.
\]

With \(x_i(k) = x_i^0 + k\lambda\) for all \(i = 1, 2, 3, 4, 5\) and \(k \geq 0\), it then follows that for all non-negative integers \(k\)

\[
\max(x_1(k) + a_{31} + \delta, x_2(k) + a_{32} + \delta) \leq x_3(k+1) \quad \text{and} \quad x_3(k+1) \leq x_5(k) + a_{35}.
\]

1.3. **Outline.** In section 2 we present some notation and preliminaries. Section 3 contains the statements of the problems that we treat in this paper. In section 4 and 5 we present our main results. Section 4 contains graph theoretic conditions for the solvability of each of the problems of this paper. Section 5 contains some algorithmic issues and in section 6 we illustrate the results by means of an example. We end the paper with some concluding remarks in section 7 and the references.
2. Preliminaries

We recall some of the well-known \((\min, \max, +)\) notation, see for instance [1], that we use in combination with traditional notation. For scalars \(a\) and \(b\) we write \(a \oplus b\) for the maximum of \(a\) and \(b\), and \(a \ominus b\) for the minimum of \(a\) and \(b\).

When \(\lambda\) is a scalar and \(x\) is a vector, the expression \(x = \lambda\) means that each component of \(x\) is equal to \(\lambda\). The expression \(x + \lambda\) stands for the vector obtained by adding \(\lambda\) to each component of \(x\). Given two vectors \(x\) and \(y\) of the same dimension, \(x \leq y\) indicates that the symbol \(\leq\) holds componentwise.

When \(\lambda\) is a scalar and \(U\) is a matrix, the expression \(U + \lambda\) stands for the matrix obtained by adding \(\lambda\) to each entry of \(U\). The transpose of a matrix \(U\) is denoted by \(U^T\). The expression \(-U^T\) stands for the transposed of \(U\) multiplied by \(-1\).

If \(U\) is an \(n \times m\) matrix with entries \(u_{ij} \in \mathbb{R} \cup \{-\infty\}\) and \(x\) is a vector with \(m\) components \(x_j \in \mathbb{R}\), the expression \(U \otimes x\) equals a new vector with \(n\) components. The \(i\)-th component of \(U \otimes x\) is given by \((u_{i1} \otimes x_1) \oplus (u_{i2} \otimes x_2) \oplus \ldots \oplus (u_{iM} \otimes x_M)\). The latter will also be denoted as \(\bigoplus_{j=1}^M (u_{ij} \otimes x_j)\). Note that it is possible that some of the entries of \(U\) are \(-\infty\). It is clear that if \(u_{ij} = -\infty\), then \(x_j\) does not play a role in the \(i\)-th component of \(U \otimes x\).

Similarly, when \(V\) is an \(n \times m\) matrix with entries \(v_{ij} \in \mathbb{R} \cup \{+\infty\}\) and \(x\) is a vector with \(m\) components \(x_j \in \mathbb{R}\), the expression \(V \ominus x\) equals a new vector with \(n\) components. The \(i\)-th component of \(V \ominus x\) is given by \((v_{i1} \ominus x_1) \ominus \ldots \ominus (v_{iM} \ominus x_M)\). The latter will also be denoted as \(\bigoplus_{j=1}^M (v_{ij} \ominus x_j)\). Note that it is possible that some of the entries of \(V\) are \(+\infty\). It is clear that if \(v_{ij} = +\infty\), then \(x_j\) does not play a role in the \(i\)-th component of \(V \ominus x\).

Given an \(n \times n\) matrix \(U\), the graph of \(U\), denoted as \(G_U\), consists of the vertex set \(V_U = \{1, 2, \ldots, n\}\) and the edge set \(E_U = \{(i, j) | u_{ij}\text{ is finite}\}\). A circuit in the graph \(G_U\) corresponding to the matrix \(U\) consists of a ordered list of vertices \(i_1, \ldots, i_t \in V_U\) with \(i_{t+1} = i_1\) (the vertices \(i_1\) and \(i_{t+1}\) are said to be identified), such that \((i_j, i_{j+1}) \in E_U\) for \(j = 1, \ldots, t\). We occasionally denote the circuit as \(i_1 \rightarrow \ldots \rightarrow i_t \rightarrow i_{t+1}\). The circuit is called an "elementary" circuit if in addition to the previous the vertices \(i_1, \ldots, i_t\) are mutually distinct. In both cases the weight of the circuit is then given by \(\sum_{j=1}^t u_{i_j, i_{j+1}}\).

If \(U\) and \(V\) are two matrices of the same dimensions with entries in \(\mathbb{R} \cup \{-\infty\}\), then \(U \oplus V\) stands for the \((\max, +)\) sum of \(U\) and \(V\), i.e. the matrix obtained by taking component wise the maximum of \(U\) and \(V\). Further, if the number of columns of \(U\) and the number of rows of \(V\) are the same, say \(l\), then \(U \oplus V\) stands for the \((\max, +)\) product of \(U\) and \(V\), i.e. the matrix of which the \((i, j)\)-th entry is defined as \(\bigoplus_{k=1}^l (u_{ik} \ominus v_{kj})\) if \(U\) is a matrix and \(\lambda\) a scalar, then \(U + \lambda\) may be also expressed as \(U \ominus \lambda\).

In this paper the following property will be used abundantly. To describe the property, suppose that \(U\) is an \(n \times m\) matrix with entries in \(\mathbb{R} \cup \{-\infty\}\) and that \(x\) is an \(n\) vector with entries \(x_j \in \mathbb{R}\) and \(y\) is an \(m\) vector with entries \(y_i \in \mathbb{R}\). Then, compare [1] page 112,

\[ U \otimes x \leq y \text{ if and only if } x \leq -U^T \ominus y. \]

Indeed, \(U \otimes x \leq y\) means that \(\bigoplus_{j=1}^m (u_{ij} \otimes x_j) \leq y_i\) for all \(i = 1, \ldots, n\). As \(\bigoplus\) stands for maximization, the latter implies that \(u_{ij} \otimes x_j \leq y_i\) for all \(i = 1, \ldots, n\) and \(j = 1, \ldots, m\). In traditional notation this means that for all \(i = 1, \ldots, n\) and \(j = 1, \ldots, m\), there holds \(u_{ij} + x_j \leq y_i\), or equivalently \(x_j \leq -u_{ij} + y_i\). Recall that \(-u_{ij} + y_i = -u_{ij} \ominus y_i\), and that \(\bigoplus\) denotes mininimization. Then the previous implies that \(x_j \leq \bigoplus_{j=1}^m (-u_{ij} \ominus y_i)\) for all \(j = 1, \ldots, m\), implying that \(x \leq -U^T \ominus y\). The converse implication follows by reversing the above arguments.

Note that if a finite vector \(x \in \mathbb{R}^n\) is to be found such that \(U \otimes x \leq x\) and \(V \otimes x \leq x\), where \(U\) and \(V\) are two \(n \times n\) matrices with entries in \(\mathbb{R} \cup \{-\infty\}\), this is equivalent with the existence of a finite vector \(x \in \mathbb{R}^n\) such that \((U \oplus V) \otimes x \leq x\).

Let \(\eta\) be a finite vector in \(\mathbb{R}^n\). Then denote \(\Delta(\eta)\) for the \(n \times n\) matrix with on its diagonal the entries from \(\eta\) and with its off-diagonal entries equal to \(-\infty\). Note that if \(x\) and \(y\) are finite vectors in \(\mathbb{R}^n\) and \(U\)
is an $n \times n$ matrix with entries in $\mathbb{R} \cup \{-\infty\}$, then $U \odot (x + y)$ can also be seen as $(U \odot \Delta(x)) \odot y$. If, in addition $x = -\lambda$ with $\lambda$ a scalar, then $(U - \lambda) \odot y = (U \odot (-\lambda)) \odot y = (U \odot \Delta(x)) \odot y$. Finally, note that $-\Delta(-\eta)$ is a diagonal matrix with on its diagonal the entries from $\eta$ and with its off-diagonal entries equal to $+\infty$. If $V$ is an $n \times n$ matrix with entries in $\mathbb{R} \cup \{+\infty\}$ and $x, y$ are finite vectors in $\mathbb{R}^n$, then $V \odot' (x + y)$ can also be seen as $(V \odot' -\Delta(-x)) \odot' y$.

3. Problem statements

Inspired by the examples in the introduction we are going to study in this paper the problems formulated below. In the problem formulations we assume that $A$ is a given $n \times n$ matrix with entries in $\mathbb{R} \cup \{-\infty\}$ and that $B$ is a given $n \times n$ matrix with entries in $\mathbb{R} \cup \{+\infty\}$.

**Problem 1:** Find, if possible, a finite vector $x \in \mathbb{R}^n$ such that

$$A \odot x \leq B \odot' x.$$

**Problem 2:** Find, if possible, a finite vector $x \in \mathbb{R}^n$ and a finite scalar $\lambda \in \mathbb{R}$ such that

$$A \odot x \leq x + \lambda \leq B \odot' x.$$

**Problem 3:** Find, if possible, finite vectors $x, \eta \in \mathbb{R}^n$ such that for all integers $k \geq 0$

$$A \odot (x + k\eta) \leq x + (k + 1)\eta \leq B \odot' (x + k\eta).$$

We note that the first problem is inspired by our first example of the introduction and is concerned with the possibility that a process can happen at all. The second problem is inspired by the second problem of the introduction and is concerned with the question whether or not a process can be repeated with one overall period. The third problem can be seen as a generalization of the second problem and is inspired by the approach in [2].

4. Solvability conditions

In order to develop conditions for the solvability of the problems formulated in the previous section we recall the following known result.

**Lemma** Let $C$ be an $n \times n$ matrix with entries in $\mathbb{R} \cup \{-\infty\}$. There exists a finite vector $x \in \mathbb{R}^n$ such that $C \odot x \leq x$ if and only if the graph of $C$ only contains circuits with non-positive weight.

**Proof** (Sufficiency) Assume that the graph of the matrix $C$ only contains circuits with non-positive weight. The according to [1], theorem 3.17, there exists a finite solution $x \in \mathbb{R}^n$ to the following equation

$$(C \odot x) \odot d = x,$$

where $d$ is some finite vector in $\mathbb{R}^n$. Clearly, the solution $x$ also satisfies $C \odot x \leq x$.

(Necessity) Assume there exists a finite vector $x \in \mathbb{R}^n$ such that $C \odot x \leq x$ and consider an elementary circuit in the graph of the matrix $C$ given by the ordered list $i_1, \ldots, i_t, i_{t+1}$ with $i_1, \ldots, i_t$ mutually distinct and $i_{t+1} = i_1$. Then it follows from $C \odot x \leq x$ that $x_{i_j} + c_{i_j+i_{j+1}} \leq x_{i_{j+1}}$, for all $j = 1, \ldots, t$, and that

$$\sum_{j=1}^t x_{i_j} + c_{i_{j+1}, i_j} \leq \sum_{j=1}^t x_{i_{j+1}}.$$ Since, $i_{t+1} = i_1$ it follows that $\sum_{j=1}^t c_{i_{j+1}, i_j} \leq 0$. Hence, the weight of any elementary circuit in the graph of the matrix $C$ is non-positive. As any circuit is composed of one or more elementary circuits, it follows that also the weight of any circuit is non-positive. \( \square \)

The above lemma is useful in the proof of the next theorem. This theorem can be seen as the main result of this paper.

**Theorem**

- Problem 1 is solvable if and only if the graph of $-B^T \odot A$ only contains circuits with non-positive weight.
- Problem 2 is solvable if and only if there exists a finite $\lambda \in \mathbb{R}$ such that the graph of $(A - \lambda) \oplus (\lambda - B^T)$ only contains circuits with non-positive weight.
• Problem 3 is solvable if and only if there exists a finite vector \( \eta \in \mathbb{R}^n \) such that the graph of \( (A \odot \Delta(-\eta)) \oplus (\Delta(\eta) \odot (-B^T)) \) only contains circuits with non-positive weight.

Proof

• Problem 1 is solvable if and only if there exists a finite vector \( x \in \mathbb{R}^n \) such that \( A \odot x < B \odot' x \), or equivalently, such that \( -B^T \odot (A \odot x) = (-B^T \odot A) \odot x < x \). By the above lemma the latter is equivalent to the statement that the graph of \( -B^T \odot A \) only contains circuits with non-positive weight. Note that an equivalent statement for this solvability is that the graph of \( -A^T \odot B \) only contains circuits with non-negative weights.

• Problem 2 is solvable if and only if there exists a finite vector \( x \in \mathbb{R}^n \) and a finite \( \lambda \) such that \( A \odot x \leq x + \lambda \leq B \odot' x \), or equivalently, such that \( (A - \lambda) \odot x \leq x \) and \( \lambda \leq x \). By the above lemma the latter is equivalent to the statement that the graph of \( (A - \lambda) \odot (\lambda - B^T) \) only contains circuits with non-positive weight for a certain \( \lambda \).

• Problem 3 is solvable if and only if there exist finite vectors \( x, \eta \in \mathbb{R}^n \) such that \( A \odot (x + k \eta) \leq x + (k + 1) \eta \leq B \odot' (x + k \eta) \), for all integers \( k \geq 0 \), or equivalently, such that \( (A \odot \Delta(-\eta)) \odot (x + (k + 1) \eta) \leq (B \odot' - \Delta(\eta)) \odot' (x + (k + 1) \eta) \), for all integers \( k \geq 0 \). Like in the proof problem 2 the latter is equivalent to the existence of a finite vector \( \eta \in \mathbb{R}^n \) such that the graph of the matrix \( (A \odot \Delta(-\eta)) \oplus (\Delta(\eta) \odot (-B^T)) \) only contains circuits with non-positive weight. \( \square \)

5. Computational aspects

The solvability of the problems stated in section 3 can be formulated in terms of linear programs as follows.

• Problem 1 is solvable if and only if the next linear program has a non-positive solution \( \lambda \):

\[
\begin{align*}
\text{minimize} & \quad \lambda \\
\text{subject to} & \quad c_{ij} + x_j \leq x_i + \lambda \quad \forall \ (i, j) \in E_C,
\end{align*}
\]

where we have written \( C = -B^T \odot A \).

• Problem 2 is solvable if and only if the next linear program has a non-positive solution \( \tau \):

\[
\begin{align*}
\text{minimize} & \quad \tau \\
\text{subject to} & \quad a_{ij} + x_j \leq x_i + \lambda \\
& \quad x_i + \lambda \leq b_{ij} + x_j + \tau \quad \forall \ (i, j) \in E_A.
\end{align*}
\]

• Problem 3 is solvable if and only if the next linear program has a non-positive solution \( \tau \):

\[
\begin{align*}
\text{minimize} & \quad \tau \\
\text{subject to} & \quad \frac{\eta_j}{a_{ij} + x_j} \leq \frac{\eta_i}{x_i + \eta_i} \\
& \quad \frac{\eta_i}{x_i + \eta_i} \leq b_{ij} + x_j + \tau \quad \forall \ (i, j) \in E_B.
\end{align*}
\]

Proof

• Problem 1 has a solution if and only if there exists a finite vector \( x \in \mathbb{R}^n \) such that \( (-B^T \odot A) \odot x < x \). The latter is equivalent with the existence of a finite vector \( x \in \mathbb{R}^n \) such that \( (-B^T \odot A) \odot x \leq x + \lambda \) for some \( \lambda \leq 0 \). With \( C = -B^T \odot A \), this in turn is equivalent to the existence of finite \( x_i \in \mathbb{R} \) such that \( c_{ij} + x_j \leq x_i + \lambda \) for all \( (i, j) \in E_C \) with \( \lambda \leq 0 \).

• Problem 2 has a solution if and only if there exists a finite vector \( x \in \mathbb{R}^n \) and a finite scalar \( \lambda \in \mathbb{R} \) such that \( A \odot x \leq x + \lambda \) and \( x + \lambda \leq B \odot' x \). The latter is equivalent to the existence of a finite vector \( x \in \mathbb{R}^n \) and finite scalars \( \lambda, \mu \in \mathbb{R} \) such that \( A \odot x \leq x + \lambda \) and \( x + \mu \leq B \odot' x \) with \( \lambda \leq \mu \). Writing \( \mu = \lambda - \tau \), the previous is equivalent to the existence of \( x_i \in \mathbb{R} \) and \( \lambda \in \mathbb{R} \) such that \( a_{ij} + x_j \leq x_i + \lambda \) for all \( (i, j) \in E_A \) and \( x_i + \lambda \leq b_{ij} + x_j + \tau \) for all \( (i, j) \in E_B \) with \( \tau \leq 0 \).
• Note that problem 3 is equivalent to the existence of finite vectors \( x, \eta \in \mathbb{R}^n \) such that

\[
\begin{align*}
(\alpha) & \quad a_{ij} + x_j \leq x_i + \eta_j \quad \forall (i, j) \in E_A \\
(\beta) & \quad \eta_j \leq \eta_i \quad \forall (i, j) \in E_A \quad \text{and} \quad x_i + \eta_i \leq b_{ij} + x_j \quad \forall (i, j) \in E_B \\
\end{align*}
\]

Indeed, assume that \( x \) and \( \eta \) are finite vectors in \( \mathbb{R}^n \) such that \( A \otimes (x + k\eta) \leq x + (k + 1)\eta \leq B \otimes' (x + k\eta) \) for all integers \( k \geq 0 \). Then taking \( k = 0 \) it follows that \( A \otimes x \leq x + \eta \leq B \otimes' x \), or equivalently that \( a_{ij} + x_j \leq x_i + \eta_j \) for all \( (i, j) \in E_A \), and \( x_i + \eta_i \leq b_{ij} + x_j \) for all \( (i, j) \in E_B \). Next, considering the limit for \( k \) to infinity of \( \frac{1}{k}(A \otimes (x + k\eta)) \leq \frac{1}{k}(x + (k + 1)\eta) \) \( \leq \frac{1}{k}(B \otimes' (x + k\eta)) \), it is not difficult to prove that \( \eta_j \leq \eta_i \) for all \( (i, j) \in E_A \) and \( \eta_i \leq \eta_j \) for all \( (i, j) \in E_B \).

Conversely, assume that \( (\alpha) \) and \( (\beta) \) are satisfied for certain finite vectors \( x_i, \eta_i \in \mathbb{R}^n, i = 1, \ldots, n \). Then it follows that \( (a_{ij} + x_j) + k\eta_j \leq (x_i + \eta_i) + k\eta_i \) for all \( (i, j) \in E_A \) and \( (x_i + \eta_i) + k\eta_i \leq (b_{ij} + x_j) + k\eta_j \) for all \( (i, j) \in E_B \). The latter is equivalent to the existence of finite vectors \( x, \eta \in \mathbb{R}^n \) such that \( A \otimes (x + k\eta) \leq x + (k + 1)\eta \leq B \otimes' (x + k\eta) \) for all integers \( k \geq 0 \).

The proof that the linear program (3) has a non-positive solution \( r \) if and only if there exist finite vectors \( x, \eta \in \mathbb{R}^n \) such that \( (\alpha) \) and \( (\beta) \) are satisfied can now be completed in the much same way as the proof that the existence of a non-positive solution of the linear program (2) is equivalent to the solvability of problem 2. \( \square \)

The solvability of problems 1, 2 and 3 is expressed here in terms of linear programs because for those programs efficient algorithms, for instance based on the primal-dual approach, exist such that the "structure" of the matrices \( A \) and \( B \) can be taken into account.

The solution set consisting of all vectors \( x \) and scalars \( r \) and \( \lambda \) which satisfy (2) is obviously convex and therefore the set of \( \lambda \)'s for which a solution of problem 2 exists is a closed interval on the real line.

6. Example

Consider the matrices \( A \) and \( B \), with real parameters \( a \) and \( b \), given below

\[
A = \begin{pmatrix} -\infty & 1 & -\infty \\ 1 & -\infty & -\infty \\ a & -\infty & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & +\infty & b \\ +\infty & +\infty & 7 \\ +\infty & 7 & +\infty \end{pmatrix}.
\]

Note that the graphs of both \( A \) and \( B \) are not strongly connected, see for instance [1] for more details. This means that the eigenvalue of the matrix \( A \) in the \( \text{max,}+\text{-sense} \) does not exist and the same holds for the eigenvalue of the matrix \( B \) in the \( \text{min,}+\text{-sense} \). The previous also follows from the corresponding cycle time vectors. See for instance [2]. The cycle time vector of the matrix \( A \) (in the \( \text{max,}+\text{-sense} \)) is given by \( (1, 1, 2)^T \) and the cycle time vector of the matrix \( B \) (in the \( \text{min,}+\text{-sense} \)) is given by \( (6, 7, 7)^T \).

Now we investigate the combination of the matrices \( A \) and \( B \) as in the context of this paper. To that end, consider the matrix

\[
(A - \lambda) \oplus (\lambda - B^T) = \begin{pmatrix} \lambda - 6 & 1 - \lambda & -\infty \\ 1 - \lambda & -\infty & 1 - \lambda \\ (a - \lambda) \oplus (\lambda - b) & \lambda - 7 & 2 - \lambda \end{pmatrix}.
\]

Note that that matrix \((A - \lambda) \oplus (\lambda - B^T)\) has entries in \( \mathbb{R} \cup \{-\infty\} \). The graph of \((A - \lambda) \oplus (\lambda - B^T)\) is given below.
It follows that the graph of the matrix $(A - \lambda) \oplus (\lambda - B^T)$ is strongly connected. Hence, the eigenvalue of $(A - \lambda) \oplus (\lambda - B^T)$ exists in the \((\max, +)\)-sense. If the graph for a certain value of \(\lambda\) only contains (elementary) circuits with non-positive weight, the eigenvalue is non-positive. Here the elementary circuits are given by

\[1 \rightarrow 1, \quad 3 \rightarrow 3, \quad 1 \rightarrow 2 \rightarrow 1, \quad 2 \rightarrow 3 \rightarrow 2, \quad 1 \rightarrow 3 \rightarrow 2 \rightarrow 1,\]

with weight \(\lambda - 6, \quad 2 - \lambda, \quad 2 - 2\lambda, \quad 2\lambda - 14\) and \((a - 6 - \lambda) \oplus (\lambda - 6 - b)\), respectively. For all weights to be non-positive it follows that \(\lambda\) must satisfy

\[(a - 6) \oplus 2 \leq \lambda \leq (6 + b) \oplus 6.\]

For instance, with \(a = 10\) and \(b = -2\) only \(\lambda = 4\) is possible. In that case, it can be computed easily that a vector \(x\) such that \(A \odot x \leq x + \lambda \leq B \odot' x\) has components \(x_i, i = 1, 2, 3\), with \(x_1 = x_3 - 3\) and \(x_3 = x_2 + 3\), where \(x_3\) can be chosen freely. Hence, the vector \(x\) is uniquely determined modulo a shift. Such a shift corresponds to a multiplicative factor in the \((\max, +)\) algebra.

However, with \(a = 8\) and \(b = 0\) it follows that \(\lambda \in [2, 6]\). Hence, \(\lambda\) is not uniquely determined anymore. Moreover, given a feasible \(\lambda \in [2, 6]\), a vector \(x\) such that \(A \odot x \leq x + \lambda \leq B \odot' x\) is not uniquely determined either, not even modulo a shift. For instance, take \(\lambda = 5\). Then both the vectors \(x = (0, 4, 5)^T\) and \(x = (6, 4, 6)^T\) satisfy the inequalities \(A \odot x \leq x + \lambda \leq B \odot' x\). However, the two vectors can not be obtained from each other by a shift.

References


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