Non–Commutative Heat Kernel

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Abstract

We consider a natural generalisation of the Laplace type operators for the case of non-commutative (Moyal star) product. We demonstrate existence of a power law asymptotic expansion for the heat kernel of such operators on $T^n$. First four coefficients of this expansion are calculated explicitly. We also find an analog of the UV/IR mixing phenomenon when analysing the localised heat kernel.

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1 Introduction

Quantum field theory on non-commutative spaces (see reviews [1]) is a very fast developing topic. Although the heat kernel expansion is an essential ingredient of some earlier approaches to the non-commutative field theory [2] properties of the heat expansion of the operators on non-commutative spaces remain a relatively neglected subject. We can mention a recent work [3] which studied $q$-deformed zeta functions (without a relation to a particular operator, however).

At the same time, the heat kernel expansion [4, 5] is a very powerful instrument of ordinary (commutative) quantum field theory. In particular, coefficients of this expansion define the one-loop counterterms, quantum anomalies, and various expansions of the effective action (e.g., the large mass expansion).

The aim of this paper is to study the asymptotics of the heat kernel for a natural non-commutative generalisation of the Laplace type operator. Roughly speaking, this generalisation is achieved by replacing ordinary product by the

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Moyal star (cf. eqs. (1) and (2) below). Because of the presence of the Moyal star
the operator becomes pseudodifferential with rather unusual oscillatory behaviour
of the symbol at large momenta. Operators of this type do not fall into the
category considered by Seeley [6] (see [7] for a recent review). Therefore, very
little is known about the behaviour of main spectral functions for these operators.
In the next section we show that somewhat surprisingly the heat kernel on a torus
admits a power law asymptotic expansion for small proper time \( t \). The coefficients
of this expansion can be easily calculated. We present explicit expression for first
four coefficients. Our main message is that the heat kernel coefficients for the non-
commutative case are fully defined by the the heat kernel expansion for ordinary
("commutative") but non-abelian operators. In section 3 we analyse the localised
heat kernel and find an analog of the so-called UV/IR mixing phenomenon.

2 Heat trace asymptotics

Let us consider an \( n \)-dimensional torus \( T^n \) with the coordinates \( 0 \leq x^j < 2\pi r_j, \)
\( j = 1, \ldots, n \). The Moyal star product is defined by the equation

\[
f \ast g = f(x) \exp \left( \frac{i}{2} \theta^{\mu \nu} \left\langle \partial_\mu, \partial_\nu \right\rangle g(x) \right)
\]

with some constant antisymmetric matrix \( \theta^{\mu \nu} \). It arises in the context of deformations of flat Poisson manifolds [8].

In this paper we consider a natural generalisation of the Laplace type operators for the non-commutative case:

\[
D \phi = - (\delta^{\mu \nu} \partial_\mu \partial_\nu + a^\mu \partial_\mu + b) \ast \phi.
\] (2)

We shall call the operator (2) the star-Laplacian. This operator acts on sufficiently smooth sections of a vector bundle over \( T^n \). We assume (for technical reasons) that this bundle is trivial. Therefore, one can simply think of \( a^\mu \) and \( b \) as being some matrix valued functions. The coefficient in front of the second derivative term defines a Riemannian metric on \( T^n \) which is the unit one in the present case. Consequently, there is no distinction between upper and lower vector indices. The operator (2) can be represented in the canonical form:

\[
D \phi = - (\delta^{\mu \nu} \nabla_\mu \ast \nabla_\nu + E) \ast \phi,
\] (3)

where

\[
\nabla_\mu = \partial_\mu + \omega_\mu, \quad \omega_\mu = \frac{1}{2} a_\mu, \quad E = b - \partial_\mu \omega_\mu - \omega^\mu \ast \omega_\mu.
\] (4)

It is convenient to introduce the field strength of \( \omega \):

\[
\Omega_{\mu \nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + \omega_\mu \ast \omega_\nu - \omega_\nu \ast \omega_\mu.
\] (5)
The inner product in the space of fields is not sensitive to the non-commutativity parameter:
\[
\langle \psi, \phi \rangle = \int d^n x \psi^\dagger \phi = \int d^n x \psi^\dagger \phi
\]
for smooth $\psi$ and $\phi$. Therefore, the $L^2$ structure is identical for commutative and non-commutative fields. One can easily check that the operator $D$ is hermitian if $E(x)$ is a hermitian matrix, and $\omega_\mu(x)$ is an anti-hermitian matrix at each point.

Let us now turn to the heat kernel which is defined as a functional trace in the space of the square integrable functions on $M$:
\[
K(t, D) = \text{Tr}_{L^2}(\exp(-tD)).
\]
(7)

If $D = D_0$ is a partial differential operator of Laplace type (which is achieved in the limit $\theta \to 0$) the heat kernel is well defined for positive $t$ and there is an asymptotic series as $t \to +0$:
\[
K(t, D_0) \cong \sum_{k \geq 0} t^{(k-n)/2} a_k(D_0).
\]
(8)

If $M$ has no boundaries, odd-numbered coefficients vanish, $a_{2j+1} = 0$. Moreover, the coefficients $a_k$ are locally computable, i.e. they can be presented as integrals over $M$ of some local invariants constructed from $\nabla_\mu$ and $E$. In Quantum Field Theory language this corresponds to locality of the counterterms. We stress, that all these properties hold only in the limit $\theta \to 0$.

For non-zero values of $\theta$ the operator $D$ contains arbitrary large powers of the derivatives (which are contained in the star product). Therefore, $D$ is a pseudodifferential operator (ψdo) rather than a differential one. The study of spectral geometry of ψdo’s was initiated by Seeley [6] (see [7] for an overview). In particular, it was shown that ln $t$ terms can appear in the heat kernel expansion and the heat kernel coefficients become, in general, non-local. However, even these results are not applicable to our case since the symbol of $D$ does not belong to the so-called standard symbol space\(^1\).

It appears, nevertheless, that the heat kernel expansion for $D$ has a rather simple structure and the coefficients can be effectively calculated. To proceed further we need the following definition. We call a functional of $\nabla_\mu$ and $E$ a star-local polynomial functional if it is an integral over $M$ of a finite sum of monomials each consisting of a star product of a finite number of $\nabla_\mu$ and $E$ taken in an arbitrary order. For example, integrals of $E$ and of $\Omega_\mu^{\nu} \Omega_\nu^{\mu}$ are star local polynomial functionals, while that of $E^3$ is not.

\(^1\)Roughly speaking, the symbol $A(x, \xi)$ is obtained from a ψdo $D$ by replacing all partial derivatives by $i\xi$, like in doing the Fourier transform. For the symbols belonging to the standard symbol space $\partial_\xi A(x, \xi) = O((1 + |\xi|^2)^{m-a/2})$, where $m$ is the order of $D$ and $a$ is a multi-index. In other words, each derivative with respect to $\xi$ improves behaviour of $A$ at $\xi \to \infty$. This condition clearly excludes oscillatory behaviour of the symbol.
Now we can formulate our main result. Let $D$ be a star-Laplacian (3) on $T^n$. Then

1. There is a power-law asymptotic expansion (8) of the heat kernel for the operator $D$. The coefficients $a_k(D)$ are star-local polynomial functionals of $\nabla$ and $E$ which have no explicit dependence on $\theta$, i.e. all theta dependence is hidden in the star product.

2. In particular,
\begin{align}
    a_0 &= (4\pi)^{-n/2} \operatorname{tr}(I) \text{ volume } T^n, \\
    a_2 &= (4\pi)^{-n/2} \int d^n x \operatorname{tr}(E), \\
    a_4 &= (4\pi)^{-n/2} \int d^n x \operatorname{tr} \left( (6E + \Omega^\mu_{\nu} \star \Omega_{\mu \nu}) \right), \\
    a_6 &= (4\pi)^{-n/2} \frac{1}{360} \int d^n x \operatorname{tr} \left( 60E \star E + 30E \star E_{;\mu \nu} \\
                           &+ 30E \star \Omega_{\mu \nu} \star \Omega_{\mu \nu} - 4\Omega_{\mu \nu ;\rho} \star \Omega_{\mu \nu ;\rho} + 2\Omega_{\mu \nu ;\rho} \star \Omega_{\mu \nu ;\rho} \\
                           &- 12\Omega_{\mu \nu} \star \Omega_{\mu \nu} \star \Omega_{\mu \nu} \right),
\end{align}

where semicolon denotes covariant differentiation, $E_{;\mu} := \partial_{\mu} E + \omega_{\mu} \star E - E \star \omega_{\mu}$. $\operatorname{tr}$ is the matrix trace.

The proof consists in a rather straightforward evaluation of the asymptotic behaviour of (7) (cf sec. 4.1 of ref. [5]). To calculate the trace in (7) we need a basis in $L^2$ on the torus. Let $\{u^a\}$ be a basis in the “internal” space. Then the functions
\begin{equation}
    \phi_k^a(x) = \frac{u^a e^{ikx}}{(2\pi)^{n/2}(r_1 r_2 \ldots r_n)^{1/2}}
\end{equation}

with $\{\tilde{k}_\mu\} = \{k_\mu r_\mu\} \in \mathbb{Z}^n$ (no summation over $\mu$) form an orthonormal system on the torus. One can represent the heat kernel in the form:
\begin{equation}
    K(t, D) = \int d^n x \sum_a \sum_k \phi_k^a(x) \exp(-tD)\phi_k^a(x).
\end{equation}

Here the integral over $x$ comes from the scalar product (6) needed to calculate diagonal matrix elements, the trace is then taken by summing over $a$ and $k$. As a next step, we expand the exponential in (14) and push all derivatives to the right. Typical monomial obtained in this way reads:
\begin{equation}
    a \star b \star \cdots \star c \star (\partial)^{(\alpha)},
\end{equation}

where $(\alpha)$ is a multi-index. $a$, $b$, $c$ stay instead of $E$ or $\omega$, or their derivatives. Let now the operator (15) act on $e^{ikx}$. One obtains:
\begin{equation}
    e^{ikx} (a \star_k (b \star_k \cdots \star_k (c \star_k 1)) \cdots ) (ik)^{(\alpha)},
\end{equation}
where
\[ f \ast_k g = f(x) \exp \left( \frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \left( \partial_{\nu} + i k_{\nu} \right) \right) g(x), \] (17)

Since this product is not associative we have had to put brackets in (16).

Let us expand \( a, b, c \) in Fourier series: \( a(x) = \sum q^{[i]} \alpha_{q^{[i]}} e^{i q^{[i]} x} \) etc. If the monomial (15) is sandwiched between \( \phi_k^1(x) \) and \( \phi_k(x) \) the exponential \( e^{ikx} \) in (16) is cancelled, and the whole \( x \)-dependence resides in the phase factor:

\[ \exp \left( i \sum q^{[i]} + q^{[i]} + \cdots + q^{[i]} x^\mu \right). \] (18)

If now we integrate over \( x \) as prescribed by (14), we obtain a delta-symbol:

\[ \delta \left( q^{[i]} + q^{[i]} + \cdots + q^{[i]} \right) \] (19)

Next we note that the only effect of the modification (17) of the star product in (16) is the phase factor

\[ \exp \left( -\frac{i}{2} \theta^{\mu\nu} (q^{[i]} + q^{[i]} + \cdots + q^{[i]} k_{\nu}) \right) = 1, \] (20)

where we have used (19). Therefore, we can as well delete the subscript \( k \) in the star products in (16). This technical observation turns out to be very important.

Next we collect again all monomials to an exponent and perform summation over \( a \) in (14). This yields:

\[ K(t, D) = (2\pi)^{-n} \int \frac{d^n x}{r_1 r_2 \cdots r_n} \sum_{k \in \mathbb{Z}^n} \text{tr} \exp \left[ t \left( (\nabla_\mu + i k_\mu) \ast (\nabla_\nu + i k_\nu + E) \right) \right]. \] (21)

We stress again that the star product in (21) does not depend on \( k \). Therefore, the asymptotic behaviour of (21) can be evaluated rather straightforwardly. One has to isolate \( e^{-t k^2} \) and expand the rest of the exponential in a power series. Then one sums over \( k \) by using the formulae:

\[ \sum_{k \in \mathbb{Z}^n} e^{-t k^2} \approx \left( \frac{\pi}{t} \right)^{n/2} (r_1 r_2 \cdots r_n), \]
\[ \sum_{k \in \mathbb{Z}^n} k_\mu k_\nu e^{-t k^2} \approx \delta_{\mu\nu} \frac{1}{2t} \left( \frac{\pi}{t} \right)^{n/2} (r_1 r_2 \cdots r_n), \] (22)
\[ \sum_{k \in \mathbb{Z}^n} k_\mu k_\nu k_\rho k_\sigma e^{-t k^2} \approx \left( \delta_{\mu\nu} \delta_{\rho\sigma} + \delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho} \right) \frac{1}{4t^2} \left( \frac{\pi}{t} \right)^{n/2} (r_1 r_2 \cdots r_n), \]

eq \text{etc. Corrections to (22) are exponentially small if } t \to +0. \text{ Clearly, in this way one obtains a power-law asymptotic expansion (8). If, as usual, one assigns}\n
5
dimension one to $\nabla$ and dimension two to $E$, simple power counting arguments show that each coefficient $a_m$ is a star-local polynomial functional of $E$ and $\nabla$ with an integrand of the dimension $\mu$, $\theta^{\mu \nu}$ never appears explicitly in these calculations. This proves the first assertion formulated above in this section.

To illustrate how the procedure works, consider a simplified case $E = 0$ and calculate the heat kernel coefficients up to $a_4$.

\begin{equation}
K(t, D) = (2\pi)^{-n} \int \frac{d^n x}{r_1 r_2 \ldots r_n} \sum_{k \in \mathbb{Z}^n} \text{tr} e^{-i t^2} \exp \left[ t \left( \nabla_\mu \ast \nabla_\mu + \frac{2i}{t} k_\mu \nabla_\mu \right) \ast \right]. \tag{23}
\end{equation}

Now we expand the second exponential keeping only the terms of dimension four or lower:

\begin{equation}
K(t, D) \cong (2\pi)^{-n} \int \frac{d^n x}{r_1 r_2 \ldots r_n} \sum_{k \in \mathbb{Z}^n} \text{tr} e^{-ik^2} \left[ 1 + t \nabla_\mu \ast \nabla_\mu - 2t^2 (k \nabla) \ast (k \nabla) + \frac{t^2}{2} \nabla_\mu \ast \nabla_\mu \ast \nabla_\nu \ast \nabla_\nu \right. \\
- \frac{2t^3}{3} \left( (k \nabla) \ast (k \nabla) \ast \nabla_\mu \ast \nabla_\mu + \nabla_\mu \ast \nabla_\mu \ast (k \nabla) \ast (k \nabla) \right)
+ \left. (k \nabla) \ast \nabla_\mu \ast \nabla_\mu \ast (k \nabla) + \frac{2t^4}{3} (k \nabla) \ast (k \nabla) \ast (k \nabla) \ast (k \nabla) \right],
\end{equation}

where $(k \nabla) := k_\mu \nabla_\mu$. Next we use (22) to perform summation over $k$. All covariant derivatives combine into commutators yielding the final result:

\begin{equation}
K(t, D) \cong (4\pi t)^{-n/2} \int d^n x \text{tr} \left[ 1 + \frac{t^2}{12} \Omega_{\mu \nu} \ast \Omega^{\mu \nu} + \mathcal{O}(t^3) \right]. \tag{25}
\end{equation}

This result confirms (9) - (11) in the particular case considered. In principle one can go on and compute the rest of (9) - (12). However, this is not needed. The crucial fact is that the calculations go exactly the same way as in the commutative case (cf. sec. 4.1 of [5]). The reason is that even in a theory is commutative, both $\omega$ and $E$ are matrix-valued, and, therefore, commutativity is not being used in the course of the calculations. As a consequence, the heat kernel coefficients (9) - (12) can be read off from the commutative but non-abelian results presented e.g. in [4, 5].

3 Localised heat kernel and UV/IR mixing

In ordinary commutative case the “global” heat kernel (7) is sometimes replaced by a more general (localised) expression

\begin{equation}
K(f; t, D_0) = \text{Tr}_{L^2} \left( f \exp(-t D_0) \right), \tag{26}
\end{equation}
where $f$ is a function. Obviously, $K(t, D_0) = K(1; t, D_0)$. By varying (26) with respect to $f(x)$ one obtains matrix elements of $\exp(-tD_0)x_y$ at coinciding arguments, $x = y$. This modification proves convenient for technical reasons [4]. More important is that (26) describes local quantum anomalies (cf. [5] and references therein).

A natural generalisation of (26) to the noncommutative case reads:

$$K_*(f; t, D) = \text{Tr}_{L^2} (f \star \exp(-tD)) .$$

(27)

Let us consider the $t \to +0$ asymptotic expansion of this expression. Clearly, all methods used in the previous section work also for this case. The only modification is to replace $a$ by $f$ in (15) and (16). Therefore, we conclude that there is an asymptotic expansion

$$K_*(f; t, D) \cong \sum_{k=0,2,4,...} t^{(k-n)/2} a_k(f, D) ,$$

(28)

where the coefficients $a_k(f, D)$ are star-local polynomial functionals without any explicit $\theta$-dependence. Again, commutative non-abelian heat kernel coefficients define uniquely the heat kernel expansion for the noncommutative heat kernel. One only has to remember to take matrix-valued smearing function $f$ in the commutative case in order to preserve all relevant invariants. In particular, by comparing with non-abelian commutative heat kernel coefficients given in [4, 9] one easily derive first three coefficients in the expansion (28):

$$a_0(f, D) = (4\pi)^{-n/2} \text{tr}(f) \text{ volume } T^n ,$$

$$a_2(f, D) = (4\pi)^{-n/2} \int d^n x \text{ tr}(f \star E) ,$$

(29)

$$a_4(f, D) = (4\pi)^{-n/2} \frac{1}{12} \int d^n x \text{ tr} \left[ f \star (2E_{\mu\nu} + 6E \star E + \Omega^{\mu\nu} \star \Omega_{\mu\nu}) \right]$$

Of course, the same expressions may be obtained by more direct methods (cf. equations (23) - (25) above). One can make a simple but important observation that the “zero-momentum” limit $f \to 1$ commutes with the asymptotic expansion in $t$ so that $a_k(D) = a_k(1, D)$.

One can also consider a different generalisation of (26) to the non-commutative case:

$$K(f; t, D) = \text{Tr}_{L^2} (f[\exp(-tD)]) ,$$

(30)

where there is no star between $f$ and the exponent. One has to put additional brackets in (30) since mixed products are not associative.

To evaluate an asymptotic expansion of (30) one can use again the same plane wave basis as in (14), but there is one important difference. Now we cannot

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2One can calculate the heat kernel expansion even if $f$ is a differential operator [9].
replace the modified product $\ast_k$ by $\ast$ under the trace. The reason is that the momentum corresponding to $f$ enters, of course, the momentum conservation delta function (cf. (19)) but does not appear in an analog of the phase factor (20). Therefore, one cannot guarantee existence of the power-law asymptotics, star-locality, or absence of explicit $\theta$-dependence. As we will see in a moment, all these properties are indeed violated.

To illustrate this point let us calculate (30) in the case of zero connection $\omega_\mu = 0$ to the linear order in $E$ neglecting also all derivatives of $E$ which are not coupled directly to $\theta$. In this approximation

$$K(f; t, D) = \int d^n x \sum_{\alpha} \sum_k \phi_k^\alpha(x) f(x) \exp(-t k^2) (t E \ast \phi_k^\alpha(x)) \right), \quad (31)$$

After expanding $f$ and $E$ in Fourier series, summing over $k$, and returning back to the coordinate representation one obtains:

$$K(f; t, D) = \frac{t}{(4\pi t)^{n/2}} \int d^n x \text{tr} \left[ E(x) \exp \left( \frac{\theta^{\mu\nu} \theta^{\rho\sigma} \partial_\mu \partial_\nu}{16 t} \right) f(x) \right] + \ldots \quad (32)$$

where dots denote the higher order terms which we dropped in this calculation.

The differential operator in the exponential in (32) is non-positive. Therefore, for a non-constant $f$ there is a very strong exponential damping at $t \to 0$. At first glance this exponential looks as a regulator of the heat kernel. However, the whole effect disappears for $f = \text{const}$. This is a manifestation of the so-called UV/IR mixing [10] which is a characteristic feature of non-commutative field theories. In the heat-kernel context this mixing is rather a consequence of the way we have made the localisation. The exponential factor in (32) is similar to the typical exponent $\exp(-(x - y)^2 / 4t)$ which appears in matrix elements of the heat kernel of a “commutative” operator $D_0$ between non-coinciding points $x$ and $y$.

### 4 Conclusions

In this paper we considered a natural generalisation of the Laplace type operators for the non-commutative case (the so-called star-Laplacians). We have demonstrated that both global (7) and localised (28) heat kernels for these operators on $T^n$ admit a power-law asymptotic expansion for $t \to +0$. The coefficients of these expansions are star-local functionals of the $\omega$ and $E$ which do not depend explicitly on the non-commutativity parameter $\theta$. They can be easily calculated. Expressions for first several coefficients have been given explicitly. We have also considered a different way (30) to localise the heat kernel an observed an analog of of the UV/IR mixing phenomenon. Most of the results of this paper may be reformulated for $\mathbb{R}^n$ by imposing suitable fall-off conditions on the fields and on the function $f$. 

8
We note that the star product is a natural object in the operator theory which describes composition of symbols of $\psi$do's (see [7, 11]). It has been used in calculations of the effective action in \textit{commutative} field theories [12].

Our results “confirm” in a way the spectral action principle [2]: the $\Omega$-term in $a_4$ (cf. (11)) is just the action for non-commutative Yang-Mills theory.

Our results suggest that the background field formalism and spectral regularization methods (like, e.g., the zeta function regulation) are efficient tools to study divergences in non-commutative theories. Implications for renormalization of non-commutative field theories will be analysed in a separate publication.

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\section*{References}


