Issues related to Rubio de Francia's Littlewood–Paley Inequality: A Survey

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October 24, 2003

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at
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Abstract

Let $S_\omega f = \int_\omega \hat{f}(\xi) e^{i\xi} \, d\xi$ be the Fourier projection operator to an interval $\omega$ in the real line. Rubio de Francia’s Littlewood Paley inequality [31] states that for any collection of disjoint intervals $\Omega$, we have

$$\left\| \left[ \sum_{\omega \in \Omega} |S_\omega f|^2 \right]^{1/2} \right\|_p \lesssim \|f\|_p, \quad 2 \leq p < \infty.$$ 

We survey developments related to this inequality, including the higher dimensional case, and consequences for multipliers.

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*This work has been supported by an NSF grant. These notes were prepared as part of a course presented at the Erwin Schrödinger Institute in Vienna Austria. The author is indebted to the Institute for the opportunity to present those lectures.
1 Introduction

We give a survey of topics related to Rubio de Francia’s extension [31] of the classical Littlewood Paley inequality. We are especially interested in presenting a proof that highlights an approach in the language of time–frequency analysis, and addresses the known higher dimensional versions of this Theorem. It is hoped that this approach will be helpful in conceiving of new versions of these inequalities. These inequalities yield interesting consequence for multipliers, and these are reviewed as well.

Define the Fourier transform by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} \, dx.$$ 

In one dimension, the projection onto the positive frequencies

$$P_+ f(x) := \int_0^\infty \hat{f}(\xi) e^{i\xi x} \, d\xi$$

is a bounded operator on all $L^p(\mathbb{R})$, $1 < p < \infty$. The typical proof of this fact first establishes the $L^p$ inequalities for the Hilbert transform, given by,

$$H f(x) := \lim_{\epsilon \to \infty} \int_{|y| > \epsilon} f(x - y) \frac{dy}{y}$$
The Hilbert transform is given in frequency by a constant times \( \int \hat{f}(\xi) \text{sign}(\xi) e^{i \xi \cdot x} \, d\xi \). And thus \( P_+ \) is linear combination of the identity and \( H \). In particular \( P_+ \) and \( H \) enjoy the same mapping properties.

In this paper, we will take the view that \( L^p(\mathbb{R}^d) \) is the tensor product of \( d \) copies of \( L^p(\mathbb{R}) \). A particular consequence is that the projection onto the positive quadrant

\[
P_+ f(x) := \int_{[0, \infty]^d} f(\xi) e^{i \xi \cdot x} \, d\xi
\]

is a bounded operator on all \( L^p(\mathbb{R}^d) \), as it is merely a tensor product of the one dimensional projections.

A rectangle in \( \mathbb{R}^d \) is denoted by \( \omega \). Define the Fourier restriction operator to be

\[
S_\omega f(x) = \int_{\omega} e^{i \xi \cdot x} \hat{f}(\xi) \, d\xi,
\]

This operator is bounded on all \( L^p(\mathbb{R}^d) \), with constant bounded independently of \( \omega \). To see this, define the modulation operators by

\[
(1.1) \quad \text{Mod}_\xi f(x) := e^{i \xi \cdot x} f(x)
\]

Observe that for \( \xi = (\xi_1, \ldots, \xi_d) \), the interval \( \omega = \prod_{j=1}^d [\xi_j, \infty) \), we have \( S_\omega = \text{Mod}_{-\xi} P_+ \text{Mod}_{\xi} \). Hence this projection is uniformly bounded. And any rectangle is a linear combination of projections of this type.

The Theorem we wish to explain is

**1.2 Theorem.** Let \( \Omega \) be any collection of disjoint rectangles. Then the square function below maps \( L^p(\mathbb{R}^d) \) into itself for \( 2 \leq p < \infty \).

\[
S_\Omega^\Omega f(x) := \left[ \sum_{\omega \in \Omega} |S_\omega f(x)|^2 \right]^{1/2}.
\]

In one dimension this is Rubio de Francia’s Theorem [31]. And his proof pointed to the primacy of a \( BMO \) estimate in the proof of the Theorem. The higher dimensional form was investigated by J.-L. Journé [22]. His original argument has been reshaped by F. Soria [29], S. Sato, [33], and Xue Zhu [35]. In this instance, the product \( BMO \) is essential, in the theory as developed
by S.-Y. Chang and R. Fefferman [7, 8, 15]. A geometric Lemma of Journé [21] concerning dyadic rectangles in the plane plays a decisive role here as well.

We begin our discussion with the one dimensional case, and then move to the higher dimensional case. The same pattern is adopted for the multiplier questions. The paper concludes with notes and comments.

We do not keep track of the value of generic absolute constants, instead using the notation $A \lesssim B$ iff $A \leq KB$ for some constant $K$. And $A \simeq B$ iff $A \lesssim B$ and $B \lesssim A$. For a rectangle $\omega$ and scalar $\lambda$, $\lambda\omega$ denotes the rectangle with the same center as $\omega$ but each side length is $\lambda$ times the same side length of $\omega$. We use the notation $1_A$ to denote the indicator function of the set $A$, and

$$\int_A f \, dx := |A|^{-1} \int_A f \, dx.$$ 

For an operator $T$, $\|T\|_p$ denotes the norm of $T$ as an operator from $L^p(\mathbb{R}^d)$ to itself. In addition to the Modulation operator defined above, we will also use the translation operator

$$\text{Tr}_y f(x) := f(x - y).$$

We shall assume the reader is familiar with the norm bounds for the one dimensional maximal function

$$Mf(x) = \sup \int_{\mathbb{R}} |f(x - y)| \, dt$$

And in particular that it maps $L^p$ into itself for $1 < p < \infty$. In $d$ dimensions, the strong maximal function refers to the maximal function

$$Mf(x) = \sup_{t_1, \ldots, t_d > 0} \int_{[-t_1, t_1] \times \cdots \times [-t_d, t_d]} |f(x_1 - y_1, \ldots, x_d - y_d)| \, dy_1 \cdots dy_d$$

Note that this maximal function is less than the one dimensional maximal function applied in each coordinate in succession.
2 The One Dimensional Argument

In this setting, we give the proof in one dimension, as it is very much easier in this case. In addition, some of the ideas in this case will extend immediately to the higher dimensional case.

2.1 Classical Theory

We should take some care to recall the classical theory of Littlewood and Paley. Let \( \Delta \) denote the dyadic intervals

\[
\Delta := \{ \epsilon[2^k, 2^{k+1}) : \epsilon \in \{\pm 1\}, \ k \in \mathbb{Z} \}.
\]

The classical Theorem is that

2.1 Theorem. For all \( 1 < p < \infty \), we have

\[
\|S^\Delta f\|_p \simeq \|f\|_p
\]

(2.2)

We will not prove this here, but will make comments about the proof. If one knows that

\[
\|S^\Delta f\|_p \lesssim \|f\|_p, \quad 1 < p < \infty
\]

(2.3) then a duality argument permits one to deduce the reverse inequality for \( L^{p'} \) norms, \( p' = p/(p-1) \). Indeed, for \( g \in L^{p'} \), choose \( f \in L^p \) of norm one so that \( \|g\|_{p'} = \langle f, g \rangle \). Then

\[
\|g\|_{p'} = \langle f, g \rangle
\]

(2.4)

\[
= \int \sum_{\omega \in \Delta} S_\omega f \overline{S_\omega g} \, dx
\]

\[
\leq \langle S^\Delta f, S^\Delta g \rangle
\]

\[
\leq \|S^\Delta f\|_p \|S^\Delta g\|_{p'}
\]

\[
\lesssim \|S^\Delta g\|_{p'}
\]
And so one only need prove the upper inequality for the full range of $1 < p < \infty$. In so doing, we are faced with a common problem in the subject. Sharp frequency cutoff produces long range kernels, as is evidenced by the Hilbert transform, which has a single jump in frequency, and a non-integrable kernel. The operator $S^\Delta$ has infinitely many frequency cutoffs. We want to study an operator with smoother frequency behavior, as they are more easily susceptible to the standard singular integral theory. And so it is our purpose to introduce a class of operators which mimic the behavior of $S^\Delta$, but have smoother frequency behavior.

Consider a smooth function $\psi_+$ which satisfies $1_{[1,2]} \leq \tilde{\psi_+} \leq 1_{[2,5/2]}$. Notice that $\psi * f$ is a smooth version of $S_{[1,2]} f$. Let $\psi_- = \tilde{\psi_+}$. Define the dilation operators by

\begin{equation}
\text{Dil}\frac{\partial^p}{\lambda} f(x) := \lambda^{-1/p} f(x/\lambda), \quad 0 < p \leq \infty, \quad \lambda > 0.
\end{equation}

And consider distributions of the form

\begin{equation}
K = \sum_{k \in \mathbb{Z}} \sum_{\sigma \in \{\pm\}} \varepsilon_{k,\sigma} \text{Dil}\frac{\partial^p}{2^k} \psi_{\sigma}, \quad \varepsilon_{k,\sigma} \in \{\pm 1\}.
\end{equation}

and the operators $T f = K * f$. This class of distributions satisfy the standard estimates of Calderón–Zygmund theory, with constants independent of the choices of signs above. In particular, these estimates would be

$$\sup_{\xi} |\hat{K}(\xi)| < C,$$

$$|K(y)| < C|y|^{-1},$$

$$|\frac{d}{dy} K(y)| < C$$

for a universal constant $C$. These inequalities imply that the $L^p$ of $T$ are bounded by constants that depend only on $p$. The uniformity of the constants permits us to average over the choice of signs, and apply the Khintchine inequalities to conclude that

\begin{equation}
\left\| \left[ \sum_{k \in \mathbb{Z}} \sum_{\sigma \in \{\pm\}} \text{Dil}\frac{\partial^p}{2^k} \psi_{\sigma} * f \right]^2 \right\|_p \leq \|f\|_p, \quad 1 < p < \infty.
\end{equation}

This is nearly the upper half of the inequalities in Theorem 1.2. For historical reasons, “smooth” square functions such as the one above, are referred to as “$G$ functions.”
To conclude the Theorem as stated, one method uses an extension of the boundedness of the Hilbert transform to a vector valued setting. The particular form needed concerns the extension of the Hilbert transform to functions taking values in $\ell^q$ spaces. In particular, we have the inequalities

\begin{equation}
\|H f_k\|_{\ell^q} \lesssim C_{\epsilon, q} \|f_k\|_{\ell^p}, \quad 1 < p, q < \infty.
\end{equation}

Vector valued inequalities are strongly linked to weighted inequalities, and one of the standard approaches to these inequalities depends upon the beautiful inequality of C. Fefferman and E.M. Stein [12]

\begin{equation}
\int |H f|^q g \, dx \lesssim \int |f|^q (M|g|^{1+\epsilon})^{1/(1+\epsilon)} \, dx, \quad 1 < q < \infty, \ 0 < \epsilon < 1.
\end{equation}

The implied constant depends only on $q$ and $\epsilon$. While we stated this for the Hilbert transform, it is important for our purposes to further note that this inequality continues to hold for a wide range of Calderón–Zygmund operators, including those that occur in (2.6).

The proof that (2.9) implies (2.8) follows. Note that we need only prove the vector valued estimates for $1 < q \leq p < \infty$, as the remaining estimates follow by duality, namely the dual estimate of $H : L^p(\ell^q) \to L^p(\ell^q)$ is $H : L^p(\ell^q) \to L^p(\ell^q)$, in which the primes denote the conjugate index. The cases of $q = p$ are trivial. For $1 < q < p < \infty$, and $\{f_k\} \in L^p(\ell^q)$ of norm one, it suffices to show that

$$
\left\| \sum_k |H f_k|^q \right\|_{p/q} \lesssim 1.
$$

To do so, by duality, we can take $g \in L^{(p/q)'}$ of norm one, and estimate

$$
\sum_k \int |H f_k|^q g \, dx \lesssim \sum_k \int |f_k|^q (M|g|^{1+\epsilon})^{1/(1+\epsilon)} \, dx
$$

$$
\lesssim \left\| \sum_k |f_k|^q \right\|_{p/q} \left\| (M|g|^{1+\epsilon})^{1/(1+\epsilon)} \right\|_{(p/q)'}
$$

$$
\lesssim 1
$$

provided we take $1 + \epsilon < (p/q)'$. 

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Clearly the estimate (2.8) extends to the projections onto intervals, as such projections are linear combinations of modulations of Hilbert transforms. Namely, we have the estimate

\[ \| S \omega f \omega \|_{L^p(\Omega)} \lesssim \| f \omega \|_{L^p(\Omega)}, \quad 1 < p < \infty. \]

This is valid for all collections of intervals \( \Omega \). Applying it to (2.7), with \( \Omega = \Delta \), and using the fact that \( S_{\sigma[2^k,2^{k+1}]} f = S_{\sigma[2^k,2^{k+1}]} \text{Dil}_{2^k}^{(1)} \psi_\sigma * f \) proves the upper half of the inequalities of Theorem 2.1, which is all that need be done.

For our subsequent use, we note that the vector valued extension of the Hilbert transform depends upon structural estimates that continue to hold for a wide variety of Calderón–Zygmund kernels. In particular, the Littlewood–Paley inequalities also admit a vector valued extension,

\[ \| \| S^\Delta f_k \|_{\ell^q} \|_p \simeq \| \| f_k \|_{\ell^q} \|_p \|, \quad 1 < p, q < \infty. \]

\[ (2.10) \]

### 2.2 Well–Distributed Collections

We begin the main line of argument for Rubio’s inequality in one dimension. Say that a collection of intervals \( \Omega \) is well distributed if

\[ (2.11) \quad \left\| \sum_{\omega \in \Omega} 1_{2^\omega} \right\|_\infty < 100. \]

The well distributed collections allow one to smooth out \( S_\omega \), just as one does \( S_{[1,2]} \) in the proof of the classical Littlewood–Paley inequality. And the main fact we should observe here is that

**2.12 Lemma.** For each collection of intervals \( \Omega \), we can define a well distributed collection \( \text{Well}(\Omega) \) for which

\[ \| S \Omega f \|_p \simeq \| S \text{Well}(\Omega) f \|_p, \quad 1 < p < \infty. \]

We define the collection \( \text{Well}(\Omega) \) by first considering the interval \([ -\frac{1}{2}, \frac{1}{2} ] \). Set

\[ \text{Well}\left( \left[ -\frac{1}{2}, \frac{1}{2} \right] \right) = \left\{ \left[ -\frac{1}{32}, \frac{1}{32} \right], \pm \left[ \frac{1}{2} - \frac{1}{3} \left( \frac{1}{2} \right)^k, \frac{1}{2} - \frac{1}{3} \left( \frac{1}{2} \right)^{k+1} \right] : k \geq 0 \right\}. \]
It is straightforward to check that all the intervals in this collection have a distance to the boundary of $[-\frac{1}{2}, \frac{1}{2}]$ that is four times their length. In particular, this collection is well distributed, and for each $\omega \in \text{Well}([-\frac{1}{2}, \frac{1}{2}])$ we have $2\omega \subset [-\frac{1}{2}, \frac{1}{2}]$.

It is an extension of the usual Littlewood–Paley inequality that

$$ \| S_{[-1/2,1/2]} f \|_p \approx \| S^{\text{Well}([-1/2,1/2])} S_{[-1/2,1/2]} f \|_p, \quad 1 < p < \infty. $$

This inequality continues to hold in the vector valued setting of (2.10). We define $\text{Well}(\omega)$ by affine invariance. For an interval $\omega$, select an affine function $\alpha : [-\frac{1}{2}, \frac{1}{2}] \rightarrow \omega$, we set $\text{Well}(\omega) := \alpha(\text{Well}([-\frac{1}{2}, \frac{1}{2}]))$. And we define $\text{Well}(\Omega) := \bigcup_{\omega \in \Omega} \text{Well}(\omega)$. It is clear that $\text{Well}(\Omega)$ is well distributed for collections of disjoint intervals $\Omega$. By a vector valued Littlewood–Paley inequality, we have

$$ \| S^\Omega f \|_p \approx \| S^{\text{Well}(\Omega)} f \|_p, \quad 1 < p < \infty. $$

And this is the proof of our Lemma.

And to prove the estimate of Lemma 2.12, we need only consider a smooth version of this square function. The assumption of well distributed is critical to boundedness of the smooth operator on $L^2$. Let $\varphi$ be a Schwartz function so that

$$ 1_{[-1/2,1/2]} \leq \hat{\varphi} \leq 1_{[-1,1]} \quad (2.13) $$

Set $\varphi^\omega = \text{Mod}(\omega, \omega_0) \text{Di}_{[\omega, \omega]} \varphi$. And set

$$ G^\Omega f = \left[ \sum_{\omega \in \Omega} |\varphi^\omega * f|^2 \right]^{1/2}. $$

We need only show that

$$ \| G^\Omega f \|_p \lesssim \| f \|_p, \quad 2 \leq p < \infty, \quad (2.14) $$

for well distributed collections $\Omega$. Note that the well distributed assumption and the assumptions about $\varphi$ make the $L^2$ inequality obvious.
2.3 The Tile Operator

We use the previous Lemma to pass to an operator that is easier to control than the projections \(S_\omega\) or \(\varphi \ast f\). And we do so in the time frequency plane. Let \(D\) be the dyadic intervals in \(\mathbb{R}\). That is

\[ D := \{ [j2^k, (j + 1)2^k) : j, k \in \mathbb{Z} \}. \]

Say that \(s = I_s \times \omega_s\) is a tile if \(I_s \in D\), \(\omega_s\) is an interval, and \(1 \leq |s| = |I_s| \cdot |\omega_s| < 2\). Note that for any \(\omega_s\), there is one choice of \(|I_s|\) for which \(I_s \times \omega_s\) will be a tile. We fix a Schwartz function \(\varphi\), and define

\[ \varphi_s := \text{Mod}_{c(\omega_s)} \text{Tr} \left( I_s \right) \mathcal{D}_{\|\cdot\|_I} f, \]

where \(c(J)\) denotes the center of \(J\).

That a tile has area approximately equal to one is suggested by the Fourier uncertainty principle. With this choice of definitions, the function \(\varphi_s\) is approximately localized in the time frequency plane to the rectangle \(I_s \times \omega_s\). This localization is precise in the frequency variable. The function \(\hat{\varphi_s}\) is supported in the interval \(2\omega_s\). But, \(\varphi_s\) is only approximately supported near the interval \(I_s\). But, since \(\varphi\) is rapidly decreasing, we trivially have the estimate

\[ |\varphi_s(x)| \lesssim |I_s|^{-1/2} (1 + |I_s|^{-1} |x - c(I_s)|)^{-N}, \quad N \geq 1. \]

For a collection of intervals \(\Omega\), we set \(T(\Omega)\) to be the set of all possible tiles \(s\) such that \(\omega_s \in \Omega\). Note that for each \(\omega \in \Omega\), the set of intervals \(\{I : I \times \omega \in T(\Omega)\}\) is a a partition of \(\mathbb{R}\) into intervals of equal length. See Figure 1. Associated to \(T(\Omega)\) is a natural square function

\[ T^\Omega f = \left( \sum_{s \in T(\Omega)} \frac{|\langle f, \varphi_s \rangle|^2}{|I_s|} 1_{I_s} \right)^{1/2}. \]

Our main Lemma is that

2.15 Lemma. For any collection of well distributed intervals \(\Omega\), we have

\[ \|T^\Omega f\|_p \lesssim \|f\|_p, \quad 2 \leq p < \infty. \]
Figure 1: Example tiles in a collection $\mathcal{T}(\Omega)$

Let us argue that this Lemma proves (2.14), for a slightly different square function, and so proves Rubio’s Theorem in the one dimensional case. The main task is to pass to a square function of convolution operators. Let

$$\chi(x) := (1 + |x|)^{-10}, \quad \chi(l) = \text{Dil}_{|l|}^{11} \text{Tr}_r(l) \chi,$$

and set for $\omega \in \Omega$,

$$H_\omega f = \sum_{s \in \mathcal{T}(\Omega)} \langle f, \varphi_s \rangle \varphi_s$$

By Cauchy–Schwarz, we may dominate

$$|H_\omega f| \leq \sum_{s \in \mathcal{T}(\Omega)} |\langle f, \varphi_s \rangle | \varphi_s|$$

$$\leq \sum_{s \in \mathcal{T}(\Omega)} \sqrt{\frac{\langle f, \varphi_s \rangle}{|f_s|}} |\chi(l) * 1_{l_s}|^2$$

$$\leq \left[ \sum_{s \in \mathcal{T}(\Omega)} \frac{|\langle f, \varphi_s \rangle|^2}{|f_s|} |\chi(l) * 1_{l_s}| \right]^{1/2}.$$

We took some care to include the convolution in this inequality, so that we could use the easily verified inequality $\int |\chi(l) * f|^2 g \, dx \leq \int |f|^2 \chi(l) * g \, dx$ in the following way. The square function $\|H_\omega f\|_{\ell^2(\Omega)}$ is seen to map $L^p$ into itself, $2 < p < \infty$ by duality. For functions $g \in L^{(p/2)_c}$ of norm one, we can estimate

$$\sum_{s \in \mathcal{T}(\Omega)} \frac{|\langle f, \varphi_s \rangle|^2}{|f_s|} \int |\chi(l) * 1_{l_s}| g \, dx \leq \sum_{s \in \mathcal{T}(\Omega)} \frac{|\langle f, \varphi_s \rangle|^2}{|f_s|} \int 1_{l_s} \chi(l) * g \, dx$$

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\[
\leq \int |T^\Omega f|^2 \sup_t \chi(t) \ast g \, dx \\
\lesssim \|T^\Omega f\|_p^2 \|Mg\|_{(p/2)'} \\
\lesssim \|f\|_p^2.
\]

Here, \((p/2)’\) is the conjugate index to \(p/2\), and \(M\) is the maximal function.

Thus, we have verified that

\[
\|\|H_\omega f\|_{\mathcal{E}(\Omega)}\|_p \lesssim \|f\|_p, \quad 2 < p < \infty.
\]

We now derive a convolution inequality. By Lemma 2.16,

\[
\lim_{T \to \infty} \int_{[0,T]} \text{Tr}_{-y} H_\omega \text{Tr}_y f \, dy = \psi^{\omega} \ast f,
\]

for all \(\omega\), where \(\widehat{\psi^{\omega}} = |\widehat{\varphi^{\omega}}|^2\). Thus, we see that a square function inequality much like that of (2.14) holds, and this completes the proof of Rubio’s Theorem in the one dimensional case, aside from the proof of Lemma 2.15.

**2.16 Lemma.** Let \(\varphi\) and \(\phi\) be real valued Schwartz functions on \(\mathbb{R}\). Then,

\[
\int_{[0,1]} \sum_{m \in \mathbb{Z}} \langle f, \text{Tr}_{y+m} \varphi \rangle \text{Tr}_{y+m} \phi \, dy = f \ast \Phi
\]

where \(\Phi(x) = \int \overline{\varphi(u)} \phi(x + u) \, du\).

In particular, \(\widehat{\Phi} = \overline{\widehat{\varphi} \phi}\).

The proof is immediate. The integral in question is

\[
\iint_{\mathbb{R}} f(z) \overline{\varphi(z - y)} \phi(x - y) \, dydz
\]

and one changes variables, \(u = z - y\).
2.4 Proof of Lemma 2.15

The well distributed assumptions make the estimate on $L^2$ obvious. And so we seek an appropriate endpoint estimate. That of $BMO$ is very useful. Namely for $f \in L^\infty$, we show that

\begin{equation}
\|T^\Omega f\|_{BMO} \lesssim \|f\|_\infty
\end{equation}

Here, by $BMO$ we mean dyadic $BMO$, which has this definition.

\begin{equation}
\|g\|_{BMO} = \sup_{I \in D} \left( \int_I \left| g - \int_I g \right|^2 \, dx \right)^{1/2}.
\end{equation}

The usual definition of $BMO$ is formed by taking a supremum over all intervals. It is a useful simplification for us to restrict the supremum to dyadic intervals. The $L^p$ inequalities for $T^\Omega$ are deduced by an interpolation argument, which we will summarize below.

There is a closely related notion, one that in the one parameter setting coincides with the $BMO$ norm. We distinguish it here. For a map $\alpha : D \to \mathbb{R}$, set

\begin{equation}
\|\alpha\|_{CM} = \sup_{J \in D} |J|^{-1} \sum_{I \subseteq J} |\alpha(I)|.
\end{equation}

“CM” is for Carleson measure. And the inequality (2.17) is in this notation

\begin{equation}
\left\| \left\{ \sum_{s \in T(\Omega)} \langle f, \varphi_s \rangle^2 : J \in D \right\} \right\|_{CM} \lesssim \|f\|_\infty^2.
\end{equation}

Our proof of (2.17) follows a familiar pattern of argument. Fix a function $f$ of $L^\infty$ norm one. We fix a dyadic interval $I$ on which we check the $BMO$ norm. As we subtract off the mean value of $T^\Omega f$ on $I$, all tiles $s \in T(\Omega)$ with $I_s \supset I$ may be disregarded. And it suffices to check that

\[ \int I \sum_{s : I_s \cap t I} |\langle f, \varphi_s \rangle|^2 \lesssim |I|. \]
Truncate $g_0 = f_{1_{2I}}$, and take

$$g_k = f_{1_{2^{k+1}I-2^kI}}, \quad k \geq 1.$$ 

The inequality (2.17) then follows from the estimate below valid for all integers $k$.

\begin{equation}
\int \sum_{s : 1_s \subset I} |\langle g_k, \varphi_s \rangle|^2 \lesssim 2^{-k}|I|.
\end{equation}

The bound

$$\int \sum_{s : 1_s \subset I} |\langle g_0, \varphi_s \rangle|^2 \lesssim \|g_0\|_2^2 \lesssim |I|$$

follows from the $L^2$ inequality that clearly holds.

For the terms corresponding to $g_k$, we should note the following fact. Let $J$ be an interval of length $2^j$ for some integer $j$. Fix $I \in \mathbb{D}$ of length $2^j$. Then, we have the estimate

\begin{equation}
\sum_{s \in \mathcal{T}(\Omega) \backslash \{1_s = I\}} |\langle f_{1_s}, \varphi_s \rangle|^2 \lesssim \|f_{1_s}\|_2 \left(1 + \frac{\text{dist}(I, J)}{|J|}\right)^{-10}.
\end{equation}

To check this, by dilation invariance, we may assume that $J$ has measure one. By invariance under translations by integers, we may assume that the origin is in $I$. The inequality becomes

$$\sum_{n \in \mathbb{Z}} |\langle f_{1_s}, \text{Mod}_{\lambda(n)} \varphi \rangle|^2 \lesssim \|f_{1_s}\|_2^2 \left(1 + \text{dist}(0, J)\right)^{-10}.$$ 

Here, we take $\{\lambda_n : n \in \mathbb{Z}\} \subset \mathbb{R}$, such that for $n \neq n'$ we have $|\lambda_n - \lambda_{n'}| \geq 1/2$. Under this assumption, the exponentials $\{e^{i\lambda_n x} : n \in \mathbb{Z}\}$ are equivalent to an orthonormal basis on intervals of length one, and the rapid decay of the function $\varphi$ provides the decay on the right.

The inequality (2.22) immediately gives for integers $n \geq 1$

$$\sum_{s : 1_s \subset I} \sum_{n \in \mathbb{Z}} |\langle g_k, \varphi_s \rangle|^2 \lesssim 2^{-10n - k}.$$
This is summed over \( n \geq 1 \) to prove (2.21). This completes the proof of the BMO estimate.

Let us indicate how to prove the \( L^p \) inequalities of restricted type without appealing to a more general interpolation Theorem. This fact is based upon a critical property linking the BMO condition with \( L^p \) inequalities, namely the inequality of F. John and L. Nirenberg. We state this lemma in the language of the \( CM \) norm of (2.19).

**2.23 Lemma.** For each \( 1 < p < \infty \), we have the estimate below valid for all dyadic intervals \( J \),

\[
\left\| \sum_{I \subseteq J} \frac{\alpha(I)}{|I|} 1_I \right\|_p \lesssim \|\alpha\|_{CM} |J|^{1/p}.
\]

**Proof.** It suffices to prove the inequality for \( p \) an integer, as the remaining values of \( p \) are available by Hölder’s inequality. The case of \( p = 1 \) is the definition of the Carleson measure norm. Assuming the inequality for \( p \), consider

\[
\int_J \left[ \sum_{I \subseteq J} \frac{\alpha(I)}{|I|} 1_I \right]^{p+1} dx \leq 2 \sum_{J' \subseteq J} \frac{|\alpha(J')|}{|J'|} \int_{J'} \left[ \sum_{I \subseteq J'} \frac{\alpha(I)}{|I|} 1_I \right]^p dx
\]

\[
\lesssim \|\alpha\|_{CM} \sum_{J' \subseteq J} |\alpha(J')|
\]

\[
\lesssim \|\alpha\|_{CM}^{p+1} |J|.
\]

Notice that we are strongly using the grid property of the dyadic intervals, namely that for \( I, J \in D \) we have \( I \cap J \in \{\emptyset, I, J\} \).

We can then prove the restricted type estimate, for \( 2 < p < \infty \), which states that

\[
(2.24) \quad \|T^\Omega 1_F\|_p \lesssim |F|^{1/p}, \quad 2 < p < \infty,
\]

for all sets \( F \subset \mathbb{R} \) of finite measure. The \( L^p \) inequality above is obtained by considering subsets of tiles, \( T \subset T(\Omega) \), for which we will need to the notation

\[
T^\mathcal{T} 1_F := \left[ \sum_{s \in \mathcal{T}} \frac{|1_{F \cap \mathcal{T}}|^2}{|I_s|} 1_{I_s} \right]^{1/2}
\]
As well, take \( \text{sh}(\mathcal{T}) := \bigcup_{s \in \mathcal{T}} I_s \) to be the *shadow of \( \mathcal{T} \).*

The critical step is to decompose \( \mathcal{T}(\Omega) \) into subsets \( \mathcal{T}_k \) for which

\[
\| T^{T_k} 1_F \|_{BMO} \lesssim 2^{-k}, \quad |\text{sh}(\mathcal{T}_k)| \lesssim 2^{2k} |F|, \quad k \geq 1.
\]

Since the \( BMO \) norm is at most a constant, we need only consider \( k \geq 1 \) above. Then, by the John–Nirenberg inequality,

\[
\| T^{T_k} 1_F \|_p \lesssim 2^{-k(1-2/p)} |F|.
\]

This is summable in \( k \) for \( p > 2. \)

The decomposition (2.25) is achieved inductively. Suppose that \( \mathcal{T} \subset \mathcal{T}(\Omega) \) satisfies

\[
\| T^{T} 1_F \|_{BMO} \lesssim \beta
\]

We show how to write it as a union of \( \mathcal{T}_{\text{big}} \) and \( \mathcal{T}_{\text{small}} \) where

\[
|\text{sh}(\mathcal{T}_{\text{big}})| \lesssim \beta^{-2}, \quad \| T^{T_{\text{small}}} 1_F \|_{BMO} \lesssim \beta/2.
\]

The decomposition is achieved in a recursive fashion. Initialize

\[
\mathbf{J} := \emptyset, \quad \mathcal{T}_{\text{big}} := \emptyset, \quad \mathcal{T}_{\text{small}} := \emptyset, \quad \mathcal{T}_{\text{stock}} := \mathcal{T}.
\]

While \( \| T^{T_{\text{stock}}} 1_F \|_{BMO} \geq \beta/2 \), there is a maximal dyadic interval \( J \in \mathbf{D} \) for which

\[
\sum_{I_s \subseteq J} |(1_F, \varphi_s)|^2 \geq \frac{\beta^2}{4} |J|.
\]

Update

\[
\mathbf{J} := \mathbf{J} \cup \{ J \}, \quad \mathcal{T}_{\text{big}} := \mathcal{T}_{\text{big}} \cup \{ s \in \mathcal{T}_{\text{stock}} : I_s \subseteq J \}, \quad \mathcal{T}_{\text{stock}} := \mathcal{T}_{\text{stock}} \setminus \{ s \in \mathcal{T}_{\text{stock}} : I_s \subseteq J \}.
\]

Upon completion of the While loop, update \( \mathcal{T}_{\text{small}} := \mathcal{T}_{\text{stock}} \) and return the values of \( \mathcal{T}_{\text{big}} \) and \( \mathcal{T}_{\text{small}} \).

Observe that by the orthogonality of the functions \( \{ \varphi_s : s \in \mathcal{T}(\Omega) \} \), we have

\[
\beta^2 |\text{sh}(\mathcal{T}_{\text{big}})| \lesssim \beta^2 \sum_{J \in \mathbf{J}} |J|
\]
\[
\lesssim \sum_{x \in T_{\text{lat}}} |\langle f, \varphi_x \rangle|^2 \lesssim |F|.
\]

This completes the proof of (2.25).

3 Rubio in Higher Dimensions

We give the proof of Theorem 1.2 in higher dimensions. The tensor product structure permits us to adapt many of the arguments of the one dimensional case. For instance, one can apply the classical Littlewood Paley inequality in each variable separately. This would yield a particular instance of a Littlewood Paley inequality in higher dimensions. Namely, for all dimensions \(d\),

\begin{equation}
\| S^{\Delta^d} f \|_p \simeq \| f \|_p, \quad 1 < p < \infty,
\end{equation}

where \( \Delta^d = \prod_i \Delta \) is the \(d\)-fold tensor product of the lacunary intervals \(\Delta\), as in Theorem 1.2.

Considerations of this type apply to many of the arguments made in the one dimensional case of Theorem 1.2. In particular the definition of well distributed, and the Lemma 2.12 continues to hold in the higher dimensional setting.

As before, the well distributed assumption permits one to define a “smooth” square function that is clearly bounded on \(L^2\). The definition of this square function, as well as the associated tiles, requires a little more care. For positive quantities \(t = (t_1, \ldots, t_d)\), set

\[
\text{Dil}_{t}^{(p)} f(x_1, \ldots, x_d) = \left[ \prod_{j=1}^{d} t_j^{1/p} \right] f(x_1/t_1, \ldots, x_j/t_j), \quad 0 < p \leq \infty.
\]

For any rectangle \(R\) we write it as a product using the notation \(R = R_{(1)} \times \cdots \times R_{(d)}\). Set

\[
\text{Dil}_{R}^{(p)} = \text{Tr}_{R} \text{Dil}_{(1)}^{(p)} \cdots \text{Dil}_{(d)}^{(p)}
\]

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Figure 2: An example frequency rectangle $\omega$ on the right, and dual dyadic rectangles on left.

For a Schwartz function $\varphi$ on $\mathbb{R}^d$, satisfying

$$1_{[-1/2,1/2]^d} \leq \hat{\varphi} \leq 1_{[-1,1]^d}$$

we set

$$\varphi^\omega = \text{Mod}_{\epsilon}(\omega) \text{ Dil}^{(1)}_{|\omega_{(j)}|^{-1}, \ldots, |\omega_{(n)}|^{-1}} \varphi$$

For a collection of well distributed rectangles $\Omega$, we should show that the inequality (2.14) holds. It will be convenient for us to specify in a little more detail the function $\varphi$ we take. Choose two Schwartz functions $\alpha$ and $\beta$ on $\mathbb{R}^d$ satisfying

$$1_{[-3/5,3/5]^d} \leq \hat{\alpha} \leq 1_{[-4/5,4/5]^d}$$

$$101_{[-1/20,1/20]^d} \leq \hat{\beta} \leq 201_{[-1/10,1/10]^d}, \quad \int_{\mathbb{R}^d} \hat{\beta} \, d\xi = 1$$

In fact, we take $\beta$ to be $\tilde{\beta}(\xi_1, \ldots, \xi_d) = \prod_{j=1}^d \tilde{\beta}(\xi_j)$, where $\tilde{\beta}$ is a Schwartz function on $\mathbb{R}$. Set $\varphi = \alpha \beta$, so that $\hat{\varphi} = \hat{\alpha} \ast \hat{\beta}$.

We substitute the “smooth” convolution square function for a sum over tiles. Say that $R \times \omega$ is a tile if both $\omega$ and $R$ are rectangles and for all $1 \leq j \leq d$, $1 \leq |\omega_{(j)}| |R_{(j)}| < 2$, and $R_{(j)}$ is a dyadic interval. Write $s = R_s \times \omega_s$. As before, let $\mathcal{T}(\Omega)$ be the set of all tiles $s$ such that $\omega_s \in \Omega$.

Define

$$\varphi_s = \text{Mod}_{\epsilon(\omega_s)} \text{ Dil}^{(2)}_{R_s} \varphi$$

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\[ T^\Omega f = \left[ \sum_{s \in T(\Omega)} \frac{|\langle f, \varphi_s \rangle|^2}{|R_s|} 1_{R_s} \right]^{1/2} \]

The main Lemma of the higher dimensional case is the analog of (2.17) and (2.20).

**3.2 Lemma.** We have the inequality

\[ \left\| \left\{ \sum_{R \in \mathbb{D}^d} |\langle f, \varphi_s \rangle|^2 : R \in \mathbb{D}^d \right\} \right\|_{CM} \lesssim \|f\|_\infty^2. \]

Here, the norm \( \|g\|_{CM} \) is defined in (3.4).

Just as in the one dimensional case, this Lemma and the John Nirenberg inequality in Lemma 3.5, imply that \( T^\Omega \) satisfies a restricted type \( L^p \) inequality (2.24), for \( 2 < p < \infty \). Thus, by standard interpolation methods, it maps \( L^p \) into itself for the same range of \( p \)'s. These inequalities imply the inequalities (2.14). The details are omitted.

The Lemma requires us to show that for \( f \in L^\infty(\mathbb{R}^d) \) of norm one, and sets \( U \) of finite measure in \( \mathbb{R}^d \)

\[ \sum_{s \in T(\Omega) \cap R \subset U} |\langle f, \varphi_s \rangle|^2 \lesssim |U|. \]

Using the notations (3.8) and (3.9), set

\[ V_k = \bigcup_{R \subset U} 2^k \text{emb}_d(R)R, \quad k \geq 0 \]

and \( f_0 = f 1_{V_0}, f_k = f(1_{V_k} - 1_{V_{k-1}}) \), for \( k \geq 1 \). Observe that \( |\text{En}_d(U)| \lesssim |U| \), and \( V_k \subset \{ M 1_{\text{En}_d(U)} \geq 2^{-dk} \} \), where \( M \) is the strong maximal function. And so \( |V_k| \lesssim 2^{2dk}|U| \). We should verify that

\[ \sum_{s \in T(\Omega) \cap R \subset U} |\langle f_k, \varphi_s \rangle|^2 \lesssim 2^{-k}|U|, \quad k \geq 0 \]
The well distributed assumption assures us that the functions \( \{ \varphi_s : s \in \mathcal{T}(\Omega) \} \) satisfy a Bessel inequality, and this fact supplies the inequality in the case \( k = 0 \) immediately.

The cases \( k > 0 \) require Lemma 3.10, and in particular Corollary 3.11. Using the notations for that Lemma, we should verify the following. For all dyadic rectangles \( R \subset U \),

\[
\sum_{s \in \mathcal{T}(\Omega) \atop R_s \subseteq R} \langle f_k, \varphi_s \rangle^2 \lesssim \mu^{-1} 2^{-10dk} |R|, \quad k \geq 1,
\]

where \( \mu = \text{emb}_d(R) \). Fix a dyadic rectangle \( R' \subset R \) with \( |R'| = 2^{-n} |R| \). Note that that there are \( \lesssim dn 2^n \) such rectangles \( R' \). Therefore, the inequality above will follow from

\[
\sum_{s \in \mathcal{T}(\Omega) \atop R_s = R'} \langle f_k, \varphi_s \rangle^2 \lesssim \mu^{-1} 2^{-10d(n+k)} |R'|, \quad k \geq 1.
\]

Recall that we have taken \( \varphi = \alpha \beta \), where \( \alpha \) and \( \beta \) are Schwartz functions. It follows from the definition of \( \varphi_s \), that for \( s \in \mathcal{T}(\Omega) \) with \( R_s = R' \), we have

\[
\langle f_k, \varphi_s \rangle = \langle g_{R',k}, \alpha_s \rangle, \quad g_{R',k} = f_k \text{ Dil}^{(\infty)}_{R'} \beta
\]

Moreover, the functions \( \{ \alpha_s : s \in \mathcal{T}(\Omega) \} \) satisfy a Bessel inequality, and so

\[
\sum_{s \in \mathcal{T}(\Omega) \atop R_s = R'} |\langle f_k, \varphi_s \rangle|^2 \lesssim \| g_{R',k} \|_2^2
\]

\[
\lesssim \| \text{Dil}_{R'}^{(\infty)} \beta \|_{L^2(\mathbb{R}^d \setminus V_{k-1})}^2.
\]

Some of the \( R'_{(j)} \subset \neq R_{(j)} \). And in particular, the set

\[
2^{k-1} \mu \prod_{j=1}^d \frac{|R_{(j)}|'}{|R'_{(j)}|'} \frac{|R'_{(j)}|}{|R_{(j)}|}
\]

does not meet \( V_{k-1} \). Recalling that \( \beta \) is a product of one dimensional functions \( \tilde{\beta} \), we see that

\[
\| \text{Dil}_{R'}^{(\infty)} \beta \|_{L^2(\mathbb{R}^d \setminus V_{k-1})} \lesssim \prod_{j=1}^d 2^{k-1} \mu \left( \frac{|R_{(j)}|'}{|R'_{(j)}|'} \right)^{-10d} \sqrt{|R'|}.
\]
This estimate completes the proof of (3.3). Our proof of Theorem 1.2 in the higher dimensional setting is finished aside from a discussion of the Carleson measure condition, and the Journé Lemma, which are taken up below.

3.1 Carleson Measures in the Product Setting

The product Carleson measure applies to maps from the dyadic rectangles $D^d$ of $\mathbb{R}^d$ given by $\alpha : D^d \to \mathbb{R}_+$. This norm is

\begin{equation}
\| \alpha \|_{CM} = \sup_{U \subset \mathbb{R}^d} \left| U \right|^{-1} \sum_{R \subset U} \alpha(R).
\end{equation}

What is most important is that the supremum is taken over all sets $U \subset \mathbb{R}^d$ of finite measure. It would of course be most natural to restrict the supremum to rectangles, and while this is not an adequate definition, it nevertheless plays an important role in the theory. Thus, let $\| \alpha \|_{CM_{\text{rect}}}^p$ the supremum as above, but restricted to a supremum over dyadic rectangles $U$.

Of importance here is the analog of the John–Nirenberg inequality in this setting.

3.5 Lemma. We have the inequality below, valid for all sets $U$ of finite measure.

$$
\left\| \sum_{R \subset U} \frac{\alpha(R)}{|R|} 1_R \right\|_p \lesssim \| \alpha \|_{CM} |U|^{1/p}, \quad 1 < p < \infty.
$$

Proof. Let $\| \alpha \|_{CM} = 1$. Define

$$
F_U := \sum_{R \subset V} \frac{\alpha(R)}{|R|} 1_R
$$

We shall show that for all $U$, there is a set $V$ satisfying $|V| < \frac{1}{2}|U|$ for which

\begin{equation}
\| F_U \|_p \lesssim |U|^{1/p} + \| F_V \|_p
\end{equation}

Clearly, inductive application of this inequality will prove our Lemma.
The argument for (3.6) is by duality. Thus, for a given $1 < p < \infty$, and conjugate index $p'$, take $g \in L^{p'}$ of norm one so that $\|F_U\|_p = \langle F_U, g \rangle$. Set

$$V = \{Mg > K|U|^{-1/p'}\}$$

where $M$ is the strong maximal function and $K$ is sufficiently large so that $|V| < \frac{1}{2}|U|$. Then,

$$\langle F_U, g \rangle = \sum_{R \subset U \cap V} \alpha(R) \int_R g \, dx + \langle F_V, g \rangle$$

The second term is at most $\|F_V\|_p$ by Hölder’s inequality. For the first term, note that the average of $g$ over $R$ can be at most $K|U|^{-1/p'}$. So by the definition of Carleson measure norm, it is at most

$$\sum_{R \subset U \cap V} \alpha(R) \int_R g \, dx \lesssim |U|^{-1/p'} \sum_{R \subset U} \alpha(R) \lesssim |U|^{1/p},$$

as required by (3.6).

These next comments are to explain the relevance of the product Carleson measure condition to product $BMO$, and are not directly relevant to the proof of Theorem 1.2 in higher dimensions. $H^1(\mathbb{C}^d_+)$ denotes the $d$-fold product Hardy space. This space consists of functions $f : \mathbb{R}^d \to \mathbb{C}$. $\mathbb{R}^d$ is viewed as the boundary of

$$\mathbb{C}^d_+ = \prod_{j=1}^d \{z \in \mathbb{C} : \Re(z) > 0\}$$

And we require that $f$ have an extension $F$ to $\mathbb{C}^d_+$ that is holomorphic in each variable separately. The norm of $f$ is taken to be

$$\|f\|_{H^1} = \lim_{y_1 \downarrow 0} \cdots \lim_{y_d \downarrow 0} \|F(x_1 + y_1, \ldots, x_d + y_d)\|_{L^1(\mathbb{R}^d)}$$

The dual of this space is $H^1(\mathbb{C}^d_+)^* = BMO(\mathbb{C}^d_+)$, the $d$-fold product $BMO$ space. It is a Theorem of S.-Y. Chang and R. Fefferman [8] that this space
has a characterization in terms of the product Carleson measure introduced above. We need the product Haar basis. Thus, set

\[ h(x) = -1_{[-\pi, 0]} + 1_{[0, \pi]}, \quad h_I = \text{Di}[^{(2)}_{||}] \text{Tr}_c(I) h, \quad I \in \mathcal{D}. \]

The functions \( \{h_I : I \in \mathcal{D}\} \) are the Haar basis for \( L^2(\mathbb{R}) \), which is closely associated with the analysis of singular integrals. For a rectangle \( R = \prod_{j=1}^d R_{(j)} \in \mathcal{D}^d \) set

\[ h_R(x_1, \ldots, x_d) = \prod_{j=1}^d h_{R_{(j)}}(x_j). \]

The basis \( \{h_R : R \in \mathcal{D}^d\} \) is the \( d \)-fold tensor product of the Haar basis. Then it is the Theorem of Chang and Fefferman that the product \( BMO \) space has the equivalent norm

\[ \|b\|_{\overset{\mathcal{B}}{BMO}} = \sup_{U \subset \mathbb{R}^d} |U|^{-1} \sum_{R \subset U} \langle b, h_R \rangle^2 \]

\[ = \| R \rightarrow \langle b, h_R \rangle^2 \|_{CM} \]

Broad experience has shown that the \( BMO \) space serves as a very useful substitute for \( L^\infty \) in issues related to singular integrals.

Let us briefly indicate the typical formulation of the John Nirenberg inequality. The Haar functions admit their own version of the Littlewood Paley inequalities

\[ \left\| \sum_{I \in \mathcal{D}} a_I h_I \right\|_p \approx \left\| \left[ \sum_{I \in \mathcal{D}} \frac{|a_I|^2}{|I|} 1_I \right]^{1/2} \right\|_p, \quad 1 < p < \infty. \]

Take a function in \( b \in BMO(\mathbb{C}^d) \), and a set \( U \subset \mathbb{R}^d \) of finite measure. Applying the Littlewood Paley inequality above in each coordinate separately and then Lemma 3.5, we see that

\[ \left(3.7\right) \left\| \sum_{R \subset U} \langle b, h_R \rangle h_R \right\|_p \leq \| b \|_{BMO} |U|^{1/p}, \quad 1 < p < \infty. \]

Our use of the term “Carleson measure” is not the standard one. Given a function \( \alpha : \mathcal{D} \rightarrow \mathbb{R}_+ \), define a measure on \( \mathbb{R}^d \times \mathbb{R}_+ \) by

\[ \mu_\alpha = \sum_{R \in \mathcal{D}} \alpha(R) \delta_{R \times \|R\|}. \]

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where \( \| R \| = (|R(1)|, \ldots, |R(d)|) \). In the instance that \( \alpha(R) = |R|^{-1} |\langle f, h_R \rangle|^2 \), the measure \( \mu_\alpha \) is of the type associated with the area integral of \( f \).

For a set \( U \subset \mathbb{R}^d \), define an associated set \( \text{Tent}(U) \subset \mathbb{R}^d \times \mathbb{R}^d_+ \) by
\[
\text{Tent}(U) := \bigcup_{R \in \mathbb{R}^d} R \times [0, |R(1)|] \times \cdots \times [0, |R(d)|]
\]

Then, the substance of the Carleson measure condition is the inequality
\[
\mu_\alpha(\text{Tent}(U)) \leq \| \alpha \|_{CM} |U|,
\]
for all sets \( U \subset \mathbb{R}^d \) of finite measure. Notice that the left hand side concerns objects of \( 2d \) dimensions, while the right hand side has only dimension \( d \).

### 3.2 Journé’s Lemma

The verification of the Carleson measure condition, phrased in terms of arbitrary open sets, is difficult to verify for explicit measures. A geometric Lemma of J.-L. Journé [21] is extraordinarily useful in checking this condition. As we explain below, it permits one to use the much simpler rectangular definition of Carleson measure, at the price of a relatively small blowup in constants.

Let \( U \) be a subset of \( \mathbb{R}^d \) of finite measure. We inductively define a sequence of enlarged sets associated to \( U \) by
\[
(3.8) \quad \text{Enl}_2(U) := \{ M1_U > 1/2 \}, \quad \text{Enl}_{j+1}(U) := \text{Enl}_2(\text{Enl}_j(U)), \ j > 2.
\]

Given a dyadic rectangle \( R \subset U \), we give measures of how deeply embedded this rectangle is inside of \( U \) by
\[
(3.9) \quad \text{emb}_j(R) := \sup \{ \mu \geq 1 : \mu R \subset \text{Enl}_j(U) \}, \ j \geq 2.
\]

**3.10 Lemma.** For all \( \epsilon > 0 \), for all open sets \( U \) of finite measure in \( \mathbb{R}^d \), \( d \geq 2 \), and collections of rectangles \( \mathcal{R} \) of rectangles contained in \( U \), which are pairwise incomparable with respect to inclusion, we have the inequality
\[
\sum_{R \in \mathcal{R}} \text{emb}_{4d-6}(R)^{-\epsilon} |R| \lesssim |\bigcup_{R \in \mathcal{R}} R|.
\]

The implied constant depends only on \( \epsilon \).
Notice that if the rectangles in $R$ were disjoint, then we could take $\epsilon = 0$ above. The heuristics of proofs of Journé’s Lemma is that the dyadic rectangles $R \subset U$ with $\text{emb}_{j}(R) \simeq \mu$ have overlap that is bounded by some power of $\log \mu$.

The next Lemma is intended to explain how this Lemma is typically applied.

**3.11 Corollary.** For all $\epsilon > 0$, $\mu > 1$, and open sets $U$ of finite measure in $\mathbb{R}^d$, let $\mathcal{R}$ be a collection of rectangles with $\text{emb}_{j}(R) \simeq \mu$ for all $R \in \mathcal{R}$. Then, we have the inequality valid for all functions $\alpha : \mathcal{R} \rightarrow \mathbb{R}_+$,

$$
\|\alpha\|_{CM} \lesssim \mu^\epsilon \|\alpha\|_{CM(R_{\epsilon\epsilon})}
$$

Note that in this last inequality we are estimating Carleson measure in terms of rectangular Carleson measure, up to a small factor of $\mu^\epsilon$, with rectangular Carleson measure being an easier quantity to control. The proof of the Corollary is immediate.

**Proof.** The proof is by induction on the dimension $d$, with the base case being $d = 2$, which we consider in detail. The term $\text{emb}_{j}(R)^{-\epsilon}$ can control any number of powers of $\log(\text{emb}_{2}(R))$. So, we may assume that $\mathcal{R}$ is a collection of rectangles that satisfies for all $R, R' \in \mathcal{R},$

\begin{align}
\mu &\leq \text{emb}_{2}(R) < 2\mu, \\
\text{if } |R_j| < |R'_j| \text{ then } 10\mu|R_j| < |R'_j|, &\text{for } j = 1, 2.
\end{align}

Observe that by the first condition, and the definition of embeddedness, the rectangles in $\mathcal{R}$ are necessarily pairwise incomparable with respect to inclusion. For such collections, we shall show that

$$
\sum_{R \in \mathcal{R}} |R| \lesssim |\bigcup_{R \in \mathcal{R}} R|
$$

The main construction of the proof is this inductive procedure. Given a subset $\mathcal{R}'$ of $\mathcal{R}$, we construct a decomposition of $\mathcal{R}'$ into “good” $\mathcal{G}(\mathcal{R}')$ and “bad” $\mathcal{B}_{j}(\mathcal{R}')$ parts, with $j = 1, 2$. Initialize

$$
\text{Stock} := \mathcal{R}', \quad \mathcal{G} = \emptyset, \quad \mathcal{B}_j = \emptyset, \quad j = 1, 2.
$$

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If Stock = ∅ we return \( \mathcal{G}(\mathcal{R}') = \mathcal{G} \), \( \mathcal{B}_j(\mathcal{R}') := \mathcal{B}_j \), for \( j = 1, 2 \).

While Stock is non-empty, select any \( R \in \text{Stock} \), and update

\[
\text{Stock} = \text{Stock} - \{ R \}, \quad \mathcal{G} = \mathcal{G} \cup \{ R \}.
\]

Continuing, for \( j = 1, 2 \), while there is an \( R' \in \text{Stock} \) so that there are \( R_1, R_2, \ldots, R_N \in \mathcal{G} \) such that the \( R_n \) are longer than \( R' \) in the \( j \)th coordinate, and

\[
(3.14) \quad |R' \cap \bigcup_{n=1}^{N} R_n| > \frac{8}{9} |R'|,
\]

update

\[
\text{Stock} = \text{Stock} - \{ R' \}, \quad \mathcal{B}_j = \mathcal{B}_j \cup \{ R' \}.
\]

By construction, it is the case that

\[
\sum_{R \in \mathcal{G}(\mathcal{R})} |R| \leq 9 |\text{sh}(\mathcal{R}^e)|.
\]

We shall argue that for \( j = 1, 2 \), we have

\[
(3.15) \quad \mathcal{B}_j(\mathcal{B}_j(\mathcal{R})) = \emptyset.
\]

And it follows that inductively applying this procedure to each of \( \mathcal{B}_j(\mathcal{G}) \) will terminate after three rounds.

Suppose by way of contradiction, that there is an \( R \in \mathcal{B}_1(\mathcal{B}_1(\mathcal{R})) \). Thus, there are \( R_1, R_2, \ldots, R_N \in \mathcal{B}_1(\mathcal{R}) \) for which each \( R_n \) is longer in the first coordinate and (3.14) holds. Then, suppose that \( R_1 \) has first coordinate \( R_{1(1)} \) that among all the \( R_n \) is shortest in the first coordinate. Since each \( R_n \) is in \( \mathcal{B}_1(\mathcal{R}) \) each of these rectangles are themselves nearly covered by rectangles in \( \mathcal{R} \) that are longer in the first coordinate. By our initial assumption (3.13), these rectangles are themselves much longer than \( R_{1(1)} \). Hence, we take \( I \) to be the dyadic interval of length \( 10\mu|R_{1(1)}| \leq |I| < 20\mu|R_{1(1)}| \) that contains \( R_{1(1)} \). Let \( J \) be the second coordinate of \( R \). Then, it is necessarily the case that

\[
|I \times J \cap \text{sh}(\mathcal{R}^e)| \geq \left( \frac{8}{9} \right)^2 |I \times J|.
\]
But then, $\mathcal{G}(I \times J) \subset \text{Enl}_2(U)$.

By (3.13), $I$ is much larger than $R_1$ in the first coordinate. For the same reason, $J$ is much larger than $R_1$ in the second coordinate. Hence, we see that $3\mu R_1 \subset \mathcal{G}(I \times J)$. But this contradicts (3.12), and so completes the proof of the Lemma.

We now present the inductive stage of the proof. We are deliberately extravagant in the number of compositions of $\text{Enl}_1(U)$ we call, in order to simplify the presentation of the proof. The number of compositions we call is $\beta(d) = 4d - 6$.

Assume that the Lemma holds for $\mathbb{R}^d$, for $d \geq 2$. We demonstrate that it holds on $\mathbb{R}^{d+1}$. Given a set $U \subset \mathbb{R}^{d+1}$, a collection $\mathcal{R}$ of dyadic rectangles contained in $U$, let us first consider a slightly different notion of embeddedness. For a rectangle $R \in \mathcal{R}$, let us write it as a product of a two dimensional rectangle $R_{(2)}$ in the first two coordinates, and a rectangle of $d-1$ dimensions, $R_{(d-1)}$. For each $x_{(d-1)} \in R_{(d-1)}$, we set

$$R_{x_{(d-1)}} = \begin{cases} R_{(d-1)}, & x_{(d-1)} \in R_{(d-1)} \\ \emptyset, & \text{otherwise} \end{cases}$$

$$\bar{\text{enl}}(R; x_{(d-1)}) := \sup \{ \mu \geq 1 : \mu R_{(2)} \times R_{(d-1)} \subset \text{Enl}_2(U) \}$$

Comparing this notion of embeddedness to the one we have considered in detail in the two dimensional case, we certainly see that

$$\sum_{R \in \mathcal{R}} \bar{\text{enl}}(R; x_{(d-1)})^{-\epsilon} |R_{x_{(d-1)}}| \lesssim \left| \bigcup_{R \in \mathcal{R}} R_{x_{(d-1)}} \right|.$$  

(3.16)

For those rectangles $R$ for which it is the case that

$$\left| \left\{ x_{(d-1)} \in R_{(d-1)} : \bar{\text{enl}}(R; x_{(d-1)}) < 100 \text{emb}_{\beta(d+1)}(R) \right\} \right| > \frac{1}{8} |R_{(d-1)}|,$$

then we can integrate the inequality (3.16) to conclude the lemma.

In the next stage of the proof, assume that (3.17) fails for all rectangles $R \in \mathcal{R}$. We will now slice rectangles in the first coordinate. For rectangles
\( R \in \mathcal{R} \), let \( R_{(1)} \) be the projection of \( R \) onto the 1st coordinate. Write
\[ R = R_{(1)} \times R_{(d)}. \]
Our assumption implies that
\[ 50 \text{emb}_{\beta(d+1)}(R) R_{(1)} \times R_{(d)} \subset \text{Enl}_3(U). \]

We move to apply the induction hypothesis. Set
\[ \text{enl}(R; x_{(1)}) := \sup \{ \mu : x_{(1)} \times \mu R_{(d)} \subset \text{Enl}_{\beta(d)}(\text{Enl}_3(U)) \} \]
if \( x_{(1)} \in 50 \text{emb}_{\beta(d)}(R) R_{(1)} \) and set this to be infinity otherwise. It follows
from the induction hypothesis that we have for all \( x_{(1)} \),
\[
\sum_{R \in \mathcal{R}} \text{enl}(R; x_{(1)})^{-\epsilon} |R_{(d)}| \leq \bigcup_{\substack{R \in \mathcal{R} \\text{such that}}} R_{(d)} \bigg|_{x_{(1)} \in 50 \text{emb}_{\beta(d+1)}(R) R_{(1)}}
\]
Following the argument used in (3.17), we observe that if it is the case that
\[
\left| \left\{ x_{(1)} \in 50 \text{emb}_{\beta(d+1)}(R) R_{(1)} : \text{enl}(R; x_{(1)}) < 100 \text{emb}_{\beta(d+1)}(U) \right\} \right|
\geq \frac{1}{8} 50 \text{emb}_{\beta(d+1)}(R) R_{(1)}.
\]
then we may integrate (3.18) to conclude the Lemma.

Consider a rectangle \( R \) for which this last inequality fails. Then it is the case that
\[ 50 \text{emb}_{\beta(d+1)}(R) \geq \text{emb}_{\beta(d)+1}(R). \]
This is a contradiction for \( \beta(d) = 4d - 6 \). The inductive stage of the proof is complete.

\[ \square \]

4 Implications for Multipliers

Let us consider a bounded function \( m \), and define
\[ A_m f(x) := \int m(\xi) \hat{f}(\xi) \xi. \]
This is the multiplier operator given by \( m \), and clearly the \( L^2 \) norm of \( A \) is given by \( \| m \|_\infty \). It is of significant interest to have a description of the the norm of \( A \) as an operator on \( L^p \) only in terms of properties of the function \( m \).

Littlewood–Paley inequalities have implications here, as is recognized through the proof of the classical Marcinciewcz Theorem. Coifman, Rubio de Francia and Semmes [11] found a beautiful extension of this classical Theorem with a proof that is a pleasing application of Rubio’s inequality. We work first in one dimension. To state it, for an interval \([a, b]\), and index \(0 < q < \infty\), we set the \( q \) variation norm of \( m \) on the interval \([a, b]\) to be

\[
\| m \|_{\text{Var}_q ([a, b])} := \sup \left\{ \left[ \sum_{k=1}^{K} \left| m(\xi_{k+1}) - m(\xi_k) \right|^q \right]^{1/q} \right\}
\]

where the supremum is over all finite sequences \( a = \xi_0 < \xi_1 < \xi_2 < \ldots < \xi_{K+1} = b \). Set \( \| v \|_{V_q ([a, b])} := \| m \|_{L^\infty ([a, b])} + \| m \|_{\text{Var}_q ([a, b])} \) Note that if \( q = 1 \), this norm coincides with the classical bounded variation norm.

**4.2 Theorem.** Suppose that \( 1 < p, q < \infty \), satisfying \( \frac{1}{2} - \frac{1}{p} < \frac{1}{q} \). Then for all functions \( m \in L^\infty (\mathbb{R}) \), we have

\[
\| A_m \|_p \lesssim \sup_{I \in \mathcal{D}} \| m \|_{V_q (I)}
\]

Note that the right hand side is a supremum over the Littlewood–Paley intervals \( I \in \mathcal{D} \). The Theorem above is as in the Marcinciewcz Theorem, provided one takes \( q = 1 \). But the Theorem of Coifman, Rubio de Francia and Semmes states that even for the much rougher case of \( q = 2 \), the right hand side is an upper bound for all \( L^p \) operator norms of the multiplier norm \( A_m \). In addition, as \( q \) increases to infinity, the \( V_q \) norms approach that of \( L^\infty \), which is the correct estimate for the multiplier norm at \( p = 2 \).

### 4.1 Proof of Theorem 4.2

The first Lemma in the proof is a transparent display of the usefulness of the Littlewood Paley inequalities in decoupling scales.
4.3 Lemma. Suppose that the multiplier $m$ is of the form $m = \sum_{\omega \in \mathcal{D}} a_\omega 1_\omega$, for a sequence of reals $a_\omega$. Then,

$$\|A_m\|_p \lesssim \|a_\omega\|_{\ell^\infty(\mathcal{D})}, \quad 1 < p < \infty.$$ 

Suppose that for an integer $n$, that $\mathcal{D}_n$ is a partition of $\mathbb{R}$ that refines the partition $\mathcal{D}$, and partitions each $\omega \in \mathcal{D}$ into at most $n$ subintervals. Consider a multiplier of the form

$$m = \sum_{\omega \in \mathcal{D}_n} a_\omega 1_\omega.$$

For $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{4}$, we have

$$(4.4) \quad \|A_m\|_p \lesssim n^{1/2} \|a_\omega\|_{\ell^\infty(\mathcal{D}_n)}.$$ 

Proof. In the first claim, for each $\omega \in \mathcal{D}$, we have $S_\omega A_m = a_\omega S_\omega$, so that for any $f \in L^p$, we have by the Littlewood Paley inequalities

$$\|A_m f\|_p \simeq \|S^\mathcal{D} A_m f\|_p$$

$$\simeq \left\| \left[ \sum_{\omega \in \mathcal{D}} |a_\omega|^2 |S_\omega f|^2 \right]^{1/2} \right\|_p$$

$$\lesssim \|a_\omega\|_{\ell^\infty(\mathcal{D})} \|S^\mathcal{D} f\|_p.$$ 

The proof of (4.4) is by interpolation. Let us presume that $\|a_\omega\|_{\ell^\infty(\mathcal{D}_n)} = 1$. We certainly have $\|A_m\|_2 = 1$. On the other hand, with an eye towards applying the classical Littlewood Paley inequality and Rubio’s extension of it, for each $\omega \in \mathcal{D}$, we have

$$|S_\omega A_m f| \leq n^{1/2} \left[ \sum_{\omega \in \mathcal{D}_n \omega' \subset \omega} |S_{\omega'} f|^2 \right]^{1/2}.$$ 

Therefore, we may estimate for any $1 < r < \infty$,

$$(4.5) \quad \|A_m f\|_r \lesssim \|S^\mathcal{D} A_m f\|_r$$

$$\lesssim n^{1/2} \|S^\mathcal{D}_n f\|_r$$

$$\lesssim n^{1/2} \|f\|_r.$$ 

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To conclude (4.4), let us first note the useful principle that \( \|A_m\|_p = \|A_m\|_{p'} \), where \( p' \) is the conjugate index. So we can take \( p > 2 \). For the choice of \( \frac{1}{2} - \frac{1}{p} < \frac{1}{r} < \frac{1}{q} \), take a value of \( r \) that is very large, in fact

\[
\frac{1}{2} - \frac{1}{p} < \frac{1}{r} < \frac{1}{q}
\]

and interpolate (4.5) with the \( L^2 \) bound.

Since our last inequality is so close in form to the Theorem we wish to prove, the most expedient thing to do is to note a slightly technical lemma about functions in the \( V_q \) class.

**4.6 Lemma.** If \( m \in V_q(I) \) is of norm one, we can choose partitions \( \Pi_j \), \( j \in \mathbb{N} \), of \( I \) into at most \( 2^j \) subintervals and functions \( m_j \) that are measurable with respect to \( \Pi_j \), so that

\[
m = \sum_j m_j, \quad \|m_j\|_\infty \lesssim 2^{-j}.
\]

**Proof.** Let \( P_j = \{(k2^{-j}, (k + 1)2^{-j}) : 0 \leq k < 2^j \} \) the be standard partition of \([0, 1]\) into intervals of length \( 2^{-j} \). Consider the function \( \mu : I = [a, b] \to [0, 1] \) given by

\[
\mu(x) := \|m\|_{V_q([a, x])}.
\]

This function is monotone, non-decreasing, hence has a well defined inverse function. And we take the \( \Pi_j = \mu^{-1}(P_j) \). We define the functions \( m_j \) so that

\[
\sum_{j=1}^j m_j = \sum_{\omega \in \Pi_j} 1_\omega \int_\omega m \, d\xi.
\]

That is, the \( m_j \) are taken to be a martingale difference sequence with respect to the increasing sigma fields \( \Pi_j \). Thus, it is clear that \( m = \sum m_j \). The bound on the \( L^\infty \) norm of the \( m_j \) is easy to deduce from the definitions.

We can prove the Theorem 4.2 as follows. For \( \frac{1}{2} - \frac{1}{p} < \frac{1}{r} < \frac{1}{q} \), and \( m \) such that

\[
\sup_{\omega \in \mathcal{D}} \|m 1_\omega\|_{V_q(\omega)} \leq 1,
\]

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we apply Lemma 4.6 and (4.4) to each \( m \mathbf{1}_\omega \) to conclude that we can write
\[ m = \sum_j m_j, \]
so that \( m_j \) is a multiplier satisfying \( \| A_{m_j} \|_p \lesssim 2^{j/p - j/q} \). But this estimate is summable in \( j \), and so completes the proof of the Theorem.

### 4.2 The Higher Dimensional Form

The extension of the theorem above to higher dimensions was made by Q. Xu [34]. His point of view was to take an inductive and vector valued approach. Some of his ideas were motivated by prior work of G. Pisier and Q. Xu [26, 27] in which interesting applications of \( q \)-variation spaces are made.

The definition of the \( q \) variation in higher dimensions is done inductively. For a function \( m : \mathbb{R}^d \to \mathbb{C} \), define difference operators by

\[
\text{Diff}(m, k, h, x) = m(x + h \epsilon_k) - m(x), \quad 1 \leq k \leq d,
\]
where \( \epsilon_k \) is the \( k \)th coordinate vector. For a rectangle \( R = \prod_{k=1}^d [x_k, x_k + h_k] \), set

\[
\text{Diff}_R(m) = \text{Diff}(m, 1, h_1, x) \cdots \text{Diff}(m, d, h_d, x), \quad x = (x_1, \ldots, x_d).
\]

Define

\[
\| m \|_{\text{Var}_q(Q)} = \sup_{\mathcal{P}} \left[ \sum_{R \in \mathcal{P}} \| \text{Diff}_R(M) \|_q \right]^{1/q}, \quad 0 < q < \infty.
\]

The supremum is formed over all partitions \( \mathcal{P} \) of the rectangle \( Q \) into sub-rectangles.

Given \( 1 \leq k < d \), and \( y = (y_1, \ldots, y_k) \in \mathbb{R}^k \), and a map \( \alpha : \{1, \ldots, k\} \to \{1, \ldots, d\} \), let \( m_{y, \alpha} \) be the function from \( \mathbb{R}^{d-k} \) to \( \mathbb{C} \) obtained from \( m \) by restricting the \( \alpha(j) \)th coordinate to be \( y_j, 1 \leq j \leq k \). Then, the \( V_q(Q) \) norm is

\[
\| m \|_{V_q(Q)} = \| m \|_\infty + \| m \|_{\text{Var}_q(\mathbb{R}^q)} + \sup_{k, \alpha} \sup_{y \in \mathbb{R}^k} \| m_{y, \alpha} \|_{\text{Var}_q(Q, \alpha)}.
\]

Here, we let \( Q_{y, \alpha} \) be the cube obtained from \( Q \) by restricting the \( \alpha(j) \)th coordinate to be \( y_j, 1 \leq j \leq k \).

Recall the notation \( \Delta^d \) for the lacunary intervals in \( d \) dimensions, and in particular (3.1).
4.7 Theorem. Suppose that $|\frac{1}{2} - \frac{1}{p}| < \frac{1}{q}$. For functions $m : \mathbb{R}^d \rightarrow \mathbb{C}$, we have the estimate on the multiplier norm of $m$

$$\|A_m\|_p \lesssim \sup_{R \in \Delta^d} \|m1_R\|_{\mathcal{V}_q(\mathbb{R}^d)}$$

4.3 Proof of Theorem 4.7

The argument is a reprise of that for the one dimensional case. We begin with definitions in one dimension. Let $B$ be a linear space with norm $\| \|_B$. For an interval $I$ let $\mathcal{E}(I, B)$ be the linear space of step functions $m : I \rightarrow B$ with finite range. Thus,

$$m = \sum_{j=1}^J b_j 1_{I_j}$$

for a finite partition $\{I_1, \ldots, I_J\}$ of $I$ into intervals, and a sequence of values $b_j \in B$. If $B = \mathbb{C}$, we write simply $\mathcal{E}(I)$. There are a family of norms that we impose on $\mathcal{E}(I, B)$.

$$\langle m \rangle_{\mathcal{E}(I, B), q} = \left[ \sum_{j=1}^J \|b_j\|_B^q \right]^{1/q}, \quad 1 \leq q \leq \infty.$$

For a rectangle $R = R_1 \times \cdots R_d$, set

$$\mathcal{E}(R) := \mathcal{E}(R_1, \mathcal{E}(R_2, \cdots, \mathcal{E}(R_d, \mathbb{C}) \cdots))$$

The following Lemma is a variant of Lemma 4.3.

4.8 Lemma. Let $m : \mathbb{R}^d \rightarrow \mathbb{C}$ be a function such that $m1_R \in \mathcal{E}(R)$ for all $R \in \Delta^d$. Then, we have these two estimates for the multiplier $A_m$.

$$\|A_m\|_p \lesssim \sup_{R \in \Delta^d} \langle m \rangle_{\mathcal{E}(R), 2}, \quad 1 < p < \infty,$$

$$\|A_m\|_p \lesssim \sup_{R \in \Delta^d} \langle m \rangle_{\mathcal{E}(I, B), q}, \quad 1 < p < \infty, \quad |\frac{1}{2} - \frac{1}{p}| < \frac{1}{q}.$$

Proof. The first claim, the obvious bound at $L^2$, and complex interpolation prove the second claim.
As for the first claim, take a multiplier $m$ for which the right hand side in (4.9) is 1. To each $R \in \Delta^d$, there is a partition $\Omega_R$ of $R$ into a finite number of rectangles so that

$$m 1_R = \sum_{\omega \in \Omega_R} a_\omega 1_\omega,$$

$$\sum_{\omega \in \Omega_R} |a_\omega|^2 \leq 1.$$

This conclusion is obvious for $d = 1$, and induction on dimension will prove it in full generality.

Then observe that by Cauchy-Schwarz,

$$|S_R A_m| \leq S^\Omega_R.$$

Set $\Omega = \bigcup_{R \in \Delta^d} \Omega_R$. Using the Littlewood-Paley inequality (3.1), and Rubio's inequality in $d$ dimensions, we may estimate

$$\|A_m f\|_p \approx \|S^\Delta f A_m f\|_p$$

$$\lesssim \|S^\Omega f\|_p$$

$$\lesssim \|f\|_p.$$

The last step requires that $2 \leq p < \infty$, but the operator norm $\|A_m\|_p$ is invariant under conjugation of $p$, so that we need only consider this range of $p$'s.

\[ \square \]

We extend the notion of $\mathcal{E}(I, B)$. Let $B$ be a Banach space, and set $\mathcal{U}_q(I, B)$ to be the Banach space of functions $m : I \rightarrow B$ for which the norm below is finite.

$$\|m\|_{\mathcal{U}_q} := \inf \left\{ \sum_j \langle m_j \rangle_{\mathcal{E}(I, B), q} : m = \sum_j m_j, m_j \in \mathcal{E}(I, B) \right\}.$$

For a rectangle $R = R_1 \times \cdots \times R_d$, set

$$\mathcal{U}_q(R) := \mathcal{U}_q(R_1, \mathcal{U}_q(R_2, \cdots, \mathcal{U}_q(R_d, \mathbb{C}), \cdots)).$$
By convexity, we clearly have the inequalities
\[ \|A_m\|_p \leq \sup_{R \in \Delta^d} \|m 1_R\|_{\mathcal{U}_q(R)}, \quad 1 < p < \infty, \]
\[ \|A_m\|_p \leq \sup_{R \in \Delta^d} \|m 1_R\|_{\mathcal{U}_q(R)}, \quad 1 < p < \infty, \quad \left| \frac{1}{p} - \frac{1}{q} \right| < \frac{1}{q}. \]

As well, we have the inclusion \( \mathcal{U}_q(R) \subseteq \text{Var}_q(R) \), for \( 1 \leq q < \infty \). The reverse inclusion is not true in general, nevertheless the inclusion is true with a small perturbation of indices.

Let us note that the definition of the \( q \) variation space on an interval, given in (4.1), has an immediate extension to a setting in which the functions \( m \) take values in a Banach space \( B \). Let us denote this space as \( V_q(I, B) \).

**4.11 Lemma.** For all \( 1 \leq p < q < \infty \), all intervals \( I \), and Banach spaces \( B \), we have the inclusion
\[ V_p(I, B) \subseteq \mathcal{U}_q(I, B). \]

For all pairs of intervals \( I, J \), we have
\[ V_q(I \times J, B) = V_q(I, V_q(J, B)). \]

And for all rectangles \( R \), we have
\[ V_p(R) \subseteq \mathcal{U}_q(R). \]

In each instance, the inclusion map is bounded.

The first claim of the Lemma is proved by a trivial modification of the proof of Lemma 4.6. (The martingale convergence theorem holds for all Banach space valued martingales.) The second claim is an easy to verify, and the last claim is a corollary to the first two.

## 5 Notes and Remarks

5.1. L. Carleson [5] first noted the possible extension of the Littlewood Paley inequality, proving in 1967 that Theorem 1.2 holds in the special case that
\[ \Omega = \{ [n, n+1) : n \in \mathbb{Z} \}. \] He also noted that the inequality does not extend to \( 1 < p < 2 \). A corresponding extension to homothetic parallelepipeds was given by A. Córdoba [9], who also pointed out the connection to multipliers.

5.2. Rubio de Francia’s paper [31] adopted an approach that we could outline this way. The reduction to the well distributed case is made, and we have borrowed that line of reasoning from him. This permits to define a “smooth” operator \( G^\Omega \) in (2.14). That \( G^\Omega \) is bounded on \( L^p \), for \( 2 < p < \infty \), is a consequence of a bound on the sharp function. In our notation, that sharp function estimate would be

\[ (G^\Omega f)^1 \lesssim (M|f|^2)^{1/2}. \]

The sharp denotes the function

\[ g^1 = \sup J \left[ \frac{1}{f} \left| g - \int g \right|^2 \right]^{1/2}, \]

the supremum being formed over all intervals \( J \). It is known that \( \| g \|_p \approx \| g^1 \|_p \) for \( 1 < p < \infty \). Notice that our proof can be adapted to prove a dyadic version of (5.3) for the tile operator \( T^\Omega \) if desired. The sharp function estimate has the advantage of quickly giving weighted inequalities. It has the disadvantage of not easily generalizing to higher dimensions. On this point, see R. Fefferman [13].

5.4. The weighted version of Rubio’s inequality states that for all weights \( w \in A^1 \), one has

\[ \int |S^\Omega f|^2 w \, dx \lesssim \int |f|^2 w \, dx. \]

There is a similar conclusion for multipliers.

\[ \int A_m f|^2 w \, dx \lesssim \sup_{R \in \Delta^a} \| m 1_R \|_{L^2(R)} \int |f|^2 w \, dx. \]

See Coifman, Rubio de Francia, and Semmes [11], for one dimension and Q. Xu [34] for dimensions greater than 1.

5.5. Among those authors who made a contribution to Rubio’s one dimensional inequality, P. Sjölin [30] provided an alternate derivation of Rubio’s
sharp function estimate (5.3). In another direction, observe that (2.4), Rubio’s inequality has the dual formulation \( \|f\|_p \lesssim \|S^\Omega f\|_p \), for \( 1 < p < 2 \). J. Bourgain [6] established a dual endpoint estimate for the unit circle:

\[
\|f\|_{H^1} \lesssim \|S^\Omega f\|_1
\]

This inequality in higher dimensions seems to be open.

5.6. Rubio’s inequality does not extend below \( L^2 \). While this is suggested by the duality estimates, an explicit example is given in one dimension by \( f = 1_{[0,N]} \), for a large integer \( N \), and \( \Omega = \{(n,n+1) : n \in \mathbb{Z}\} \). Then, it is easy to see that

\[
N^{1/2} 1_{[0,1]} \lesssim S^\Omega f, \quad \|f\|_p \simeq N^{p/(p-1)}, \quad 1 < p < \infty.
\]

It follows that \( 1 < p < 2 \) is not permitted in Rubio’s inequality.

5.7. In considering an estimate below \( L^2 \), in any dimension, we have the following interpolation argument available to us for all well distributed collections \( \Omega \). We have the estimate

\[
(5.8) \quad \sup_{s \in \mathbb{T}^\Omega} 1_{R_s} \frac{\langle f, \varphi_s \rangle}{\sqrt{|R_s|}} \lesssim M f
\]

where \( M \) denotes the strong maximal function. Thus, the right hand side is a bounded operator on all \( L^p \). By taking a value of \( p \) very close to one, and interpolating with the \( L^2 \) bound for \( S^\Omega \), we see that

\[
(5.9) \quad \left\| 1_{R_s} \frac{\langle f, \varphi_s \rangle}{\sqrt{|R_s|}} \right\|_{L^p(\Omega)} \lesssim \|f\|_p, \quad 1 < p < 2, \quad \frac{p}{p-1} < q < \infty.
\]

By (2.10), this inequality continues to hold without the well distributed assumption. Namely, using (2.8), for all disjoint collections of rectangles \( \Omega \),

\[
\|S_\omega f\|_{L^q(\Omega)} \lesssim \|f\|_p \quad 1 < p < 2, \quad \frac{p}{p-1} < q < \infty.
\]

5.10. Observe that (5.8) has the endpoint estimate, in one dimension, of

\[
\left\| 1_{R_s} \frac{\langle f, \varphi_s \rangle}{\sqrt{|R_s|}} \right\|_{L^\infty(\Omega)} \lesssim \|f\|_1
\]

The possibility of interpolating this weak \( L^1, \ell^\infty \) valued estimate with the \( L^2 \) estimate arises. This would in particular imply that (5.9) holds for \( q \) equal to the conjugate index \( p/(p-1) \). But this is not in general possible, due to an example of D. Müller and C. Thiele (personal communication).
5.11. Despite the fact that the general interpolation argument is not available, (5.9) may nevertheless hold for $q = p/(p - 1)$. M. Cowling and T. Tao [10] have constructed an example which shows that this is not the case.

5.12. The higher dimensional formulation of Rubio’s inequality did not admit an immediately clear formulation. J.-L. Journé [22] established the Theorem in the higher dimensional case, but used a very sophisticated proof. Simpler arguments, very close in spirit to what we have presented, were given by F. Soria [29] in two dimensions, and in higher dimensions by S. Sato [33] and X. Zhu [35]. These authors continued to focus on the $G$ function (2.14), instead of the time frequency approach we have used.

5.13. We should mention that if one is considering the higher dimensional version of Theorem 1.2, with the simplification that the collection of rectangles consists only of cubes, then the method of proof need not invoke the difficulties of the $BMO$ theory of Chang and Fefferman. The usual one parameter $BMO$ theory will suffice. The same comment holds if all the rectangles in $\Omega$ are homeothetic under translations and application of a power of a fixed expanding matrix.

5.14. It would be of interest to establish variants of Rubio’s inequality for other collections of sets in the plane. A. Córdoba has established a preliminary result in this direction for finite numbers of sectors in the plane.

5.15. The inequality (2.8) is now typically seen as a consequence of the general theory of weighted inequalities. In particular, if $h \in L^1(\mathbb{R})$, and $\epsilon > 0$, it is the case that $(Mh)^{1-\epsilon}$ is a weight in the Mockenhoupt class $A^1$. In particular, this observation implies (2.8). See the material on weighted inequalities in E.M. Stein [32].

5.16. The proof of Lemma 3.5 is due to Chang and Fefferman [8].

5.17. Critical to the proof of Rubio’s inequality is the $L^2$ boundedness of the tile operator $T^\Omega$. This is of course an immediate consequence of the well distributed assumption. It would be of some interest to establish a reasonable geometric condition on the tiles which would be sufficient for the $L^2$ boundedness of the operator $T^\Omega$. In this regard, one should consult the inequality of J. Barrionuevo and the author [1]. This inequality therein is of a weak type, but is sharp.
5.18. V. Olevskii [25] independently established a version of Theorem 4.2 on the integers.

5.19. Observe that, in a certain sense, the multiplier result Theorem 4.2 is optimal. In one dimension, let $\psi$ denote a smooth bump function $\psi$ with frequency support in $[-1/2, 1/2]$, and for random choices of signs $\varepsilon_k$, $k \geq 1$, and integer $N$, consider the multiplier

$$m = \sum_{k=1}^{N} \varepsilon_k \text{Tr}_k \widehat{\psi}$$

Apply this multiplier to the function $\hat{f} = 1_{\mathcal{P}, N}$. By an application of Khintchine’s inequality,

$$\mathbb{E}\|A_m f\|_{p} \simeq \sqrt{N}, \quad 1 < p < \infty.$$ 

On the other hand, it is straightforward to verify that $\|f\|_{q} \simeq N^{1/(q-1)}$. We conclude that

$$\|A_m\|_{p} \gtrsim N^{1/(q-1)}.$$ 

Clearly $\mathbb{E}\|m\|_{V_q} \simeq N^{1/q}$. That is, up to arbitrarily small constant, the values of $q$ permitted in Theorem 4.2 are optimal.

5.20. V. Olevskii [23] refines the notion in which Theorem 4.2 is optimal. The argument is phrased in terms of multipliers for $\ell^p(\mathbb{Z})$. It is evident that the $q$ variation norm is preserved under homeomorphisms. That is let $\phi : \mathbb{T} \to \mathbb{T}$ be a homeomorphism. Then $\|m\|_{V_q(\mathbb{T})} = \|m \circ \phi\|_{V_q(\mathbb{T})}$. For a multiplier $m : \mathbb{T} \to \mathbb{R}$, let

$$\|m\|_{M_p^q} = \sup_{\phi} \sup_{\|f\|_{\ell^p(\mathbb{Z})} = 1} \left\| \int_{\mathbb{T}} \hat{f}(\tau) m \circ \phi(\tau) e^{i\tau} d\tau \right\|_p$$

Thus, $M_p^q$ is the supremum over multiplier norms of $m \circ \phi$, for all homeomorphisms $\phi$. Then, Olevskii shows that if $\|m\|_{M_p^q} < \infty$, then $m$ has finite $q$ variation for all $|\frac{1}{2} - \frac{1}{p}| < \frac{1}{q}$.

5.21. E. Berkson and T. Gillespie [2-4] have extended the Coifman, Rubio, Semmes result to a setting in which one has an operator with an appropriate spectral representation.
5.22. The Rubio de Francia inequalities are in only one direction. K.E. Hare and I. Klemeš [17–19] have undertaken a somewhat general study of necessary and sufficient conditions on a class of intervals to satisfy a the inequality that is reverse to that of Rubio. A theorem from [19] concerns a collection of intervals $\Omega = \{\omega_j : j \in \mathbb{Z}\}$ which are assumed to partition $\mathbb{R}$, and satisfy $|\omega_{j+1}|/|\omega_j| \to \infty$. Then one has
\[ \|f\|_p \lesssim \|\sum_{\Omega} f\|_p, \quad 2 < p < \infty. \]

What is important here is that the locations of the $\omega_j$ are not specified. The authors conjecture that it is enough to have $|\omega_{j+1}|/|\omega_j| > \lambda > 1$.

5.23. It might be of interest to establish versions of the Coifman Rubio Semmes Theorem for non convolution operators. In so doing, one would want to use the time frequency square functions of this paper.

5.24. The proof of Lemma 3.10 in the two dimensional case is an easier form of the appendix of a paper by S. Ferguson and M. Lacey [16]. The higher dimensional proof arose from discussions with C. Cabrelli and U. Molter.

5.25. Journé’s Lemma is typically stated with a larger version of “embeddedness” than the one that we used. In Journé’s original paper, that measure is
\[ \mu(R_{(1)} \times R_{(2)}) := \sup \{ \mu \geq 1 : (\mu R_{(1)}) \times R_{(2)} \subset \text{Emb}_2(U) \}. \]

The quantity we use $\text{emb}_2(R)$ is always smaller than $\mu(R)$. One may construct examples in which this quantity $\mu(R)$ is in many cases very much bigger than $\text{emb}_2(R)$. It is of course of some interest to have formulations of Journé’s Lemma in which the measure of embeddedness is as small as possible.

5.26. In a similar vein, in higher dimensions, a statement of Journé’s Lemma may use a notion of embeddedness of this type. For simplicity, in three dimensions, set
\[ \mu_1(R) := \sup \{ \mu_1 : \mu_1 R_{(1)} \times R_{(2)} \times R_{(3)} \subset \text{Emb}_3(U) \}. \]

Here, the rectangle $R$ is a product of the intervals $R_{(j)}$, and there is a similar definition of $\mu_2(R)$. Then, the measure of embeddedness is $\mu_1(R)\mu_2(R)$. See J. Pipher [28]. Again, ignoring the issue of exactly which set $R$ is embedded in, our measure of embeddedness $\text{Emb}_3(R)$ is always smaller, and one can construct examples in which it is very much smaller for many rectangles $R$. 

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5.27. There is also some interest in have the set $\text{En}_2(U)$ be, in some sense, as small as possible. Indeed, given $\delta > 0$, one can choose $\text{En}_2(U)$ so that it has measure at most $< (1 + \delta)|U|$, and Journé’s Lemma still holds. See the appendix of S. Ferguson and M. Lacey [16].

6 References


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