Quantization of Quasi-Presymplectic Groupoids and their Hamiltonian Spaces

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Dedicated to Alan Weinstein on the occasion of his 60th birthday

Abstract

We study the prequantization of quasi-presymplectic groupoids and their Hamiltonian spaces using $S^1$-gerbes. We give a geometric description of the integrality condition. As an application, we study the prequantization of the quasi-Hamiltonian $G$-spaces of Alekseev–Malkin–Meinrenken.

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1 Introduction

Quasi-symplectic groupoids are natural generalizations of symplectic groupoids [6, 19]. The main motivation of [19] in studying quasi-symplectic groupoids was to be able to introduce a single, unified momentum map theory in which ordinary Hamiltonian G-spaces, Lu’s momentum maps of Poisson group actions, and the group-valued momentum maps of Alekseev–Malkin–Meinrenken can be understood under a uniform framework. An important feature of this unified theory is that it allowed one to understand the diverse theories in such a way that techniques in one could be applied to the others.

It turns out that much of the theory of Hamiltonian spaces of a symplectic groupoid can be generalized to quasi-symplectic groupoids. In particular, one can perform reduction and prove that $J^{-1}(O)/\Gamma$ is a symplectic manifold, where $O \subset M$ is an orbit of the groupoid $\Gamma \rightrightarrows M$. More generally, one can introduce the classical intertwiner spaces $\Sigma_2 \times_\Gamma X_1$ between two Hamiltonian $\Gamma$-spaces $X_1$ and $X_2$, generalizing the notion studied by Guillemin-Sternberg [9] for ordinary Hamiltonian G-spaces. One shows that this is a symplectic manifold (whenever it is a smooth manifold). In particular, when $\Gamma$ is the AMM quasi-symplectic groupoid, this reduced space describes the symplectic structure on the moduli space of flat connections on a surface [2].

As is the case for symplectic groupoids [17], one can introduce Morita equivalence for quasi-symplectic groupoids. In particular, it has been proven [19] that (i) Morita equivalent quasi-symplectic groupoids give rise to equivalent momentum map theories, in the sense that there is a bijection between their Hamiltonian spaces; (ii) the classical intertwiner space $\Sigma_2 \times_\Gamma X_1$ is independent of the Morita equivalence class of $\Gamma$. This Morita invariance principle accounts for various well-known results concerning the equivalence of momentum maps, including the Alekseev–Ginzburg–Weinstein linearization theorem and the Alekseev–Malkin–Meinrenken equivalence theorem for group-valued momentum maps [19].

One important feature of Hamiltonian G-spaces is the Guillemin–Sternberg theorem which states that “$[Q, R] = 0$”: quantization commutes with reduction [9, 12]. One expects that “$[Q, R] = 0$” should be a general guiding principle for all momentum map theories. To carry out such a quantization program, the first important step is the construction of prequantum line bundles. In this paper, we study the prequantization of Hamiltonian spaces for quasi-symplectic groupoids. Our method uses the theory of $S^1$-bundles and $S^1$-gerbes over a groupoid along with their characteristic classes, as developed in [3, 4]. Roughly, our construction can be described
as follows. A prequantization of a quasi-symplectic groupoid \((\Gamma \cong M, \omega + \Omega)\) is an \(S^1\)-central extension \(\hat{R} \rightarrow \Gamma\) of the groupoid \(\Gamma \cong M\) (or an \(S^1\)-gerbe over the groupoid) equipped with a pseudo-connection having \(\omega + \Omega\) as the pseudo-curvature. Such a prequantization exists if and only if \(\omega + \Omega\) is a de Rham integer 3-cocycle and \(\Omega\) is exact (assuming that \(\Gamma\) is a proper groupoid). A prequantization of a Hamiltonian space is then an \(S^1\)-bundle \(L \rightarrow X\) over \(R \cong M\) together with a compatible pseudo-connection, where the \(R\)-action on \(L\) is \(S^1\)-equivariant. A prequantization of the symplectic intertwiner space \(\mathbb{X}_2 \times \Gamma X_1\) can be constructed using these data.

Indeed one can show that \(\hat{R}\backslash(\hat{L}_1 \times_M L_2)\) is a prequantization of the symplectic intertwiner space \(\mathbb{X}_2 \times \Gamma X_1\), and the natural 1-form on \(\hat{L}_1 \times_M L_2\) induced by the connection forms on \(L_1\) and \(L_2\) descends to a prequantization connection on the quotient space \(\hat{R}\backslash(\hat{L}_1 \times_M L_2)\). When \(\Omega\) is not exact, one must pass to a Morita equivalent quasi-symplectic groupoid first. Then the Morita invariance principle guarantees that the resulting quantization does not depend on the particular choice of Morita equivalent quasi-symplectic groupoid. As a special case, when \(\Gamma\) is the AMM quasi-symplectic groupoid, our construction yields the prequantization of quasi-Hamiltonian \(G\)-spaces of Alekseev–Malkin–Meinrenken and their symplectic reductions, and our quantization condition coincides with that of Alekseev–Meinrenken [1].

Quantization of Hamiltonian spaces for symplectic groupoids was studied in [18]. Note that in the usual Hamiltonian case, since the symplectic 2-form defines a zero class in the third cohomology group of the groupoid \(T^*G \cong \mathfrak{g}^*\), which is the equivariant cohomology \(H^3_{eq}(\mathfrak{g}^*)\), gerbes do not appear explicitly. However, for a general quasi-symplectic groupoid (for instance the AMM quasi-symplectic groupoid), since the 3-cocycle \(\omega + \Omega\) may define a nontrivial class, gerbes are inevitable in the construction. Also note that no nondegeneracy condition is needed in the quantization construction, so we drop this assumption in the present paper to assure full generality.

This paper is organized as follows. In Section 2, we review some basic results concerning quasi-presymplectic groupoids and their Hamiltonian spaces. In Section 3, we gather some important results on \(S^1\)-bundles and \(S^1\)-central extensions. We give a simple formula for the index of an \(S^1\)-bundle over a central extension in terms of the Chern class. In Section 4, we introduce prequantizations of quasi-presymplectic groupoids and discuss compatible prequantizations of their Hamiltonian spaces. Section 5 is devoted to the description of a geometric integrality condition of pre-Hamiltonian \(G\)-spaces. The application to quasi-Hamiltonian \(G\)-spaces are discussed.

Unless specified, by a groupoid in this paper, we always mean a Lie groupoid whose orbit space is connected.

Prequantization of symplectic groupoids was first studied by Alan Weinstein and the second author in [16], when the second author was his PhD student. In the same paper, \(S^1\)-central extensions of Lie groupoids were also first systematically investigated. Undoubtedly, Alan Weinstein’s work and insights have had a tremendous impact on the development of this subject in the last two decades. It is our great pleasure to dedicate this paper to him.

Acknowledgments. The second author would like to thank the Erwin Schrödinger Institute and the University of Geneva for their hospitality while work on this project was being done. We would like to thank Anton Alekseev, Kai Behrend, Eckhard Meinrenken, and Jim Stasheff for useful discussions.
2 Pre-Hamiltonian $\Gamma$-spaces and classical intertwiner spaces

2.1 Quasi-presymplectic groupoids and their pre-Hamiltonian spaces

First, let us recall the definition of the de-Rham double complex of a Lie groupoid.

Let $\Gamma \rightrightarrows M$ be a Lie groupoid. Define for all $p \geq 0$

$$\Gamma_p = \underbrace{\Gamma \times M \cdots \times M}_{p \text{ times}} \Gamma,$$

i.e. $\Gamma_p$ is the manifold of composable sequences of $p$ arrows in the groupoid $\Gamma \rightrightarrows M$ (and $\Gamma_0 = M$). We have $p+1$ canonical maps $\Gamma_p \to \Gamma_{p-1}$ (each leaving out one of the $p+1$ objects involved a sequence of composable arrows), giving rise to a diagram

$$\ldots \Gamma_2 \longrightarrow \Gamma_1 \longrightarrow \Gamma_0.$$

Consider the double complex $\Omega^\ast (\Gamma_\ast)$:

$$\begin{array}{ccc}
\ldots & \ldots & \ldots \\
\Omega^1 (\Gamma_0) & \overset{\partial}{\longrightarrow} & \Omega^1 (\Gamma_1) & \overset{\partial}{\longrightarrow} & \Omega^1 (\Gamma_2) & \overset{\partial}{\longrightarrow} & \ldots \\
\downarrow & & & \uparrow & & & \downarrow \\
\Omega^0 (\Gamma_0) & \overset{d}{\longrightarrow} & \Omega^0 (\Gamma_1) & \overset{d}{\longrightarrow} & \Omega^0 (\Gamma_2) & \overset{d}{\longrightarrow} & \ldots \\
\end{array}$$

Its boundary maps are $d : \Omega^k (\Gamma_p) \to \Omega^{k+1} (\Gamma_p)$, the usual exterior derivative of differentiable forms and $\partial : \Omega^k (\Gamma_p) \to \Omega^k (\Gamma_{p+1})$, the alternating sum of the pull-back maps of (1). We denote the total differential by $\delta = (-1)^p d + \partial$. The cohomology groups of the total complex $C^\ast_{dR} (\Gamma_\ast)$

$$H^k_{dR} (\Gamma_\ast) = H^k (\Omega^\ast (\Gamma_\ast))$$

are called the de Rham cohomology groups of $\Gamma \rightrightarrows M$.

**Definition 2.1** A quasi-presymplectic groupoid is a Lie groupoid $\Gamma \rightrightarrows M$ equipped with a two-form $\omega \in \Omega^2 (\Gamma)$ and a three form $\Omega \in \Omega^3 (M)$ such that

$$d\Omega = 0, \quad d\omega = \partial\Omega \quad \text{and} \quad \partial\omega = 0.$$  \hspace{1cm} (3)

In other words, $\omega + \Omega$ is a 3-cocycle of the total de-Rham complex of the groupoid $\Gamma \rightrightarrows M$.

A quasi-presymplectic groupoid $(\Gamma \rightrightarrows M , \omega + \Omega)$ is said to be exact if $\Omega$ is exact three-form on $M$.

A quasi-symplectic groupoid is a quasi-presymplectic groupoid $(\Gamma \rightrightarrows M , \omega + \Omega)$ where $\omega$ satisfies certain non-degenerate condition [6, 19]. Quasi-symplectic groupoids are natural generalization of symplectic groupoids, whose momentum map theory unifies various momentum map theories, including the ordinary Hamiltonian $G$-spaces, Lu’s momentum maps of Poisson group actions, and group valued momentum maps of Alekseev–Malkin–Meinrenken.
**Definition 2.2** Given a quasi-presymplectic groupoid $(\Gamma \rightrightarrows M, \omega + \Omega)$, a pre-Hamiltonian $\Gamma$-space is a (left) $\Gamma$-space $J : X \to M$ (i.e., $\Gamma$ acts on $X$ from the left) with a compatible two-form $\omega_X \in \Omega^2(X)$:

1. $d\omega_X = J^*\Omega$;

2. the graph of the action $\Lambda = \{(r, x, rx) | f(r) = J(x)\} \subset \Gamma \times X \times \overline{X}$ (where $\overline{X}$ is the manifold $X$ endowed with the form $-\omega_X$) is isotropic with respect to the two-form $(\omega, \omega_X, -\omega_X)$.

To illustrate the intrinsic meaning of the above compatibility condition, let us elaborate it in terms of groupoids. Let $\Gamma \times_M X \rightrightarrows X$ be the transformation groupoid corresponding to the $\Gamma$-action, and, by abuse of notation, $J : \Gamma \times_M X \to \Gamma$ the natural projection. It is simple to see that

$$
\begin{array}{ccc}
\Gamma \times_M X & \xrightarrow{J} & \Gamma \\
\downarrow & & \downarrow \\
X & \xrightarrow{J} & M
\end{array}
$$

is a Lie groupoid homomorphism. Therefore it induces a map, i.e., the pull-back map, on the level of de-Rham complex:

$$
J^* : \Omega^* (\Gamma_\bullet) \to \Omega^* ((\Gamma \times_M X)_\bullet).
$$

**Proposition 2.3** [19] Let $(\Gamma \rightrightarrows M, \omega + \Omega)$ be a quasi-presymplectic groupoid and $J : X \to M$ a left $\Gamma$-space. A two-form $\omega_X \in \Omega^2(X)$ is compatible with the action if and only if

$$
J^*(\omega + \Omega) = \delta\omega_X.
$$

### 2.2 Classical intertwiner spaces

Given a quasi-presymplectic groupoid $(\Gamma \rightrightarrows M, \omega + \Omega)$, and pre-Hamiltonian $\Gamma$-spaces $(X_1 \xrightarrow{\phi} M, \omega_1)$, and $(X_2 \xrightarrow{\phi} M, \omega_2)$. Assume that $\Gamma \backslash (\overline{X_2} \times_M X_1)$ is a smooth manifold, and denote by

$$
p : \overline{X_2} \times_M X_1 \to \Gamma \backslash (\overline{X_2} \times_M X_1)
$$

the natural projection. Note that $i^*(-\omega_2, \omega_1)$, where $i : \overline{X_2} \times_M X_1 \to X_2 \times X_1$ is the natural embedding, is a closed two-form on $\overline{X_2} \times_M X_1$.

We need a technical lemma.

**Lemma 2.4** [10] Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $X \to M$ a $\Gamma$-space. Assume that $\Gamma \backslash X$ is a smooth manifold. Then a differential form $\omega \in \Omega^*(X)$ descends to a differential form on the quotient $\Gamma \backslash X$ if and only if $\partial \omega = 0$, where $\partial$ is with respect to the transformation groupoid $\Gamma \times_M X \rightrightarrows X$.

**Proposition 2.5** The two-form $i^*(\omega_2, \omega_1)$ descends to a closed 2-form on $\Gamma \backslash (\overline{X_2} \times_M X_1)$. Therefore $\Gamma \backslash (\overline{X_2} \times_M X_1)$ is a presymplectic manifold.
\textbf{Proof.} Note that $J : X_2 \times_M X_1 \to M, J(x_2, x_1) = J_1(x_1) = J_2(x_2)$, is a naturally $\Gamma$-space, where $\Gamma \rightrightarrows M$ acts on $X_2 \times_M X_1$ diagonally. Then

$$
\partial [i^*(-\omega_2, \omega_1)] = i^*(-\partial \omega_2, \partial \omega_1) = i^*(-J_2^*\omega, J_1^*\omega) = ((J_2 \times J_1) \circ i)^*(-\omega, \omega) = 0,
$$

where $J_1, J_2$ and $i$ are respectively the groupoid homomorphisms:

$$
\begin{array}{c}
\Gamma \times_M X_1 \xrightarrow{J_i} \Gamma \\
\downarrow \downarrow \\
X_i \xrightarrow{J_i} M
\end{array} i = 1, 2. \quad (6)
$$

and

$$
\begin{array}{c}
\Gamma \times_M (X_1 \times_M X_2) \xrightarrow{i} (\Gamma \times_M X_1) \times (\Gamma \times_M X_2), \\
\downarrow \downarrow \downarrow \downarrow \\
X_1 \times_M X_2 \xrightarrow{i} X_1 \times X_2
\end{array} \quad (7)
$$

and $\partial [i^*(-\omega_2, \omega_1)]$ and $\partial \omega_i$, $i = 1, 2$, are with respect to the groupoids on the left hand side of Eqs. (7) and (6) respectively.

The conclusion thus follows from Lemma 2.4. \hfill \Box

The presymplectic manifold $\Gamma \backslash (X_2 \times_M X_1)$ is called the \textit{classical intertwiner space}, and is denoted by $\overline{X_2 \times_M X_1}$ for simplicity. In particular, if $(\Gamma \rightrightarrows M, \omega + \Omega)$ is a quasi-symplectic groupoid, and $(X_1 \xrightarrow{J_1} M, \omega_1)$ and $(X_2 \xrightarrow{J_2} M, \omega_2)$ are Hamiltonian $\Gamma$-spaces, and if $J_1 : X_1 \to M$ and $J_2 : X_2 \to M$ are clean, then $\overline{X_2 \times_M X_1}$ becomes a symplectic manifold. See [19] for detail.

\section{3 \textit{S}\textsuperscript{1}-bundles and \textit{S}\textsuperscript{1}-central extensions}

In this section we recall some basic results concerning $S^1$-bundles and $S^1$-central extensions over a groupoid. For details, consult [3, 4, 15].

\subsection{3.1 Integer de-Rham cocycles}

Let us recall some basic facts concerning singular homology. For any manifold $N$, we denote by $(C_\bullet(N, \mathbb{Z}), d)$ the piecewise smooth singular chain complex, and $Z_k(N, \mathbb{Z})$ the space of smooth $k$-cycles. For a smooth map $\phi : M \to N$, we denote by $\phi_\ast$ both the chain map from $(C_\bullet(M, \mathbb{Z}), d)$ to $(C_\bullet(N, \mathbb{Z}), d)$ and the morphism of singular homology $H_\ast(M, \mathbb{Z}) \to H_\ast(N, \mathbb{Z})$ induced by $\phi$. 

6
For any groupoid $\Gamma \equiv M$, consider the double complex $C_\ast(\Gamma_\ast, \mathbb{Z})$:

$$
\begin{array}{ccc}
\cdots & \cdots & \cdots \\
\downarrow d & \downarrow d & \downarrow d \\
C_1(\Gamma_0, \mathbb{Z}) & \overset{\partial}{\leftarrow} & C_1(\Gamma_1, \mathbb{Z}) \overset{\partial}{\leftarrow} C_1(\Gamma_2, \mathbb{Z}) \\
\downarrow d & \downarrow d & \downarrow d \\
C_0(\Gamma_0, \mathbb{Z}) & \overset{\partial}{\leftarrow} & C_0(\Gamma_1, \mathbb{Z}) \overset{\partial}{\leftarrow} C_0(\Gamma_2, \mathbb{Z}),
\end{array}
$$

where $\partial : C_k(\Gamma_p, \mathbb{Z}) \rightarrow C_k(\Gamma_{p-1}, \mathbb{Z})$ is the alternating sum of the chain maps induced by the face maps. We denote the total differential by $\delta = (-1)^p d + \partial$. Its homology will be denoted by $H_k(\Gamma_\ast, \mathbb{Z})$. By $Z_k(\Gamma_\ast, \mathbb{Z})$, we denote the space of $k$-cycles and by $[C] \in H_k(\Gamma_\ast, \mathbb{Z})$ the class of a given cycle $C$. Note that $C_k(\Gamma_p, \mathbb{Z})$ is the free Abelian group generated by the piecewise differentiable maps $\Delta_k \rightarrow \Gamma_p$.

The construction above can be carried out in exactly the same way replacing $\mathbb{Z}$ by $\mathbb{R}$. The corresponding homology groups are denoted by $H_k(\Gamma_\ast, \mathbb{R})$. According to the universal-coefficient formula (see, for example, [14]), there is a canonical isomorphism

$$H_k(\Gamma_\ast, \mathbb{R}) \simeq H_k(\Gamma_\ast, \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{R}.$$ 

There is a natural pairing between $C_\ast(\Gamma_\ast, \mathbb{R})$ and $C_\ast(\Gamma_\ast)$ given as follows. For any generator $C : \Delta_k \rightarrow \Gamma_p$ in $C_\ast(\Gamma_\ast, \mathbb{Z})$,

$$\langle C, \omega \rangle = \begin{cases} 
\int_{\Delta_k} C^\ast \omega & \text{if } \omega \in \Omega^k(\Gamma_p) \\
0 & \text{otherwise.}
\end{cases} \quad (8)$$

For simplicity, we will denote this pairing by $\int_C \omega$. With this notation, the pairing satisfies the following identities:

$$
\int_C \omega = \int_{\partial C} \omega \\
\int_C \omega = \int_{\partial C} \omega \\
\int_C \omega = \int_{\partial C} \omega \\
\int_C \omega = \int_{\partial C} \omega
$$

Moreover, if $\phi : G \rightarrow H$ is a strict homomorphism of groupoids, then for any $C \in C_\ast(G_\ast, \mathbb{R})$ and $\omega \in C_\ast(\Omega^k(\Gamma_\ast))$

$$\int_C \omega = \int_{\phi^\ast(C)} \phi^\ast \omega. \quad (9)$$

**Proposition 3.1 (Proposition 6.1 [9])** The pairing $H_k(\Gamma_\ast, \mathbb{R}) \otimes H_k(\Gamma_\ast) \rightarrow \mathbb{R}$, $([C], [\omega]) \rightarrow \int_C \omega$ is non-degenerated.
Recall that
\[ Z^k_{dR}(\Gamma_\ast, \mathbb{Z}) = \{ \omega \in C^k_{dR}(\Gamma_\ast) \text{ such that } \int_C \omega \in \mathbb{Z} \text{ for any cycle } C \in Z_k(\Gamma_\ast, \mathbb{Z}) \}. \] (10)

Elements in \( Z^k_{dR}(\Gamma_\ast, \mathbb{Z}) \) are called integer de-Rham coycles, or simply integer cocycles.

### 3.2 \( S^1 \)-bundles and \( S^1 \)-central extensions

**Definition 3.2** Let \( \Gamma \rightrightarrows M \) be a Lie groupoid. A (right) \( S^1 \)-bundle over \( \Gamma \rightrightarrows M \) is a (right) \( S^1 \)-bundle \( P \) over \( M \), together with a (left) action of \( \Gamma \) on \( P \), which respects the \( S^1 \)-action, i.e., we have \((\gamma \cdot x) \cdot t = \gamma \cdot (x \cdot t)\), for all \( t \in S^1 \) and all compatible pairs \((\gamma, x) \in \Gamma \times_M P\).

Let \( Q : \Gamma \times_M P \rightrightarrows P \) be the transformation groupoid of the \( \Gamma \)-action. There is a natural homomorphism of groupoids \( \pi \) from \( Q \rightrightarrows P \) to \( \Gamma \rightrightarrows M \). Of course, \( Q \) is an \( S^1 \)-bundle over \( \Gamma \).

A pseudo-connection is a one-cochain \( \theta \in C^2_{dR}(Q_\ast) \), where \( \theta \in \Omega^1(P) \) is a connection form for the \( S^1 \)-bundle \( P \rightarrow M \). One checks that \( \delta \theta \in C^2_{dR}(Q_\ast) \) descends to a 2-cocycle in \( Z^2_{dR}(\Gamma_\ast) \). In other words, there exist unique \( \omega \in \Omega^1(\Gamma) \) and \( \Omega \in \Omega^2(M) \) such that
\[ \delta \theta = \pi^*(\omega + \Omega). \]

Then \( \omega + \Omega \) is called the pseudo-curvature, which is an integer 2-cocycle. Its class \([\omega + \Omega] \in H^2(\Gamma_\ast, \mathbb{Z})\) is called the Chern class of the \( S^1 \)-bundle \( P \).

**Proposition 3.3** [3, 4] Let \( \Gamma \rightrightarrows M \) be a proper Lie groupoid. Assume that \( \omega + \Omega \in \Omega^1(\Gamma) \oplus \Omega^2(M) \subset C^2_{dR}(\Gamma_\ast) \) is an integer 2-cocycle. Then there exists an \( S^1 \)-bundle \( P \) over \( \Gamma \rightrightarrows M \) and a pseudo-connection \( \theta \in \Omega^1(P) \) for the bundle \( P \rightarrow M \), whose pseudo-curvature equals to \( \omega + \Omega \).

**Definition 3.4** Let \( \Gamma \rightrightarrows M \) be a Lie groupoid. An \( S^1 \)-central extension of \( \Gamma \rightrightarrows M \) consists of

1) a Lie groupoid \( R \rightrightarrows M \), together with a morphism of Lie groupoids \((\pi, \text{id}) : [R \rightrightarrows M] \rightarrow [\Gamma \rightrightarrows M])\),

2) a left \( S^1 \)-action on \( R \), making \( \pi : R \rightarrow \Gamma \) a (left) principal \( S^1 \)-bundle. These two structures are compatible in the sense that \((s \cdot x)(t \cdot y) = st \cdot (xy)\), for all \( s, t \in S^1 \) and \((x, y) \in R \times_M R\).

Given a central extension \( R \) of \( \Gamma \rightrightarrows M \), a pseudo-connection is a two-cochain \( \theta + B \in C^2_{dR}(R_\ast) \), where \( \theta \in \Omega^1(R) \) is a connection form for the bundle \( R \rightarrow \Gamma \) and \( B \in \Omega^2(M) \). It is simple to check that \( \delta(\theta + B) \) descends to a 3-cocycle in \( Z^3(\Gamma_\ast) \), i.e.,
\[ \delta(\theta + B) = \pi^*(\eta + \omega + \Omega), \]

where \( \eta + \omega + \Omega \in Z^3(\Gamma_\ast) \). Then \( \eta + \omega + \Omega \) is an integer cocycle in \( Z^3_{dR}(\Gamma_\ast, \mathbb{Z}) \), and is called the pseudo-curvature. Its class \([\eta + \omega + \Omega] \in H^3(\Gamma_\ast, \mathbb{Z})\) is called the Dixmier-Douady class of \( R \).

**Proposition 3.5** [3, 4] Assume that \( \Gamma \rightrightarrows M \) is a proper Lie groupoid. Given any 3-cocycle \( \eta + \omega + \Omega \in Z^3_{dR}(\Gamma_\ast) \) satisfying

1) \([\eta + \omega + \Omega] \) is integer, and
2) \( \Omega \) is exact,

there exists a groupoid \( S^1 \)-central extension \( R \rightrightarrows M \) of the groupoid \( \Gamma \rightrightarrows M \), and a pseudo-connection \( \theta + B \in \Omega^1(R) \oplus \Omega^2(M) \) such that its pseudo-curvature equals to \( \eta + \omega + \Omega \).
3.3 Index of an $S^1$-bundle over a central extension

Let $R \xrightarrow{\pi} \Gamma \cong M$ be an $S^1$-central extension, and $S^1 \to L \xrightarrow{p} M$ a principal $S^1$-bundle over the groupoid $R \cong M$, which defines a class $[L] \in H^1(R_\ast, S^1)$. The example below will be useful in the future.

**Example 3.6** Consider, for any $k \in \mathbb{Z}$, the principal $S^1$-bundle $B_k$ over $S^1 \cong \ast$ where the groupoid $S^1 \cong \ast$ acts on $B_k = S^1 \to \ast$ by

$$\lambda \cdot z = \lambda^k z \quad \forall \lambda \in S^1 \cong \ast \quad \text{and} \quad \forall z \in S^1 \to \ast.$$

It is well-known that $H^1(S^1_\ast, S^1) \cong \mathbb{Z}$. Under this isomorphism, the class $[B_k]$ is simply equal to $k$.

It is also simple to see that the Chern class of $B_k$ can be represented by

$$e = k \frac{dt}{2\pi} \in \Omega^1(S^1)$$

where $\frac{dt}{2\pi}$ is the normalized Haar measure on $S^1$.

For any $m \in M$, there exists a strict homomorphism of groupoids $f_m$ from $S^1 \cong \ast$ to $R \cong M$ defined by

$$f_m(\lambda) = \lambda \cdot 1_m \quad \forall \lambda \in S^1,$$

where $1_m \in R$ is the unit element over $m \in M$.

This strict homomorphism induces a map

$$f_m^* : H^1(R_\ast, S^1) \to H^1(S^1_\ast, S^1) \cong \mathbb{Z}.$$  \hfill (13)

For a principal $S^1$-bundle $L$ over $R \cong M$, we define the index by

$$\text{Ind}_m(L) = f_m^*([L]) \in H^1(S^1_\ast, S^1) \cong \mathbb{Z}.$$  \hfill (13)

Let us list some of its properties.

**Proposition 3.7** Let $R \xrightarrow{\pi} \Gamma \cong M$ be an $S^1$-central extension, and $S^1 \to L \xrightarrow{p} M$ a principal $S^1$-bundle over the groupoid $R \cong M$. Then

1. the index is characterized by the relation

$$f_m(\lambda) \cdot l = l \cdot \lambda^{\text{Ind}_m(L)} \quad \forall \lambda \in S^1, \ l \in p^{-1}(m),$$

where the $\cdot$ on the left hand side denotes the $\Gamma$-action on $L$, while the $\cdot$ on the right hand side refers to the $S^1$-action on $L$;

2. for any $m \in M$, the pull-back $f_m^* L$ is isomorphic to $B_{\text{Ind}_m(L)}$;

3. $\text{Ind}_m(L)$ is constant on the groupoid orbits;

4. $\text{Ind}_m(L)$ is constant on any connected component of $M$, and
5. if $M/\Gamma$ is path connected, then the index $\text{Ind}_m(L)$ is independent of $m \in M$.

**Proof.** 1) and 2) Let $l$ be any point in the fiber $L_m = p^{-1}(m)$. For any $\lambda \in S^1$, there exists an unique $\phi(\lambda) \in S^1$ such that

$$f_m(\lambda) \cdot l = l \cdot \phi(\lambda). \quad (14)$$

The map $\lambda \to \phi(\lambda)$ does not depend on the choice of $l$ in the fiber $p^{-1}(m)$ and is a group homomorphism from $S^1$ to $S^1$. It is therefore of the form $\phi(\lambda) = \lambda^k$ for some $k \in \mathbb{Z}$.

By the definition of $f_m$, the class $f_m^*[L] \in H^1(S^1, S^1)$ is the class associated to the pull-back of $L$ by $f_m$. According to Eq. (14), this pull-back $f_m^*L$ is isomorphic (as a principal $S^1$-bundle over $S^1 \cong \cdot$) to $B_k$. Therefore $k = \text{Ind}_m(L)$ and Eq. (14) leads to

$$f_m(\lambda) \cdot l = l \cdot \lambda^{\text{Ind}_m(L)}, \quad \forall l \in p^{-1}(m). \quad (15)$$

This proves 1) and 2).

3) For any $\gamma \in \hat{\mathcal{R}}$ with $s(\gamma) = n$ and $t(\gamma) = m$, we have $\gamma 1_m = 1_n \gamma$. It follows from Eq. (12) that $\gamma f_m(\lambda) = f_n(\lambda) \gamma$. Now for any $l \in p^{-1}(m)$, we have $(\gamma f_m(\lambda)) \cdot l = (f_n(\lambda) \gamma) \cdot l$. On the one hand, we have

$$(\gamma f_m(\lambda)) \cdot l = (\gamma \cdot l) \cdot \lambda^{\text{Ind}_m(L)}, \quad (16)$$

and

$$(f_n(\lambda) \gamma) \cdot l = f_n(\lambda)(\gamma \cdot l) = (\gamma \cdot l) \cdot \lambda^{\text{Ind}_n(L)}. \quad (17)$$

From Eqs. (16) and (17) it follows that $\text{Ind}_m(L) = \text{Ind}_n(L)$.

4) It is clear from Eqs. (15) that $\text{Ind}_m(L)$ depends continuously on $m \in M$. Since it a $\mathbb{Z}$-valued function, we have $\text{Ind}_m(L) = \text{Ind}_n(L)$ for any pair of points $(m, n) \in M \times M$ that are in the same connected component of $M$.

5) follows from 3) and 4) immediately. □

### 3.4 Index and Chern class

From now on, we will assume that the space of orbits $M/\Gamma$ is connected and denote the index of $L$ by $\text{Ind}(L)$. Therefore, we have a group homomorphism:

$$\text{Ind}(L) : H^1(R^*, S^1) \to \mathbb{Z}.$$  

From the commutativity of the diagram

$$
\begin{array}{ccc}
H^1(R^*, S^1) & \to & H^2(R^*, \mathbb{Z}) \\
\downarrow & & \uparrow \\
H^1(S^1, S^1) & \to & H^2(S^1, \mathbb{Z})
\end{array}
$$

we see that $\text{Ind}(L)$ factorizes through $H^2(R^*, \mathbb{Z}) \to \mathbb{Z}$. In the following proposition, we give an explicit formula of $\text{Ind}(L)$ in terms of the Chern class.
Proposition 3.8 Assume that $L \to M$ is a principal $S^1$-bundle over $R \cong M$ with the Chern class $[\theta + \omega] \in H^2(R, \mathbb{Z})$, where $R \xrightarrow{\pi} \Gamma \cong M$ is an $S^1$-central extension, and $\theta + \omega \in \Omega^1(R) \oplus \Omega^2(M)$. The index of $L$ is given by

\[ \text{Ind}(L) = \int_{\pi^{-1}(\epsilon(m))} \theta, \]

where $\epsilon : M \to \Gamma$ is the unit map.

Proof. Let $L'$ be the pull-back of the principal $S^1$-bundle $L$ via the strict homomorphism $f_m : S^1 \to R$. The Chern class of $L'$ is the pull-back of the Chern class of $L$, i.e. the class defined by $f_m^* \theta + f_m^* \omega \in C^2_{dR}(S^1)$. Since $f_m^* \omega$ is a 2-form over a point, it is equal to zero and therefore the Chern class of $L'$ is represented by $f_m^* \theta \in \Omega^1(S^1)$.

By Proposition 3.7, $L'$ is isomorphic to $B_{\text{Ind}(L)}$. According to Eq. (11), the identity $f_m^* \theta = \text{Ind}(L) \frac{dt}{2\pi} + \delta g$ holds for some function $g \in C^\infty(S^1, \mathbb{R})$.

Now since $f_m$ is a bijection from $S^1$ to $\pi^{-1}(\epsilon(m))$, we have

\[ \int_{\pi^{-1}(\epsilon(m))} \theta = \int_{S^1} f_m^* \theta. \]

Therefore

\[ \int_{\pi^{-1}(\epsilon(m))} \theta = \int_{S^1} f_m^* \theta = \text{Ind}(L) \int_{S^1} \frac{dt}{2\pi} + \int_{S^1} dg = \text{Ind}(L). \]

\[ \square \]

Recall that a line bundle $L \to M$ over $R \cong M$ is called a $(\Gamma, R)$-twisted line bundle if $\ker \pi \cong M \times S^1$ acts on $L$ by scalar multiplication, where $S^1$ is identified with the unit circle of $\mathbb{C}$ [15]. The following corollary is an immediate consequence of Proposition 3.8 and Proposition 3.7.

Corollary 3.9 Under the same hypothesis as in Proposition 3.8, $L \to M$ defines a twisted line bundle if and only if

\[ \int_{\pi^{-1}(\epsilon(m))} \theta = 1. \]

4 Prequantization of classical intertwiner spaces

4.1 Compatible prequantizations

Definition 4.1 A prequantization of a quasi-presymplectic groupoid $(\Gamma \cong M, \omega + \Omega)$ consists of an $S^1$-central extension $R \xrightarrow{\pi} \Gamma \cong M$ together with a pseudo-connection $\theta + B \in \Omega^1(R) \oplus \Omega^2(M)$ such that

\[ \delta(\theta + B) = \pi^*(\omega + \Omega). \] (18)

According to Proposition 3.3, if $\Gamma \cong M$ is a proper Lie groupoid, a prequantization exists if and only if $(\Gamma \cong M, \omega + \Omega)$ is exact and $\omega + \Omega$ is an integer cocycle. A quasi-presymplectic groupoid $(\Gamma \cong M, \omega + \Omega)$ is said to be integral if $\omega + \Omega$ is an integer cocycle.
**Definition 4.2** Let \((\Gamma \rightrightarrows M, \omega + \Omega)\) be an exact quasi-presymplectic groupoid, and \((R \to \Gamma \rightrightarrows M, \Theta + B)\) a prequantization. Assume that \((X \rightrightarrows M, \omega_X)\) is a pre-Hamiltonian \(\Gamma\)-space. A compatible prequantization of \(X\) is an \(S^1\)-bundle \(\phi: L \to X\) with a connection one-form \(\theta_L \in \Omega^1(L)\) such that

1. \(J = J_\phi: L \to M\) is a left \(R\)-space and the action satisfies:
   \[(s \cdot k)(l \cdot x) = st \cdot (kx),\]
   for all \(s, t \in S^1\) and \((k, x) \in R \times_M X\) a compatible pair,

2. the one-form \((\theta, \theta_L, -\theta_L) \in \Omega^1(R \times L \times \Gamma)\) vanishes on the graph of the action
   \[\Xi = \{(k, l, n) | k \in R, l \in L \text{ compatible pairs}\},\]
   and

3. \(d\theta_L = \phi^* (J^* B - \omega_X)\).

Note that the second condition above is equivalent to saying that \((R \times L \times \Gamma)/T^2 \xrightarrow{\pi} \Gamma \times X \times X\) with \(p([k, l, m]) = (\pi(k), \phi(l), \phi(m))\) is a flat \(S^1\)-bundle with the connection \(\Theta\), which is the one-form on \((R \times L \times \Gamma)/T^2\) naturally induced from \(\Theta = (\theta, \theta_L, -\theta_L) \in \Omega^1(E \times L \times L)\) (see [18]).

**Example 4.3** If \(\Gamma\) is the symplectic groupoid \((T^* G \rightrightarrows g^*, \omega)\), where \(\omega \in \Omega^2(T^* G)\) is the canonical cotangent symplectic 2-form, a prequantization of \(\Gamma\) can be taken \(R \cong T^* G \times S^1 \to T^* G\), the trivial \(S^1\)-bundle and \(\theta = (\theta_{T^* G}, dt)\), where \(\theta_{T^* G} \in \Omega^1(T^* G)\) is the Liouville one-form and \(t\) is the natural coordinate on \(S^1\). A Hamiltonian \(\Gamma\)-space is a Hamiltonian \(G\)-space \(J: X \to g^*\) in the usual sense. It is simple to see that a compatible prequantization is a \(G\)-equivariant prequantization of \(X\), which always exists when \(G\) is connected and simply connected [9].

More generally, the following result was proved in [18] (the theorem was stated for the symplectic case, but it is valid for the presymplectic case as well).

**Theorem 4.4** Let \((\Gamma \rightrightarrows M, \omega)\) be an \(s\)-connected and \(s\)-simply connected pre-symplectic groupoid, and \((X \rightrightarrows M, \omega_X)\) a pre-Hamiltonian space. If both \(\omega\) and \(\omega_X\) represent integer cohomology class in \(H^2_{dR}(\Gamma)\) and \(H^2_{dR}(X)\) respectively, then there exists a compatible prequantization.

For a given quasi-presymplectic groupoid \((\Gamma \rightrightarrows M, \omega + \Omega)\) and a prequantization \((R \to \Gamma \rightrightarrows M, \Theta + B)\), let \(\Gamma \times_M X \rightrightarrows X\) be the transformation groupoid as in Eq. (4). By pulling back the central extension \(R \to \Gamma \rightrightarrows M\) via \(J\), one obtains a central extension of groupoids \(R \times_M X \to \Gamma \times_M X \equiv X\). Here \(R \times_M X\) is again a transformation groupoid, where \(R\) acts on \(X\) by projecting \(R\) to \(\Gamma\) and using the given \(\Gamma\)-action on \(X\).

By abuse of notation, we still use \(J\) to denote the projection \(R \times_M X \to R\). Therefore we have the following morphism of \(S^1\)-central extensions of groupoids.

\[
\begin{array}{ccc}
R \times_M X & \xrightarrow{J} & R \\
\Gamma \times_M X & \xrightarrow{J} & \Gamma \\
X & \xrightarrow{J} & M
\end{array}
\]
Remark 4.5 Note that Proposition 2.3 implies that the Dixmier-Douady class of $R \times_M X \to \Gamma \times_M X \cong X$ vanishes. If $\Gamma \cong M$ is a proper groupoid, so is $\Gamma \times_M X \cong X$. Therefore $R \times_M X \to \Gamma \times_M X \cong X$ defines a trivial gerbe. According to Proposition 4.2 [3], there exists an $S^1$-bundle $E \to X$ such that

$$R \times_M X \cong s^*E \oplus t^*E$$

as central extensions.

Proposition 4.6 Let $(\Gamma \cong M, \omega + \Omega)$ be an exact quasi-presymplectic groupoid, and $(R \to \Gamma \cong M, \theta + B)$ its prequantization. Assume that $(X \xrightarrow{\theta} M, \omega_X)$ is a pre-Hamiltonian $\Gamma$-space. Then $(L \xrightarrow{\phi} X, \theta_L)$ is a compatible prequantization of $X$ if and only if the associated line bundle of $\phi : L \to X$ is a twisted line bundle over $R \times_M X \to \Gamma \times_M X \cong X$ with the pseudo-connection and pseudo-curvature being given by $\theta_L$ and $J^*\theta + (J^*B - \omega_X) \in \Omega^1(R \times_M X) \oplus \Omega^2(X)$ respectively.

PROOF. Given a compatible prequantization $L \xrightarrow{\phi} X$, define an action of $R \times_M X \cong X$ on $L$ by $(\kappa, \phi(l)) \cdot l = \kappa l$, where $\kappa \in R$ and $l \in L$ are compatible pairs. It is simple to check that all the compatible conditions are satisfied so that $L \xrightarrow{\phi} X$ is a twisted line bundle over $R \times_M X \to \Gamma \times_M X \cong X$. It is simple to see that the corresponding transformation groupoid $(\Gamma \times_M X) \times_X L \cong L$ is isomorphic to the transformation groupoid $R \times_M L \cong L$. Moreover, it is simple to see that Condition (2) of Definition 4.2 implies that

$$\partial \theta_L = \phi^* J^* \theta,$$

where, by abuse of notation, we use $\phi$ to denote the Lie groupoid homomorphism:

$$\begin{array}{c}
R \times_M L \\
\phi
\end{array} \xrightarrow{\phi} \begin{array}{c}
R \times_M X \\
\phi
\end{array}$$

and $\partial \theta_L$ is with respect to the groupoid $R \times_M L \cong L$. Therefore we have

$$\delta \theta_L = \partial \theta_L + d\theta_L = \phi^*(J^* \theta + J^*B - \omega_X).$$

The converse can be proved by working backwards. □

As an immediate consequence, we have

Corollary 4.7 Under the same hypothesis as in Proposition 4.6 and assume that $\Gamma \cong M$ is proper, for a pre-Hamiltonian $\Gamma$-space $(X \xrightarrow{\theta} M, \omega_X)$, a compatible prequantization exists if and only if $J^*(\theta + B) - \omega_X$ is an integer 2-cocycle in $Z^2_{\text{int}}((\Gamma \times_M X)_*, \mathbb{Z})$.

PROOF. One direction is obvious by Proposition 4.6.

For the other direction, note that Proposition 2.3 implies that $J^*(\theta + B) - \omega_X$ is always a two-cocycle since

$$\delta(J^*(\theta + B) - \omega_X) = J^*\delta(\theta + B) - \pi^*\delta \omega_X = J^*\pi^*(\omega + \Omega) - \pi^*J^*(\omega + \Omega) = 0.$$
Here we have used Eqs. (18) and (6). If $J^*(\theta + B) - \omega_X$ is an integer cocycle in $\mathbb{Z}^2_{\text{H}}((\mathbb{R} \times M) \times, \mathbb{Z})$, according to Proposition 3.3, there exists an $S^1$-bundle $L \to X$ over $R \equiv X$ and a pseudo-connection $\theta_L \in \Omega^1(L)$ whose pseudo-curvature equals to $J^*\theta + (J^*B - \omega_X)$. According to Corollary 3.9, one sees that the associated line bundle of $L$ is indeed a twisted line bundle over $R \times_M X \to \Gamma \times_M X \equiv X$. Then $L \to X$ is a compatible prequantization by Proposition 4.6. □

4.2 Prequantization of classical intertwiner spaces

We are now ready to state the main theorem of this section.

**Theorem 4.8** Let $(\Gamma \equiv M, \omega + \Omega)$ be an exact quasi-presymplectic groupoid, and $(R \to \Gamma \equiv M, \theta + B)$ a prequantization. Assume that $(X_i \xrightarrow{j_i} M, \omega_i^1)$, $i = 1, 2$, are pre-Hamiltonian $\Gamma$-spaces, $\Gamma \equiv M$ acts freely on $\mathbb{X}_2 \times_M X_1$, and $\mathbb{X}_2 \times_\Gamma X_1 = \Gamma \backslash \mathbb{X}_2 \times_M X_1$ is a smooth manifold. Assume that $(L_i \xrightarrow{\phi_i} X_i, \theta_i)$, $i = 1, 2$, are compatible prequantizations of $X_i$, respectively. Then

$$\phi : R \backslash (L_2 \times_M L_1) \to \mathbb{X}_2 \times_\Gamma X_1, \; \phi[l_2, l_1] = [\phi_2(l_2), \phi_1(l_1)],$$

with the $S^1$-action $\lambda \cdot [l_2, l_1] = [\lambda \cdot l_2, l_1]$, $\lambda \in S^1$ is an $S^1$-principal bundle. Moreover, $i^*(\theta_2, -\theta_1)$ descends to a connection one-form on $R \backslash (L_2 \times_M L_1)$, which defines a prequantization of the classical intertwiner space $\mathbb{X}_2 \times_\Gamma X_1$. Here $i : L_2 \times_M L_1 \to L_2 \times L_1$ is the natural embedding.

**Proof.** One checks directly that $\phi : R \backslash (L_2 \times_M L_1) \to \mathbb{X}_2 \times_\Gamma X_1$ is an $S^1$-bundle. Now let $R \equiv M$ act on $L_2 \times_M L_1$ diagonally. We have

$$\partial i^*(\theta_2, -\theta_1) = i^*(\partial \theta_2, -\partial \theta_1) = i^*(\phi_i^* J_2^* \theta, -\phi_1^* J_1^* \theta) = 0.$$ 

Hence $i^*(\theta_2, -\theta_1)$ descends to a 1-form on the quotient space $R \backslash (L_2 \times_M L_1)$, which can be easily seen to be a connection form. Now

$$d(i^*(\theta_2, -\theta_1)) = i^*(d\theta_2, -d\theta_1) = i^*(\phi_1^* (J_1^* B - \omega_1), \phi_2^* (J_2^* B - \omega_2)) = i^*(\phi_2 \times \phi_1)^*(\omega_2, -\omega_1),$$

where in the last equality we used the relation $J_1 \phi_1 = J_2 \phi_2$ on $L_2 \times_M L_1$. Here $\phi_i$ and $J_i$, $i = 1, 2$ are groupoid homomorphisms:

$$\begin{array}{ccc}
R \times_M L_1 & \xrightarrow{\phi_i} & R \times_M X_i \\
\downarrow & & \downarrow J_i \\
L_i & \xrightarrow{\phi_i} & X_i \\
\downarrow & & \downarrow J_i \\
& & M \\
\end{array} \quad (22)$$

and $i$ is the groupoid homomorphism:

$$R \times_M (L_2 \times_M L_1) \xrightarrow{i} (R \times_M L_2) \times (R \times_M L_1).$$

This completes the proof. □
4.3 Morita equivalence

Definition 4.9 Quasi-presymplectic groupoids \((G \rightrightarrows G_0, \omega_G + \Omega_G)\) and \((H \rightrightarrows H_0, \omega_H + \Omega_H)\) are said to be Morita equivalent if there exists a Morita equivalence bimodule \(G_0 \overset{\epsilon}{\rightarrow} X \overset{\sigma}{\leftarrow} H_0\) between the Lie groupoids \(G\) and \(H\), together with a two-form \(\omega_X \in \Omega^2(X)\) such that \((X, \epsilon \otimes \sigma \circ \rho) \rightarrow (G_0 \times H_0, \omega_X)\) is a pre-Hamiltonian \(G \times H\)-space, where the \(G \times H\)-action on \(X\) is given by \((g, h) \cdot x = gxh^{-1}\), for any compatible triples \(g \in G, \ h \in H\) and \(x \in X\).

One easily checks that this is indeed an equivalence relation among quasi-presymplectic groupoids.

Let \(Q \rightrightarrows X\) be the transformation groupoid

\[Q : (G \times H) \times_{(G_0 \times H_0)} X \rightrightarrows X.\]

Then the natural projections \(p_1 : Q \rightarrow G\) and \(p_2 : Q \rightarrow H\) are groupoid homomorphisms. As an immediate consequence of Proposition 2.3, we have the following identity

\[p_1^*(\omega_G + \Omega_G) - p_2^*(\omega_H + \Omega_H) = \delta \omega_X.\]

Note that the axioms of Morita equivalence of Lie groupoids assure that, as groupoids, \(Q \rightrightarrows G[X]\) and \(Q \rightrightarrows H[X]\) (see the proof of Proposition 4.5 \([19]\)).

Recall that for a given Lie groupoid \(\Gamma \rightrightarrows M\), two cohomologous 3-cocycles \(\omega_i + \Omega_i \in \Omega^2(\Gamma) \oplus \Omega^3(M), \ i = 1, 2\), are said to differ by a gauge transformation of the first type if

\[(\omega_1 + \Omega_1) - (\omega_2 + \Omega_2) = \delta B\]

for some \(B \in \Omega^2(M)\).

By a Morita morphism from the quasi-presymplectic groupoid \((\Gamma' \rightrightarrows M', \omega' + \Omega')\) to \((\Gamma \rightrightarrows M, \omega + \Omega)\), we mean a Morita morphism of Lie groupoid \(p : \Gamma' \rightarrow \Gamma\) (i.e \(\Gamma'\) is isomorphic to the pullback groupoid \(\Gamma[M'] \rightrightarrows M'\)) such that \(\omega' + \Omega'\) and \(p^*\omega + p^*\Omega\) differ by a gauge transformation of the first type.

The following result gives a more intuitive meaning of Morita equivalence.

Proposition 4.10 Two quasi-presymplectic groupoids are Morita equivalent if and only if there exists a third quasi-presymplectic groupoid together with a Morita morphism to each of them.

Corollary 4.11 For two Morita equivalent quasi-presymplectic groupoids, if one is integral, so is the other.

Therefore Morita equivalence can be passed to an equivalence relation among integral quasi-presymplectic groupoids.

One of the most important feature of Morita equivalent quasi-presymplectic groupoids is the following

Theorem 4.12 Suppose that \((G \rightrightarrows G_0, \omega_G + \Omega_G)\) and \((H \rightrightarrows H_0, \omega_H + \Omega_H)\) are Morita equivalent quasi-presymplectic groupoids with an equivalence bimodule \(G_0 \overset{\epsilon}{\rightarrow} X \overset{\sigma}{\leftarrow} H_0\). Then,
1. corresponding to any pre-Hamiltonian $G$-space $J_F : F \to G_0$, there is a unique (up to isomorphism) pre-Hamiltonian $H$-space $J_E : E \to H_0$ such that $F$ and $E$ are a pair of related pre-Hamiltonian spaces and vice versa.

2. Let $J_i : F_i \to G_0$, $i = 1, 2$, be pre-Hamiltonian $G$-spaces and $J_i : E_i \to H_0$, $i = 1, 2$, their related pre-Hamiltonian $H$-spaces. Then $F_1 \times_G F_2$ and $E_1 \times_H E_2$ are diffeomorphic as presymplectic manifolds (in the sense that one is smooth so is the other).

Proof. This was proved in [19] for quasi-symplectic groupoids and their Hamiltonian spaces. One can prove this theorem in a similar fashion (in fact in a simpler way by using Proposition 2.5). We will leave the detail to the reader. □

We now can introduce Morita equivalence for prequantization of quasi-presymplectic groupoids.

**Definition 4.13** Let $(G \cong G_0, \omega_G + \Omega_G)$ and $(H \cong H_0, \omega_H + \Omega_H)$ be Morita equivalent integral exact quasi-presymplectic groupoids with an equivalence bimodule $(G_0 \xleftarrow{\omega} X \xrightarrow{\omega} H_0, \omega_X)$. We say their prequantizations $(R_G \to G \cong G_0, \theta_G + B_G)$ and $(R_H \to H \cong H_0, \theta_H + B_H)$ are Morita equivalent if $X$ admits a compatible prequantization $(Z \to X, \theta_Z)$ with respect to the natural product: $(R_G \times \overline{R_H})/S^1 \to G \times \overline{H} \cong G_0 \times \overline{H}_0, (\theta_G + \overline{\theta_H}) + (B_G + \overline{B_H})$.

It is simple to see that $G_0 \leftarrow Z \to H_0$ is an equivalence bimodule of central extensions in the sense of Definition 2.11 [15].

**Remark 4.14**

1. Note that prequantizations can be Morita equivalent as central extensions, but not Morita equivalent as prequantizations. The earlier one simply means that they correspond to isomorphic $S^1$-gerbes, and up to a torsion, are determined by their Dixmier-Douady class.

2. It is interesting to investigate the following question: given two Morita equivalent quasi-presymplectic groupoids and a prequantization of one of them, is it possible to construct a Morita equivalent prequantization for the other quasi-presymplectic groupoid?

A useful feature of Morita equivalence is to give us a recipe which allows us to construct compatible prequantizations.

**Theorem 4.15** For Morita equivalent prequantizations of quasi-presymplectic groupoids, there is a bijection between their compatible prequantizations of pre-Hamiltonian spaces.

Proof. Let $(G \cong G_0, \omega_G + \Omega_G)$ and $(H \cong H_0, \omega_H + \Omega_H)$ be Morita equivalent integral exact quasi-presymplectic groupoids with an equivalence bimodule $(G_0 \op X \to H_0, \omega_X)$, and $(R_G \to G \cong G_0, \theta_G + B_G)$ and $(R_H \to H \cong H_0, \theta_H + B_H)$ are Morita equivalent prequantizations given by $(Z \to X, \theta_Z)$. Assume that $J : F \to G_0$ is a pre-Hamiltonian $G$-space and $(L \to F, \omega_L)$ a compatible prequantization. It is known that the corresponding pre-Hamiltonian $H$-space is

$E := X \times_G F \to H_0$, where $J' : E \to H_0$ and the $H$-action on $E$ are defined by $J'([x, f]) = \sigma(x)$ and $h \cdot [x, f] = [x \cdot h^{-1}, f]$.

Let $L' = Z \times_{R_G} L$. Then it is clear that $L'$ is an $S^1$-bundle over $E$, and $R_H$ acts on $L'$ equivariantly. It is simple to check that $i^*(\theta_Z, -\theta_L)$, where $i : Z \times_{G_0} L \to Z \times L$, descends to a one-form on the quotient space $Z \times_{R_G} L$, which is indeed a connection one-form $\theta_{L'}$ on $L'$. It is.
routine to check that $(L' \to E, \theta_{L'})$ is a compatible prequantization of the pre-Hamiltonian $H$-space $J': E \to H_0$.

The inverse functor can be constructed in a similar fashion. □

**Remark 4.16** The above theorem indicates a method of converting prequantizations of Hamiltonian $LG$-spaces and of quasi-Hamiltonian $G$-spaces of AMM to each other. The latter is understood as a compatible prequantization corresponding to the quasi-symplectic groupoid $(G \times G)[U] = \bigsqcup U_i$, which is the pullback quasi-symplectic groupoid of the AMM quasi-symplectic groupoid using an open covering $U = (U_i)_{i \in I}$ of $G$ (see [13], for instance, for an explicit construction). It is known that $(G \times G)[U] = \bigsqcup U_i$ is Morita equivalent to the symplectic groupoid $(LG \times L\mathfrak{g} \cong L\mathfrak{g}, \omega_{LG \times L\mathfrak{g}})$ according to Proposition 4.26 [19]. The question is, therefore boiled down to the construction of a compatible prequantization of the Morita equivalence Hamiltonian bimodule.

## 5 Integral pre-Hamiltonian $\Gamma$-spaces

The main purpose of this section is to give a geometric integrality condition which guarantees the existence of a prequantization of a pre-Hamiltonian $\Gamma$-space.

### 5.1 Integrality condition

**Lemma 5.1** Let $J : G \to H$ be a strict homomorphism of groupoids. By $\text{Ker}(J_*)$ we denote the kernel of $J_* : H_2(G_*, \mathbb{Z}) \to H_2(H_*, \mathbb{Z})$. Let $\omega \in Z^2_{\mathbb{Z}}(G_*)$. The following conditions are equivalent:

1. there exists $\Xi \in Z^2_{\mathbb{Z}}(H_*)$ such that

$$\omega + J^*\Xi \in Z^2_{\mathbb{Z}}(G_*, \mathbb{Z});$$

2. for any $C \in Z_2(G_*, \mathbb{Z})$ with $[C] \in \text{Ker}(J_*)$, we have

$$\int_C \omega \in \mathbb{Z}.$$ 

**Proof.** 1) $\Rightarrow$ 2). By definition, we have for any $C \in Z_2(G_*, \mathbb{Z})$,

$$\int_C (\omega + J^*\Xi) \in \mathbb{Z}. \quad (24)$$

From Eq. (9), we also have

$$\int_C (\omega + J^*\Xi) = \int_C \omega + \int_{J_* (C)} \Xi.$$ 

If $[J_*(C)] = 0$, i.e. $J_*(C) = \delta D$ for some $D \in C_3(H_*, \mathbb{Z})$, then

$$\int_C (\omega + J^*\Xi) = \int_C \omega + \int_D \Xi = \int_C \omega + \int_D \delta \Xi = \int_C \omega.$$ 

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since $\delta \Xi = 0$. Therefore $\int_C \omega \in \mathbb{Z}$.

2) $\Rightarrow$ 1). Since there exists a $\mathbb{Z}$-submodule $H$ in $H_2(G, \mathbb{Z})$ such that $H_2(G, \mathbb{Z}) = H \oplus \text{Ker}(J_*)$, the $\mathbb{Z}$-map

$$f : \text{Ker}(J_*) \to \mathbb{Z}, \quad f([C]) = \int_C \omega, \quad \forall [C] \in \text{Ker}(J_*),$$

can be extended to a $\mathbb{Z}$-map $\bar{f} : H_2(G, \mathbb{Z}) \to \mathbb{Z}$. According to Proposition 3.1, there exists $\omega' \in Z^2_{dR}(G)$ such that

$$\bar{f}([C]) = \int_C \omega', \quad \forall [C] \in H_2(G, \mathbb{Z}).$$

By Eq. (10), $\omega'$ is an integer cocycle in $Z^2_{dR}(G, \mathbb{Z})$. Moreover, we have

$$\int_C (\omega' - \omega) = 0, \quad \forall \text{ cycle } C \text{ with } [C] \in \text{Ker}(J_*). \quad (25)$$

Since $J_* : H_2(G, \mathbb{R}) \to H_2(H, \mathbb{R})$ is dual to $J^* : \text{H}^2_{dR}(H) \to H^2_{dR}(G)$, we have Ker$(J_*^*) = \text{Im} J^*$. Therefore $[\omega' - \omega] = J^* [\Xi]$ for some $\Xi \in Z^2_{dR}(\Gamma, \mathbb{Z})$. This proves 1). □

**Definition 5.2** Let $(\Gamma \rightrightarrows M, \omega + \Omega)$ be a quasi-presymplectic groupoid. A pre-Hamiltonian $\Gamma$-space $(X \to M, \omega_X)$ is said to satisfy the **integrality condition** if for any $C \in Z_2((\Gamma \times_M X), \mathbb{Z})$ and any $D \in C_3(\Gamma, \mathbb{Z})$,

$$\delta D = J_*(C) \quad \Rightarrow \quad \int_C \omega_X - \int_D (\omega + \Omega) \in \mathbb{Z}. \quad (26)$$

In this case, we also say that the pair $(\omega_X, \omega + \Omega)$ satisfies the integrality condition.

**Remarks 5.3**

1. By taking $C = 0$, Eq. (26) implies that $\int_D (\omega + \Omega) \in \mathbb{Z}, \quad \forall D \in Z_3(\Gamma, \mathbb{Z})$.

That is, $\omega + \Omega$ must be an integer 3-cocycle and therefore $(\Gamma \rightrightarrows M, \omega + \Omega)$ is an integral quasi-presymplectic groupoid.

2. If $\omega + \Omega$ is a 3-coboundary $\delta K$, then the integrality condition is equivalent to

$$\int_C (\omega_X - J^* K) \in \mathbb{Z}, \quad \forall C \in Z_2((\Gamma \times_M X), \mathbb{Z}) \text{ such that } J_* [C] = 0. \quad (27)$$

From now on, we shall always assume that $(\Gamma \rightrightarrows M, \omega + \Omega)$ is an integral quasi-presymplectic groupoid. The following lemma indicates that it is sufficient to require that Eq. (26) hold for a representative $(C, D)$ in any class of Ker$J_*$. 

**Lemma 5.4** Let $(\Gamma \rightrightarrows M, \omega + \Omega)$ be an integral quasi-presymplectic groupoid. A pre-Hamiltonian $\Gamma$-space $(X \to M, \omega_X)$ satisfies the integrality condition if and only if for any class $c \in \text{Ker} J_*$, there exists $C \in Z_2((\Gamma \times_M X), \mathbb{Z})$ and $D \in C_3(\Gamma, \mathbb{Z})$ with $c = [C]$ and $J_*(C) = \delta D$ such that

$$\int_C \omega_X - \int_D (\omega + \Omega) \in \mathbb{Z}.$$
Proof. Let $C' \in Z_2((\Gamma \times_M X)_\ast, \mathbb{Z})$ and $D' \in C_3(\Gamma_\ast, \mathbb{Z})$ be any pair satisfying $J_\ast(C') = \delta D'$. Then $[C'] \in \ker J_\ast$. By assumption, there exists a pair $(C, D)$ such that $[C] = [C']$, $J_\ast(C) = \delta D$ and $\int_C \omega_X - \int_D (\omega + \Omega) \in \mathbb{Z}$. Assume that $C = C' + \delta E$ for some $E \in C_3((\Gamma \times_M X)_\ast, \mathbb{Z})$. Then we have

$$
\int_C \omega_X - \int_D (\omega + \Omega) - \left( \int_C \omega_X - \int_D (\omega + \Omega) \right) \\
= - \int_C \omega_X + \int_D (\omega + \Omega) - \int_D (\omega + \Omega) \\
= - \int_E \delta \omega_X + \int_D (\omega + \Omega) - \int_D (\omega + \Omega) \\
= - \int_E J_\ast(\omega + \Omega) + \int_D (\omega + \Omega) - \int_D (\omega + \Omega) \\
= - \int_{D - J_\ast(E) - D'} (\omega + \Omega),
$$

Since $\delta(D - J_\ast(E) - D') = J_\ast(C - C' - \delta E) = 0$ and $\omega + \Omega$ is an integer cocycle, it follows that $\int_{D - J_\ast(E) - D'} (\omega + \Omega) \in \mathbb{Z}$. This completes the proof. □

Assume now that $(\Gamma \rightrightarrows M, \omega + \Omega)$ is an integral exact quasi-pseudoplectic groupoid, and $R \rightarrow \Gamma \rightrightarrows M$ is a prequantization. Let $\theta + B \in \Omega^1(R) \oplus \Omega^2(M)$ be a pseudo-connection as in Eq. (18). In order to fix the notations, recall that we have the following commutative diagram of groupoid homomorphisms

$$
\begin{array}{ccc}
R \times_M X & \xrightarrow{J} & R \\
\downarrow \pi & & \downarrow \pi \\
\Gamma \times_M X & \xrightarrow{J} & \Gamma
\end{array}
$$

(28)

where the horizontal arrows are the groupoid homomorphisms induced by $J : X \rightarrow M$ and the vertical arrows are projections of $S^1$-central extensions onto the underlying groupoids.

Lemma 5.5 Assume that $C' \in Z_2((R \times_M X)_\ast, \mathbb{Z})$ satisfies $J_\ast(C') = kZ + \delta D'$ for some $D' \in C_3(R_\ast, \mathbb{Z})$. Let $C = \pi_\ast(C')$ and $D = \pi_\ast(D')$. Then

$$
\int_C \omega_X - \int_D (\omega + \Omega) = k + \int_C (\omega_X - J_\ast(\theta + B)),
$$

(29)

where $Z \in Z_1(R_\ast, \mathbb{Z})$ is the 1-cycle defined by Eq. (41).

Proof. First, since the restriction of $\pi : R \times_M X \rightarrow \Gamma \times_M X$ to the unit manifolds is the identity map, we have

$$
\int_C \omega_X = \int_{\pi_\ast(C')} \omega_X = \int_C \omega_X.
$$

(30)
Now by Eq. (9), we have
\[
\int_{C'} J^*(\theta + B) = \int_{J_*([C'])} (\theta + B) = k \int_{Z} (\theta + B) + \int_{\partial D'} (\theta + B).
\] (31)

According to Lemma 6.1, \( \int_{Z} (\theta + B) = \int_{Z} \theta = 1 \). Therefore
\[
\int_{C'} J^*(\theta + B) = k + \int_{\partial D'} (\theta + B) = k + \int_{D'} \delta(\theta + B) \quad \text{(by Eq. (18))}
\]
\[
= k + \int_{D'} \pi^*(\omega + \Omega) \quad \text{(by Eq. (9))}
\]
\[
= k + \int_{D} (\omega + \Omega).
\]

Hence it follows that
\[
\int_{C} \omega_X - \int_{D} (\omega + \Omega) = k + \int_{C'} (\omega_X - J^*(\theta + \Omega)).
\]

\( \square \)

**Theorem 5.6** Let \((\Gamma \rightrightarrows M, \omega + \Omega)\) be an integral quasi-presymplectic groupoid, and \((R \to \Gamma \rightrightarrows M, \theta + B)\) a prequantization. Assume that \((X \to M, \omega_X)\) is a pre-Hamiltonian \(\Gamma\)-space. Then the following conditions are equivalent.

1. There exists a 2-cocycle \(\Xi \in Z^2_d R(\Gamma_*)\) such that
\[
\omega_X - J^*(\theta + B) - J^* \pi^* \Xi \in Z^2_d R((R \times_M X)_*, \mathbb{Z}).
\]

2. For any cycle \(C' \in Z^2((R \times_M X)_*, \mathbb{Z})\) such that \([C'] \in \text{Ker}(\pi_* J_*), \) we have
\[
\int_{C'} (\omega_X - J^*(\theta + B)) \in \mathbb{Z}.
\]

3. The pair \((\omega_X, \omega + \Omega)\) satisfies the integrality condition.

**Proof.** 1) \( \iff \) 2), follows from Lemma 5.1.

2) \( \implies \) 3). Any class in \(\text{Ker} J_* \subset H^2((R \times_M X)_*, \mathbb{Z})\) can be represented by a 2-cocycle of the form \(C = \pi_*(C')\) where \(C' \in Z^2((R \times_M X)_*, \mathbb{Z})\). Then \([C'] \in \text{Ker}(\pi_* J_*), \) and it thus follows that \([J_*(C')] \in \text{Ker}(\pi_*)\). By Lemma 6.1, \([J_*(C')] = k[Z]\) for some \(k \in \mathbb{Z}\) where
$Z \in C_1(R, \mathbb{Z})$ is defined by Eq. (41). In other words, there exists $D' \in C_3(R, \mathbb{Z})$ such that $J_\ast(C') = kZ + \delta D'$. Let $D = \pi_\ast(D')$. One can easily see that $\delta D = \pi_\ast(J_\ast(C')) = J_\ast(C)$. Then by Lemma 5.5, we have $\int_X \omega_X - \int_D (\omega + \Omega) \in \mathbb{Z}$. By Lemma 5.4, this implies that the pair $(\omega_X, \omega + \Omega)$ satisfies the integrality condition.

3) $\Rightarrow$ 2). Let $C' \in Z_2((R \times X)_\ast, \mathbb{Z})$ be any cycle whose class is in the kernel of $\pi_\ast - J_\ast$. Since $[J_\ast(C')] \in \text{Ker}(\pi_\ast)$, Lemma 6.1 implies that there exists $k \in \mathbb{Z}$ and $D' \in C_3(R, \mathbb{Z})$ such that

$$J_\ast(C') = kZ + \delta D'.$$

(32)

Therefore, by Eq. (29), we have $\int_{C'} (\omega_X - J^\ast(\theta + B)) = -k + \int_C \omega_X - \int_D (\omega + \Omega)$, where $C = \pi_\ast(C')$ and $D = \pi_\ast(D')$. By composing Eq. (32) with $\pi_\ast$, one finds that $J_\ast(C) = \delta D$. Since $(\omega_X, \omega + \Omega)$ satisfies the integrality condition, it thus follows that $\int_{C'} (\omega_X - J^\ast(\theta + B)) \in \mathbb{Z}$. □

As an immediate consequence, we obtain the following main result of the section.

**Theorem 5.7** Let $(\Gamma \rightrightarrows M, \omega + \Omega)$ be an exact proper quasi-presymplectic groupoid, and $(X \rightrightarrows M, \omega_X)$ a pre-Hamiltonian $\Gamma$-space. Then there exists a compatible prequantization $R \to \Gamma \rightrightarrows M$ and $L \to X$ if and only if the pair $(\omega_X, \omega + \Omega)$ satisfies the integrality condition: Eq. (26).

**Proof.** Assume that $(R \to \Gamma \rightrightarrows M, \theta + B)$ and $(L \to X, \theta_L)$ are a pair of compatible prequantizations. Then by Corollary 4.7, we have $\omega_X - J^\ast(\theta + B) \in Z^2_{dR}(\Gamma \times_M X)_\ast, \mathbb{Z})$. Hence $(\omega_X, \omega + \Omega)$ satisfies the integrality condition by Theorem 5.6.

Conversely, assume that $(\omega_X, \omega + \Omega)$ satisfies the integrality condition. Then $\omega + \Omega$ must be an integer cocycle. Let $(R \to \Gamma \rightrightarrows M, \theta + B)$ be a prequantization, which always exists since $\Gamma$ is proper. Again according to Theorem 5.6, there exists a 2-cocycle $\Xi \in Z^2_{dR}(\Gamma_\ast)$ such that $\omega_X - J^\ast(\theta + B) - J^\ast(\pi^\ast \alpha) \in Z^2_{dR}(\Gamma \times_M X)_\ast, \mathbb{Z})$. Since $\Gamma$ is proper, $\Xi$ is cohomologous to $\omega_X - J^\ast(\theta + B) + \pi^\ast \omega_X + J^\ast(\pi^\ast \omega_X) - J^\ast(\pi^\ast \omega_X)$ is clearly a pseudo-connection and $\omega_X - J^\ast(\theta + B') \in Z^2_{dR}(\Gamma \times_M X)_\ast, \mathbb{Z})$. Hence from Corollary 4.7, it follows that there exists a compatible prequantization $(L \to X, \theta_L)$ for $(X \rightrightarrows M, \omega_X)$. □

### 5.2 Integral quasi-Hamiltonian $G$-spaces

In this subsection, $G$ is a compact connected and simply-connected Lie group and 1 denotes the unit of $G$. We intend to study the case where $\Gamma$ is the AMM quasi-symplectic groupoid.

There is a natural map $i : H_2(X, \mathbb{Z}) \to H_2((G \times X)_\ast, \mathbb{Z})$ induced by the inclusion of $C_2(X, \mathbb{Z}) \subset C_2((G \times X)_\ast, \mathbb{Z})$. The following lemma indicates that $i$ is an isomorphism.

**Lemma 5.8** If $G$ is a connected and simply-connected Lie group, then the map

$$i : H_2(X, \mathbb{Z}) \to H_2((G \times X)_\ast, \mathbb{Z})$$

is bijective.

**Proof.** This is a standard result. For completeness, we sketch a proof below. Let $G \to EG \to BG$ be the usual $G$-bundle over the classifying space $BG$ of $G$ and $X_G = G \setminus (EG \times X)$. We have the fibration $G \to EG \times X \to X_G$.

21
The second term of the homology Leray-Serre spectral sequence is \( E^2_{p,q} = H_p(X_G, \mathcal{H}_q(G, \mathbb{Z})) \), i.e. the homology of \( X_G \) with local coefficients in \( \mathcal{H}_q(G, \mathbb{Z}) \) (see [11]). Since \( G \) is simply-connected, we have \( H_1(G, \mathbb{Z}) = H_2(G, \mathbb{Z}) = 0 \) and \( E^2_{p,q} \) has the following form for \( 0 \leq p \leq 3 \) and \( 0 \leq q \leq 2 \):

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
H_0(X_G, \mathbb{Z}) & H_1(X_G, \mathbb{Z}) & H_2(X_G, \mathbb{Z}) & H_3(X_G, \mathbb{Z})
\end{array}
\]  

(33)

According to Leray-Serre Theorem, this spectral sequence converges to \( H_*(EG \times X, \mathbb{Z}) \). It is clear from Eq. (33) that in particular, we have

\[
H_2(EG \times X, \mathbb{Z}) \cong H_2(X_G, \mathbb{Z}).
\]

Since \( EG \) is contractible we get

\[
H_2(X_G, \mathbb{Z}) \cong H_2(X, \mathbb{Z}).
\]

The lemma now follows from the well-known isomorphism \( H_2((G \times X)_*, \mathbb{Z}) \cong H_2(X_G, \mathbb{Z}) \). \( \square \)

By Lemma 5.8, since \( H_2(g^*, \mathbb{Z}) = 0 \), we have

\[
H_2((T^*G)_*, \mathbb{Z}) = 0.
\]  

(34)

Since any simply-connected Lie group \( G \) satisfies \( H_2(G, \mathbb{Z}) = 0 \), we have also

\[
H_2((G \times G)_*, \mathbb{Z}) = 0.
\]  

(35)

Recall that the AMM quasi-symplectic groupoid is \( (G \times G \rightrightarrows G, \omega + \Omega) \) [5, 19], where \( G \) is a compact Lie group equipped with an ad-invariant non-degenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \). Here \( G \times G \rightrightarrows G \) is the transformation groupoid, where \( G \) acts on itself by conjugation, and \( \omega \) and \( \Omega \) are defined as follows.

Following [2], we denote by \( \theta \) and \( \Theta \) the left and right Maurer-Cartan forms on \( G \) respectively, i.e., \( \theta = g^{-1}dg \) and \( \Theta = dg g^{-1} \). Let \( \Omega \in \Omega^3(G) \) denote the bi-invariant 3-form on \( G \) corresponding to the Lie algebra 3-cocycle \( \frac{1}{12} \langle \cdot, [\cdot, \cdot] \rangle \in \Lambda^3 g^* \):

\[
\Omega = \frac{1}{12} \langle \theta, [\theta, \theta] \rangle = \frac{1}{12} \langle \Theta, [\theta, \theta] \rangle
\]  

(36)

and \( \omega \in \Omega^2(G \times G) \) the two-form:

\[
\omega|_{(g,x)} = -\frac{1}{2} [ (Ad_x pr_1^* \theta, pr_1^* \theta) + (pr_1^* \theta, pr_2^* (\theta + \theta))] ,
\]  

(37)

where \( (g,x) \) denotes the coordinate in \( G \times G \), and \( pr_1 \) and \( pr_2 : G \times G \to G \) are natural projections.

A triple \( (X, \omega_X, J) \) (where \( X \) is a manifold, \( \omega_X \) a \( G \)-invariant two-form and \( J : X \to G \) a smooth map) is a quasi-Hamiltonian \( G \)-space in the sense of [2] if

\begin{itemize}
  \item[(B1)] the differential of \( \omega_X \) is given by:
  \[
  d\omega_X = J^* \Omega ,
  \]
\end{itemize}
(B2) the map $J$ satisfies
\[
\hat{\xi} \omega_X = \frac{1}{2} J^* (\xi, \theta + \theta),
\]
(B3) at each $x \in X$, the kernel of $\omega_X$ is given by
\[
\text{ker} \omega_X = \{ \hat{\xi}(x) | \xi \in \text{ker} \left( \text{Ad}_J(x) + 1 \right) \},
\]
where $\theta$ and $\theta$ are the left and right Maurer-Cartan forms and $\hat{\xi}$ is the vector field on $X$ associated to the infinitesimal action of $\xi \in g$ on $X$.

It is known [19] that these conditions are equivalent to that $(X \xrightarrow{J} G, \omega_X)$ is a compatible $\Gamma$-space where $\Gamma$ is the AMM quasi-symplectic groupoid $(G \times G \rightrightarrows G, \omega + \Omega)$.

**Proposition 5.9** Let $\Gamma$ be the AMM quasi-symplectic groupoid $(G \times G \rightrightarrows G, \omega + \Omega)$, where $G$ is a connected and simply-connected Lie group equipped with an ad-invariant non-degenerate symmetric bilinear form. Let $(X \xrightarrow{J} G, \omega_X)$ be a quasi-Hamiltonian $G$-space. Assume that $\omega + \Omega$ is an integer $3$-cocycle in $Z_3^{dr}((G \times G)_*, \mathbb{Z})$. Then the pair $(\omega_X, \omega + \Omega)$ satisfies the integrality condition if and only if $\forall C \in Z_2(X, \mathbb{Z})$ and $D \in C_3(G, \mathbb{Z})$ such that $dD = J_* (C)$,
\[
\int_C \omega_X - \int_D \Omega \in \mathbb{Z}.
\]

Note that such $D$ always exists for any $C \in Z_2(X, \mathbb{Z})$.

**Proof.** Note that we have the following commuting diagram of groupoid homomorphisms:
\[
\begin{array}{ccc}
X_* & \xrightarrow{J} & (G \times X)_* \\
\downarrow J & & \downarrow J \\
G_* & \xrightarrow{J} & (G \times G)_* \\
\end{array}
\]

where $X_*$ and $G_*$ are spaces $X$ and $G$ considered as groupoids, while $(G \times X)_*$ and $(G \times G)_*$ are the transformation groupoids. Thus one direction is obvious.

Conversely, according to Eq. (35), we have $H_2((G \times M)_*, \mathbb{Z}) = 0$. Therefore $\text{Ker}(J_*) = H_2((G \times M)_*, \mathbb{Z})$. By Lemma 5.8, for any class $\mathcal{C} \in H_2((G \times X)_*, \mathbb{Z})$, there exists $C \in Z_2(X, \mathbb{Z})$ such that $\mathcal{C} = i_* [C]$. Since $H_2(G, \mathbb{Z}) = 0$, there always exists $D \in C_3(G, \mathbb{Z})$ such that $J_* (C) = dD$. Hence
\[
J_*(i_* C) = i_*(J_* C) = i_* dD = \delta (i_* D).
\]

Now it is clear that
\[
\int_C \omega_X - \int_D (\omega + \Omega) = \int_C i^* \omega_X - \int_D i^*(\omega + \Omega) = \int_C \omega_X - \int_D \Omega.
\]

The conclusion thus follows from Lemma 5.4. □

As an immediate consequence of Proposition 5.9, we have the following:
Corollary 5.10 Let \( \Gamma \) be the AMM quasi-symplectic groupoid \((G \times G \cong G, \omega + \Omega)\). Then \( 1 \in G \), considered as a quasi-Hamiltonian \( G \)-space, satisfies the integrality condition.

Applying Theorem 5.7 to the case of AMM quasi-presymplectic groupoid, we are lead to

Corollary 5.11 Let \((X \xrightarrow{J} G, \omega_X)\) be a quasi-Hamiltonian \( G \)-space. Assume that Eq. (38) holds. Then there exists a compatible prequantization \( \prod R_i \to (G \times G)[H] \cong \prod U_i \) and \( \prod L_i \to \prod X | U_i \), where \((G \times G)[H] \cong \prod U_i \) is the pullback quasi-symplectic groupoid of the AMM groupoid using an open covering of \( G \) such that \( \Omega | U_i, \forall i \) are all exact.

Remark 5.12
1. Note that Eq. (38) coincides with the quantization condition of Alekseev–Meinrenken [1].
2. It would be interesting to investigate when a conjugacy class satisfies the integrality condition.

Let us consider the case of Example 4.3 where \( \Gamma \) is the symplectic groupoid \( T^*(G) \cong g^* \). In this case we recover a well-known result of Guillemin–Sternberg [9].

Proposition 5.13 Let \( \Gamma \) be the symplectic groupoid \((T^*(G) \cong g^*, \omega)\), where \( G \) is a connected and simply-connected Lie group. Let \( J : X \to g^* \) be a momentum map for a Hamiltonian \( G \)-space \((X, \omega_X)\) as in Example 4.3. The pair \((\omega_X, \omega)\) satisfies the integrality condition if and only if \( \omega_X \) is an integer two form.

Proof. According to Eq. (34) we have \( H_2(T^*(G), Z) = 0 \). Therefore for any \( C \in Z_2((G \times X), Z) \) there exists \( D \in C_3(T^*(G), Z) \) such that \( J_*(C) = \delta D \).

By Lemma 5.8, we can assume that \( C \in Z_2(X, Z) \). Since \( H_2(g^*, Z) = 0 \), we can assume that \( D \in C_3(g^*, Z) \).

Since \( \Omega = 0 \), thus the integrality condition of Eq. (26) reads \( \int_C \omega_X \in Z \). \( \square \)

In particular, a coadjoint orbit \( O \subset g^* \), endowed with the Kirillov-Kostant-Souriau symplectic structure \( \omega_O \), satisfies the integrality condition if and only if \( \omega_O \) is an integer two-form.

5.3 Integrality condition and Morita equivalence

In general, a quasi-presymplectic groupoid may not be exact, for instance, the AMM-quasi-symplectic groupoid. In such a case, one has to pass to a Morita equivalent quasi-presymplectic groupoid in order to construct a prequantization. According to Theorem 4.12, Morita equivalent quasi-(pre)symplectic groupoids yield equivalent momentum map theories in the sense that there is a bijection between their (pre)-Hamiltonian \( \Gamma \)-spaces, and the classical intertwiner spaces are independent of Morita equivalence [19].

More precisely, given a quasi-presymplectic groupoid \((\Gamma \cong M, \omega + \Omega)\), where \( \Omega \) may not be exact, one can choose a surjective submersion \( N \xrightarrow{\pi} M \), and consider the pull-back groupoid \( \Gamma[N] \cong N \) of \( \Gamma \cong M \) via \( \pi \). Then \( \Gamma[N] \cong N, p^*\omega + p^*\Omega \) is again a quasi-presymplectic groupoid. Moreover, if \((X \xrightarrow{J} M, \omega_X)\) is a pre-Hamiltonian \( \Gamma \)-space, then \((X_N \xrightarrow{J_N} N, p^*\omega_X)\) is a pre-Hamiltonian \( \Gamma[N] \)-space, where \( X_N = X \times_M N \), and \( p : X_N \to X \) and \( J_N : X_N \to N \) are the projections to the first and second components, respectively. The following proposition indicates that integrality condition is preserved under this pull-back procedure.
Lemma 5.14 The pair \((\omega_X, \omega + \Omega)\) satisfies the integrality condition if and only if \((p^*\omega_X, p^*\omega + p^*\Omega)\) satisfies the integrality condition.

Proof. By abuse of notation, we use the same letter \(p\) to denote the groupoid homomorphisms from \(\Gamma[N] \times_N X_N \cong X_N\) to \(\Gamma \times M \cong M\), and from \(\Gamma[N] \cong N\) to \(\Gamma \cong M\), both of which are indeed Morita morphisms.

For any \(C' \in Z_2((\Gamma[N] \times_N X_N)\times, \mathbb{Z}) \) and \(D' \in C_3(\Gamma[N]\times, \mathbb{Z})\) with \(J_*(C') = \delta D'\), we have

\[
\int_C p^*\omega_X - \int_D p^*(\omega + \Omega) = \int_C \omega_X - \int_D (\omega + \Omega),
\]

(40)

where \(C = p_*(C')\) and \(D = p_*(D')\) clearly satisfies \(J_*(C) = \delta D\).

Assume that the pair \((\omega_X, \omega + \Omega)\) satisfies the integrality condition, Eq. (40) implies immediately that so does the pair \((p^*\omega_X, p^*\omega + p^*\Omega)\).

Conversely, if \((p^*\omega_X, p^*\omega + p^*\Omega)\) satisfies the integrality condition, then \(\omega + \Omega\) must be an integer cocycle. Now we have the following commutative diagram:

\[
\begin{array}{ccc}
H_2((\Gamma[N] \times N X_N)\times, \mathbb{Z}) & \xrightarrow{p_*} & H_2((\Gamma \times M X)\times, \mathbb{Z}) \\
\downarrow J_{N*} & & \downarrow J_* \\
H_2(\Gamma[N]\times, \mathbb{Z}) & \xrightarrow{p_*} & H_2(\Gamma\times, \mathbb{Z}),
\end{array}
\]

where the horizontal arrows are isomorphisms. Therefore \(p_*: H_2((\Gamma[N] \times N X_N)\times, \mathbb{Z}) \to H_2((\Gamma \times M X)\times, \mathbb{Z})\) induces an isomorphism from \(\text{Ker} J_{N*}\) to \(\text{Ker} J_*\). This implies that any class in \(\text{Ker} J_*\) has a representative of the form \(C = p_*(C')\) where \(p_*(C') = \delta D'\) for some \(D' \in C_3((\Gamma[N] \times N X_N)\times, \mathbb{Z})\). Let \(D = p_*(D')\). By Eq. (40), we see that if the pair \((p^*\omega_X, p^*\omega + p^*\Omega)\) satisfies the integrality condition then \(\int_C \omega_X - \int_D (\omega + \Omega) \in \mathbb{Z}\). By Lemma 5.4, we conclude that \((\omega_X, \omega + \Omega)\) satisfies the integrality condition. \(\square\)

Corollary 5.15 Let \((G \cong G_0, \omega_G + \Omega_G)\) and \((H \cong H_0, \omega_H + \Omega_H)\) be Morita equivalent quasi-presymplectic groupoids. Assume that \((F \to G_0, \omega_F)\) and \((E \to H_0, \omega_E)\) are a pair of corresponding pre-Hamiltonian spaces. Then \((\omega_F, \omega_G + \Omega_G)\) satisfies the integrality condition if and only if \((\omega_E, \omega_H + \Omega_H)\) satisfies the integrality condition.

Proof. It suffices to prove this assertion for a Morita morphism of quasi-presymplectic groupoids. By Lemma 5.14, it remains to prove that the integrality condition is preserved by gauge transformations of the first type, which can be easily checked. \(\square\)

As a consequence, given a quasi-presymplectic groupoid \((\Gamma \cong M, \omega + \Omega)\), where \(\Omega\) may not be exact, one can choose a surjective submersion \(N \twoheadrightarrow M\) such that \(p^*\Omega \in \Omega^3(N)\) is exact, and replace \((\Gamma \cong M, \omega + \Omega)\) by a Morita equivalent exact quasi-presymplectic groupoid \((\Gamma[N] \cong N, p^*\omega + p^*\Omega)\). Usually, one takes \(N := \coprod U_i \to M\), where \(U = (U_i)\) is an open cover of \(M\). Then the pullback quasi-presymplectic groupoid is \((\Gamma[U] \cong \coprod U_i, \omega|_{\Gamma[U]} + \Omega|_{U_i})\), where \(\Gamma[U]\) as a manifold, can be identified with the disjoint union \(\coprod \Gamma[U_i]\). Lemma 4.7 guarantees that the integrality condition always holds no matter which surjective submersion (or an open covering) \(N \to M\) is taken as long as the initial pair \((\omega_X, \omega + \Omega)\) satisfies the integrality condition, and therefore one can always construct a compatible prequantization.
5.4 Strong integrality condition

**Definition 5.16** Let \((\Gamma \equiv M, \omega + \Omega)\) be a quasi-presymplectic groupoid. A pre-Hamiltonian \(\Gamma\)-space \((X \rightarrow M, \omega_X)\) is said to satisfy the strong integrality condition if

1. it satisfies the integrality condition and
2. the map \(J^* : H^2_{dR}(\Gamma, \ast) \rightarrow H^2_{dR}(\Gamma \times M X, \ast)\) vanishes.

The following result follows from Theorem 5.7.

**Proposition 5.17** Let \((\Gamma \equiv M, \omega + \Omega)\) be an exact proper quasi-presymplectic groupoid, and \((X \rightarrow M, \omega_X)\) a pre-Hamiltonian \(\Gamma\)-space. Then \((X \rightarrow M, \omega_X)\) satisfies the strong integrality condition if and only if for any prequantization of \((\Gamma \equiv M, \omega + \Omega)\), \(X\) admits a compatible prequantization.

**Proof.** If \((X \rightarrow M, \omega_X)\) satisfies the strong integrality condition, it is clear from Theorem 5.7 that \(X\) admits a compatible prequantization for any prequantization of \((\Gamma \equiv M, \omega + \Omega)\).

Conversely, \(J^* (\theta + B) - \omega_X\) must be an integer two cocycle in \(Z^2_{dR}(\Gamma \times M X, \ast, \mathbb{Z})\) for any pseudo-connection \(\theta + B\). If \(\theta + B\) is a pseudo-connection, so is \(\theta + B + \pi^* \Xi\), \(\forall \Xi \in Z^2_{dR}(\Gamma, \ast)\). Since the set of integer classes \(Z^2_{dR}(\Gamma \times M X, \ast, \mathbb{Z})\) is discrete, then \(J^* (\theta + B) - \omega_X + J^* \pi^* \Xi\) being an integer cocycle for all \(\Xi\) implies that \([J^* \pi^* (\Xi)] = 0\). In other words, the map \(J^* \circ \pi^* : H^2_{dR}(\Gamma, \ast) \rightarrow H^2_{dR}(\Gamma \times M X, \ast)\) is the zero map. From the identity \(J^* \circ \pi^* = \pi^* \circ J^*\) and the fact that \(\pi^*\) is injective, it follows that the map \(J^* : H^2_{dR}(\Gamma, \ast) \rightarrow H^2_{dR}(\Gamma \times M X, \ast)\) must vanish. \(\square\)

The following proposition is an analogue of Corollary 5.15.

**Proposition 5.18** Let \((G \equiv G_0, \omega_G + \Omega_G)\) and \((H \equiv H_0, \omega_H + \Omega_H)\) be Morita equivalent quasi-presymplectic groupoids. Assume that \((F \rightarrow G_0, \omega_F)\) and \((E \rightarrow H_0, \omega_E)\) are a pair of corresponding pre-Hamiltonian spaces. Then \((\omega_F, \omega_G + \Omega_G)\) satisfies the strong integrality condition if and only if \((\omega_E, \omega_H + \Omega_H)\) satisfies the strong integrality condition.

**Proof.** By Corollary 5.15, we just have to check that the condition (2) in the definition of strong integrality condition is invariant by Morita equivalence. The latter follows immediately from the commutativity of the diagram

\[
\begin{array}{ccc}
H^2_{dR}(G, \ast) & \simeq & H^2_{dR}(H, \ast) \\
\downarrow & & \downarrow \\
H^2_{dR}((G \times_{G_0} F), \ast) & \simeq & H^2_{dR}((H \times_{H_0} E), \ast),
\end{array}
\]

where the horizontal arrows are the natural isomorphism between the de Rham cohomologies of two Morita equivalent groupoids. \(\square\)

**Remark 5.19** 1. If the groupoid \(\Gamma\) satisfies \(H^2_{dR}(\Gamma, \ast, \mathbb{Z}) = 0\) then Condition (2) in the definition of the strong integrality is satisfied for any pre-Hamiltonian \(\Gamma\)-space. In this case, a pre-Hamiltonian space satisfies the integrality condition if and only if it satisfies the strong integrality condition.
2. If $G$ is a connected and simply-connected Lie group, then $H_2((G \times G)_*, \mathbb{Z}) = 0$. Therefore, any quasi-Hamiltonian $G$-space satisfying the integrality condition must satisfy the strong integrality condition.

The following proposition summarizes the results of this section.

**Proposition 5.20** Let $(\Gamma \rightrightarrows M, \omega + \Omega)$ be an exact proper quasi-presymplectic groupoid, and $(X_1 \rightrightarrows M, \omega_1)_i$, $i = 1, 2$, pre-Hamiltonian $\Gamma$-spaces. Assume that $(X_1 \rightrightarrows M, \omega_1)$ satisfies the integrality condition while $(X_2 \rightrightarrows M, \omega_2)$ satisfies the strong integrality condition. Then there exists a prequantization of $(\Gamma \rightrightarrows M, \omega + \Omega)$ and compatible prequantizations of both $X_1$ and $X_2$. Therefore the classical intertwiner space $\Sigma_2 \times_\Gamma X_1$ is quantizable.

Applying this result to the case of AMM quasi-symplectic groupoid, we have the following

**Corollary 5.21** Let $G$ be a connected and simply-connected compact Lie group equipped with an ad-invariant non-degenerate symmetric bilinear form, and $(X \rightrightarrows G, \omega_X)$ a quasi-Hamiltonian $G$-space. Assume that $\omega_X$ satisfies the integrality condition as in Eq. (38). Then the reduced symplectic manifold $J_0(1)/G$ is prequantizable, and the prequantization can be constructed using the prequantization of the AMM quasi-symplectic groupoid $(G \times G)[U] \rightrightarrows \prod U_i$ (in fact more precisely the pullback groupoid of the AMM quasi-symplectic groupoid) together with a compatible prequantization of the Hamiltonian space $(\prod X|_{U_i} \rightarrow \prod U_i, \omega_X|_{U_i})$, where $U = (U_i)_{i \in 1}$ is some open covering of $G$ such that $\Omega|_{U_i}$, $\forall i$ are all exact.

6 **Appendix**

We denote by $C_{S^1}$ the canonical cycle in $C_1(S^1, \mathbb{Z})$ that generates $H_1(S^1, \mathbb{Z}) = \mathbb{Z}$. If we consider $C_{S^1}$ as an element of $C_2(S^1, \mathbb{Z})$, $[C_{S^1}]$ generates $H_2(S^1, \mathbb{Z}) \simeq \mathbb{Z}$. For any point $p$ in a manifold $N$, we denote by $C_p$ the constant map from $S^1$ to $p$ and consider it as an element of $C_1(N, \mathbb{Z})$. Assume that $R \rightarrow \Gamma \rightrightarrows M$ is an $S^1$-central extension of groupoids. For any $m \in M$, let

$$Z_m = f_m^*(C_{S^1}) \in C_2(R_*, \mathbb{Z}),$$

where $f_m : S^1 \rightarrow R$ is defined by Eq. (12). More generally, for any $r \in R$, let $f_r : S^1 \rightarrow R$ be the map $\lambda \rightarrow \lambda \cdot r$, and set

$$Z_r = f_r^*(C_{S^1}) - C_r \in C_2(R_*, \mathbb{Z}).$$

**Proposition 6.1** Let $R \rightarrow \Gamma \rightrightarrows M$ be an $S^1$-central extension. Assume that $M/\Gamma$ is connected.

1. The class $[Z_m] \in H_2(R_*, \mathbb{Z})$ does not depend on the choice of $m \in M$. For this reason, we will drop the subscript $m$ and denote this class simply by $[Z]$;

2. For any $r \in R$, $Z_r$ is a cycle and $[Z_r] = [Z]$;

3. the natural map $\pi_* : H_2(R_*, \mathbb{Z}) \rightarrow H_2(\Gamma_*, \mathbb{Z})$ is surjective;

4. its kernel $\text{Ker}(\pi_*)$ is generated by $[Z]$;
5. The following identity holds
\[ \int_{\mathcal{Z}_m} \theta = 1 \]

Before we prove this proposition, we need a lemma first. Given any point \( p \in N \), we will denote by \( C_{p(k)} \) the chain in \( C_k(N, \mathbb{Z}) \) defined by the constant path \( \Delta_k \to p \).

**Lemma 6.2** Let \( R \to \Gamma \to M \) be an \( S^1 \)-central extension.

1. Any element \( E \) in \( C_0(R, \mathbb{Z}) \) with \( \pi_* \cdot (E) = 0 \) can be written of the form \( E = \delta D' \), where \( D' \in C_1(R, \mathbb{Z}) \) satisfies \( \pi_* D' = 0 \).

2. \( \pi_* : C_*(R, \mathbb{Z}) \to C_*(\Gamma, \mathbb{Z}) \) is a surjective map.

3. Any element in the kernel of \( \pi_* : H_k(R, \mathbb{Z}) \to H_k(\Gamma, \mathbb{Z}) \) has a representative \( C \in Z_k(R, \mathbb{Z}) \) with \( \pi_* (C) = 0 \).

4. Any element \( C \in C_0(R_2, \mathbb{Z}) \) with \( \pi_* (C) = 0 \) is of the form \( C = dD' \) where \( D' \in C_1(R_2, \mathbb{Z}) \) satisfies \( \pi_* (D') = 0 \).

5. For any cycle \( C' \in C_1(R, \mathbb{Z}) \) such that \( \delta C' = 0 \) and \( \pi_* (C') = 0 \), we have
\[ [C'] = \sum_{i \in I} k_i [Z_{r_i}] \]

for some finite set \( I \), \( k_i \in \mathbb{Z} \), and \( r_i \in R \).

**Proof.**

1) The kernel of \( \pi_* : C_0(R, \mathbb{Z}) \to C_0(\Gamma, \mathbb{Z}) \) is generated by elements of the form \( p - q \), where \( p \) and \( q \) are two points in the same fibre of \( R \to \Gamma \). Hence, it suffices to prove the claim for such a generator.

Let \( D : \Delta_1 \to R \) be a path in the fibre over \( p \) satisfying \( dD = p - q \). Set \( D' = D - C_{p(1)} \). Clearly, the identities \( dD' = p - q \) and \( \pi_* (D') = 0 \) hold. Moreover, from \( \pi_* (D') = 0 \), it follows that \( \partial D' = (\pi_* \cdot \partial) D' = (\partial \pi_*) D' = 0 \).

2) Since the projections \( Q_n \to \Gamma_n \) are surjective submersions, all the maps \( \pi_* : C_i(Q_n, \mathbb{Z}) \to C_i(\Gamma_n, \mathbb{Z}) \) are onto for any \( k, i \in \mathbb{N} \).

3) Let \( C' \in Z_k(R, \mathbb{Z}) \) be a cycle with \( \pi_* (C') = 0 \). By definition, there exists \( D \in C_{k+1}(R, \mathbb{Z}) \) such that \( \partial D = \pi_* (C') \). By 2), there exists \( D' \in C_{k+1}(R, \mathbb{Z}) \) such that \( \pi_* (D') = D \). Set \( C := C' - \delta D' \). Thus we have \( [C] = [C'] \) and \( \pi_* (C) = 0 \).

4) The kernel of \( \pi_* : C_0(R_2, \mathbb{Z}) \to C_0(\Gamma_2, \mathbb{Z}) \) is generated by elements of the form \( p - q \), where \( p \) and \( q \) are two points on the same fibre of \( R_2 \to \Gamma_2 \). It thus suffices to show this property for such generators.

Let \( D : \Delta_1 \to R_2 \) be a path in the fibre over \( p \) such that \( dD = p - q \). Let \( D' = D - C_{p(1)} \). Thus
\[ \pi_* (D') = \pi_* (D - C_{p(1)}) = C_{\pi(p)(1)} - C_{\pi(p)(1)} = 0 \]
and \( dD' = p - q \).

5) For simplicity, we call fibered 1-chain those chains in \( C_1(R, \mathbb{Z}) \) of the form \( C - C_{p(1)} \), where \( p \in R \) is a point and \( C : \Delta_1 \to \pi^{-1} (\pi(p)) \) is a path in the fibre through the point \( p \). Any fibered 1-chain is in the kernel of \( \pi_* \) and hence lies in the kernel of \( \partial \). If a fibered 1-chain \( E \) in a given
fiber satisfies \( dE = 0 \), then \( [E] = k[Z_r] \) for some \( r \in R \) and \( k \in \mathbb{Z} \). As a consequence, if a linear combination \( F \) of fibered 1-chains is a cycle in \( C_1(R, \mathbb{Z}) \), then it is clear that \( [F] = \sum_{i \in I} k_i[Z_{r_i}] \) for some finite set \( I \), \( k_i \in \mathbb{Z} \) and \( r_i \in R \).

Now the kernel of \( \pi_* : C_1(R, \mathbb{Z}) \to C_1(\Gamma, \mathbb{Z}) \) is generated by elements of the form \( C_0 - C_1 \), where \( C_i, i = 0, 1 \) are paths \( \Delta_i \to R \) satisfying \( \pi_*(C_0) = \pi_*(C_1) \). There is a map \( \gamma : \Delta_1 \to S^1 \) such that \( C_0(t) = \gamma(t)C_1(t) \forall t \in \Delta_1 \). Let \( \gamma : [0,1] \times \Delta_1 \to S^1 \) be any map with \( \gamma(0,t) = 1 \) and \( \gamma(1,t) = \gamma(t) \). Let us define two maps \( D_1 \) and \( D : [0,1] \times \Delta_1 \to R \) by \( D_1(s,t) = \gamma(s,t) \cdot C_0(t) \) and \( D(s,t) = C_0(t) \). Set \( D := D_1 - D \). We have \( \pi_*(D) = 0 \) and therefore \( \partial D = 0 \). Moreover, by construction, \( C_0 - C_1 + \delta D = C_0 - C_1 + \delta D \) is the sum of two 1-fibered chains: one in the fiber through \( C_0(0) \) and another in the fiber through \( C_0(1) \). The conclusion thus follows. \( \square \)

**Proof of Proposition 6.1.** 1) and 2). It is clear that if \( m \) and \( n \) are in the same connected component of \( M \), then \( [Z_m] = [Z_n] \). Now by the definition of \( Z_r \), we have

\[
\begin{align*}
dZ_r &= d\left((f_*(C_{S^1})) - dC_r = 0, \\
s_*(Z_r) &= s_*(f_*(C_{S^1}) - C_r) = C_{s(r)} - C_{s(r)} = 0, \text{ and} \\
t_*(Z_r) &= t_*(f_*(C_{S^1}) - C_r) = C_{t(r)} - C_{t(r)} = 0.
\end{align*}
\]

Therefore \( \delta(Z_r) = 0 \). Consider the map \( D : S^1 \to R_2 \) defined by \( \lambda \to (f_*(\lambda), f_*(\lambda)\lambda^{-1}) \). We have \( dD = 0 \) and \( \delta D = f_*(C_{S^1}) - C_r - f_*(C_{S^1}) \). Hence we have \( [Z_r] = [Z_{\gamma}] \), \( \forall \gamma \in R \). Therefore 2) follows.

3). Let \( C \in Z_2(\Gamma, \mathbb{Z}) \) be any 2-cycle. According to Lemma 6.2 (2), there exists \( D \in C_2(R, \mathbb{Z}) \) with \( \pi_*(D) = C \).

In general, \( \delta D \neq 0 \). However since the restriction of \( \pi \) to \( M \) is the identity map, we have \( \partial D_1 - d D_2 = 0 \) and thus \( \delta D = \delta D_0 + \delta D_1 \), where \( D = D_0 + D_1 + D_2, D_i \in C_i(R_{2-i}, \mathbb{Z}) \). Therefore \( \delta D \) is an element of \( C_0(R, \mathbb{Z}) \) and \( \pi_*(\delta D) = \delta \pi_*(D) = \delta C = 0 \). By Lemma 6.2 (1), there exists \( D' \in C_2(R, \mathbb{Z}) \) with \( \pi_*(D') = 0 \) and \( \delta D' = \delta D \). Therefore it follows that \( D - D' \) is a cycle in \( Z_2(R, \mathbb{Z}) \) and

\[
\pi_*([D - D']) = [\pi_*(D)] - [\pi_*(D')] = [C] - [0] = [C].
\]

4) According to Lemma 6.2 (3), any class in \( \text{Ker}(\pi_*) \) has a representative \( C \) such that \( \pi_*(C) = 0 \) and therefore can be written of the form \( C_0 + C_1 \), where \( C_0 \in C_0(R_2, \mathbb{Z}) \) and \( C_1 \in C_1(R, \mathbb{Z}) \) satisfy \( \pi_*(C_0) = 0 \) and \( \pi_*(C_1) = 0 \). According to Lemma 6.2 (4), there exists \( D' \in C_1(R_2, \mathbb{Z}) \) with \( \pi_*(D') = 0 \) such that \( C_0 = dD' \). Consider now \( C' = C - \delta D' \in C_1(R, \mathbb{Z}) \). Thus we have

\[
\delta C' = \delta C - \delta^2 D' = 0, \quad [C'] = [C], \quad \pi_*(C') = 0.
\]

According to Lemma 6.2 (5), we have

\[
[C'] = \sum_{i \in I} k_i[Z_{r_i}]
\]

for some finite set \( I \), \( k_i \in \mathbb{Z} \) and \( r_i \in R \). From Eq. (42), it follows that \( [C] = [C'] = \sum_{i \in I} k_i[Z_{r_i}] \).

By Lemma 6.1 (2), we have \( [C] = (\sum_{i \in I} k_i)[Z] \). \( \square \)
References


