Second Order Periodic Differential Operators. 
Threshold Properties and Homogenization

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THRESHOLD PROPERTIES AND HOMOGENIZATION

M. Sh. Birman, T. A. Suslina

Abstract. In $L_2(\mathbb{R}^d)$, we consider vector periodic differential operators (DO's) $A$ admitting a factorization $A = X \ast X$, where $X$ is a first order homogeneous DO. Many operators of mathematical physics have this form. The effects that depend only on a rough behavior of the spectral decomposition of $A$ in a small neighborhood of zero are called threshold effects at the point $\lambda = 0$. An example of a threshold effect is the behavior of a DO in the small period limit (the homogenization effect). Another example is related to the negative discrete spectrum of the operator $A = a V$, $a > 0$, where $V(x) \geq 0$ and $V(x) \to 0$ as $|x| \to \infty$. The "effective characteristic", namely, the homogenized medium, the effective mass and Hamiltonian, etc., arise in these problems. We propose a general approach to these problems based on the spectral perturbation theory for operator-valued functions admitting analytic factorization. A great deal of considerations is fulfilled in abstract terms. As for applications, the main attention is paid to the homogenization of DO's.

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\[ \text{§0. Introduction} \]

\[ \text{6.1. Periodic differential operators (DO's), acting on } \mathbb{R}^d, d \geq 1, \text{ can be partially diagonalized with the help of the Gelfand transformation (cf. Chapter 2). Then, the initial DO is represented as the direct integral of family of DO's, each of the latter acts on the cell of periods } \Omega \text{ glued in a torus. This family depends on the parameter } k \in \mathbb{R}^d \text{ (called the quasi-momentum). We consider the semi-bounded selfadjoint DO's.} \]

Further analysis (the Floquet-Bloch decomposition) is possible if an operator acting in } L_2(\Omega) \text{ has compact resolvent that depends on } k \text{ continuously. This condition is valid for many DO's of mathematical physics. Then, the spectrum of the initial DO has a band structure. It is convenient to assume that the lower edge of the spectrum coincides with the point } \lambda = 0. \text{ The solution of some questions may happen to require the knowledge of the approximate spectral expansion of the initial DO only near the lower edge of the spectrum. In such cases we talk about the threshold effects at the point } \lambda = 0. \text{ Threshold effects may be related to the edges of internal spectral gaps, but we shall speak only about the threshold } \lambda = 0. \text{ Usually, it is not an easy task to detect whether or not a given effect is threshold. For instance, we mention a question about the discrete spectrum to the left of the point } \lambda = 0 \text{ for a periodic DO perturbed by a negative potential vanishing at infinity. If the potential does not decay too rapidly, then the threshold effect dominates; to the contrary, if a perturbation is rapidly decaying, then the high-energy part of the initial DO that corresponds to large values of } \lambda \text{ plays the main role. Another} \]
important example of the threshold effect at the point \( \lambda = 0 \) is the behavior of a
periodic DO in \( L_2(\mathbb{R}^d) \) in the small period limit. Below, we pay the main attention
to this problem.

6.2. One of our goals is to give a description, concise and convenient for applica-
tions, of the spectral characteristics of periodic DO’s near the threshold \( \lambda = 0 \).
By using partial diagonalization, we reduce the problem to some questions of the
perturbation theory for the discrete spectrum. The difficulties are related to the
fact that usually the unperturbed eigenvalue is \textit{multiple} and the parameter \( k \) is
\textit{multidimensional}. Such cases cannot be treated by means of the classical pertur-
bation theory, and one should look for a roundabout way. (If at least one of the
two above reasons does not occur, the problem simplifies.) We take \( t = |k| \) as the
perturbation parameter; at the same time, we have to make our constructions and
estimates uniform in \( \theta = t^{-1}k \).

We start with an abstract operator theory method (cf. Chapter 1), distin-
guishing the case where the family in question admits an \textit{analytic factorization}. Taking
account of this additional structure allows us to advance unexpectedly far in ab-
tract terms. Often, the operators in applications admit the required factorization
from the very beginning. In some other cases we can introduce the factorization
into the problem forcedly. If the operator does not admit the required factoriza-
tion, it is much more difficult to study the threshold properties, and it is useful to
understand this clearly.

Keeping in mind the applications to (vector) second order DO’s, in the abstract
method, we restrict ourselves to the \textit{quadratic} dependence on the parameter. The
crucial place for us is to distinguish and study the notion of the \textit{spectral germ of an oper-
ator family at the point} \( t = 0 \) (cf. §1.1). In general, we think that
an effect should be classified as threshold one if it can be described in terms of
the corresponding spectral germ. In the abstract form, the threshold effects are
considered in §1.5. For instance, the estimate (1.5.10) gives an abstract solution of
the question of the behavior of periodic DO’s in the small period limit. To a great
extent, the paper is devoted to realization of this approach as applied to DO’s.
Herewith, the specific properties of DO’s are not employed much.

6.3. The threshold effects bring about the so called \textit{effective characteristics}. We
believe that the mechanism of their appearance is the following. Since a threshold
effect is determined only by the spectral germ, the initial periodic DO can be re-
placed (in the description of this effect) by any other periodic DO with the same
spectral germ. Among these "equivalent" DO’s, there may be simple operators,
often, DO’s with constant coefficients. That is why the notions of effective mass,
effective Hamiltonian, etc., appear in the quantum mechanics problems, and the
notions of effective (homenized) medium, homenized DO appear in the problems
concerning periodic structures with vanishing period. Advantages of the description
via the effective characteristics are obvious. Usually, the very idea of the existence
of effective characteristics has a physical origin. At the same time, when using the
effective characteristics, one should remember that, in fact, they contain only the
information about the spectral germ, moreover, in a disguised form.

6.4. In Chapter 2, we distinguish a comparatively wide class of elliptic periodic
second order DO’s acting in \( L_2(\mathbb{R}^d; \mathbb{C}) \). This class includes a number of operators
of mathematical physics, though it does not satisfy all needs of applications. Under
the Gelfand transformation, each operator $A$ of this class generates an operator family $A(t, \Theta)$, $\Theta = k$, in $L^2(\Omega; \mathbb{C}^n)$, which admits a factorization analytic in $t = |k|$. Now, the spectral germ $S(\Theta)$ for $A(t, \Theta)$ at $t = 0$ depends on the parameter $\Theta$; this is essential for further considerations.

In Chapter 3, on the basis of general results of Chapter 1, we introduce the \textit{effective characteristics} for each operator $A$. They are defined directly via the corresponding germ $S(\Theta)$. In Chapter 4, it is shown that these characteristics are responsible for the homogenization procedure for DO in the small period limit.

6.5. In our days, the study of periodic problems with rapidly varying medium parameters (with small period) is a broad field of theoretical and applied science. There is a variety of methods specific for this field, and a lot of significant results. The limit procedures have been studied for the boundary value problems in bounded domains; methods have been developed for constructing full asymptotic (with respect to the small period) expansions; the homogenization procedures have been analyzed for non-selfadjoint operators, for non-stationary operators, for nonlinear problems. A lot of surveys and monographs is devoted to these topics. It was especially useful for us to acquaint with the remarkable books [BaPa, BeLP, ZhKO]. We are far from the idea to revise all this large area by our methods. At the same time, we would like to attract the reader’s attention to the following.

6.6. In the homogenization theory, finding the effective characteristics is the initial problem, which is usually solved by direct methods. Even in the cases where the Floquet-Bloch decomposition and the analytic perturbation theory (usually, for a simple eigenvalue) are employed for this purpose, these methods are viewed as purely technical and are used in no relation to the general notion of a threshold effect. The essence of our approach is distinguishing the notion of the spectral germ at $\lambda = 0$. The germ directly determines the effective characteristics, which inevitably appear in any specific threshold effect. The homogenization in the small period limit is one of such effects, another one corresponds to the problem about the negative discrete spectrum mentioned above. In Chapters 5–7, we illustrate our unified approach by the traditional operators of mathematical physics. Many facts distinguished there are new for the corresponding concrete operators, though they follow from the general scheme of Chapters 1–4 almost directly. Apparently, the possibilities of our approach are not exhausted by the given applications.

6.7. Formally speaking, the present text is written as a survey. This means more care about the reader’s interests than it is usual in the ordinary articles. In this connection, we note that what was said in the Introduction is supplemented by the adjusting §4.1 and §7.1, comments on Chapters 4–7 and concluding remarks. There one can find most of references and comparisons.

We clearly understand that the present text is in no way a survey on the homogenization theory. This is impossible not only because of the "quantitative" reasons, but also because the authors are not specialists in the homogenization theory. However, pondering the spectral effects near the thresholds during a number of years, we realized that the homogenization is one of the more pronounced threshold effects. Thus, the proposed text is a sequential presentation of our point of view on the homogenization as a threshold effect. Of course, this view of the problem was used by other authors; we mention, e. g., the articles [Zh1] and [Se]. However, we hold the "threshold" point of view more sequentially and we use the operator theory approach as the basis for our considerations.
The authors' paper [BSu2] is the initial version of the proposed exposition. As compared with [BSu2], we refined the abstract part, bridged a number of technical gaps and extended the collection of applications. In the first place, the latter concerns Chapter 7. The structure of the paper is clear from the table of contents.

6.8. Notation. Let \( \mathcal{H} \) and \( \mathcal{G} \) be two separable Hilbert spaces. The symbols \( \langle \cdot , \cdot \rangle_{\mathcal{H}} \) and \( \| \cdot \|_{\mathcal{H}} \) stand for the inner product and the norm in \( \mathcal{H} \); the symbol \( \| \cdot \|_{\mathcal{H} \to \mathcal{G}} \) stands for the norm of a bounded operator from \( \mathcal{H} \) to \( \mathcal{G} \). If \( \mathcal{H} = \mathcal{G} \), we may write only one index in the notation of the operator norm. Sometimes we omit the indices, if this does not lead to confusion. Let \( I = I_{\mathcal{H}} \) denote the unit operator in \( \mathcal{H} \). If \( \mathcal{N} \) is a subspace in \( \mathcal{H} \), then \( \mathcal{N}^\perp := \mathcal{H} \ominus \mathcal{N} \). If \( P \) is the orthoprojector of \( \mathcal{H} \) onto \( \mathcal{N} \), then \( P^\perp \) is the orthoprojector onto \( \mathcal{N}^\perp \). For a closed operator \( T \) in \( \mathcal{H} \), \( \sigma(T) \) denotes its spectrum, and \( \rho(T) \) denotes the set of its regular points. The symbol \( \langle \cdot , \cdot \rangle \) stands for the standard inner product in \( \mathbb{C}^n \). \( | \cdot | \) denotes the norm of a vector in \( \mathbb{C}^n \); the unit \((n \times n)\)-matrix is denoted by \( I_n \). Next, we use the notation \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), \( iD_j = \partial_j = \partial / \partial x_j \), \( j = 1, \ldots, d \), \( \nabla = \text{grad} = (\partial_1, \ldots, \partial_d) \), \( D = -i\nabla = (D_1, \ldots, D_d) \), \( \nabla^* = -\text{div} \). The \( L_p \)-classes of \( \mathbb{C}^n \)-valued functions in a domain \( \Omega \subseteq \mathbb{R}^d \) are denoted by \( L_p(\Omega; \mathbb{C}^n) \), \( 1 \leq p \leq \infty \). The Sobolev classes of order \( s \) with integrability index \( p \) of \( \mathbb{C}^n \)-valued functions in a domain \( \Omega \subseteq \mathbb{R}^d \) are denoted by \( W^s_p(\Omega; \mathbb{C}^n) \). For \( p = 2 \) we abbreviate this to \( H^s(\Omega; \mathbb{C}^n) \), \( s \in \mathbb{R} \). If \( n = 1 \), we write simply \( W^s_p(\Omega) \), \( H^s(\Omega) \). Within one chapter, we use the two-index enumeration of subsections, statements and formulas. When referring to other chapters, we use the three-index enumeration.

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Chapter 1. Operator families admitting factorization

The material of this chapter concerns the spectral perturbation theory for selfadjoint operator families. Our goal is to specify the case where the positive operator family admits a factorization. The considerations are adapted to the study of the threshold effects near the lower edge of the spectrum. The obtained estimates are efficient for applications and the constants in estimates are well controlled.

§1. The quadratic pencils of the form \( X(t)^*X(t) \)

1.1. The pencils of the form \( X(t)^*X(t) \) and \( X(t)^*X(t) \). Let \( \mathcal{H}_1, \mathcal{H}_2 \) be complex separable Hilbert spaces. Suppose that the operator \( X_0 : \mathcal{H}_1 \to \mathcal{H}_2 \) is densely defined and closed, and that the operator \( X_1 : \mathcal{H}_1 \to \mathcal{H}_2 \) is bounded. Then the operator (the linear operator pencil)

\[
X(t) = X_0 + tX_1, \quad t \in \mathbb{R},
\]
is closed on the domain $\text{Dom } X(t) := \text{Dom } X_\delta$. We have $X(t)^* = X_\delta^* + t X^*_\delta$ on the domain $\text{Dom } X^*_\delta$ (which is dense in $\mathfrak{H}_\delta$). The selfadjoint positive operator family

$$A(t) := X(t)^* X(t)$$

in $\mathfrak{H}$ is our main object. The operator (1.2) is generated by the closed quadratic form $\|X(t)u\|_{\mathfrak{H}_\delta}^2$, $u \in \text{Dom } X_\delta$. For the operator $A_\delta := A(0) = X_\delta^* X_\delta$, we put

$$\mathfrak{N} := \text{Ker } A_\delta = \text{Ker } X_\delta.$$

In what follows, we suppose that the next condition is true.

**Condition 1.1.** The point $\lambda = 0$ is an isolated point of the spectrum of the operator $A_\delta$, and

$$(0 <) n := \dim \mathfrak{N} < \infty.$$

We denote the spectral projector of the operator $A(t)$ for a closed interval $[0, s]$ by $F(t, s)$, and put $\mathfrak{N}(t, s) := F(t, s)\mathfrak{N}$. We fix a number $\delta > 0$ such that $8\delta < d^0$, where $d^0$ is the distance from the point $\lambda = 0$ to the rest of the spectrum of $A_\delta$. We often write $F(t)$ in place of $F(t, \delta)$ and $\mathfrak{N}(t)$ in place of $\mathfrak{N}(t, \delta)$. The following statement is an easy consequence of the spectral theorem.

**Proposition 1.2.** We have

$$F(t, \delta) = F(t, 3\delta), \quad \text{rank } F(t, \delta) = n, \quad |t| \leq t^0 = t^0(\delta) := \delta^{1/2} \|X_1\|^{-1}.$$  

**Proof.** Assume that $\text{rank } F(t, \delta) < n$. Then there exists an element $f \in \mathfrak{N}$ such that $\|f\| = 1$ and $f \perp \mathfrak{N}(t, \delta)$. Hence, $\|X(t)f\|^2 > \delta$. From the other side, $\|X(t)f\|^2 = \|X_\delta f + t X_1 f\|^2 \leq t^2 \|X_1\|^2 \leq \delta$. The obtained contradiction shows that

$$\text{rank } F(t, \delta) \geq n.$$  

Now, assume that $\text{rank } F(t, 3\delta) > n$. Then there exists an element $f \in \mathfrak{N}(t, 3\delta)$ such that $\|f\| = 1$ and $f \perp \mathfrak{N}$. Therefore, $\|X_\delta f\|^2 > d^0 > 8\delta$. But, we have

$$\|X_\delta f\|^2 \leq 2\|X(t)f\|^2 + 2t^2 \|X_1\|^2 \leq 6\delta + 2\delta = 8\delta.$$  

The arising contradiction shows that

$$\text{rank } F(t, 3\delta) \leq n.$$  

Comparing (1.4) with (1.5) implies that $\text{rank } F(t, \delta) = \text{rank } F(t, 3\delta) = n$, which is equivalent to (1.3). \qed

Simultaneously with (1.2), it is convenient to consider the selfadjoint operator family

$$A_\ast(t) := X(t) X(t)^*$$

in $\mathfrak{H}_\delta$. We put $A_\ast_\delta := X_\delta^* X_\delta^*$,

$$\mathfrak{N}_\ast := \text{Ker } A_\ast_\delta = \text{Ker } X_\delta^*, \quad n_\ast := \dim \mathfrak{N}_\ast.$$

The operators $A(t)$ and $A_\ast(t)$ have the same nonzero spectrum. In general, the numbers $n$ and $n_\ast$ are distinct. We suppose that

$$n \leq n_\ast \leq \infty.$$  

Let $P$ and $P_\ast$ denote the orthoprojectors onto $\mathfrak{N}$ and $\mathfrak{N}_\ast$ correspondingly.
1.2. The operator \( R \). We introduce the notation \( \mathcal{D} := \text{Dom}X_0 \cap \mathfrak{M}^\perp \). Since the point \( \lambda = 0 \) is isolated in \( \sigma(A_0) \), the form \( (X_0 \varphi, X_0 \zeta)_{\mathcal{D}}, \varphi, \zeta \in \mathcal{D} \), defines an inner product in \( \mathcal{D} \) converting \( \mathcal{D} \) into the Hilbert space. Let \( z \in \mathfrak{M}_* \). Let \( \hat{\varphi} \in \mathcal{D} \) satisfy the equation \( X_0^*(X_0 \hat{\varphi} - z) = 0 \) which is understood in the weak sense. In other words, we look for an element \( \hat{\varphi} \in \mathcal{D} \) satisfying the identity
\[
(X_0 \hat{\varphi}, X_0 \zeta)_{\mathcal{D}} = (z, X_0 \zeta)_{\mathcal{D}}, \quad \zeta \in \mathcal{D}.
\]
Since the right-hand side of (1.7) is an antilinear continuous functional of \( \zeta \in \mathcal{D} \), there exists the unique solution \( \hat{\varphi} \). Moreover, \( \|X_0 \hat{\varphi}\|_{\mathcal{D}} \leq \|z\|_{\mathcal{D}} \). Note also that \( X_0 \hat{\varphi} - z \in \mathfrak{M}_* \).
Now, let
\[
\omega \in \mathfrak{M}_*, \quad z := -X_1 \omega.
\]
We denote the element \( \hat{\varphi} \) satisfying (1.7) (with such \( z \)) by \( \hat{\varphi}(\omega) \), and put
\[
\omega_* := X_0 \hat{\varphi}(\omega) + X_1 \omega \in \mathfrak{M}_*.
\]
In accordance with (1.7)–(1.9), we introduce the linear operator \( R \) which takes \( \omega \) into \( \omega_* \):
\[
R : \mathfrak{M} \to \mathfrak{M}_*, \quad R \omega = \omega_*.
\]
It is clear that \( \omega_* \in \mathfrak{M}_* \), and \( X_0 \hat{\varphi}(\omega) \in \text{Ran}X_0 \subset \mathfrak{M}^\perp \). Hence, \( \omega_* = P_* X_1 \omega \). This yields another representation for \( R \):
\[
R = P_* X_1|_{\mathfrak{M}_*},
\]
which is equivalent to
\[
P_* X_1 P = R \oplus \mathbb{0};
\]
here we extend \( R \) onto \( \mathfrak{M}^\perp \) by zero. The continuous operator \( R_* : \mathfrak{M}_* \to \mathfrak{M} \) is defined by analogy. Then
\[
R_* = P X_1^*|_{\mathfrak{M}_*}, \quad P X_1^* P_* = R_* \oplus \mathbb{0}.
\]
By (1.11)–(1.13), it is clear that
\[
R_* = R^*.
\]
1.3. The spectral germ \( S \) of a family \( A(t) \) at \( t = 0 \).

**Definition 1.3.** The selfadjoint operator
\[
S := R^* R : \mathfrak{M} \to \mathfrak{M}
\]
is called the spectral germ of the operator family (1.2) at \( t = 0 \).

From (1.12)–(1.14) it follows that
\[
S = P X_1^* P_* X_1|_{\mathfrak{M}_*}, \quad S \oplus \mathbb{0} = P X_1^* P_* X_1 P.
\]
Note also that
\[
(S \zeta, \zeta)_{\mathfrak{M}} = \|R \zeta\|_{\mathfrak{M}}^2 = \|P_* X_1 \zeta\|_{\mathfrak{M}}^2, \quad \zeta \in \mathfrak{M}.
\]
For the family (1.6), the role of the spectral germ is played by \( S_* := RR^* : \mathfrak{M}_* \to \mathfrak{M}_* \). The part of the operator \( S \) acting in \( (\text{Ker} \, S)^\perp \) is unitarily equivalent to the part of \( S_* \) acting in \( (\text{Ker} \, S_*)^\perp \). Below (cf. Subsection 1.6), we shall clarify the role of the operator \( S \). In particular, we shall show that \( S \) is independent of the concrete choice of factorization (1.2) for \( A(t) \).
**Definition 1.4.** The spectral germ $S$ of the operator family (1.2) at $t = 0$ is called non-degenerate if $\ker S = \{0\}.

Obviously, the non-degeneracy of $S$ is equivalent to the condition $\ker R = \{0\}$, or, equivalently, to the condition rank $R = n$.

Let $A(t) = \hat{X}(t)^* \hat{X}(t)$ be one more operator family in $\mathfrak{H}$, subject to the same conditions (cf. Subsection 1.1) as $A(t)$. (Hereafter, $\mathfrak{H}$, may be different from $\mathfrak{H}_*$.) Let $\mathfrak{R} := \ker \hat{A}(0)$ and $S : \mathfrak{R} \to \overline{\mathfrak{R}}$ be the spectral germ of the family $A(t)$ at $t = 0$.

**Definition 1.5.** Operator families $A(t)$ and $\hat{A}(t)$ are called threshold equivalent if $\mathfrak{R} = \mathfrak{R}$ and $S = \overline{S}$.

The defined relation is an equivalence relation on the set of operator families of the form (1.2).

**1.4. Estimates for the operator $S$**. Upper estimates. Let $\mathfrak{N} \subset \mathfrak{G} \subset \mathfrak{G}_*$, where $\mathfrak{G}_*$ is a subspace in $\mathfrak{G}_*$, and let $\Pi_*$ be the orthoprojector onto $\mathfrak{G}_*$. Obviously, (1.16) implies that

$$ S \oplus \mathfrak{N} \leq PX_1^* \Pi_* X_1 P, $$

or, equivalently, (cf. also (1.17))

$$ (S \zeta, \zeta)_{\mathfrak{N}} \leq \|X_1 \zeta\|^2_{\mathfrak{G}_*}, \quad \zeta \in \mathfrak{N}. $$

In particular, if $\mathfrak{G}_* = \mathfrak{G}_*$, we obtain

$$ (S \zeta, \zeta)_{\mathfrak{G}_*} \leq \|X_1 \zeta\|^2_{\mathfrak{G}_*}, \quad \zeta \in \mathfrak{G}_*. $$

Finally, (1.19) (or (1.16)) implies the estimate $\|S\| \leq \|X_1\|^2$. The estimate (1.18) gives the opportunity for refining estimates of the operator $S$ on the basis of the Ritz method.

Lower estimates for $S$. Let $\mathfrak{G}_*$ be a subspace in $\mathfrak{N}_*$ and let $\Pi'_*$ be the orthoprojector onto $\mathfrak{G}'_*$. Then $S \geq \mathfrak{N}_* \geq PX_1^* \Pi'_* X_1 P$, or

$$ (S \zeta, \zeta)_{\mathfrak{N}_*} \geq \|X_1 \zeta\|^2_{\mathfrak{G}_*}, \quad \zeta \in \mathfrak{N}. $$

**1.5. The case of a family** $X(t) = \hat{X}(t)M$. Let $\mathfrak{H}$ be one more Hilbert space, and let $\hat{X}(t) = \hat{X}_0 + t\hat{X}_1 : \mathfrak{H} \to \mathfrak{H}$ be a family of the form (1.1) which satisfies the assumptions of Subsection 1.1. We emphasize that the space $\mathfrak{H}_*$ is the same as before. Suppose that $M : \mathfrak{H} \to \hat{\mathfrak{H}}$ is an isomorphism. Let $M \operatorname{Dom} X_0 = \operatorname{Dom} \hat{X}_0$, $X(t) = \hat{X}(t)M : \mathfrak{H} \to \mathfrak{H}^*; X_0 = \hat{X}_0 M$, $X_1 = \hat{X}_1 M$,

$$ A(t) = M^* \hat{A}(t) M. $$

In what follows, all the objects related to $\hat{X}(t)$ are marked by the index "e". We note that

$$ \hat{\mathfrak{H}} = M\mathfrak{H}, \quad \hat{n} = n, \quad \hat{\mathfrak{N}} = \mathfrak{N}, \quad \hat{n}_* = n_*, \quad \hat{\Pi}_* = \Pi_*.$$

Besides, $R = P_* X_1 |_{\mathfrak{N}} = \hat{P}_* \hat{X}_1 M|_{\mathfrak{N}}$, i. e.,

$$ R = \hat{R} M|_{\mathfrak{N}}.
Next, according to (1.17), (1.21), for \( \zeta \in \mathfrak{M} \) and \( \widehat{\zeta} = M\zeta(\in \mathfrak{N}) \), we have

\[
(\hat{S}\zeta, \zeta)_\mathfrak{N} = \|R\zeta\|_\mathfrak{N}^2, \quad \|R\hat{\zeta}\|_\mathfrak{N}^2 = (\hat{S}\hat{\zeta}, \hat{\zeta})_\mathfrak{N} = (M^* \hat{S}M\zeta, \zeta)_\mathfrak{N}.
\]

Hence,

\[
(1.23) \quad S = PM^* \hat{S}M|_\mathfrak{N}, \quad S \oplus \mathbb{D} = PM^* \hat{S}M \mathbb{P},
\]

\[
(1.24) \quad \hat{S} = \tilde{P}(M^*)^{-1}SM^{-1}|_\mathfrak{N}, \quad \hat{S} \oplus \mathbb{D} = \tilde{P}(M^*)^{-1}SM^{-1} \tilde{P}.
\]

Formulas (1.24) follow from (1.23), by interchanging the roles of \( X(t) \) and \( \hat{X}(t) \).

From (1.21) it follows that \( \operatorname{rank} R = \operatorname{rank} \hat{R} \), and, therefore, \( S \) and \( \hat{S} \) are non-degenerate or degenerate simultaneously. From (1.22) it is clear that

\[
(1.25) \quad \frac{(S\zeta, \zeta)_\mathfrak{N}}{\|\zeta\|_\mathfrak{N}^2} = \frac{(\hat{S}\zeta, \hat{\zeta})_\mathfrak{N}}{\|\hat{\zeta}\|_\mathfrak{N}^2} \frac{\|M\zeta\|_\mathfrak{N}^2}{\|\zeta\|_\mathfrak{N}^2} \leq \|M\| \frac{(\hat{S}\hat{\zeta}, \hat{\zeta})_\mathfrak{N}}{\|\hat{\zeta}\|_\mathfrak{N}^2}.
\]

and, similarly, \( (S\hat{\zeta}, \hat{\zeta})_\mathfrak{N} \|\hat{\zeta}\|_\mathfrak{N}^{-2} \leq \|M^{-1}\|^2 (S\zeta, \zeta)_\mathfrak{N} \|\zeta\|_\mathfrak{N}^{-2}. \) Hence, the eigenvalues \( \gamma \) of the operator \( S \) and the eigenvalues \( \hat{\gamma} \) of \( \hat{S} \) satisfy the relations

\[
\|M^{-1}\|^{-1} \hat{\gamma}_l \leq \gamma \leq \|M\|^2 \hat{\gamma}_l, \quad l = 1, \ldots, n.
\]

Now, let \( n = 1 \). Then (1.25) implies the following relation for \( \gamma = \gamma_1, \hat{\gamma} = \hat{\gamma}_1 \):

\[
(1.26) \quad \gamma = \hat{\gamma} \|M\zeta\|_\mathfrak{N}^2 \|\zeta\|_\mathfrak{N}^{-2}, \quad \zeta \in \mathfrak{M}.
\]

1.6. The eigenvalue problem for the operator \( S \). According to the general analytic perturbation theory [Ka], for \( |t| \leq t^0 \), there exist real-analytic functions \( \lambda(t) \) (the branches of eigenvalues) and real-analytic \( \mathfrak{F}\)-valued functions \( \varphi(t) \) (the branches of eigenvectors) such that

\[
(1.27) \quad A(t)\varphi(t) = \lambda(t)\varphi(t), \quad l = 1, \ldots , n, \quad |t| \leq t^0 = t^0(\delta),
\]

and \( \varphi(t), l = 1, \ldots, n, \) form an orthonormal basis in \( \mathfrak{F}(t) \). Moreover, for sufficiently small \( t^* \), we have the convergent power series expansions

\[
(1.28) \quad \lambda(t) = \gamma t^2 + \cdots, \quad \gamma_l \geq 0, \quad l = 1, \ldots, n, \quad |t| \leq t^* \leq t^0,
\]

\[
(1.29) \quad \varphi(t) = \omega_l + t^{(1)} \varphi_l^{(1)} + t^2 \varphi_l^{(2)} + \cdots, \quad l = 1, \ldots, n, \quad |t| \leq t^* \leq t^0.
\]

The elements \( \omega_l := \varphi_l(0), l = 1, \ldots, n, \) form an orthonormal basis in \( \mathfrak{M} \), and \( P = \sum_{l=1}^n \langle \cdot, \omega_l \rangle \omega_l \). Relations (1.27) are equivalent to

\[
(1.30) \quad (X(t)\varphi(t), X(t)\zeta)_\mathfrak{M} = \lambda_l(t)\varphi_l(t), \zeta)_\mathfrak{N}, \quad \zeta \in \operatorname{Dom} X_0.
\]

The values \( \gamma_l \) and the elements \( \omega_l, \varphi_l^{(1)}, \varphi_l^{(2)}, \ldots, \) can be found from (1.30) and from the normalization condition \( \|\varphi_l(t)\|_\mathfrak{N}^2 = 1 \), by comparing the coefficients with the same power of \( t \). Comparing the terms of first order, we obtain (cf. (1.7)-(1.9))

\[
(1.31) \quad \varphi_l^{(1)} = \hat{\varphi}(\omega_l) \in \mathfrak{M}, \quad l = 1, \ldots, n.
\]
Comparing the terms with $t^2$, we see that

\[(1.32)\]
\[(X_0 \varphi_1^1 + X_1 \omega_i, X_1 \zeta)_{\mathcal{E}_n} + (X_0 \varphi_1^2 + X_1 \varphi_1^1, X_0 \zeta)_{\mathcal{E}_n} = \gamma(\omega_i, \zeta)_{\mathcal{E}_n}, \quad \zeta \in \text{Dom } X_0.\]

According to (1.9), (1.10), (1.31), $X_0 \varphi_1^1 + X_1 \omega_i = R \omega_i$. For $\zeta \in \mathfrak{N}$, relation (1.32) takes the form

\[(1.33)\]
\[(R \omega_i, X_1 \zeta)_{\mathcal{E}_n} = \gamma(\omega_i, \zeta)_{\mathcal{E}_n}, \quad \zeta \in \mathfrak{N}.\]

By (1.11), (1.16),

\[(R \omega_i, X_1 \zeta)_{\mathcal{E}_n} = (P X_1^* R \omega_i, \zeta)_{\mathcal{E}_n} = (S \omega_i, \zeta)_{\mathcal{E}_n}.

Then (1.33) yields the following proposition.

**Proposition 1.6.** The numbers $\gamma_i$ and the elements $\omega_i$ defined by (1.27)-(1.29) are eigenvalues and eigenvectors of the operator $S$:

\[(1.34)\]
\[S \omega_i = \gamma_i \omega_i, \quad i = 1, \ldots, n.\]

This clarifies the meaning of the eigenvalue problem for the operator $S$. Relations (1.34) show that

\[(1.35)\]
\[SP = \sum_{i=1}^{n} \gamma_i(\cdot, \omega_i)_{\mathcal{E}_n} \omega_i.

Series (1.28), (1.29) are determined by the family $A(t)$ directly and independently of the factorization (1.2). This shows that $S$ does not depend on the choice of factorization. At the same time, relation (1.15) can be interpreted as "inheritance" of the chosen factorization for $A(t)$ by the operator $S$. The non-degeneracy of $S$ is equivalent to the inequalities

\[(1.36)\]
\[\gamma_i \geq c_i > 0, \quad i = 1, \ldots, n,\]

for the numbers $\gamma_i$ defined by (1.28).

If all the eigenvalues $\gamma_i$ are simple, then (1.34) determines (up to a phase factor) the initial elements $\omega_i$ in (1.29). If there are multiple eigenvalues among $\gamma$, then, in general, the knowledge of $S$ is insufficient for this purpose.

### 1.7. Operator-valued functions $F(t)$ and $A(t)F(t)$

For what follows, it is important to find "good" approximations for the operator-valued functions $F(t)$ and $A(t)F(t)$. Both functions are real-analytic for $|t| \leq \delta$. From (1.29) and Proposition 1.2 it follows that, for sufficiently small $t_\ast \leq \delta$, we have

\[(1.37)\]
\[F(t, \delta) = F(t) = P + t F_1 + \cdots, \quad |t| \leq t_\ast,

where $F_1 = \hat{F}_1 + \hat{F}_1^*$, $\hat{F}_1 = \sum_{i=1}^{n} (\cdot, \omega_i)_{\mathcal{E}_n} \varphi_1^1$. Next, according to (1.27) and Proposition 1.2,

\[A(t)F(t) = \sum_{i=1}^{n} \lambda_i(t)(\cdot, \varphi_i(t))_{\mathcal{E}_n} \varphi_i(t), \quad |t| \leq \delta.\]
Combining this with (1.28), (1.29) and (1.35), we obtain

\[(1.38) \quad A(t) F(t) = t^2 S P + \ldots, \quad |t| \leq t_* \]

The significance of (1.38) is in the fact that \( S \) admits representations (1.15), (1.16), which do not require the knowledge of the eigenvectors \( \omega_1, \ldots, \omega_n \).

However, power series expansions (1.37), (1.38) are not completely sufficient for our purposes. We need only the estimates for \( F(t) - P, A(t) F(t) - t^2 S P \), but on the wider interval \( |t| \leq t^0(\delta) \), and with explicitly controlled constants in estimates. Such estimates can be obtained by integrating the difference of the resolvents for \( A(t) \) and \( A_0 \) over an appropriate contour \( \Gamma \). However, the difficulty is that, under our assumptions, the ordinary resolvent identity \( (A(t) - zI)^{-1} - (A_0 - zI)^{-1} = (A_0 - zI)^{-1}(A_0 - A(t))(A(t) - zI)^{-1} \) is not applicable. Indeed, in general, the difference \( A(t) - A_0 \) makes no sense. It remains to use the version of the resolvent identity for operators having the same domain of the quadratic forms. (In our case, the domain of the quadratic form coincides with \( \text{Dom}X(t) = \text{Dom}X_0 \).) The corresponding relations are not of common use. So, in the next §2, we present the necessary auxiliary material.

1.8. Families of the block structure

\[ X(t) = \begin{pmatrix} 0 & X(t)^* \\ X(t) & 0 \end{pmatrix} \]

act in the space

\[ \mathcal{H} = \mathfrak{H} \oplus \mathfrak{H}_r. \]

Now, it is natural to assume that \( n = n_* \). Let \( \mathbf{w} = \text{col}(u, v) \), \( s = \text{col}(q, r) \in \mathcal{H} \) and let

\[(1.39) \quad (X(t) - i\kappa I)w = s, \quad \kappa > 0.\]

We write the solution \( w \) of the equation (1.39) in the form

\[ w = w_q + w_r, \]

where \( w_q \) is the solution of (1.39) with \( r = 0 \), and \( w_r \) is the solution of (1.39) with \( q = 0 \). We have \( w_q = \text{col}(u_q, v_q) \), \( w_r = \text{col}(u_r, v_r) \), and

\[(1.40) \quad X(t)^*v_q - i\kappa u_q = q, \quad X(t)u_q - i\kappa v_q = 0,\]

\[(1.41) \quad X(t)^*v_r - i\kappa u_r = 0, \quad X(t)u_r - i\kappa v_r = r.\]

From (1.40), (1.41) it follows that

\[ u_q = i\kappa(A(t) + \kappa^2 I)^{-1}q, \quad v_q = (A_*(t) + \kappa^2 I)^{-1}X(t)q, \]
\[ u_r = (A(t) + \kappa^2 I)^{-1}X(t)^*r, \quad v_r = i\kappa(A_*(t) + \kappa^2 I)^{-1}r.\]

The functions \( v_q, u_r \) also admit another representations:

\[ v_q = X(t)(A(t) + \kappa^2 I)^{-1}q, \quad u_r = X(t)^*(A_*(t) + \kappa^2 I)^{-1}r.\]

Note that the expressions for \( u_q, v_r \) are simpler than for \( v_q, u_r \).
2.1. The resolvent identity. Let $\mathcal{H}$ be a Hilbert space. The symbols $(\cdot, \cdot)$ and $|| \cdot ||$ stand for the inner product and the norm in this space. Let $a$ and $b$ be two sesquilinear non-negative closed forms in $\mathcal{H}$. Suppose that

(2.1) \quad \text{Dom} a = \text{Dom} b =: \mathcal{D},

and $\mathcal{D}$ is dense in $\mathcal{H}$. By $A$ (respectively, $B$) we denote the selfadjoint operator in $\mathcal{H}$ generated by the form $a$ (respectively, $b$). We put

(2.2) \quad a_\gamma [u, v] = a[u, v] + \gamma (u,v), \quad \gamma > 0.

The notation $b_\gamma$ has similar meaning. The linear set $\mathcal{D}$ is a complete Hilbert space $\mathcal{D}(a_\gamma)$ with respect to the inner product (2.2). We denote the norm in $\mathcal{D}(a_\gamma)$ by $|| \cdot ||_{\mathcal{D}}$. Obviously,

(2.3) \quad ||u|| \leq \gamma^{-1/2} ||u||_{\mathcal{D}}.

By (2.1), the form $b_\gamma$ is continuous in $\mathcal{D}(a_\gamma)$ and generates a metric in this space equivalent to the standard one. We put

(2.4) \quad a^* = \sup_{\| f \| \neq 0} \frac{a_\gamma [f, f]}{b_\gamma [f, f]}.

Obviously, the real-valued form $t = b - a$ is $(a_\gamma)$-continuous, and, therefore, it generates a continuous selfadjoint operator $T_\gamma$ in $\mathcal{D}(a_\gamma)$. Thus,

(2.5) \quad t[u, v] = a_\gamma [T_\gamma u, v], \quad u, v \in \mathcal{D},

(2.6) \quad ||T_\gamma||_{\mathcal{D}} = \sup_{\| u \| \neq 0} \frac{|t[u, u]|}{a_\gamma [u, u]}.

The operator $T_\gamma$ can be viewed as an operator from $\mathcal{D}(a_\gamma)$ to $\mathcal{H}$. Then, by (2.3),

(2.7) \quad ||T_\gamma||_{\mathcal{D} \rightarrow \mathcal{H}} \leq \gamma^{-1/2} ||T_\gamma||_{\mathcal{D}}.

Now, we consider the equations

$$(A + \gamma I)x = f, \quad (B + \gamma I)y = f,$$

which are equivalent to the relations

$$a_\gamma [x, v] = (f, v), \quad b_\gamma [y, v] = (f, v), \quad v \in \mathcal{D}.$$

From (2.5) it follows that

$$b_\gamma [y, v] = a_\gamma [y, v] + t[y, v] = a_\gamma [(I + T_\gamma)y, v], \quad v \in \mathcal{D},$$

whence $x = y + T_\gamma y$. Thus,

(2.8) \quad (B + \gamma I)^{-1} - (A + \gamma I)^{-1} = -T_\gamma (B + \gamma I)^{-1}.$
By \( R_z(A) \), \( R_z(B) \) we denote the corresponding resolvents. Using the Hilbert identity \( R_z(B) - R_{-\gamma}(B) = (z + \gamma)R_{-\gamma}(B)R_z(B) \) and the similar identity for \( R_z(A) \), we see that, by (2.8),

\[
R_z(B) - R_z(A) = -T_z R_{-\gamma}(B) (I + (z + \gamma)R_z(B)) + (z + \gamma)R_{-\gamma}(A)(R_z(B) - R_z(A)).
\]

We introduce the notation

\[
\Omega_z(A) := I + (z + \gamma)R_z(A)
\]

and the similar notation \( \Omega_z(B) \) for \( B \). Since \((I - (z + \gamma)R_{-\gamma}(A))^{-1} = \Omega_z(A)\), (2.9) implies that

\[
R_z(B) - R_z(A) = -\Omega_z(A)T_z R_{-\gamma}(B)\Omega_z(B), \quad z \in \rho(A) \cap \rho(B).
\]

Next, by the identity

\[
R_z(B) = R_{-\gamma}(B)\Omega_z(B),
\]

(2.11) yields that

\[
R_z(B) - R_z(A) = -\Omega_z(A)T_z R_z(B), \quad z \in \rho(A) \cap \rho(B).
\]

Relation (2.13) is the analogue of the ordinary resolvent identity under the conditions that (2.1) is fulfilled, but, probably, \( \text{Dom } A \neq \text{Dom } B \). Relation (2.13) will be also called the \textit{resolvent identity}.

\[ \textbf{2.2. Estimates for operators of the form } LR_z(B) \text{ (where } L \text{ is a continuous operator in } \mathfrak{d}(a_r) \). \]

We write the identity

\[
LR_{-\gamma}(B) = L(A + \gamma I)^{-1/2}((A + \gamma I)^{1/2}(B + \gamma I)^{-1/2})|R_{-\gamma}(B)|^{1/2}.
\]

Using (2.3) (cf. also (2.7)) and the isometry of the mapping \((A + \gamma I)^{-1/2} : \mathfrak{H} \to \mathfrak{d}(a_r)\), we obtain

\[
\|L(A + \gamma I)^{-1/2}\|_\mathfrak{d} \leq \|(A + \gamma I)^{-1/2}\|_{\mathfrak{d} \to \mathfrak{d}}\|L\|_\mathfrak{d} \leq \gamma^{-1/2}\|L\|_\mathfrak{d}.
\]

Next, \(\|R_{-\gamma}(B)\|^{1/2}_\mathfrak{d} \leq \gamma^{-1/2}\), and it is easily seen that \(\|(A + \gamma I)^{1/2}(B + \gamma I)^{-1/2}\|_\mathfrak{d} = \alpha\), where \(\alpha = \alpha(\gamma)\) is defined by (2.4). Thus,

\[
\|LR_{-\gamma}(B)\|_\mathfrak{d} \leq \alpha^{-1}\|L\|_\mathfrak{d},
\]

and, by (2.12),

\[
\|LR_z(B)\|_\mathfrak{d} \leq \alpha^{-1}\|L\|_\mathfrak{d}\|\Omega_z(B)\|_\mathfrak{d}, \quad z \in \rho(B).
\]

If we replace \( B \) by \( A \) in (2.15), then we have to take \(\alpha = 1\), whence

\[
\|LR_z(A)\|_\mathfrak{d} \leq \gamma^{-1}\|L\|_\mathfrak{d}\|\Omega_z(A)\|_\mathfrak{d}, \quad z \in \rho(A).
\]

We also need to estimate operators of the form \( L_1\Omega_z(A)L_2 R_z(B) \), where \( L_1 \), \( L_2 \) are continuous operators in \( \mathfrak{d}(a_r) \). According to (2.10),

\[
\|L_1\Omega_z(A)L_2 R_z(B)\|_\mathfrak{d} \leq \|(L_1 L_2) R_z(B)\|_\mathfrak{d} + \|z + \gamma\|L_1 R_z(A)\|_\mathfrak{d}\|L_2 R_z(B)\|_\mathfrak{d},
\]

Now, (2.15), (2.16) imply that, for \( z \in \rho(A) \cap \rho(B) \),

\[
\|L_1\Omega_z(A)L_2 R_z(B)\|_\mathfrak{d} \leq \alpha^{-1}\|\Omega_z(B)\|_\mathfrak{d}(1 + \|z + \gamma\gamma^{-1}\|\Omega_z(A)\|_\mathfrak{d})\|L_1\|_\mathfrak{d}\|L_2\|_\mathfrak{d}.
\]
§3. Estimates for the difference of the resolvents on the contour

3.1. The contour \( \Gamma \). We need to integrate the difference of the resolvents \( R_z(A(t)) - R_z(A(0)) \) over the contour \( \Gamma \), which envelopes the real interval \([0, \delta]\) equidistantly with the distance \( \delta \). Recall that \( \delta \) is a fixed number such that \( 8\delta < d^0 \), where \( d^0 \) is the distance between the non-zero spectrum of \( A_0 \) and zero point. Parameter \( t \) is subject to the condition

\[
|t| \leq t^0 = t^0(\delta) = \delta^{1/2}||X_1||^{-1},
\]

i.e., condition (1.3). Below, we shall write \( R_z(t) \) in place of \( R_z(A(t)) \), \( \Omega_z(t) \) in place of \( \Omega_z(A(t)) \), etc. Also, we shall not use the lower indices \( \delta_0 \) or \( \delta_* \), in the notation of the norm and the inner product. By Proposition 1.2, under condition (3.1), the distance between \( \Gamma \) and \( \sigma(A(t)) \) is greater or equal than \( \delta \), whence,

\[
||R_z(t)|| \leq \delta^{-1}, \quad z \in \Gamma, \quad |t| \leq t^0(\delta).
\]

3.2. Incorporation into the scheme of §2. Using the results of §2, assume that \( \gamma = 2\delta \). To begin with, mention the estimate for the operator-valued function \( \Omega_z(t) = I + (z + 2\delta)R_z(t) \) of the form (2.10). By (3.2) and the inequality \( |z| \leq 2\delta \) for \( z \in \Gamma \), we have

\[
||\Omega_z(t)|| \leq 5, \quad z \in \Gamma, \quad |t| \leq t^0(\delta).
\]

Now, the role of the forms \( b \) and \( a \) from §2 is played by the forms \( a(t) \) and \( a(0) \) respectively:

\[
a(t)[u, u] = ||X(t)u||^2, \quad a(0)[u, u] = ||X_0u||^2, \quad u \in \text{Dom } X_0 =: \mathcal{D}.
\]

First of all, we estimate the number \( a \) defined by (2.4). By (3.1), we have

\[
||X_0u||^2 = ||X(t) - tX_1)u||^2 \leq 2||X(t)u||^2 + 2t^2||X_1||^2||u||^2 \leq 2(||X(t)u||^2 + \delta||u||^2), \quad u \in \mathcal{D},
\]

which corresponds to the estimate

\[
a^2 \leq 2.
\]

Now, the difference of the forms \( a(t) - a(0) \) is equal to the form

\[
2t \text{Re}(X_0u, X_1u) + t^2||X_1u||^2, \quad u \in \mathcal{D}.
\]

This form generates the operator \( T_\gamma \) in the space \( \mathcal{D} \) with the metric form \( a_\gamma(0) \), \( \gamma = 2\delta \). From (3.5) it follows that the operator \( T_\gamma = T_\gamma(t) \) can be represented as

\[
T_\gamma(t) = tT_\gamma^{(1)} + t^2T_\gamma^{(2)},
\]

where \( T_\gamma^{(1)} \), \( T_\gamma^{(2)} \) already do not depend on \( t \). Let us estimate their norms in \( \mathcal{D} \). Since

\[
2|\text{Re}(X_0u, X_1u)| \leq \kappa||X_0u||^2 + \kappa^{-1}||X_1||^2||u||^2, \quad \kappa > 0,
\]
then, for \( x = (2\delta)^{-1/2} \| X_1 \| \), we obtain
\[
2 | \text{Re}(X_0 u, X_1 u) | \leq (2\delta)^{-1/2} \| X_1 \| (\| X_0 u \|^2 + 2\delta \| u \|^2),
\]
or, according to (2.6),
\[
\| T_{\gamma}^{(1)} \| \leq (2\delta)^{-1/2} \| X_1 \|.
\]  
(7.3)

Next, \( \| X_1 u \|^2 \leq (2\delta)^{-1} \| X_1 \|^2 (\| X_0 u \|^2 + 2\delta \| u \|^2) \), i.e.,
\[
\| T_{\gamma}^{(2)} \| \leq (2\delta)^{-1} \| X_1 \|^2.
\]  
(7.4)

Finally, we estimate the norm of the operator (3.6). Relations (7.3), (7.4) imply that
\[
\| T_{\gamma} (t) \| \leq t (2\delta)^{-1/2} \| X_1 \| + t^2 (2\delta)^{-1} \| X_1 \|^2.
\]
Combining this with (3.1), we obtain
\[
\| T_{\gamma} (t) \| \leq \frac{\sqrt{2} + 1}{2} t \delta^{-1/2} \| X_1 \|,
\]  
(9.1)

3.3. Estimate for the norm of the difference of the resolvents. We start with representation (2.11). According to (2.14), (3.4) and (9.1),
\[
\| T_{\gamma} (t) R_{\gamma} (t) \| \leq 4^{-1} (1 + \sqrt{2} t \delta^{-3/2}) \| X_1 \|.
\]
Combining this with (3.3), we obtain
\[
\| R_{\gamma} (t) - R_{\gamma} (0) \| \leq \beta_{\gamma}^0 t \delta^{-3/2} \| X_1 \|, \quad | t | \leq t^0 (\delta), \quad z \in \Gamma,
\]
where \( \beta_{\gamma}^0 = 5^2 2^{-1} (1 + 2^{-1/2}) \). In what follows we shall not write down the cumbersome explicit expressions for the absolute constants; these constants will be denoted by \( \beta \) or \( \beta^0 \) with indices. One should keep in mind that it is possible to give concrete numerical bounds for these constants.

3.4. The difference of the resolvents. Separation of the main part. The estimate (3.10) is not sufficient for our purposes. We separate from the resolvent the terms up to the order \( t^2 \) and give the estimate of order \( t^3 \) for the reminder term. For this, we use representation of the form (2.13):
\[
R_{\gamma} (0) - R_{\gamma} (t) = \Omega_{\gamma} (0) T_{\gamma} (t) R_{\gamma} (t), \quad z \in \Gamma.
\]  
(11.1)

Here \( | t | \leq t^0 (\delta), \gamma = 2\delta \), and the operator \( T_{\gamma} (t) \) is defined by (3.6). The difference (11.1) can be written as
\[
R_{\gamma} (0) - R_{\gamma} (t) = t I_1 + t^2 I_2,
\]  
(11.2)

\[
I_k = \Omega_{\gamma} (0) T_{\gamma}^{(k)} R_{\gamma} (t), \quad k = 1, 2.
\]  
(11.3)

We start with the operator \( I_2 \). Iterating (11.1), we obtain
\[
I_2 = \Omega_{\gamma} (0) T_{\gamma}^{(2)} R_{\gamma} (0) = \Omega_{\gamma} (0) T_{\gamma}^{(2)} \Omega_{\gamma} (0) T_{\gamma} (t) R_{\gamma} (t) = I_2^0 - I_2^{(1)}.
\]

The operator
\[
I_2^0 = \Omega_{\gamma} (0) T_{\gamma}^{(2)} R_{\gamma} (0)
\]
do not depend on \( t \), and for \( I_2^{(1)} \) it suffices to prove the estimate of order \( | t | \). This estimate is a direct consequence of the inequality of the form (2.17). One should only use (7.4), (9.1), and take account of (3.3), (3.4) and of the inequality \( | z + \gamma |^{-1} \leq 2 \) for \( z \in \Gamma \). As a result, we obtain \( \| I_2^{(1)} \| \leq \beta_{\gamma}^2 t \delta^{-5/2} \| X_1 \|^3 \), whence
\[
t^2 I_2 = t^2 I_2^0 + \Psi_{\gamma} (t), \quad \| \Psi_{\gamma} (t) \| \leq \beta_{\gamma}^2 t \delta^{-5/2} \| X_1 \|^3
\]  
(15.1)
3.5. **Investigation of the operator** $I_1$ is somewhat more cumbersome. We have to separate from $I_1$ the terms of zero and first order in $t$, and for the remainder term give the controlled estimate. For this, we need to iterate formula (3.11) twice. Namely, according to (3.13),

$$
I_1 = \Omega_z(0)T^{(1)}_{\gamma} R_z(0) - \Omega_z(0)T^{(1)}_{\gamma} \Omega_z(0)T^{(1)}_{\gamma} R_z(t) = I^0_1 - I^{(1)}_1.
$$

Here

$$
(3.16) \quad I^0_1 = \Omega_z(0)T^{(1)}_{\gamma} R_z(0)
$$

does not depend on $t$. For the analysis of $I^{(1)}_1$, we need to iterate once more:

$$
I^{(1)}_1 = \Omega_z(0)T^{(1)}_{\gamma} \Omega_z(0)T^{(1)}_{\gamma} R_z(0) - \Omega_z(0)T^{(1)}_{\gamma} \Omega_z(0)T^{(1)}_{\gamma} \Omega_z(0)T^{(1)}_{\gamma} R_z(t) + t^2 \Omega_z(0)T^{(1)}_{\gamma} \Omega_z(0)T^{(1)}_{\gamma} R_z(0) - \Omega_z(0)T^{(1)}_{\gamma} T^{(1)}_{\gamma} R_z(t) - (z + 2\delta)\Omega_z(0)(T^{(1)}_{\gamma} R_z(0))(T^{(1)}_{\gamma} (t)\Omega_z(0)T^{(1)}_{\gamma} R_z(t)).
$$

The first summand on the right is equal to $tI^0_2$, where

$$
(3.17) \quad I^0_2 = \Omega_z(0)T^{(1)}_{\gamma} \Omega_z(0)T^{(1)}_{\gamma} R_z(0),
$$

and other summands can be estimated with the help of (2.16), (2.17) and (3.7)-(3.9). All three summands give contributions of the same type. As a result, we obtain

$$
(3.18) \quad \left\{ \begin{array}{l}
    tI_1 = tI^0_1 - t^2 I^0_2 + \Psi^0(t), \\
    \|\Psi^0(t)\| \leq \beta_2^2 t^2 \delta^{-3/2} \|X_1\|^3.
\end{array} \right.
$$

3.6. **The final result** now directly follows from relations (3.15), (3.18) and representation (3.12).

**Theorem 3.1.** Suppose that condition (3.1) is fulfilled, $z \in \Gamma$ and $\gamma = 2\delta$. Then

$$
(3.19) \quad R_z(0) - R_z(t) = tI^0_1 - t^2 I^0_2 + \Psi^0(t).
$$

Here $I^0_2 = I^0_1 - I^0_2$. The operators $I^0_1$, $I^0_2$, $I^0_2$ are defined by (3.16), (3.14), (3.17) respectively. We have

$$
\|\Psi^0(t)\| \leq \beta_2^2 t^2 \delta^{-3/2} \|X_1\|^3.
$$

§4. **Threshold approximations**

4.1. **The difference of the spectral projectors.** We start from the representation

$$
(4.1) \quad F(t, \delta) = -\frac{1}{2\pi i} \int_{\Gamma} R_z(t) \ dz.
$$

If $t = 0$, the left-hand side of (4.1) is equal to $P$, whence

$$
(4.2) \quad F(t, \delta) - P = -\frac{1}{2\pi i} \int_{\Gamma} (R_z(t) - R_z(0)) \ dz.
$$

Now, (3.16) and (4.2) directly imply that

$$
(4.3) \quad F(t, \delta) - P = \Phi(t), \quad \|\Phi(t)\| \leq \beta_1 \delta^{-1/2} \|X_1\|, \quad |t| \leq t^0(\delta).
$$

Thus, we have proved the following theorem.
Theorem 4.1. Let $X(t)$ and $A(t)$ be operator families introduced in Subsection 1.1. Suppose that $8 \delta < d^0$, where $d^0$ is the distance between the point $\lambda = 0$ and the rest of the spectrum of $A_0 = A(0)$. Finally, suppose that condition (3.1) is satisfied. If $F(t, \delta)$ is the spectral projector for the operator $A(t)$, corresponding to the interval $[0, \delta]$, and $P$ is the orthoprojector onto $\text{Ker} A_0$, then the estimate (4.3) is true.

Remark 4.2. 1) Along with (4.3), we have the trivial estimate $\| \Phi(t) \| \leq 2$. 2) The right-hand side of (4.3) can be represented as $\beta_1 |t|/t^0(\delta)$.

4.2. Approximation for $A(t)F(t, \delta)$. If $t = 0$, the left-hand side of the representation

$$A(t)F(t, \delta) = -\frac{1}{2\pi i} \int \frac{zR_z(t)}{\Gamma} dz$$

is equal to $A_0 P = 0$, whence

(4.4) $$A(t)F(t, \delta) = \frac{1}{2\pi i} \int \frac{z(R_z(0) - R_z(t))}{\Gamma} dz.$$

From (4.4), by (3.19), we obtain

(4.5) $$A(t)F(t, \delta) = tl_1 + t^2 l_2 + \Psi(t), \quad |t| \leq t^0(\delta),$$

where the bounded operators $l_1, l_2$ do not depend on $t$, and

(4.6) $$\| \Psi(t) \| \leq \beta_2 |t|^3 \delta^{-1/2} \| \bar{X}_1 \|^3, \quad |t| \leq t^0(\delta).$$

The coefficients $l_1, l_2$ can be expressed via the integrals over $\Gamma$ of $zI_1^0, zI_3^+$. However, it is easier to find these coefficients by comparing (4.5) and (1.38). Thus, $l_1 = 0, l_2 = SP$ and the following theorem is true.

Theorem 4.3. Under the assumptions of Theorem 4.1, we have

(4.7) $$A(t)F(t, \delta) - t^2 SP = \Psi(t), \quad |t| \leq t^0(\delta),$$

where $\Psi(t)$ satisfies the estimate (4.6).

Remark 4.4. In the conditions of Theorem 4.3, the germ $S$ is not assumed to be non-degenerate.

Now, let $\bar{A}(t)$ be one more operator family in $\mathcal{F}$ of the same type as $A(t)$. All the objects related to the family $\bar{A}$ will be marked by the upper index $\bar{\cdot}$. Relations (4.6), (4.7) directly imply the following theorem.

Theorem 4.5. Suppose that $8 \delta \leq \min(d^0, \bar{d}^0)$, and let the families $A$ and $\bar{A}$ be threshold equivalent, i.e., $\mathcal{F} = \mathcal{F}$ and $S = \bar{S}$. Then

$$\| A(t)F(t, \delta) - \bar{A}(t)\bar{F}(t, \delta) \| \leq \beta_2 |t|^3 \delta^{-1/2} (\| \bar{X}_1 \|^3 + \| \bar{X}_1 \|^3),$$

$$|t| \leq \delta^{1/2} \min(\| \bar{X}_1 \|^{-1}, \| \bar{X}_1 \|^{-1}).$$
4.3. Approximation for the imaginary exponent. For \( \tau \in \mathbb{R} \), consider the group \( \exp \{-i\tau A(t)\} \). We put

\[
(4.8) \quad E(\tau) := (\exp \{-i\tau A(t)\}) F(t, \delta) - (\exp \{-i\tau^2 SP\}) P, \\
\Sigma(\tau) := (\exp (i\tau^2 SP)) E(\tau) = (\exp (i\tau^2 SP)) F(t, \delta) (\exp \{-i\tau A(t)\}) - P.
\]

Note that \( \Sigma(0) = F(t, \delta) - P = \Phi(t) \). Next,

\[
\frac{d\Sigma(\tau)}{d\tau} = i e^{i\tau^2 SP} \left( t^2 SP F(t, \delta) - F(t, \delta) A(t) \right) e^{-i\tau A(t)},
\]

whence \( \|\Sigma'(\tau)\| = \|\Psi(t) F(t, \delta)\| \leq \|\Psi(t)\| \) and \( \|E(\tau)\| = \|\Sigma(\tau)\| \leq \|\Phi(t)\| + |\tau| \|\Psi(t)\| \), where \( \Phi(t) \) and \( \Psi(t) \) satisfy the estimates (4.3), (4.6). Thus, we have proved the following theorem.

**Theorem 4.6.** Under the assumptions of Theorem 4.1, the operator-valued function (4.8) satisfies the estimate

\[
\|E(\tau)\| \leq \beta_1 \|\delta^{-1/2}\| X_1 + \beta_2 |\tau| t^3 \delta^{-1/2} \|X_1\|^3, \quad |t| \leq t^3(\delta).
\]

Theorem 4.6 automatically implies the statement analogous to Theorem 4.5. We shall not formulate this statement.

§ 5. Approximation for the operator-valued function \((A(t) + \varepsilon^2 I)^{-1}\)

5.1. Statement of the problem. Our goal is to approximate the resolvent of \( A(t) \) by the resolvent of the corresponding germ \( S \). As compared with assumptions of Theorems 4.1, 4.3, now we need an additional condition. Namely, assume that

\[
(5.1) \quad A(t) F(t, \delta) \geq c_* t^2 F(t, \delta), \quad c_* > 0, \quad |t| \leq t^3(\delta).
\]

Condition (5.1) is equivalent to the following estimate for the eigenvalues \( \lambda_l(t) \), \( l = 1, \ldots, n \), of the family \( A(t) \) (introduced in (1.27)):

\[
(5.2) \quad \lambda_l(t) \geq c_* t^2, \quad l = 1, \ldots, n, \quad c_* > 0, \quad |t| \leq t^3(\delta).
\]

Clearly, under this condition, the germ \( S \) of the family \( A(t) \) is non-degenerate, and the inequalities (1.36) are valid with the same constant \( c_* \) as in (5.2). In other words, \( S \geq c_* I_{\mathcal{M}} \), where \( I_{\mathcal{M}} \) is the unit operator in \( \mathcal{M} \).

**Remark 5.1.** In fact, by (1.28), inequalities (1.36) yield inequalities of the form (5.2), but with smaller constant \( c_* \) and for \( t \) belonging to a narrower interval. In applications, it is often more convenient to check the validity of (5.2) directly.

We have to estimate the norm of the operator-valued function

\[
(5.3) \quad G(\varepsilon, t, \delta) := (A(t) + \varepsilon^2 I)^{-1} F(t, \delta) - (t^2 SP + \varepsilon^2 I)^{-1} P, \quad \varepsilon > 0, \quad |t| \leq t^3(\delta).
\]

First, we note that

\[
(t^2 SP + \varepsilon^2 I)^{-1} P = (t^2 S + \varepsilon^2 I_{\mathcal{M}})^{-1} P,
\]
where $t^2 S + \varepsilon^2 I_{2n}$ is viewed as an operator in $\mathcal{H}$.

The following estimates are direct consequences of (5.1), (1.36):

$$\| (A(t) + \varepsilon^2 I)^{-1} \| \leq (c_3 t^2 + \varepsilon^2)^{-1}, \quad |t| \leq t^0(\delta),$$

$$\| (t^2 S + \varepsilon^2 I_{2n})^{-1} \| \leq (c_3 t^2 + \varepsilon^2)^{-1}, \quad |t| \leq t^0(\delta).$$

For a while, we shall use the abridged notation

$$A(t) = A, \quad F(t, \delta) = F, \quad G(\varepsilon, t, \delta) = G, \quad |t| \leq t^0(\delta).$$

Now it is convenient to separate the factor depending on $t$ in the estimates (4.3), (4.6). Correspondingly, we have

$$\| \Phi(t) \| \leq c_1 t, \quad C_1 = \beta_1 t^{-1/2} \| X_1 \| = \beta_1 (t^0(\delta))^{-1}, \quad |t| \leq t^0(\delta),$$

$$\| \Psi(t) \| \leq C_2 |t|, \quad C_2 = \beta_2 t^{-1/2} \| X_1 \|^{1/2} = \beta_2 (t^0(\delta))^{-1}, \quad |t| \leq t^0(\delta).$$

Finally, we write the obvious identity

$$\frac{A + \varepsilon^2 I}{A + \varepsilon^2 I} = \frac{AF + \varepsilon^2 I}{AF + \varepsilon^2 I} F.$$

### 5.2. Approximation of the resolvent near the threshold

Our nearest goal is the proof of the following theorem.

**Theorem 5.2.** Under condition (5.2), the operator-valued function (5.3) satisfies the estimate

$$\| G(\varepsilon, t, \delta) \| \leq C_0 |t| (c_3 t^2 + \varepsilon^2)^{-1}, \quad \varepsilon > 0, \quad |t| \leq t^0(\delta), \quad C_0 = 2C_1 + c_3^{-1} C_2,$$

whence,

$$2\varepsilon \| G(\varepsilon, t, \delta) \| \leq C, \quad C = c_3^{-1} C_0, \quad \varepsilon > 0, \quad |t| \leq t^0(\delta).$$

Here $c_3$ is the constant from (5.2), and $C_1, C_2$ are defined by (5.6), (5.7).

**Proof.** We start with the identity

$$F(\frac{AF + \varepsilon^2 I}{AF + \varepsilon^2 I}) = \frac{AF + \varepsilon^2 I}{AF + \varepsilon^2 I} F - \frac{AF + \varepsilon^2 I}{AF + \varepsilon^2 I},$$

which, by (5.8), implies that $G = G_1 + G_2 + G_3$, where

$$G_1 = (A + \varepsilon^2 I)^{-1} F \Phi(t),$$

$$G_2 = \Phi(t) (t^2 S P + \varepsilon^2 I)^{-1} P,$$

$$G_3 = -F(A + \varepsilon^2 I)^{-1} \Psi(t) (t^2 S P + \varepsilon^2 I)^{-1} P.$$

Recall that the operators $\Phi(t) = F - P$ and $\Psi(t) = AF - t^2 SP$ satisfy the estimates (5.6) and (5.7). From (5.4)–(5.7) it follows that, for $\varepsilon > 0$ and $|t| \leq t^0(\delta)$,

$$\| G_1 \| + \| G_2 \| \leq 2C_1 (c_3 t^2 + \varepsilon^2)^{-1} |t|,$$

$$\| G_3 \| \leq C_2 (c_3 t^2 + \varepsilon^2)^{-2} |t|^3,$$

$$\leq c_3^{-1} C_2 (c_3 t^2 + \varepsilon^2)^{-1} |t|.$$

This directly implies the estimate (5.9). \(\square\)

**Remark 5.3.** The estimate (5.9) is two-parametric. For a fixed $t$, the operator-valued function $G(\varepsilon, t, \delta)$ is bounded as $\varepsilon \to 0$. The estimate (5.10) (which follows from (5.9)) is uniform in $t$ for $|t| \leq t^0(\delta)$.

Now, let $\tilde{A}(t)$ be one more family of the form (1.2), satisfying the analogue of condition (5.1). The estimate (5.10) directly implies the following theorem.
Theorem 5.4. Let the families $A(t)$ and $\widetilde{A}(t)$ be threshold equivalent, and let $8\delta < \min\{d^i, d^0\}$. Then, for $\varepsilon > 0$ and $|t| \leq \delta^{1/2} \min\{\|X_1\|^{-1}, \|X_1\|^{-1}\} = \min\{t^0(\delta), t^0(\varepsilon)\}$, the operator-valued function

$$J = (A(t) + \varepsilon^2 I)^{-1} \widetilde{F}(t, \delta) - (\widetilde{A}(t) + \varepsilon^2 I)^{-1} \tilde{F}(t, \delta),$$

satisfies the estimate

$$2\varepsilon \|J\| \leq C + \tilde{C} = c^{-1/2}(2C_1 + c^{-1}C_2) + c^{-1/2}(2\tilde{C}_1 + \tilde{c}^{-1}\tilde{C}_2).$$

By Proposition 1.2, $\|(A(t) + \varepsilon^2 I)^{-1} F(t, \delta)^{-1}\| \leq (3\delta)^{-1}$. Hence, Theorems 5.2 and 5.4 directly imply the following two statements.

Theorem 5.5. Under the assumptions of Theorem 5.2, we have

$$\varepsilon \|(A(t) + \varepsilon^2 I)^{-1} - (t^2 S P + \varepsilon^2 I)^{-1} P\| \leq 2^{-1}C + \varepsilon(3\delta)^{-1}, \quad \varepsilon > 0, \quad |t| \leq t^0(\delta).$$

Theorem 5.6. Under the assumptions of Theorem 5.4, we have

$$\varepsilon \|(A(t) + \varepsilon^2 I)^{-1} - (\widetilde{A}(t) + \varepsilon^2 I)^{-1}\| \leq 2^{-1}(C + \tilde{C}) + 2\varepsilon(3\delta)^{-1}, \quad \varepsilon > 0, \quad |t| \leq t^0(\delta).$$

5.3. Approximation of the generalized resolvent near the threshold*

Theorem 5.5 can be extended to the case where the resolvent $(A(t) + \varepsilon^2 I)^{-1}$ is replaced by the operator $(\widetilde{A}(t) + \varepsilon^2 Q)^{-1}$. Here $Q$ is a bounded positive definite operator in $\hat{H}$. We shall deduce this useful generalization from Theorem 5.5 on the basis of Subsection 1.5. We shall use the notation and material of Subsection 1.5 without repeating explanations. We note only, that, by (1.20),

$$M (A(t) + \varepsilon^2 I)^{-1} M^* = (\widetilde{A}(t) + \varepsilon^2 Q)^{-1},$$

where

$$Q = (M^*)^{-1} M^{-1} = (M M^*)^{-1} : \hat{H} \to \hat{H}.$$  

We start with preliminary remarks. Since $\hat{P} \perp MP = 0$, then $PM^*\hat{P} \perp = 0$. Hence,

$$PM^*\hat{P} = PM^*.$$

Next, since $MP M^{-1}\hat{P} = \hat{P}$, then

$$\hat{P} = \hat{P} (M^{-1})^* PM^*.$$

Consider the block $Q_{\mathfrak{R}}$ of the operator $Q$ in the subspace $\hat{H}$:

$$Q_{\mathfrak{R}} = P Q_{\mathfrak{R}} : \hat{H} \to \hat{H}.$$

By (5.2), the germ $S$ of the family $A(t)$ is non-degenerate. Then the germ $\hat{S}$ of $\widetilde{A}(t)$ is also non-degenerate. Therefore, the operator

$$(t^2 \hat{S} + \varepsilon^2 Q_{\mathfrak{R}})^{-1} : \hat{H} \to \hat{H}$$

exists.

---

*The results of the present Subsection were obtained by the authors with participation of R. G. Shitserenberg.
Proposition 5.7. We have

\begin{equation}
M(t^2SP + \varepsilon^2 I)^{-1} P M^* = (t^2 \hat{S} + \varepsilon^2 Q) R^{-1} \hat{P}.
\end{equation}

Proof. Both operators in (5.18) act in \( \hat{H} \) and take \( \hat{H} \) into itself. The operator on the left can be represented (cf. (5.15)) as

\[ \hat{P} M(t^2SP + \varepsilon^2 I)^{-1} P M^* \hat{P}. \]

We put \( \hat{\zeta} = M(t^2SP + \varepsilon^2 I)^{-1} P M^* \hat{\eta} \). Let \( \hat{\eta} \in \hat{H} \), whence \( \hat{\zeta} \in \hat{H} \) and \( P M^* \hat{\eta} = (t^2SP + \varepsilon^2 I) M^{-1} \hat{\zeta} \). By (5.14), (5.16), (5.17) and also (1.24),

\[ \hat{\eta} = t^2 \hat{P} (M^*)^{-1} SM^{-1} \hat{\zeta} + \varepsilon^2 \hat{P} (M^*)^{-1} M^{-1} \hat{\zeta} = (t^2 \hat{S} + \varepsilon^2 Q) \hat{\zeta}. \]

Thus, \( \hat{\zeta} = (t^2 \hat{S} + \varepsilon^2 Q) \hat{\eta} \), which is equivalent to (5.18). \( \square \)

For the operator-valued function

\begin{equation}
\mathcal{Y}(\varepsilon, t) = (A(t) + \varepsilon^2 I)^{-1} - (t^2 SP + \varepsilon^2 I)^{-1} P
\end{equation}

we write the estimate (5.11) as

\begin{equation}
\varepsilon \| \mathcal{Y}(\varepsilon, t) \| \leq C, \quad C = 2^{-1} C + (3\delta)^{-1}, \quad 0 < \varepsilon \leq 1, \quad |t| \leq t^0(\delta).
\end{equation}

By (5.13), (5.18), (5.19),

\[ M\mathcal{Y}(\varepsilon, t) M^* = (\hat{A}(t) + \varepsilon^2 Q)^{-1} - (t^2 \hat{S} + \varepsilon^2 Q) R^{-1} \hat{P}. \]

Combining this with (5.20), we obtain

\begin{equation}
\varepsilon \| (\hat{A}(t) + \varepsilon^2 Q)^{-1} - (t^2 \hat{S} + \varepsilon^2 Q) R^{-1} \hat{P} \| \leq C \| M \|^2 = C \| Q \|^2.
\end{equation}

All the terms in (5.21) are expressed in terms of the family \( \hat{A}(t) \) and the operator \( Q \) acting in \( \hat{H} \). Thus, the estimate (5.21) can be treated as a generalization of the estimate (5.20) for the operator-valued function (5.19). However, the constant \( C \) remains from the initial family. One can recalculate this constant (with roughening), but, for applications, there is no need to do this.

Now, let \( \hat{A}_+(t) \) be one more family of the form \( \hat{A}(t) \) acting in \( \hat{H} \), and let \( Q_+ \) be a bounded positive definite operator in \( \hat{H} \). Suppose that the families \( \hat{A}(t) \) and \( \hat{A}_+(t) \) are threshold equivalent and the blocks of the operators \( Q \) and \( Q_+ \) in \( \hat{H} \) coincide.

Next, let \( M_+ = (Q_+)^{-1/2} \) and \( A_+(t) = M_+ A_+(t) M_+ \), and let \( C_+ \) be the analogue of the constant \( C \) (cf. (5.20)) for the family \( A_+(t) \).

Theorem 5.8. Under the above conditions, we have

\begin{equation}
\varepsilon \| (A(t) + \varepsilon^2 Q)^{-1} - (\hat{A}_+(t) + \varepsilon^2 Q_+)^{-1} \| \leq C \| Q \|^2 + C_+ \| Q_+ \|^2,
\end{equation}

where \( 0 < \varepsilon \leq 1, \delta_0 < \delta, |t| \leq t^0(\delta) \).

Finally, (5.13), (5.14) and (5.22) directly imply the following theorem.

Theorem 5.9. Under the assumptions of Theorem 5.8, we have

\[ \varepsilon \| (A(t) + \varepsilon^2 I)^{-1} - M^{-1} (\hat{A}_+(t) + \varepsilon^2 Q_+)^{-1} (M^*)^{-1} \| \leq \| M \|^2 (C \| M \|^2 + C_+ \| Q_+ \|^2). \]
Chapter 2. Periodic differential operators in $L_2(\mathbb{R}^d; \mathbb{C}^n)$

§ 1. Main definitions. Preliminaries

1.1. Factorized second order operators. We consider the selfadjoint matrix second order DO’s represented as a product of two mutually adjoint first order DO’s. Let $b(D) : L_2(\mathbb{R}^d; \mathbb{C}^n) \to L_2(\mathbb{R}^d; \mathbb{C}^n)$ be a homogeneous first order DO with constant coefficients. We always assume that

(1.1) \[ m \geq n. \]

We fix orthonormal bases $\bar{e}_1, \ldots, \bar{e}_n$ in $\mathbb{C}^n$, $e_1, \ldots, e_m$ in $\mathbb{C}^m$ and $e'_1, \ldots, e'_d$ in $\mathbb{R}^d$. Then the operator $b(D)$ is related to the symbol $b(\xi)$. This symbol is $(m \times n)$-matrix-valued linear homogeneous function of $\xi \in \mathbb{R}^d$. Suppose that

(1.2) \[ \text{rank } b(\xi) = n, \quad 0 \neq \xi \in \mathbb{R}^d. \]

Relation (1.2) yields that

(1.3) \[ \alpha_0 \mathbf{1}_n \leq b(\theta)^* b(\theta) \leq \alpha_1 \mathbf{1}_n, \quad |\theta| = 1, \quad 0 < \alpha_0 \leq \alpha_1 < \infty. \]

Note that

(1.4) \[ b(\xi) = \sum_{j=1}^d \xi^j b_j, \quad \xi = \sum_{j=1}^d \xi^j e'_j, \]

where $b_j$ are constant $(m \times n)$-matrices such that rank $b_j = n$.

Suppose that $(n \times n)$-matrix-valued function $f(x)$ and $(m \times m)$-matrix-valued function $h(x)$, $x \in \mathbb{R}^d$, are bounded and the inverse matrices are bounded:

(1.5) \[ f, f^{-1} \in L_\infty(\mathbb{R}^d); \quad h, h^{-1} \in L_\infty(\mathbb{R}^d). \]

We consider the DO

(1.6) \[ \mathcal{X} := hh(D) f : L_2(\mathbb{R}^d; \mathbb{C}^n) \to L_2(\mathbb{R}^d; \mathbb{C}^n), \]

(1.7) \[ \text{Dom} \mathcal{X} := \{ u \in L_2(\mathbb{R}^d; \mathbb{C}^n) : fu \in H^1(\mathbb{R}^d; \mathbb{C}^n) \}. \]

The operator (1.6) is closed on the domain (1.7). The selfadjoint operator

(1.8) \[ \mathcal{A} := \mathcal{X}^* \mathcal{X} \]

in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ is generated by the closed quadratic form

(1.9) \[ a[u, u] := \| \mathcal{X} u \|^2_{L_2(\mathbb{R}^d; \mathbb{C}^n)}, \quad u \in \text{Dom} \mathcal{X}. \]

Formally,\n
(1.10) \[ \mathcal{A} = f(x)^* b(D)^* g(x) b(D) f(x), \quad g(x) := h(x)^* h(x). \]

Using the Fourier transformation and conditions (1.3), (1.5), it is easily seen that

(1.11) \[ c_0 \int_{\mathbb{R}^d} |D(fu)|^2 \, dx \leq a[u, u] \leq c_1 \int_{\mathbb{R}^d} |D(fu)|^2 \, dx, \quad u \in \text{Dom} \mathcal{X}, \]

(1.12) \[ c_0 = \alpha_0 \| h^{-1} \|^2_{L_\infty}, \quad c_1 = \alpha_1 \| h \|^2_{L_\infty}. \]
1.2. The lattices $\Gamma$ and $\overline{\Gamma}$. In what follows, the functions $f$ and $h$ (and the operators $X$, $A$ with them) are assumed to be periodic with respect to some lattice $\Gamma \subset \mathbb{R}^d$:

$$f(x + a) = f(x), \quad h(x + a) = h(x), \quad a \in \Gamma.$$

Let $a_1, \ldots, a_d \in \mathbb{R}^d$ be a basis in $\mathbb{R}^d$ generating the lattice $\Gamma$:

$$\Gamma = \left\{ a \in \mathbb{R}^d : a = \sum_{j=1}^{d} \nu_j a_j, \nu_j \in \mathbb{Z} \right\},$$

and let $\Omega$ be the (elementary) cell of the lattice $\Gamma$:

$$(1.13) \quad \Omega := \left\{ x \in \mathbb{R}^d : x = \sum_{j=1}^{d} \tau_j a_j, \ 0 < \tau_j < 1 \right\}.$$

The basis $b^1, \ldots, b^d$ in $\mathbb{R}^d$ dual with respect to $a_1, \ldots, a_d$ is defined by the relations $\langle b^i, a_j \rangle = 2\pi \delta^i_j$. This basis generates the lattice $\overline{\Gamma}$ dual with respect to $\Gamma$:

$$\overline{\Gamma} = \left\{ b \in \mathbb{R}^d : b = \sum_{i=1}^{d} \mu_i b^i, \mu_i \in \mathbb{Z} \right\}.$$

The cell of the lattice $\overline{\Gamma}$ can be defined by analogy with (1.13). However, it is more convenient to denote by $\tilde{\Omega}$ the Brillouin zone

$$\tilde{\Omega} = \{ k \in \mathbb{R}^d : |k| < |k - b|, \ 0 \neq b \in \overline{\Gamma} \}.$$

As well as the cell, the domain $\tilde{\Omega}$ is a fundamental domain for $\overline{\Gamma}$. It means that all the $\Gamma$-shifted copies of $\tilde{\Omega}$ are mutually disjoint, and the union of the $\Gamma$-shifted copies of $\text{clos} \ \tilde{\Omega}$ covers $\mathbb{R}^d$. The closure $\text{clos} \ \tilde{\Omega}$ is a convex polyhedron. The points $k \in \partial \tilde{\Omega}$ are characterized by the relation $|k| = \min |k - b|, \ 0 \neq b \in \overline{\Gamma}$. We shall use the notation $|\tilde{\Omega}| = \text{mes} \ \tilde{\Omega}, \ |\overline{\Omega}| = \text{mes} \ \overline{\Omega}$. Note that $|\tilde{\Omega}|/|\overline{\Omega}| = (2\pi)^d$. Let $r_0$ be the radius of the ball inscribed into $\text{clos} \ \tilde{\Omega}$. Then

$$(1.14) \quad 2r_0 = \min |b|, \ 0 \neq b \in \overline{\Gamma}.$$

We denote

$$\mathcal{K}(r) = \{ k \in \mathbb{R}^d : |k| \leq r \}, \quad 0 < r \leq r_0.$$

The discrete Fourier transformation

$$(1.15) \quad \hat{v}(x) = |\Omega|^{-1/2} \sum_{b \in \overline{\Gamma}} v_b \exp(i \langle b, x \rangle), \quad x \in \Omega,$$

is related to the lattice $\Gamma$. This transformation unitarily maps $l_2(\overline{\Gamma})$ onto $L_2(\Omega)$:

$$\int_{\overline{\Gamma}} |v(x)|^2 \, dx = \sum_{b \in \overline{\Gamma}} |v_b|^2.$$
Below, \( \tilde{H}^1(\Omega) \) stands for the subspace of those functions in \( H^1(\Omega) \), the \( \Gamma \)-periodic extension of which to \( \mathbb{R}^d \) belongs to \( H^1_{\text{loc}}(\mathbb{R}^d) \). We have

\[
\int_{\tilde{\Omega}} |((D + k)v)(x)|^2 \, dx = \sum_{b \in \Gamma} |b + k|^2 |v_b|^2, \quad v \in \tilde{H}^1(\Omega; \mathbb{C}^n), \quad k \in \mathbb{R}^d.
\]

The convergence of the series in (1.16) is equivalent to the fact that \( v \in \tilde{H}^1(\Omega; \mathbb{C}^n) \). Let us mention more general relation. Let \( b(\xi) \) be the symbol introduced in Subsection 1.1. Then

\[
\int_{\tilde{\Omega}} |b(D + k)v|^2 \, dx = \sum_{b \in \Gamma} |b|^2 |v_b|^2, \quad v \in \tilde{H}^1(\Omega; \mathbb{C}^n), \quad k \in \mathbb{R}^d.
\]

1.3. The Gelfand transformation \( \mathcal{U} \) initially is defined for functions of the Schwartz class \( \mathcal{S} \) by the formula

\[
\tilde{v}(k, x) = (\mathcal{U}v)(k, x) = |\tilde{\Omega}|^{-1/2} \sum_{a \in \Gamma} \exp(-i(k \cdot x + a))v(x + a), \quad v \in \mathcal{S}(\mathbb{R}^d; \mathbb{C}^n), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{R}^d.
\]

Herewith,

\[
\int_{\tilde{\Omega}} \int_{\mathbb{R}^d} |\tilde{v}(k, x)|^2 \, dx \, dk = \int_{\mathbb{R}^d} |v(x)|^2 \, dx, \quad \tilde{v} = \mathcal{U}v,
\]

and \( \mathcal{U} \) is extended by continuity to a unitary mapping

\[
\mathcal{U} : L_2(\mathbb{R}^d; \mathbb{C}^n) \to \int_{\tilde{\Omega}} \oplus L_2(\Omega; \mathbb{C}^n) \, dk =: \mathcal{H}.
\]

Next, relation \( v \in H^1(\mathbb{R}^d; \mathbb{C}^n) \) is equivalent to the fact that \( \tilde{v}(k, \cdot) \in \tilde{H}^1(\Omega) \) for almost every \( k \in \tilde{\Omega} \) and

\[
\int_{\tilde{\Omega}} \int_{\mathbb{R}^d} \left( |(D + k)\tilde{v}(k, x)|^2 + |\tilde{v}(k, x)|^2 \right) \, dx \, dk < \infty.
\]

Under the transformation \( \mathcal{U} \), the operator of multiplication by a bounded periodic function in \( L_2(\mathbb{R}^d; \mathbb{C}^n) \) turns into multiplication by the same function in the fibers of the direct integral \( \mathcal{H} \) (cf. (1.18)). In the fibers of the direct integral, the operator \( b(D) \) applied to \( v \in H^1(\mathbb{R}^d; \mathbb{C}^n) \) turns into the operator \( \mathcal{U}b(D + k) \) applied to \( \tilde{v}(k, \cdot) \in \tilde{H}^1(\Omega; \mathbb{C}^n) \).

§2. Direct integral decomposition for the operator \( \mathcal{A} \)

2.1. The forms \( a(k) \) and the operators \( \mathcal{A}(k) \). We put

\[
H = L_2(\Omega; \mathbb{C}^n), \quad H_* = L_2(\Omega; \mathbb{C}^n),
\]
and consider the closed operator
\[ \mathcal{X}(k) : \mathcal{H} \to \mathcal{H}, \quad k \in \mathbb{R}^d, \]
defined on the domain
\[ \mathcal{D} := \text{Dom } \mathcal{X}(k) = \{ u \in \mathcal{H} : f u \in \overline{H}^1(\Omega; \mathbb{C}^n) \} \]
by the formula
\[ \mathcal{X}(k) = h h(D + k) f, \quad k \in \mathbb{R}^d. \]
The selfadjoint operator
\[ \mathcal{A}(k) := \mathcal{X}(k)^* \mathcal{X}(k) : \mathcal{H} \to \mathcal{H}, \quad k \in \mathbb{R}^d, \]
is generated by the closed quadratic form
\[ a(k)[u, u] := ||\mathcal{X}(k)u||^2_{\mathcal{H}}, \quad u \in \mathcal{D}, \quad k \in \mathbb{R}^d. \]
Using the Fourier series expansion (1.15), identity (1.17) and estimates (1.3), we see that
\[ c_0 \int_\Omega |(D + k)v|^2 \, dx \leq a(k)[u, u] \leq c_1 \int_\Omega |(D + k)v|^2 \, dx, \]
\[ v = f u \in \overline{H}^1(\Omega; \mathbb{C}^n), \quad k \in \mathbb{R}^d, \]
where \( c_0, c_1 \) are defined by (1.12). From (2.6) and the compactness of the embedding of \( \overline{H}^1(\Omega; \mathbb{C}^n) \) into \( \mathcal{H} \) it follows that the spectrum of \( \mathcal{A}(k) \) is discrete. Note also that the resolvent of the operator \( \mathcal{A}(k) \) is compact and depends on \( k \in \mathbb{R}^d \) continuously (in the operator norm).

Let
\[ \mathfrak{N} := \text{Ker } \mathcal{A}(0) = \text{Ker } \mathcal{X}(0). \]
Relations (2.6) with \( k = 0 \) show that
\[ \mathfrak{N} = \{ u \in L^2(\Omega; \mathbb{C}^n) : f u = c \in \mathbb{C}^n \}, \quad \dim \mathfrak{N} = n. \]
We see that \( \mathfrak{N} \) does not depend on the matrix \( g \).

### 2.2. The band functions.

Let
\[ E_1(k) \leq E_2(k) \leq \cdots \leq E_s(k) \leq \cdots, \quad k \in \mathbb{R}^d, \]
denote the consecutive eigenvalues of the operator \( \mathcal{A}(k) \) (counted with multiplicities). The band functions \( E_s(k) \) are continuous and \( \overline{\Gamma} \)-periodic.

In \( \mathfrak{N} \), we consider the closed quadratic form
\[ a^0(k)[v, v] = \int_\Omega |(D + k)v|^2 \, dx, \quad v \in \overline{H}^1(\Omega; \mathbb{C}^n). \]
By (1.16), \( a^0(k)[v, v] = \sum_{n \in \Gamma} |b + k|^2 |\nu_n|^2 \). This implies that the band functions \( E^0_j(k) \) of the corresponding operator \( A^0(k) \) are reduced to the numbers \( |b + k|^2 \), \( b \in \Gamma \). Each of these eigenvalues is of multiplicity \( n \). Let us list the properties of the functions \( E^0_j(k) \) necessary for what follows. They are quite obvious.

1°. \( E^0_j(k) = |b|^2, \quad l = 1, \ldots, n, \ k \in \text{clos } \Omega. \)

2°. \( E^0_j(k) \geq r^2, \quad k \in \text{clos } \Omega \setminus \mathcal{K}(r), \ 0 < r \leq r_0. \)

3°. \( E^0_{n+1}(k) \geq r^2_0, \quad k \in \mathbb{R}^d. \)

4°. \( E^0_{n+1}(0) = \min_{b \neq 0, \ |b| > 0} |b|^2 = 4r^2_0. \)

Now, we return to consideration of the functions (2.9). They coincide with the consecutive minima of the quotient \( q(k)[u, u]/||u||^2_0 \). By (2.6), this quotient can be estimated from below as follows: for \( u \in \mathcal{D}, \ v = f(u), \)

\[
\frac{a(k)[u, u]}{||u||^2_0} \geq c^*_0 \frac{a^0(k)[v, v]}{||v||^2_0} \geq c^*_0 \|f^{-1}\|^2_\infty \frac{a^0(k)[v, v]}{||v||^2_0}.
\]

From the variational arguments it follows that

\[
E_j(k) \geq c_* E^0_j(k),
\]

where, according to (1.12),

\[
c_* = a^0 \|f^{-1}\|^2_\infty \|h^{-1}\|^2_\infty.
\]

Combining estimates (2.10) with the properties 1°–4° of the functions \( E^0_j(k) \), we obtain the following:

\[
E_j^l(k) \geq c_* k^2, \quad l = 1, \ldots, n, \ k \in \text{clos } \Omega,
\]

\[
E_1(k) \geq c_* r^2, \quad k \in \text{clos } \Omega \setminus \mathcal{K}(r), \ 0 < r \leq r_0,
\]

\[
E_{n+1}(k) \geq c_* r^2_0, \quad k \in \mathbb{R}^d,
\]

\[
E_{n+1}(0) \geq 4c_* r^2_0.
\]

2.3. The direct integral for the operator \( A \). The operators \( A(k) \) allow us to partially diagonalize the operator \( A \) in the direct integral \( \mathcal{H} \) (cf. (1.18)). Let us compare relations (1.6), (1.7), (1.9), defining the form \( a \) of the operator \( A \) with relations (2.2), (2.3), (2.5), defining the form \( a(k) \) of the operator \( A(k) \). For \( u \in \text{Dom } \mathcal{X} = \text{Dom } a \), we denote \( v = f(u) \). Then, what was said in Subsection 1.3 for \( v \) directly implies the following for \( u \). Let \( \tilde{u} = \hat{u} + u, \ u \in \text{Dom } a \). Then

\[
\tilde{u}(k, \cdot) \in \mathcal{D} \quad \text{for a. e. } \ k \in \mathbb{R}^d,
\]

\[
a[u, u] = \int_{\hat{u}} a(k)[\tilde{u}(k, \cdot), \tilde{u}(k, \cdot)] dk.
\]

Conversely, if for \( \tilde{u} \in \mathcal{H} \) relation (2.15) is fulfilled and the integral in (2.16) is finite, then \( u \in \text{Dom } a \) and (2.16) is valid. The above arguments show that, in the direct
integral $\mathcal{H}$, the operator $\mathcal{A}$ (cf. (1.8)) turns into multiplication by the operator-valued function $\mathcal{A}(k), k \in \Omega$, defined by (2.4). Briefly, all this can be expressed by the formula

$$ (2.17) \quad \mathcal{U} \mathcal{A} \mathcal{U}^{-1} = \int_\Omega \mathcal{A}(k) \, dk. $$

From (2.17) it follows that the spectrum $\sigma(\mathcal{A})$ of the operator $\mathcal{A}$ is the union of segments (spectral bands), that is the ranges of the band functions (2.9). Relations (2.7), (2.8) imply that

$$ \min_{k \in \mathbb{R}^d} E_j(k) = E_j(0) = 0, \quad j = 1, \ldots, n, $$

whence, the lower edge of the spectrum of the operator $\mathcal{A}$ coincides with the point $\lambda = 0$: $\inf \sigma(\mathcal{A}) = 0$.

§3. Incorporation of the operators $\mathcal{A}(k)$ into the scheme of §1, Chapter 1

3.1. Agreement of notions. For $d > 1$, the operators $\mathcal{A}(k)$ depend on the multi-dimensional parameter $k$. In this case, the analytic perturbation theory gives satisfactory results only for simple eigenvalues. The role of the unperturbed operator $\mathcal{A}_0$ is played by $\mathcal{A}(0)$. Now, by (2.7), (2.8), we have $\dim \mathcal{N} = n > 1$, i.e., when $n > 1$, the eigenvalue $\lambda = 0$ for $\mathcal{A}(0)$ is multiple. To avoid this difficulty, for $k \in \mathbb{R}^d$ we put $k = t\theta, t = |k|, |\theta| = 1$, and view $t$ as the perturbation parameter. At the same time, all the constructions will depend on the additional parameter $\theta$. This dependence will be often reflected in the notation.

We shall apply the scheme of §1, Chapter 1, putting

$$ (3.1) \quad \mathcal{H} = L_2(\Omega; \mathbb{C}^n), \quad \mathcal{H}_* = L_2(\Omega, \mathbb{C}^n), $$

which coincides with (2.1). Next,

$$ (3.2) \quad X(t) = X(t, \theta) = \mathcal{X}(t\theta) = hh(D + t\theta)f, $$

$$ (3.3) \quad \begin{cases} X_0 = \mathcal{X}(0) = hh(D)f, & \text{Dom } X_0 = \mathcal{V}, \\ X_1 = X_1(\theta) = hh(\theta)f, & X(t) = X_0 + tX_1, \end{cases} $$

and, in (3.3), we can assume that $t \in \mathbb{R}$. Finally,

$$ (3.4) \quad A(t) = A(t, \theta) = \mathcal{A}(t\theta), $$

and, according to (2.7), (2.8),

$$ \mathcal{N} = \text{Ker } X_0 = \text{Ker } \mathcal{X}(0), \quad \dim \mathcal{N} = n. $$

Relation (1.1) guarantees that $n \leq n_*$. Moreover, we have

$$ (3.5) \quad \mathcal{N}_* = \text{Ker } X_0^* = \{ q \in L_2(\Omega; \mathbb{C}^n) : h^*q \in \tilde{H}^1(\Omega; \mathbb{C}^n), \ b(D)^*h(x)^*q = 0 \}. $$
Then, under condition (1.1), for \( n_* = \dim \mathfrak{N} \), the following alternative realizes: either \( n_* = \infty \) (when \( m > n \)), or \( n_* = n \) (when \( m = n \)). According to (2.14), instead of the precise value \( d^0 = E_{n+1}(0) \), we take

\[
d^0 = 4c_* r_0^2,
\]

which is, probably, understated. Here \( c_* \) is the constant (2.11). In Subsection 1.1.1, it was required to choose \( \delta \) such that \( \delta < d^0/\delta \). Now, taking account of (3.6), we fix \( \delta \) so that to have

\[
\delta < c_* r_0^2/2 = 2^{-1} r_0^2 \alpha_0 \| f^{-1} \|_{L_{\infty}} \| h^{-1} \|_{L_{\infty}}^{-2}.
\]

Next, the estimate \( \|X_1(\Theta)\| \leq \alpha_1^{1/2} \| f \|_{L_{\infty}} \| h \|_{L_{\infty}} \) allows us to choose \( t^0(\delta) \) (cf. (1.1.3)) not equal to \( \delta^{1/2} \|X_1(\Theta)\|^{-1} \), but equal to the lower number which is independent of \( \Theta \). Namely, we put

\[
t^0(\delta) = \delta^{1/2} \alpha_1^{-1/2} \| f \|_{L_{\infty}}^{-1} \| h \|_{L_{\infty}}^{-1}.
\]

Note that, by (3.7), (3.8), \( t^0(\delta) < r_0/\sqrt{2} \). Thus,

\[
\{ k \in \tilde{\Omega} : |k| = t \leq t^0(\delta) \} \subset K(2^{-1/2} r_0) \subset \tilde{\Omega}.
\]

In what follows, we always assume that \( c_* \) and \( t^0(\delta) \) are defined by (2.11) and (3.8).

3.2. The non-degeneracy of the germ of the family \( A(t, \Theta) \). The analytic in \( t \) (cf. (1.1.28), (1.1.29)) branches of eigenvalues \( \lambda_l(t, \Theta) \) and branches of eigenvectors \( \varphi_l(t, \Theta) \), \( l = 1, \ldots, n, \ |t| \leq t^0(\delta) \), depend on \( \Theta \). The first band functions (2.9) \( E_l(t\Theta) \), \( l = 1, \ldots, n \), coincide with \( \lambda_l(t, \Theta) \) only partially, since the analytic branches \( \lambda_l \) are not enumerated in the non-decreasing order. At the same time, (2.12) yields that

\[
\lambda_l(t, \Theta) \geq c_* t^2, \quad l = 1, \ldots, n, \quad t \in [0, t^0(\delta)].
\]

It is essential that, in (3.9), \( c_* \) and \( t^0(\delta) \) do not depend on \( \Theta \). From (3.9) it follows that the germ \( S(\Theta) \) of the family \( A(t, \Theta) \) (cf. (3.4)) is non-degenerate uniformly in \( \Theta \).

Chapter 3. The effective characteristics near the lower edge of the spectrum

In this chapter, we consider the spectral germ \( S(\Theta) \) of the operator family (2.3.4) \( A(t, \Theta) \). We introduce the family which is threshold equivalent to a given one simultaneously for all \( \Theta \) and such that the corresponding matrix \( g = h^* h \) for this family is constant. This constant matrix is called the effective matrix (and denoted by \( g_{\text{ef}} \)) for the initial periodic matrix \( g \). We discuss the question about the uniqueness of \( g_{\text{ef}} \); herewith, we correct the wrong statement from [BSu2] concerning this question. We distinguish the main effective matrix \( g^0 \) and discuss its properties. We introduce the effective DO for DO \( A \) (cf. (2.1.10)). The effective DO is of the same type as \( A \) with \( g \) replaced by \( g_{\text{ef}} \). On the basis of §5, Chapter 1, we study the behavior of the resolvent \( (A + \varepsilon^2 I)^{-1} \) as \( \varepsilon \to 0 \).
Below, we shall use the following notation. Let $\phi(x)$ be a $\Gamma$-periodic matrix-valued function such that $\phi \in L_{1,\text{loc}}(\mathbb{R}^d)$. We put

$$
\varphi = |\Omega|^{-1} \int_{\Omega} \phi(x) \, dx.
$$

If, besides, the matrix $\phi$ is square, non-degenerate and such that $\phi^{-1} \in L_{1,\text{loc}}(\mathbb{R}^d)$, we put

$$
\varphi = \left( |\Omega|^{-1} \int_{\Omega} \phi(x)^{-1} \, dx \right)^{-1}.
$$

Note that, if $\phi(x) > 0$, we always have $\varphi \leq \varphi$. Here the inequality turns into the equality only if $\phi$ is a constant matrix.

§1. Effective matrices and effective DO

First of all, we have to construct the operators $R = R(\theta)$ and $S = S(\theta)$ for the family $A(t, \theta)$. It is convenient to start with the case where $f = 1$, and then proceed to the general case on the basis of Subsection 1.1.5. In the case where $f = 1$, we agree to mark all the corresponding objects by the upper index “$*$”.

1.1. Operator $\hat{R}(\theta)$. We shall use the notation (2.3.1). According to (2.2.8),

$$
\hat{\mathcal{M}} = \{ u \in \mathcal{M} : u = c \in \mathbb{C}^d \}, \quad \hat{n} = n.
$$

Recall that (cf. (2.3.5)) $\mathcal{M}_n$ does not depend on $f$, whence $\hat{\mathcal{M}}_n = \mathcal{M}_n$. From the other side, $\mathcal{M}_n$ depends on $g$, and, moreover, on $h$.

Let

$$
\mathcal{M} = \{ q \in \mathcal{F}_n : q = C \in \mathbb{C}^n \}
$$

be the subspace of constant vector-valued functions. According to Subsection 1.1.2, for each $C \in \mathcal{M}$ there exists a unique element $v \in H^1(\Omega; \mathbb{C}^n)$ such that $\int_{\Omega} v \, dx = 0$ and (cf. (1.1.7) with $z = -hC$)

$$
(gb(D)v, b(D)w)_{\mathcal{M}_n} = -(gC, b(D)w)_{\mathcal{M}_n}, \quad w \in H^1(\Omega; \mathbb{C}^n).
$$

We put

$$
BC = h(b(D)v + C) \in \mathcal{M}_n \subset \mathcal{F}_n.
$$

Obviously, the operator $B : \mathcal{M} \rightarrow \mathcal{M}_n$ does not depend on $k = t\theta$. Note that $\int_{\Omega} h^{-1} BC \, dx = |\Omega|C$. This implies that $\text{Ker} \ B = \{ 0 \}$, whence $\text{rank} \ B = m$. Now, let $c \in \hat{\mathcal{M}}$ and $C = b(\theta)c$. Then (cf. (1.1.7)-(1.1.10)) $v$ plays the role of $\hat{\varphi}(c)$ and $\hat{R}(\theta)c = B(b(\theta)c)$. Thus,

$$
\hat{R}(\theta) : \hat{\mathcal{M}} = \mathcal{M}_n, \quad \hat{R}(\theta) = Bh, \quad \theta \in S^{d-1}.
$$

By condition (2.1.2), from (1.5) and from the relation $\text{Ker} \ B = \{ 0 \}$ it follows that $\text{Ker} \ \hat{R}(\theta) = \{ 0 \}$ or, equivalently,

$$
\text{rank} \ \hat{R}(\theta) = n.
$$

We emphasize that, in calculations, the operators $b(k)$, $B$, $\hat{R}(\theta)$ were represented as matrices written in fixed (cf. Subsection 2.1.1) bases. However, the final relations (e.g., (1.5)) have invariant meaning. The same is related to all what follows; in particular, below we do not distinguish the germ $\hat{S}(\theta)$ and the matrix corresponding to this germ in a fixed basis.
1.2. Operator $\mathcal{S}(\theta)$. Effective matrices. Below, it is convenient to indicate the dependence of the objects on the matrix $g$ (while the matrix $b(\xi)$ is fixed) in the notation. For instance, we shall write $\hat{A}(t, \theta; g), \hat{R}(\theta; g), \hat{S}(\theta; g)$, etc. By (1.5), it is clear that the operator $\hat{S}(\theta; g)$ (the germ of the family $\hat{A}(t, \theta; g)$),

\[(1.7) \quad \hat{S}(\theta; g) = \hat{R}(\theta; g) \ast \hat{R}(\theta; g) : \hat{R} \to \hat{R},\]

satisfies the relation $\hat{S}(\theta; g) = b(\theta) \ast (B^*B)b(\theta)$. As it follows from (1.6), (1.7) (cf. Subsection 2.3.2), the germ $\hat{S}(\theta; g)$ is non-degenerate. In a fixed basis $e_1, \ldots, e_m$, the operator $B^*B : \mathbb{R} \to \mathbb{R}$ is represented by the constant (and independent of $\theta$) positive matrix $g^0$. Thus,

\[(1.8) \quad \hat{S}(\theta; g) = b(\theta)^*g^0b(\theta), \quad g^0 = B^*B.\]

Now, along with $\hat{A}(t, \theta; g)$, we consider another family $\hat{A}(t, \theta; g^0)$, where the matrix $g^0$ is associated with the matrix-valued function $g$ as described above. We can assume that the matrix $g^0$ is somehow factorized: $g^0 = (h^0)^*h^0$; the final results do not depend on the choice of this factorization. If $g = g^0$, relation (1.8) yields $\mathbf{v} = 0$. Then $\hat{R}(\theta; g^0) = h^0b(\theta), \hat{S}(\theta; g^0) = b(\theta)^*g^0b(\theta)$, i.e., $(g^0)^0 = g^0$. Thus,

\[(1.9) \quad \hat{S}(\theta; g) = \hat{S}(\theta; g^0) = b(\theta)^*g^0b(\theta), \quad \theta \in \mathbb{T}^{d-1}.\]

In particular, (1.9) shows that the families $\hat{A}(t, \theta; g)$ and $\hat{A}(t, \theta; g^0)$ are threshold equivalent. We put

\[\hat{S}(k; g^0) := t^2\hat{S}(\theta; g^0) = b(k)^*g^0b(k), \quad k = t\theta \in \mathbb{T}^{d}.\]

Then $\hat{S}(k; g^0)$ is the symbol of the DO

\[(1.10) \quad \hat{A}^0 := b(D)^*g^0b(D)\]

with constant coefficients.

**Definition 1.1.** A constant positive $(m \times m)$-matrix $g_{\text{eff}}$ is called the effective matrix for the operator family $\hat{A}(t, \theta; g)$, if $\hat{S}(\theta; g) = \hat{S}(\theta; g_{\text{eff}})$ for all $\theta \in \mathbb{T}^{d-1}$.

If $g_{\text{eff}}$ is the effective matrix for $g$, then

\[(1.11) \quad \hat{S}(\theta; g) = \hat{S}(\theta; g_{\text{eff}}) = b(\theta)^*g_{\text{eff}}b(\theta), \quad \theta \in \mathbb{T}^{d-1}.\]

Relations (1.8), (1.9) imply the following proposition.

**Proposition 1.2.** For the operator family $\hat{A}(t, \theta; g)$, there exists at least one effective matrix. Namely, the matrix $g^0$ corresponding to the operator $B^*B$ is effective.

**Definition 1.3.** The matrix $g^0$ is called the main effective matrix for the family $\hat{A}(t, \theta; g)$. 
Definition 1.4. The operator (1.10) $\hat{A}^0$ is called the effective DO for the operator $\hat{A} = b(D)^* g^0(D)$ acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$.

We mention at once that, by (1.11), $\hat{S}(\theta; g_{eff})$ coincides with $\hat{S}(\theta; g^0)$, and then $\hat{S}(k; g_{eff}) = t^2 \hat{S}(\theta; g_{eff}) = \hat{S}(k; g^0)$. This implies that the effective operator $\hat{A}^0$ for $\hat{A} = b(D)^* g^0(D)$ is defined uniquely. In other words, $\hat{A}^0$ will not change, if in (1.10) $g^0$ is replaced by any other matrix $g_{eff}$.

Let $v_j$ be the solution of the problem (1.3) with $C = e_j$, $j = 1, \ldots, m$. We put $u_j := b(D)v_j = h(b(D)v_j + e_j)$. Then $|\Omega| g^0$ is the Gram matrix, i.e.,

\begin{equation}
|\Omega| g^0 = |\Omega|^{-1} \{(u_j, u_l)_{g^0}, j, l = 1, \ldots, m.\}
\end{equation}

If the matrices $g(x)$ and $b(\xi)$ have real-valued entries, then the solutions $v_j$ are purely imaginary, and the vector-valued functions $u_j$ are real. In this case, from (1.12) follows that the matrix $g^0$ also has real-valued entries. Then it is natural (but not necessary) to subject all other effective matrices to the condition that they have real-valued entries.

Besides representations $g^0 = B^* B$ and (1.12), we give one more useful representation for the main effective matrix $g^0$. In the homogenization theory, this representation is usually viewed as the initial.

Recall the notation (2.3.2), (2.3.3) and (1.2). For $C \in \mathfrak{M}$, we rewrite (1.4) as

\begin{equation}
hC = BC - hh(D)v, \quad C \in \mathfrak{M}.
\end{equation}

Since $BC \in \mathfrak{N}_e = \text{Ker } X_e^*$ and $hh(D)v \in \text{Ran } X_0$, the summands on the right are orthogonal to each other in $\mathfrak{N}_e$. Hence,

\begin{equation}
\|BC\|_{\mathfrak{N}_e}^2 = (BC, hC)_{\mathfrak{N}_e} = (h^* BC, C)_{\mathfrak{N}_e} = (g(C + b(D)v), C)_{\mathfrak{N}_e},
\end{equation}

and, consequently,

\begin{equation}
|\Omega| (g^0 C, C)_{\mathfrak{N}_e} = \left\langle \int_\Omega g(b(D)v + C) \, dx, C \right\rangle_{\mathfrak{N}_e}.
\end{equation}

Thus, for $g^0$ we have

\begin{equation}
g^0 C = |\Omega|^{-1} \int_\Omega g(x)(b(D)v + C) \, dx, \quad C \in \mathfrak{M},
\end{equation}

where $v$ is defined by the relation (1.3).

1.3. On the uniqueness of the effective matrix. Comparing (1.9) and (1.11) and proceeding to the quadratic forms, we obtain

\begin{equation}
\langle g^0 b(\theta) \zeta, b(\theta) \zeta \rangle = \langle g_{eff} b(\theta) \zeta, b(\theta) \zeta \rangle, \quad \zeta \in \mathbb{C}^n, \quad \theta \in \mathbb{R}^{d-1}.
\end{equation}

It follows that the matrix $g_{eff} = g^0$ is unique if and only if

\begin{equation}
\text{clos} \bigcup_0 \text{Ran } b(\theta) = \mathbb{C}^n.
\end{equation}
If the matrices \( g(\mathbf{x}) \) and \( b(\mathbf{\Theta}) \) (and then also \( g^0 \)) have real-valued entries, the uniqueness of the real matrix \( g_{\text{eff}} \) is ensured by the condition

\[
(1.16) \quad \text{clos} \left( \bigcup_{\mathbf{\Theta}} \{ b(\mathbf{\Theta}) \mathbb{R}^m \} \right) = \mathbb{R}^m.
\]

In [BSu2], the false algebraic criterion of the validity of (1.15) was formulated. In fact, the uniqueness of \( g_{\text{eff}} \) is comparatively scarce property. This is related to the fact that one and the same operator \( \hat{A} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) \) can be described by different matrices \( g(\mathbf{x}) \).

Note that, by (2.1.2), conditions (1.15), (1.16) are a fortiori valid, if \( m = n \). So, if \( m = n \), then the effective matrix is unique. Next, it is easily seen that, in the real-valued case, (1.16) is valid, if \( n = 1 \) and \( m = d \). This case corresponds to the real scalar elliptic operator \( \hat{A} \). Here the condition that \( g(\mathbf{x}) \) has real-valued entries is essential. For instance, if \( n = 1, m = d = 2, b(\xi) = e_1 \xi^1 + e_2 \xi^2 \), then any matrix of the form \( g^0 + i a \hat{g} \) will be effective. Here \( \hat{g} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( a \in \mathbb{R} \) is sufficiently small. If \( n > 1 \), the uniqueness of \( g_{\text{eff}} \) may be violated even in the real-valued case.

Since the effective DO \( \hat{A} \) does not depend on the choice of \( g_{\text{eff}} \), non-uniqueness of the latter is not very essential. Below, the main attention will be paid to the main effective matrix \( g^0 \).

### 1.4. Estimates for the matrix \( g^0 \). The Voight-Reuss bracketing.

According to (0.1), (0.2), for the matrix \( g(\mathbf{x}) \), we put

\[
(1.17) \quad \mathfrak{g} = \Omega^{-1} \int_{\Omega} g(\mathbf{x}) \, d\mathbf{x}, \quad g = \left( \int_{\Omega} |g|^{-1} \, d\mathbf{x} \right)^{-1}.
\]

Since the summands in the right-hand side of (1.13) are orthogonal to each other, then

\[
(1.18) \quad \| B C \|_{\mathfrak{g}_{\ast}}^2 \leq \| h C \|_{\mathfrak{g}_{\ast}}^2.
\]

It is easily seen that (1.18) is equivalent to the relation \( \langle B^* B C, C \rangle_{C^m} \leq \langle \mathfrak{g} C, C \rangle_{C^m} \), and, since (cf. (1.8)) \( g^0 = B^* B \), we obtain the upper estimate \( g^0 \leq \mathfrak{g} \).

In order to prove the lower estimate, we note that, by (2.3.5), \( \mathfrak{q} := (h^*)^{-1} \mathfrak{g} \mathfrak{n} \subset \mathfrak{N} \subset \mathfrak{g}_{\ast} \). We put

\[
\Pi w = |\Omega|^{-1} (h^*)^{-1} \mathfrak{g} \int_{\Omega} h^{-1} w \, d\mathbf{x}, \quad w \in \mathfrak{g}_{\ast}.
\]

It can be elementary checked that \( \Pi \mathfrak{q} \in \mathfrak{q} \), \( \Pi|_{\mathfrak{q}} = I_{\mathfrak{q}} \) and

\[
(1.19) \quad \langle \Pi w, w \rangle_{\mathfrak{g}_{\ast}} = |\Omega|^{-1} \langle \mathfrak{q} C w, C w \rangle_{C^m}, \quad C w = \int_{\Omega} h^{-1} w \, d\mathbf{x}, \quad w \in \mathfrak{g}_{\ast}.
\]

It means that \( \Pi \) is the orthoprojector of \( \mathfrak{g}_{\ast} \) onto the subspace \( \mathfrak{q} \). Now, we apply the projector \( \Pi \) to the relation (1.13). Since \( h b(\mathbf{D}) v \in \mathfrak{N}^+ \), we obtain that \( \Pi h C = \Pi h |C| \). Putting \( w = h |C| \) in (1.19), we arrive at

\[
\| \Pi h C \|_{\mathfrak{g}_{\ast}}^2 = |\Omega| \langle \mathfrak{q} C, C \rangle_{C^m}.
\]
Hence,

\[(1.20) \quad \| \langle g^0 C, C \rangle \|_{\ell^2} = \| B C \|_{\ell^2}^2 \geq \| \Pi B C \|_{\ell^2}^2 = \| \Pi h C \|_{\ell^2}^2 = \| \langle g C, C \rangle \|_{\ell^2}^2.\]

Thus, we have proved the lower estimate: \( g^0 \geq g. \)

In addition, let us discuss the case where \( m = n. \) Then \( \dim \mathcal{H}_* = n_* = n = m. \) From \( \dim \mathcal{H} = m \) it follows that \( \mathcal{H}_* = \mathcal{H}. \) Since \( BC \in \mathcal{H}_* \), the inequality (1.20) turns into equality. As a result, we see that, if \( m = n \), we always have \( g^0 = \overline{g} \).

Let us summarize.

**Theorem 1.5.** For the main effective matrix \( g^0 \) of the family \( \hat{A}(t, \theta; g) \), we have

\[(1.21) \quad g \leq g^0 \leq \overline{g}, \]

where the constant matrices \( \overline{g}, g \) are defined by (1.17). If \( m = n \), then the (unique) effective matrix \( g^0 \) is equal to \( \overline{g} \).

\[(1.22) \quad g^0 = \overline{g}, \quad m = n. \]

Now, we find the conditions under which one of the inequalities (1.21) turns into equality. We start with the case where \( g^0 = \overline{g} \). This is equivalent to the fact that in (1.18) the inequality turns into equality for any \( C \). By (1.13), the latter means that \( h b(D) v = 0 \), where \( v \) satisfies (1.3). But then (1.3) is reduced to the equality \( b(D)^* g(x) C = 0, C \in \mathbb{C}^m \). Let \( g_k(x), k = 1, \ldots, m, \) be the columns of the matrix \( g(x) \). As a result, we obtain the following statement.

**Proposition 1.6.** The equality \( g^0 = \overline{g} \) is equivalent to the relations

\[(1.23) \quad b(D)^* g_k(x) = 0, \quad k = 1, \ldots, m. \]

Note that (1.23) contains \( m n \) differential conditions imposed on the \( m^2 \) coefficients of the matrix \( g \).

It suffices to discuss the case where \( g^0 = g \) only for \( m > n \). By (1.20), the equality \( g^0 = \overline{g} \) is equivalent to the fact that \( B C \in \mathcal{H}_*, C \in \mathbb{R} \). In other words, for any \( C \in \mathbb{C}^m \) there exists such \( C_* \in \mathbb{C}^m \) that \( B C = (h^*)^{-1} C_* \). The latter means that, by (1.4),

\[(1.24) \quad g^{-1} C_* = C + b(D) v. \]

Integrating (1.24), we obtain: \( C = g^{-1} C_* \). Putting \( C_* = e_k \), we can rewrite (1.24) in terms of the columns \( l_k(x) \) of the matrix \( g(x)^{-1} \). We have proved the following proposition.

**Proposition 1.7.** The equality \( g^0 = \overline{g} \) is equivalent to the relations

\[(1.25) \quad l_k(x) = l_k^0 + b(D) v_k, \quad v_k \in H^1(\Omega; \mathbb{C}^m), \quad l_k^0 \in \mathbb{C}^m, \quad k = 1, \ldots, m, \]

for the columns \( l_k(x) \) of the matrix \( g(x)^{-1} \).

In addition, (1.25) automatically implies that \( l_k^0 = \overline{g}^{-1} e_k \) and \( v_k \) satisfies (1.3) with \( C = l_k^0 \).
Let us comment on the relations (1.21), (1.23), (1.25). The estimates (1.21) in specific cases (the acoustic operator, the operator of elasticity theory, etc.) are well known in the homogenization theory; they are called the Voight-Reuss bracketing (cf., e.g., [ZhKO]). Our proof of estimates (1.21) is parallel to the proof of the estimates for the germ $S$ given in Subsection 1.1.4 on the abstract basis. Now, by (1.9), the estimates (1.21) imply the following inequalities for the germs

\begin{equation}
\hat{S}(\theta;g) \leq \hat{S}(\theta;g^0) = \hat{S}(\theta;g) \leq \hat{S}(\theta;g^\infty).
\end{equation}

We can return from (1.26) to (1.21), but only under the additional condition (1.15).

It was this condition, under which the estimates (1.21) were proved in [BSu2] for the operators $b(D)^*g(x)b(D)$. Also, in [BSu2] the relation (1.22) with $n > 1$ was distinguished for the first time.

For the acoustic operator, i.e., when $n = 1$, $m = d$, $b(D) = D$, the results of Propositions 1.6, 1.7 are known (cf. [ZhKO]). In this case relations (1.23) mean that the columns $g_0(x)$ are solenoidal, and relations (1.25) mean that the columns $l_0(x)$ are potential (up to a constant summand; this will be meant in what follows).

1.5. The case where $f \neq 1_n$. We return to consideration of the operators $A$ of the general form (2.1.10) and the corresponding families $A(t, \theta)$ of the form (2.3.4). As before, we shall indicate the dependence on the matrix $g$ (but not on matrices $f$, $b$) in the notation. The upper index “$*” is preserved for denoting the objects corresponding to the case where $f = 1_n$ with the same $b$ and $g$.

We rely on the scheme of Subsection 1.1.5. Now

$$\hat{H} = H = L_2(\Omega; \mathbb{C}^m),$$

and the role of an isomorphism $M$ is played by the operator of multiplication by the matrix-valued function $f$. The kernel $\mathfrak{N} = \text{Ker} X_0$ is defined by (2.2.8):

$$\mathfrak{N} = \{ u \in \mathfrak{H} : fu = c \in \mathbb{C}^m \},$$

it does not depend on $b$ and $g$. According to (1.1.21), $R(\theta;g) = \hat{R}(\theta;g)f|_{\mathfrak{N}}$. From (1.1.23) it follows that

\begin{equation}
S(\theta;g) = P f^* \hat{S}(\theta;g)f|_{\mathfrak{N}},
\end{equation}

where $P$ is the orthoprojector of $\hat{H}$ onto $\mathfrak{N}$. Definition 1.1 is carried over to the general case of the family $A(t, \theta; g)$.

**Definition 1.8.** A constant positive $(m \times m)$-matrix $g_{eff}$ is called the effective matrix for the operator family $A(t, \theta; g)$, if $S(\theta;g) = S(\theta;g_{eff})$ for all $\theta \in \mathbb{S}^{d-1}$.

Relation (1.27) directly implies the following proposition.

**Proposition 1.9.** The matrix $g_{eff}$ is the effective matrix for the families $A(t, \theta; g)$ and $\hat{A}(t, \theta; g)$ simultaneously.

In particular, the main effective matrix $g^0$ corresponding to the family $\hat{A}(t, \theta; g)$ is also an effective matrix for the family $A(t, \theta; g) = f^* \hat{A}(t, \theta; g)f$. It is natural to associate the operator

\begin{equation}
A = f^* \hat{A}f
\end{equation}
with the operator

$$(1.29) \quad \mathcal{A}_0^\varepsilon = f^* \widehat{\mathcal{A}}^\varepsilon f,$$

acting in $L_3(\mathbb{R}^d; \mathbb{C}^n)$. Here, $\widehat{\mathcal{A}}^\varepsilon$ is defined by (1.10). Recall that $\widehat{\mathcal{A}}^\varepsilon$ will not change, if in (1.10) $g^\varepsilon$ is replaced by any other effective matrix $g_{\text{eff}}$. Hence, the operator $\mathcal{A}^\varepsilon_0$ also does not depend on the choice of the matrix $g_{\text{eff}}$. Thus, the operator (1.29) is defined via the operator (1.28) uniquely. We agree to call the operator $\mathcal{A}^\varepsilon_0$ the effective DO for DO $\mathcal{A}$.

We distinguish the case where $n = 1$. Then the operator $\widehat{S}(\theta; g)$ (cf. (1.9)) is the operator of multiplication by the number $\gamma(\theta) = b(\theta)^* g^\varepsilon b(\theta)$. By (1.1.26) and (1.27), the operator $\widetilde{S}(\theta; g)$ is reduced to multiplication by the number

$$(1.30) \quad \gamma(\theta) = \gamma(\theta) \Omega \| f^{-1} \|^{-2}_{L_3(\omega)}, \quad \gamma(\theta) = b(\theta)^* g^\varepsilon b(\theta).$$

§2. Behavior of the resolvent $(\mathcal{A} + \varepsilon_2 I)^{-1}$ as $\varepsilon \to 0$

As in §1, we start with the case where $f = 1_n$, and then proceed to the general case.

2.1. The resolvent of the operator $\widehat{\mathcal{A}}$. We consider the resolvent $(\widehat{\mathcal{A}} + \varepsilon^2 I)^{-1}$ of the operator $\widehat{\mathcal{A}} = b(D)^* g(x) b(D)$ acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$. We rely on Theorem 1.5.6, which is applied to the families $\widehat{\mathcal{A}}(t, \theta; g)$ and $\widehat{\mathcal{A}}(t, \theta; g^\varepsilon)$ of the form (2.3.4). Here $g^\varepsilon$ is the main effective matrix for $g(x)$.

Recall that $\alpha_0$ and $\alpha_1$ are constants from (2.1.3), and $r_0$ is the radius of the ball inscribed into the Brillouin zone ($r_0$ is defined by (2.1.14)). In the notation of the following constants we use the index $\varepsilon_0$ which distinguishes the case where $f = 1_n$. According to (2.2.11),

$$(2.1) \quad \hat{\gamma}_0 = \alpha_0 \| g^{-1} \|_{L_\infty}^{-1}.$$

According to (2.3.7), we fix the number $\hat{\delta}$:

$$(2.2) \quad \hat{\delta} = (r_0/2)^2 \alpha_0 \| g^{-1} \|_{L_\infty}^{-1}.$$

Next, in accordance with (2.3.8) and (2.2),

$$(2.3) \quad \hat{b}^\varepsilon = \hat{b}^\varepsilon(\hat{\delta}) = (r_0/2)\alpha_0 \| g^{-1} \|_{L_\infty}^{-1} \| g \|_{L_\infty} \hat{\delta}^{-1/2}.$$

Since $\hat{\delta}$ is fixed, we do not need to indicate the dependence of $\hat{b}^\varepsilon$ on $\hat{\delta}$ anymore.

Now, we have to realize the constant $\mathcal{C}$ from (1.5.12). Using (1.5.10), (1.5.9), (1.5.6), (1.5.7) and (2.1) subsequently, we find that

$$(2.4) \quad \mathcal{C} = \alpha_0^{-1/2} \| g^{-1} \|_{L_\infty}^{1/2} \left( 2\beta_1 \hat{b}^\varepsilon(\hat{\delta})^{-1} + \beta_2 \alpha_0^{-1} \| g^{-1} \|_{L_\infty} \hat{\delta} \hat{b}^\varepsilon(\hat{\delta})^{-3} \right).$$

Now, we note that, by (1.21), $\| g \|_{L_\infty} \leq \| g \|_{L_\infty}$ and $\| g^{-1} \|_{L_\infty} \leq \| g^{-1} \|_{L_\infty}$. It follows that, if $g$ is replaced by $g^\varepsilon$, the quantities (2.1)-(2.3) may only grow, and the constant (2.4) may only decrease. Hence, Theorem 1.5.6 directly yields the following result. In its formulation, we use the notation (2.3.1), $k = t \theta$ and

$$(A(k; g) = A(t, \theta; g), A(k; g^\varepsilon) = A(t, \theta; g^\varepsilon)),$$
Proposition 2.1. For the operator
\[
\hat{G}(\varepsilon, k) = (\hat{A}(k; g) + \varepsilon^2 I)^{-1} - (\hat{A}(k; g^\circ) + \varepsilon^2 I)^{-1},
\]
we have
\[
\varepsilon \| \hat{G}(\varepsilon, k) \|_{\delta \to \delta} \leq \hat{C} + 2\varepsilon (3\hat{A})^{-1}, \quad \varepsilon > 0, \quad |k| \leq \hat{r}^0.
\]
where the constants \( \hat{C}, \hat{r}^0, \hat{C} \) are defined by (2.2)-(2.4).

Besides the estimate (2.6), we need the estimates for the resolvents for \( k \in \text{clos} \, \Omega \cap \{ k : |k| > \hat{r}^0 \} \). By (2.2.13),
\[
\| (\hat{A}(k; g) + \varepsilon^2 I)^{-1} \|_{\delta \to \delta} \leq \hat{C}^{-1}(\hat{A}^h)^{-2}, \quad k \in \text{clos} \, \Omega \cap \{ k : |k| > \hat{r}^0 \},
\]
and the same estimate remains true with \( g \) replaced by \( g^\circ \). Therefore, we have the following proposition.

Proposition 2.2. The following estimates are valid, where we use the notation (2.1)-(2.5):
\[
\varepsilon \| \hat{G}(\varepsilon, k) \|_{\delta \to \delta} \leq \hat{C}_x, \quad 0 < \varepsilon \leq 1, \quad k \in \text{clos} \, \Omega,
\]
where
\[
\hat{C}_x = \max\{ \hat{C} + 2(3\hat{A})^{-1}, 2\hat{C}^{-1}((\hat{r}^0)^{-2}) \}.
\]
\[
\limsup_{\varepsilon \to 0} \varepsilon \| \hat{G}(\varepsilon, k) \|_{\delta \to \delta} \leq \hat{C}.
\]

Now, it is easy to obtain one of the main results of the paper. We estimate the difference of the resolvents of the operator \( \hat{A} = \hat{A}(g) \) and of the effective DO \( \hat{A} = \hat{A}(g^\circ) \). We use representation (2.2.17) for the operator \( \hat{A}(g) \) in the direct integral \( \mathcal{H} \) (cf. (2.1.18)). Now, necessary relations can be written as
\[
\mathcal{H} = \int_{\hat{\Omega}} \oplus \mathfrak{H} \, dk, \quad \hat{A}(g) = U^{-1} \left( \int_{\hat{\Omega}} \oplus \hat{A}(k; g) \, dk \right) U,
\]
where \( U \) is the unitary mapping of the space \( \mathfrak{H} := L_2(\mathbb{R}^d; \mathbb{C}^n) \) onto \( \mathcal{H} \). From (2.10) it follows that
\[
(\hat{A}(g) + \varepsilon^2 I)^{-1} = U^{-1} \left( \int_{\hat{\Omega}} \oplus (\hat{A}(k; g) + \varepsilon^2 I)^{-1} \, dk \right) U.
\]

The analogous representation is valid with \( g \) replaced by \( g^\circ \), and also for the difference of the resolvents. Combining this with estimates (2.8), (2.9), we obtain the following theorem.

Theorem 2.3. Let \( \hat{A}(g) = b(D)^* g(x) b(D) \), and let \( \hat{A}^0 = \hat{A}(g^\circ) = b(D)^* g^\circ b(D) \) be the effective DO for DO \( \hat{A}(g) \). Then
\[
\varepsilon \| (\hat{A}(g) + \varepsilon^2 I)^{-1} - (\hat{A}(g^\circ) + \varepsilon^2 I)^{-1} \|_{\mathfrak{H} \to \mathfrak{H}} \leq \hat{C}_x, \quad 0 < \varepsilon \leq 1,
\]
\[
\limsup_{\varepsilon \to 0} \varepsilon \| (\hat{A}(g) + \varepsilon^2 I)^{-1} - (\hat{A}(g^\circ) + \varepsilon^2 I)^{-1} \|_{\mathfrak{H} \to \mathfrak{H}} \leq \hat{C},
\]
where the constants $\hat c_\varepsilon, \hat C$ are the same as in (2.8), (2.9).

The estimates (2.11), (2.12) show that the norm of the difference of the resolvents is of order $O(\varepsilon^{-1})$. At the same time, the norm of each resolvent is $O(\varepsilon^{-2})$. The reason for the compensation in the difference is that the germ of the families $A(k;g)$ and $\hat A(k;g^0)$ coincide when $k = 0$. Thus, in Theorem 2.3, the threshold effect near the edge of the spectrum is taken into account.

**Remark 2.4.** We emphasize once more that, in fact, the effective operator $\hat A^0 = \hat A(g^0)$ does not depend on the choice of the effective matrix, but is determined by the initial operator $\hat A = \hat A(g)$. In general, the latter may be also described by non-unique $\Gamma$-periodic matrix $g(x)$.

### 2.2. The resolvent of the operator $A$

Now, we refuse the assumption that $f = 1_n$ and consider the operator

$$
A = f^\ast \hat A f = f^\ast b(D)^\ast gb(D)f.
$$

The constants (2.1)–(2.4) should be changed in order to take the dependence on $f$ into account. According to (2.2.11), (2.3.7), (2.3.8), we put

$$
(2.14) \quad c_* = \hat c_* ||f^{-1}||_{L_\infty}^2 = a_0 ||f^{-1}||_{L_\infty}^2 ||g^{-1}||_{L_\infty}^2,
$$

$$
(2.15) \quad \delta = \hat c_0 (r_0 / 2)^3 = \hat \delta ||f^{-1}||_{L_\infty}^2,
$$

$$
(2.16) \quad t^0 = t^0 (\delta) = \hat t^0 (\delta) (||f||_{L_\infty} \|f^{-1}\|_{L_\infty})^{-1}.
$$

Now, the role of the constant (2.4) is played by the constant

$$
(2.17) \quad C = c_*^{-1/2} (2 \beta_1 (t^0 (\delta))^{-1} + c_*^{-1} \beta_2 \delta (t^0 (\delta))^{-3} ) .
$$

Using the notation from Proposition 2.1, we put

$$\quad A(k;g,f) = f^\ast \hat A(k;g)f, \quad A(k;g^0,f) = f^\ast \hat A(k;g^0)f .$$

Now, Theorem 1.5.6 implies the following generalization of Proposition 2.1.

**Proposition 2.5.** For the operator

$$G(\varepsilon,k) = (A(k;g,f) + \varepsilon^2 I)^{-1} - (A(k;g^0,f) + \varepsilon^2 I)^{-1} ,$$

we have

$$
(2.18) \quad \varepsilon ||G(\varepsilon,k)||_{\varepsilon \rightarrow \delta} \leq C + 2 \varepsilon (3\delta)^{-1}, \quad \varepsilon > 0, \quad |k| \leq t^0 ,
$$

where the constants $\delta, t^0, C$ are defined by (2.15)–(2.17).

For $A(k;g^0,f)$ and $A(k;g^0,f)$ the estimate (2.7) remains true with $\hat c_*$ replaced by $c_*$ and $\hat t^0$ replaced by $t^0$. Combining this with (2.18), we obtain the generalization of Proposition 2.2.

---

*Note: The content is a mathematically rich excerpt from a higher-level mathematics text, focusing on spectral theory and operator theory.*

---
Proposition 2.6. The following estimates are valid, where we use the notation (2.14)–(2.17): 
\[
\varepsilon \| \mathcal{G}(\varepsilon, k) \|_{\mathcal{S} \rightarrow \mathcal{S}} \leq C_x, \quad 0 < \varepsilon \leq 1, \quad k \in \text{clo} \Omega, \\
C_x = \max \{ C + 2(3\delta)^{-2}, 2c^{-1}(\delta)^{-2} \}.
\]
(2.19) 
\[
\limsup_{\varepsilon \rightarrow 0} \varepsilon \| \mathcal{G}(\varepsilon, k) \|_{\mathcal{S} \rightarrow \mathcal{S}} \leq C.
\]
(2.20) 

Finally, we obtain the following theorem.

Theorem 2.7. Let \( A = A(g, f) \) be the operator (2.13), and let \( A^0 = A(g^0, f) \) be the corresponding effective DO. Then 
\[
\varepsilon \| (A(g, f) + \varepsilon^2 I)^{-1} - (A(g^0, f) + \varepsilon^2 I)^{-1} \|_{\mathcal{S} \rightarrow \mathcal{S}} \leq C_x, \quad 0 < \varepsilon \leq 1,
\]
(2.21) 
\[
\limsup_{\varepsilon \rightarrow 0} \varepsilon \| (A(g, f) + \varepsilon^2 I)^{-1} + (A(g^0, f) + \varepsilon^2 I)^{-1} \|_{\mathcal{S} \rightarrow \mathcal{S}} \leq C,
\]
(2.22) 

where the constants \( C_x, C \) are the same as in (2.19), (2.20).

Proof of Theorem follows from Proposition 2.6 in the same way as Theorem 2.3 was deduced from Proposition 2.2. \( \square \)

In (2.21), (2.22), we estimate the difference of the resolvents corresponding to \( g \) and \( g^0 \). The matrix \( f \) is fixed. It is impossible to find a good approximation for the resolvent of the operator \( A(g, f) \) with variable \( f \) by the resolvent of some operator with constant coefficients.* One can find more convenient approximation than that in the estimates (2.21), (2.22). However, the approximating operator is not a resolvent anymore. This will be done in the next section; see Theorem 3.4.

§3. Behavior of the generalized resolvent as \( \varepsilon \rightarrow 0 \)

3.1. We shall prove the analogue of Theorem 2.3 for the generalized resolvent \( \hat{\mathcal{R}}(g) + \varepsilon^2 Q)^{-1} \). Here \( Q = Q(x) \geq 0 \) is a \( \Gamma \)-periodic \( (n \times n) \)-matrix-valued function such that 
\[
Q + Q^{-1} \in L_\infty.
\]
We rely on the abstract results of Subsections 1.1.5 and 1.5.3, assuming that \( \hat{\mathcal{H}} = \hat{\mathcal{S}} = L_2(\Omega; c^n) \). Recall also (cf. (1.1)) that 
\[
\hat{\mathcal{R}} = \text{Ker} \hat{\mathcal{A}}(0; g) = \{ u \in \hat{\mathcal{S}} : u = c \in c^n \}.
\]
Intending to apply Theorem 1.5.8, we represent \( Q \) as 
\[
Q(x) = (f(x)f(x)^*)^{-1}.
\]
According to (1.5.14), now multiplication by the \( \Gamma \)-periodic matrix-valued function \( f \) plays the role of an isomorphism \( M \). From (3.1) it is directly seen that the block of the operator of multiplication by \( Q \) in the subspace \( \hat{\mathcal{R}} \) is multiplication by the

*In this case, the name "effective DO" for \( A(g^0, f) \) with variable \( f \) does not seem very appropriate.
constant matrix $\overline{Q}$, i.e., by the mean value of $Q(x)$ over $\Omega$. If we view the operator of multiplication by $\overline{Q}$ as an operator in the whole space $\overline{H}$, then its block in $\overline{H}$ also coincides with multiplication by $\overline{Q}$. We apply Theorem 1.5.8 in order to estimate the difference

$$F(\varepsilon, k) = (\hat{A}(k; g) + \varepsilon^2 Q)^{-1} - (\hat{A}(k; g^0) + \varepsilon^2 \overline{Q})^{-1}. \tag{3.3}$$

According to the scheme of Subsection 1.5.3, the constant $C = 2^{-1}C + (3\delta)^{-1}$ in (1.5.20)-(1.5.22) is expressed via constants (2.14)-(2.17) corresponding to the operator $A(g, f)$, but not to the operator $\hat{A}(g)$. Here $f$ is defined in (3.2). Note that $\|\overline{Q}\| \leq \|Q\|_{L_\infty}$. Let $\overline{Q}^{-1}$ be replaced by $Q^{-1}$ and $f_+ = (\overline{Q})^{-1/2}$. Then it is easily seen that the same constants (2.14)-(2.17) are also suitable for the operator $A(g^0, f_+)$. Hence, now in (1.5.22) one can take $C_+ = C$ and replace $\|\overline{Q}^{-1}\|$ by $\|Q^{-1}\|_{L_\infty}$. As a result, we obtain the analogue of the estimate (2.6). Namely, the following proposition is true.

**Proposition 3.1.** For the operator $F(\varepsilon, k)$ defined by (3.3), we have

$$\varepsilon \|F(\varepsilon, k)\|_{\overline{H}} \leq (C + 2(3\delta)^{-1})\|Q^{-1}\|_{L_\infty}, \quad 0 < \varepsilon \leq 1, \quad |k| \leq \theta^0,$$

where the constants $\delta, \theta^0, C$ are defined by (2.15)-(2.17).

As it has already been mentioned, the operator $\hat{A}(k; g, f)$ satisfies the estimate (2.7) with $c_* \leq c_\ast$ replaced by $c_\ast$ and $\theta^0$ replaced by $\theta^0$. Hence, by (1.5.13), we obtain the estimate

$$\|\hat{A}(k; g) + \varepsilon^2 Q\|_{\overline{H}} \leq \|Q^{-1}\|_{L_\infty} c_\ast^{-1} (\theta^0)^{-2}, \quad k \in \overline{H} \cap \{k : |k| > \theta^0\}.$$  

The same estimate is valid for $\hat{A}(k; g^0) + \varepsilon^2 \overline{Q}^{-1}$. Combining these estimates with Proposition 3.1, we obtain the following proposition.

**Proposition 3.2.** For the operator $F(\varepsilon, k)$ defined by (3.3), we have

$$\varepsilon \|F(\varepsilon, k)\|_{\overline{H}} \leq C_\ast \|Q^{-1}\|_{L_\infty}, \quad k \in \overline{H}, \quad 0 < \varepsilon \leq 1,$$

where $C_\ast$ is the constant from (2.19).

Finally, (3.4) implies (cf. deduction of Theorem 2.3 from Proposition 2.2) the following theorem.

**Theorem 3.3.** We have

$$\varepsilon \|\hat{A}(g) + \varepsilon^2 Q\|_{\overline{H}} - (\hat{A}(g^0) + \varepsilon^2 \overline{Q})^{-1} \|_{\overline{H}} \leq C_\ast \|Q^{-1}\|_{L_\infty}, \quad 0 < \varepsilon \leq 1. \tag{3.5}$$

If $Q = 1$, the estimate (3.5) turns into (2.11).

**3.2. Another approximation** for the operator

$$A(g, f) + \varepsilon^2 I)^{-1}, \tag{3.6}$$

more practical than the estimate (2.21), easily follows from Theorem 3.3. Now we assume that the matrix-valued function $f$ is given, while $Q$ is defined by the relation (3.2). Clearly,

$$A(g, f) + \varepsilon^2 I)^{-1} = f^{-1} (\hat{A}(g) + \varepsilon^2 Q)^{-1} (f^*)^{-1}. \tag{3.7}$$
We put
\begin{equation}
(3.8)
\mathcal{F}(\varepsilon) := (\mathcal{A}(g, f) + \varepsilon^2 I)^{-1} - f^{-1}(\hat{\mathcal{A}}(g) + \varepsilon^2 \overline{Q})^{-1}(f^*)^{-1}.
\end{equation}

Then (3.7) implies that
\begin{equation}
\mathcal{F}(\varepsilon) = f^{-1}((\hat{\mathcal{A}}(g) + \varepsilon^2 Q)^{-1} - (\hat{\mathcal{A}}(g^0) + \varepsilon^2 \overline{Q})^{-1})(f^*)^{-1}.
\end{equation}

The latter expression is estimated with the help of (3.5). As a result, we obtain the following theorem.

**Theorem 3.4.** For the operator $\mathcal{F}(\varepsilon)$ defined by (3.8), we have
\begin{equation}
(3.9)
\varepsilon \| \mathcal{F}(\varepsilon) \|_{\mathfrak{g} \to \mathfrak{g}} \leq C_x \| Q^{-1} \|_{L_\infty} \| Q \|_{L_\infty}, \quad 0 < \varepsilon \leq 1,
\end{equation}
where the matrix-valued function $Q$ is defined by (3.2).

**Remark 3.5.** In (3.8), (3.9), the operator (3.6) is approximated by the operator $f(x)^{-1}(\hat{\mathcal{A}}(g^0) + \varepsilon^2 \overline{Q})^{-1}(f(x^*)^{-1}$. Although this operator contains the factors depending on $x$ from both sides, but the generalized resolvent is related to the DO with constant coefficients. That is why the estimate (3.9) has a big advantage as compared with (2.21).

3.3. **One generalization of Theorem 3.4.** More closed result for the operator $\mathcal{A}(g, f)$ can be obtained, if one considers the generalized resolvent

\begin{equation}
(\mathcal{A}(g, f) + \varepsilon^2 \Omega)^{-1},
\end{equation}

instead of the resolvent (3.6). Here $\Omega(x) > 0$ is a $\Gamma$-periodic $(n \times n)$-matrix-valued function such that $\Omega + \Omega^{-1} \in L_\infty$. We represent $\Omega(x)$ in the form (3.2):

\begin{equation}
\Omega = \varphi \varphi^* - 1,
\end{equation}

and put

\begin{equation}
f = f \varphi, \quad Q = (f^*)^{-1}.
\end{equation}

The following theorem can be deduced from Theorem 3.3 by the same method as Theorem 3.4 was proved.

**Theorem 3.6.** We have
\begin{equation}
\varepsilon \| (\mathcal{A}(g, f) + \varepsilon^2 \Omega)^{-1} - f^{-1}(\hat{\mathcal{A}}(g) + \varepsilon^2 \overline{Q})^{-1}(f^*)^{-1} \|_{\mathfrak{g} \to \mathfrak{g}} \\
\leq C_x \| f^{-1} \|_{L_\infty} \| f \|_{L_\infty}^2 \| Q^{-1} \|_{L_\infty}, \quad 0 < \varepsilon \leq 1,
\end{equation}
where $C_x \{ f \}$ is constant (2.19), recalculated with $f$ replaced by $f$ in (2.14)-(2.17).

**Chapter 4. Homogenization problems for periodic DO’s.**

We proceed to the study of the homogenization problems, treating them as the threshold effects near the lower edge of the spectrum. The present Chapter contains general results about homogenization for periodic operators of the form

\begin{equation}
\mathcal{A}(g, f) = f(x)^* b(D)^* g(x)b(D)f(x) = f(x)^* \hat{\mathcal{A}}(g)f(x)
\end{equation}
acting in \( \mathcal{G} = L_2(\mathbb{R}^d; \mathbb{C}^n) \). In the next Chapters 5-7, we discuss applications of these results to specific periodic DO's of mathematical physics.

We agree to use the following notation for Hilbert spaces

\[
\mathcal{H}_s = L_2(\Omega; \mathbb{C}^s), \quad \mathcal{H}_s = L_2(\Omega; \mathbb{C}^m), \quad \mathcal{G} = L_2(\mathbb{R}^d; \mathbb{C}^n), \quad \mathcal{G}_s = L_2(\mathbb{R}^d; \mathbb{C}^m), \quad \mathcal{G}^s = H^s(\mathbb{R}^d; \mathbb{C}^n), \quad s \in \mathbb{R}.
\]

We shall also use the obvious notation like \( \mathcal{G}_{\text{loc}}, \mathcal{G}_{s, \text{loc}}, \) etc. The symbol \((\mathcal{F}, \mathcal{G})_s\) with \( \mathcal{G} \in \mathcal{G}_{\text{loc}} \), \( \mathcal{F} \in \mathcal{G}_{s, \text{loc}} \), stands for the value of the functional \( \mathcal{F} \) on the element \( \mathcal{G} \). The symbol \((\mathcal{G})_s\) means the weak limit procedure. If \( \phi \) is a measurable \( \Gamma \)-periodic function, we denote \( \phi^\varepsilon(x) = \phi(\varepsilon^{-1}x) \). Recall also the notation (3.0.1), (3.0.2).

We distinguish one elementary proposition which will be often used in weak limit procedures.

**Proposition 0.1.** 1°. Let \( \Psi \) be a \( \Gamma \)-periodic function of class \( L_{2,\text{loc}}(\mathbb{R}^d) \). Then

\[
(0.1) \quad \Psi^\varepsilon \xrightarrow{\varepsilon \to 0} \Psi \quad \text{in} \quad L_{2,\text{loc}}(\mathbb{R}^d).
\]

2°. Let \( G \in L_{\infty}(\mathbb{R}^d) \) be a \( \Gamma \)-periodic function and let \( \Phi \in L_1(\mathbb{R}^d) \). Then

\[
(0.2) \quad \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} G^\varepsilon(x) \Phi(x) \, dx = \overline{\int_{\mathbb{R}^d} \Phi(x) \, dx}.
\]

Relation (0.1) is the well known (cf., e. g., [ZhKO]) mean value property, which easily follows from the Riemann-Lebesgue Lemma. Relation (0.2) is a direct consequence of (0.1).

§1. Statement of the problem

1.1. **Concept of homogenization.** For the operators \( \mathcal{A}(g, f) \), we consider the family of \( (\varepsilon \Gamma) \)-periodic operators \( \mathcal{A}_\varepsilon(g, f) = \mathcal{A}(g^\varepsilon, f^\varepsilon), \varepsilon > 0 \). The coefficients of the operator \( \mathcal{A}_\varepsilon \) are rapidly oscillating as \( \varepsilon \to 0 \). The homogenization theory studies the behavior of the solutions \( \mathcal{u}_\varepsilon \) of the equation

\[
\mathcal{A}_\varepsilon(g, f)\mathcal{u}_\varepsilon + \mathcal{u}_\varepsilon = \mathcal{F}
\]

as \( \varepsilon \to 0 \). In other words, we deal with the behavior of the resolvent \( (\mathcal{A}_\varepsilon(g, f) + I)^{-1} \) as \( \varepsilon \to 0 \). The most successful case is that where \( f = 1_n \), i. e., the case of the equation

\[
(1.1) \quad \mathcal{A}_\varepsilon(g)\mathcal{u}_\varepsilon + \mathcal{u}_\varepsilon = \mathcal{F}.
\]

It turns out that, as \( \varepsilon \to 0 \), the solution \( \mathcal{u}_\varepsilon \) of equation (1.1) converges (in appropriate sense) to the solution \( \mathcal{u}_0 \) of the equation

\[
(1.2) \quad \mathcal{A}(g^0)\mathcal{u}_0 + \mathcal{u}_0 = \mathcal{F}.
\]

Here \( g^0 \) is the (main) effective matrix for the operator \( \mathcal{A}(g) \), and \( \mathcal{A}(g^0) \) is the (effective) DO with constant coefficients.
Besides convergence of solutions, the question of convergence of the so-called flows: \( \mathbf{p}_\varepsilon \to \mathbf{p}_\theta \), is also of interest. The flows are defined by the following relations:

\[
(1.3)\quad \mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon, \quad \mathbf{p}_\theta = g^\theta b(\mathbf{D})\mathbf{u}_\theta.
\]

Note that the operator \( \hat{A}(g^\varepsilon) \) and then the solution \( \mathbf{u}_\varepsilon \) of the equation (1.2) do not depend on the choice of the effective matrix. At the same time, the flow \( \mathbf{p}_\theta \) is defined via the main effective matrix \( g^\theta \).

Convergence of solutions and flows for the equation (1.1) with rapidly oscillating coefficients to the solution and the flow for the equation (1.2) with constant coefficients is usually interpreted as the homogenization of the medium. Usually, the medium described by the matrix \( g^0 \) is called the homogenized medium with respect to the medium described by the periodic matrix \( g^\varepsilon \). Here, it is convenient to fix the operator \( b(\mathbf{D}) \), since, as a rule, this operator is responsible for the type of the physical process, but not for its parameters. Often the effective DO \( \hat{A}(g^\varepsilon) \) is also called the homogenized DO.

For more general operators \( A(g, f) \) with variable \( f \), we did not succeed to find the operator of the same class with constant \( f \) and \( g \), such that its resolvent is the limit of the resolvent of the operator \( \mathcal{A}_\varepsilon(g, f) \). We have to approximate the family of resolvents by another (as simple as possible) operator family depending on \( \varepsilon \). The uniqueness of approximation is lost, and one can speak about homogenization of the medium only with certain reservations (cf. footnote on p. 38). However, the weaker sense of the convergence of solutions, the wider choice of the "candidate" for approximation. All these circumstances are well illustrated by the results of the present chapter.

1.2. About the nature of results. In \( \S2 \), we present theorems about approximations for the family of the resolvents

\[
(1.4)\quad (\mathcal{A}_\varepsilon(g, f) + I)^{-1}
\]

with respect to the operator norm in \( \mathcal{E} = L_2(\mathbb{R}^d; \mathbb{C}^n) \). We also consider approximations for the generalized resolvents. (After such extension of the question, the results about existence of (at least, weak) limit become "closed" (cf. Subsection 2.4). For the norm of the difference, we not only obtain the estimate of the right order \( O(\varepsilon) \), but also control the constants in estimates efficiently. We rely on Theorems 3.2.3, 3.2.7, 3.3.3, 3.3.4, 3.3.6, which automatically imply the results about homogenization. This allows us to view the homogenization as a threshold phenomenon of spectral nature. Theorems from \( \S2 \) are the most original results of the paper as applied to the homogenization theory. A part of these results has been published by the authors in [BSu2].

In particular, the estimates with respect to the operator norm in \( \mathcal{E} \) are suitable for interpolation and allow us to estimate \( \mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon \) in the Sobolev classes \( H^s \), \( 0 < s < 1 \), (cf. Subsection 2.3). However, in \( \S2 \) we do not discuss the convergence of the flows. The reason is that, in principal, the convergence of the flows can not be "too good". Therefore, in \( \S3,4 \) we discuss other, more traditional for the homogenization theory, versions of weak limit procedures. In their spirit, the corresponding proofs are also close to traditional. Although, there are some differences related to the fact that we consider rather general class of operators. However, material of \( \S3,4 \)
contains some new observations. In particular, we distinguish conditions under which convergence of solutions or convergence of flows turns out to be strong.

It is convenient to postpone further discussion till the comments in the end of this Chapter.

§2. APPROXIMATION OF THE RESOLVENT IN THE OPERATOR NORM

2.1. The case of the family \( \hat{A}_\varepsilon(g) \):

\( \hat{A}_\varepsilon(g) = b(D)^* g^\varepsilon(x) b(D), \quad \varepsilon > 0. \)

As above, \( g(x) = h(x)^* h(x) \) and \((m \times m)\)-matrix \( h(x) \) is invertible, \( \Gamma \)-periodic and such that \( h, h^{-1} \in L_\infty(\mathbb{R}^d) \). The selfadjoint operator (2.1) in \( \mathcal{S} \) is generated by the quadratic form

\[ \hat{a}_\varepsilon(g)[u, u] = (g^\varepsilon(x) b(D)u, b(D)u)_{\mathcal{S}}, \quad u \in \mathcal{S}. \]

Clearly, \( \hat{A}_1(g) \) coincides with the operator \( \hat{A}(g) \), which was discussed in Theorem 3.2.3.

By \( T_\varepsilon \) we denote the unitary scale transformation in \( \mathcal{S} \): \( (T_\varepsilon u)(\cdot) = \varepsilon^{d/2} u(\varepsilon \cdot) \).

From (2.2) it directly follows that \( \hat{A}_\varepsilon(g) = \varepsilon^{-2} T_\varepsilon^* \hat{A}(g) T_\varepsilon \), whence

\[ (\hat{A}_\varepsilon(g) + I)^{-1} = \varepsilon^{-2} T_\varepsilon^* (\hat{A}(g) + \varepsilon^2 I)^{-1} T_\varepsilon. \]

If the matrix is constant, then (2.3) turns into even simpler relation

\[ (\hat{A}(g^\varepsilon) + I)^{-1} = \varepsilon^{-2} T_\varepsilon^* (\hat{A}(g^\varepsilon) + \varepsilon^2 I)^{-1} T_\varepsilon. \]

Here \( \hat{A}(g^\varepsilon) = \hat{A}^0 \) is the effective DO for DO \( \hat{A}(g) \). Subtracting (2.4) from (2.3) and applying Theorem 3.2.3, we obtain the following theorem.

**Theorem 2.1.** Let \( \hat{A}^0 = \hat{A}(g^0) \) be the effective DO for DO \( \hat{A}(g) = b(D)^* g(x) b(D) \).

Then

\[ \| (\hat{A}_\varepsilon(g) + I)^{-1} - (\hat{A}^0 + I)^{-1} \|_{\mathcal{S} \to \mathcal{S}} \leq \hat{C}_x \varepsilon, \quad 0 < \varepsilon \leq 1, \]

\[ \limsup_{\varepsilon \to 0} \varepsilon^{-1} \| (\hat{A}_\varepsilon(g) + I)^{-1} - (\hat{A}^0 + I)^{-1} \|_{\mathcal{S} \to \mathcal{S}} \leq \hat{C}, \]

where the constants \( \hat{C} \) and \( \hat{C}_x \) are defined by (3.2.4), (3.2.8).

**Remark 2.2.** The scale transformation turns out to be efficient only because before the estimates in the operator norm have been obtained. For the study of convergence of different types, such a direct way is scarcely possible.

2.2. **Four theorems more** equally easily follow from the results of Chapter 3 with the help of the scale transformation. Let us formulate these theorems.

**Theorem 2.3.** Under the assumptions of Theorem 3.2.7, let \( \mathcal{A}_\varepsilon(g, f) = \mathcal{A}(g^\varepsilon, f^\varepsilon) \) and \( \mathcal{A}_\varepsilon(g^0, f) = \mathcal{A}(g^0, f^0) \). Then

\[ \| (\mathcal{A}_\varepsilon(g, f) + I)^{-1} - (\mathcal{A}_\varepsilon(g^0, f) + I)^{-1} \|_{\mathcal{S} \to \mathcal{S}} \leq C_x \varepsilon, \quad 0 < \varepsilon \leq 1, \]

\[ \limsup_{\varepsilon \to 0} \varepsilon^{-1} \| (\mathcal{A}_\varepsilon(g, f) + I)^{-1} - (\mathcal{A}_\varepsilon(g^0, f) + I)^{-1} \|_{\mathcal{S} \to \mathcal{S}} \leq C, \]

where the constants \( C, C_x \) are defined by (3.2.17), (3.2.19).

In the following theorems we omit the estimates for \( \limsup \). The first of these theorems carries the estimate (2.5) over to the case of a generalized resolvent. Here the matrix-valued function \( Q(x) \) is the same as in Subsection 3.3.1, and \( f \) is defined according to (3.3.2).
Theorem 2.4. We have
\[
(\hat{\mathcal{A}}_t (g) + Q^\varepsilon)^{-1} - (\hat{\mathcal{A}}(g^0) + Q)^{-1} \|_{\mathcal{O} \to \mathcal{O}} \leq \varepsilon C_{\varepsilon} \|Q^{-1}\|_{L_\infty}, \quad 0 < \varepsilon \leq 1,
\]
where \(C_{\varepsilon}\) is the constant from (3.2.19).

If \(Q = I\), the estimate (2.6) coincides with (2.5). In the following theorem, we give more convenient approximation for the resolvent (1.4). Here \(f\) and \(Q\) are still related by (3.3.2).

Theorem 2.5. Under the above assumptions, we have
\[
(\mathcal{A} (g, f) + I)^{-1} - (f^\varepsilon)^{-1} (\hat{\mathcal{A}}(g^0) + Q)^{-1} ((f^\varepsilon)^{-1})^{-1} \|_{\mathcal{O} \to \mathcal{O}} \leq \varepsilon C_{\varepsilon} \|Q^{-1}\|_{L_\infty} \|Q\|_{L_\infty}, \quad 0 < \varepsilon \leq 1,
\]
where \(C_{\varepsilon}\) is the constant from (3.2.19).

Finally, we proceed to theorem which generalizes Theorem 2.5. Here we use the notation of Subsection 3.3.3 without repeating explanations.

Theorem 2.6. Under the assumptions and the notation of Theorem 3.3.6, we have
\[
(\mathcal{A} (g, f) + Q^\varepsilon)^{-1} - (f^\varepsilon)^{-1} (\hat{\mathcal{A}}(g^0) + Q)^{-1} ((f^\varepsilon)^{-1})^{-1} \|_{\mathcal{O} \to \mathcal{O}} \leq \varepsilon C_{\varepsilon} \|Q^{-1}\|_{L_\infty} \|Q\|_{L_\infty} =: \varepsilon C_{\varepsilon}, \quad 0 < \varepsilon \leq 1.
\]

All four theorems follow from Theorems 3.2.7, 3.3.3, 3.3.4, 3.3.6 respectively, by the scale transformation. We omit the obvious arguments.

2.3. Interpolation. Along with the estimate (2.6), we have the following simple estimates for both generalized resolvents as operators from \(\mathcal{O}^{-1}\) to \(\mathcal{O}^{1}:
\[
(\hat{\mathcal{A}}_t (g) + Q^\varepsilon)^{-1} \|_{\mathcal{O}^{-1} \to \mathcal{O}^1} + (\hat{\mathcal{A}}_t (g^0) + Q)^{-1} \|_{\mathcal{O}^{-1} \to \mathcal{O}^1} \leq 2c_+,
\]
where
\[
c_+ = \min\{\alpha_\varepsilon \|g^{-1}\|_{L_\infty}^1, \|Q^{-1}\|_{L_\infty}^1\}.
\]
The standard interpolation between (2.6) and (2.9) implies the following result.

Theorem 2.7. Under the assumptions of Theorem 2.4, we have
\[
(\hat{\mathcal{A}}_t (g) + Q^\varepsilon)^{-1} - (\hat{\mathcal{A}}_t (g^0) + Q)^{-1} \|_{\mathcal{O}^{-1} \to \mathcal{O}^1} \leq \varepsilon^{1-s} (2c_+)^s \|Q^{-1}\|_{L_\infty}^{1-s}, \quad 0 \leq s < 1, \quad 0 < \varepsilon \leq 1.
\]

If \(Q = I\), the expressions for constants in (2.10), (2.11) simplify.

Interpolation under the assumptions of Theorems 2.3, 2.5, 2.6 is less natural, since only \(f^\varepsilon u_\varepsilon \in \mathcal{O}^1\). However, one can equally easily interpolate the properties of the product \(f^\varepsilon u_\varepsilon\). We formulate the corresponding result, relying on Theorem 2.5. We put
\[
W(\varepsilon) = f^\varepsilon (\mathcal{A} (g, f) + I)^{-1} - (\hat{\mathcal{A}}_t (g^0) + Q)^{-1} ((f^\varepsilon)^{-1})^{-1}.
\]
By (2.7), for the operator (2.12), we have
\[
\|W(\varepsilon)\|_{\mathcal{O} \to \mathcal{O}} \leq \varepsilon C_{\varepsilon} \|Q^{-1}\|_{L_\infty} \|Q\|_{L_\infty} =: \varepsilon C_{\varepsilon}, \quad 0 < \varepsilon \leq 1.
\]
Besides, it is easily seen that
\[
\|f^\varepsilon (\mathcal{A} (g, f) + I)^{-1} \|_{\mathcal{O} \to \mathcal{O}^1} + (\hat{\mathcal{A}}_t (g^0) + Q)^{-1} \|_{\mathcal{O} \to \mathcal{O}^1} \|((f^\varepsilon)^{-1})^{-1} \|_{\mathcal{O} \to \mathcal{O}^1} \leq 2c_+ \|f^\varepsilon\|_{L_\infty} =: c_+.
\]
Here \(c_+\) is defined by (2.10). Interpolating between (2.13) and (2.14), we arrive at the following theorem.
Theorem 2.8. Under the assumptions of Theorem 2.5, we have

$$||W(\varepsilon)||_{\mathcal{G}} \leq \varepsilon^{1-s} \varepsilon^s C_{\Delta}^{1-s}, \quad 0 \leq s < 1, \quad 0 < \varepsilon \leq 1.$$  

2.4. About weak limit for the generalized resolvent. Under the assumptions of Theorem 2.6, one can find the weak operator limit in $\mathcal{G}$ for the generalized resolvent

$$A_{\varepsilon} (g, f) + \Omega^s)^{-1}. \quad (2.15)$$

Indeed, by (2.8), it suffices to find the (w)-limit for the operator

$$R_{\varepsilon}(f) = (f^s)^{-1} (\hat{A}(g^0) + \overline{\Omega})^{-1} (f^s)^{-1}.$$  

Since the $(\mathcal{G} \to \mathcal{G})$-norm of $R_{\varepsilon}(f)$ is bounded with respect to $\varepsilon$, it suffices to find the limit

$$\lim_{\varepsilon \to 0} (R_{\varepsilon}(f), F, G)_{\mathcal{G}}, \quad F, G \in C^\infty (\mathbb{R}^d, \mathbb{C}^n). \quad (2.16)$$

By the mean value property (cf. Proposition 0.1), we have:

$$(w, \Omega)- \lim_{\varepsilon \to 0} ((f^s)^{-1} F = (f^s)^{-1} F, \quad (w, \Omega)- \lim_{\varepsilon \to 0} ((f^s)^{-1} G = (f^s)^{-1} G.$$  

Let $\eta \in C^\infty (\mathbb{R}^d)$ be such that $\eta(x) = 1$ for $x \in \text{supp} F$. Then, in (2.16), the operator $(\hat{A}(g^0) + \overline{\Omega})^{-1}$ can be replaced by the compact operator $(\hat{A}(g^0) + \overline{\Omega})^{-1} \eta$. Now, it is clear that the limit in (2.16) is equal to

$$(\hat{A}(g^0) + \overline{\Omega})^{-1} (f^s)^{-1} F, (f^s)^{-1} G)_{\mathcal{G}}.$$  

As a result, we obtain the following theorem.

Theorem 2.9. Under the assumptions of Theorem 2.6, the weak limit for the resolvent (2.15) exists:

$$\lim_{\varepsilon \to 0} (A_{\varepsilon} (g, f) + \Omega^s)^{-1} = (f^s)^{-1} (\hat{A}(g^0) + \overline{\Omega})^{-1} (f^s)^{-1} F,$$  

where the constant matrix $Q_{\Omega}$ is given by the formula $Q_{\Omega} = f^s(\overline{\Omega} f = (f^s)^{-1} f$.  

The operator in the right-hand side of (2.17) is a generalized resolvent for $A(g^0, f)$. However, $Q_{\Omega}$ is expressed via $f$ and $\Omega$ in a comparatively complicated way. Herewith, the "homogenized" matrix $Q_{\Omega}$ does not depend on $b(D)$ and $g$. Note also that, for $\Omega = 1_n$, we have $Q_{\Omega} = f^s(\overline{f} f = (f^s)^{-1} f \neq 1_n, \quad f \neq \text{const.}$ The existence of the limit for family (2.15) is related to the essential weakening of the "quality" of the limit procedure.
§3. Weak convergence of solutions and flows

3.1. Under the assumptions of Theorem 2.7, the solutions \( u_{\varepsilon} \) of the equations

\[
(\hat{A}(g) + Q'(|x|))u_{\varepsilon} = F
\]

converge in \( \mathcal{G}^s, \ s < 1 \), in a qualified way, to the solution \( u_0 \) of the homogenized equation

\[
(\hat{A}(g^0) + \overline{Q})u_0 = F.
\]

For \( s = 1 \), a fortiori, the result of such type does not take place, though the weak \( \mathcal{G}^1 \)-limit of the solutions exists, even for \( F \in \mathcal{G}^{-1} \). Besides, the weak \( \mathcal{G} \)-limit of the flows exists. These results about convergence are traditional for the homogenization theory. In this section, we prove one theorem of this type for general equations of the form (3.1). Moreover, we assume that \( F \) depends on \( \varepsilon \). The latter assumption allows us to deduce a useful consequence (cf. Subsection 4.2 below).

Let \( F_{\varepsilon}, F_0 \in \mathcal{G}^{-1} \), where the family \( \{F_{\varepsilon}\} \) is bounded in \( \mathcal{G}^{-1} \):

\[
\|F_{\varepsilon}\|_{\mathcal{G}^{-1}} \leq C^*.
\]

Besides, we assume that

\[
(\mathcal{G}^{-1})_{\text{loc}} \text{-lim }_{\varepsilon \rightarrow 0} F_{\varepsilon} = F_0.
\]

Suppose that \( Q(|x|) \) is a measurable \( \Gamma \)-periodic matrix-valued function such that \( Q(|x|) > 0 \) and

\[
Q + Q^{-1} \in L_{\text{loc}}.
\]

Consider equations (3.1) and (3.2) with \( F = F_{\varepsilon} \) and \( F = F_0 \) respectively. These equations are equivalent to the relations

\[
(\hat{g} b(D)u_{\varepsilon}, b(D)z)_{\mathcal{G}^s} + (Q'u_{\varepsilon}, z)_{\mathcal{G}} = (F_{\varepsilon}, z)_{\mathcal{G}}, \quad z \in \mathcal{G}^1,
\]

\[
(\hat{g} b(D)u_0, b(D)z)_{\mathcal{G}^s} + (\overline{Q}u_0, z)_{\mathcal{G}} = (F_0, z)_{\mathcal{G}}, \quad z \in \mathcal{G}^1.
\]

Let \( c_4 \) be the constant from (2.10). Combining (2.11) (with \( f = 1_n \)) with (3.3), (3.6), we obtain:

\[
\|u_{\varepsilon}\|_{\mathcal{G}^s} \leq c + C^*, \quad \|b(D)u_{\varepsilon}\|_{\mathcal{G}^s} \leq a_1^{1/2} c + C^*,
\]

where \( a_1 \) is the constant from (2.1.3). Next, (3.8) implies that the flows (1.3) are bounded:

\[
\|p_{\varepsilon}\|_{\mathcal{G}} \leq a_1^{1/2} c_4 C^* \|g\|_{L_\infty}.
\]

**Theorem 3.1.** Let \( u_{\varepsilon}, u_0 \in \mathcal{G}^1 \) satisfy identities (3.6) and (3.7) respectively. Under the above assumptions about \( Q, F_{\varepsilon} \) and \( F_0 \), we have the following:

1°. As \( \varepsilon \rightarrow 0 \), the solutions \( u_{\varepsilon} \) converge to \( u_0 \) weakly in \( \mathcal{G}^1 \).

2°. As \( \varepsilon \rightarrow 0 \), the flows \( p_{\varepsilon} \) converge to the flow \( p_0 = g^0 b(D)u_0 \) weakly in \( \mathcal{G}^\ast \).

The rest of §3 is the proof of this theorem.
3.2. Proof of Theorem 3.1. By (3.8), (3.9), for some sequence \( \varepsilon_j \to 0 \), the limits

\[
(\omega, \Theta^1) = \lim_{\varepsilon_j \to \varepsilon} \omega_{\varepsilon_j} = \omega \\
(\omega, \Theta^*) = \lim_{\varepsilon_j \to \varepsilon} \omega_{\varepsilon_j} = \omega
\]

exist. Our goal is to show that \( \omega = \omega_0 \), \( \omega = \omega_0 \). Then (3.10), (3.11) will directly imply both conclusions of Theorem.

Let us repeat representation (3.1.4) for \( g^\varepsilon \):

\[
g^\varepsilon C = |\varepsilon|^{-1} \int g(x)(b(D)v + C) dx, \quad C \in \mathbb{C}^n,
\]

where \( v \in \tilde{H}^1(\Omega; \mathbb{C}^n) \) is defined from (3.1.3) and from the condition \( \nabla = 0 \). We put \( \xi_\varepsilon(x) = b(D)v(x) + C \) and extend \( \xi_\varepsilon \) to a \( \Gamma \)-periodic function \( \xi_\varepsilon \in \Theta^*_\varepsilon \). By Proposition 0.1 (1'), we have

\[
(\omega, \Theta^*_\varepsilon) = \lim_{\varepsilon_j \to \varepsilon} \omega_{\varepsilon_j} = C,
\]

and, by (3.12),

\[
(\omega, \Theta^*_\varepsilon) = \lim_{\varepsilon_j \to \varepsilon} g^\varepsilon \xi_\varepsilon = g^\varepsilon C.
\]

Now we prove the following lemma.

**Lemma 3.2.** Let \( \eta \in C_0^\infty(\mathbb{R}^d) \). Then

\[
\lim_{\varepsilon_j \to \varepsilon} (\eta \xi_\varepsilon, \omega_{\varepsilon_j}) = (\eta C, \omega).
\]

**Proof of Lemma 3.2.** The under-limit expression splits into the sum

\[
(\eta C, \omega) + (\eta(b(D)v^\varepsilon), \omega_{\varepsilon_j}).
\]

By (3.11), the limit of the first summand (as \( \varepsilon_j \to 0 \)) coincides with the right-hand side of (3.14). Thus, the problem is reduced to proving that the second summand in (3.15) (we denote it by \( \Theta(\varepsilon) \)) tends to zero. We have:

\[
\Theta(\varepsilon) = \varepsilon (\eta b(D)v^\varepsilon, \omega_{\varepsilon_j}) = \varepsilon (b(D)(\eta v^\varepsilon), \eta b(D)v^\varepsilon, \omega_{\varepsilon_j}) = \varepsilon (b(D)\eta v^\varepsilon, \omega_{\varepsilon_j}) = \varepsilon(\Theta_1(\varepsilon) - \Theta_2(\varepsilon)).
\]

The matrix-valued function \( b(D) \eta \) has (cf. representation (2.1.4)) a compact support. By (0.1) and \( \nabla = 0 \), we have

\[
(\omega, \Theta_{\text{loc}}) = \lim_{\varepsilon_j \to \varepsilon} v^\varepsilon = 0.
\]

Combining this with (3.9), we obtain that \( \lim_{\varepsilon_j \to \varepsilon} \varepsilon \Theta_2(\varepsilon) = 0 \).
It remains to show that
\begin{equation}
\lim_{\varepsilon \to 0} \varepsilon \Theta_1(\varepsilon) = 0.
\end{equation}
Let $K \subset \mathbb{R}^d$ be a ball such that $K \supset \text{supp } \eta$. It is easily seen that
\begin{equation}
|\eta \psi^\varepsilon||_{L^1(K; \mathbb{C}^n)} \leq C(\eta, K) \varepsilon^{-1} |\psi||_{L^1(\Omega; \mathbb{C}^n)}.
\end{equation}
By (3.6) with $z = \eta \psi^\varepsilon$,
\[ \Theta_1(\varepsilon) = (\eta \psi^\varepsilon, F_\varepsilon)_\mathcal{S} - (\eta \psi^\varepsilon, Q^\varepsilon u_\varepsilon)_\mathcal{S} = : \Theta_{11}(\varepsilon) - \Theta_{12}(\varepsilon). \]
According to (3.4) and (3.18), we have $\lim_{\varepsilon \to 0} \varepsilon (\eta \psi^\varepsilon, F_\varepsilon - F_0)_\mathcal{S} = 0$. Next, $\lim_{\varepsilon \to 0} \varepsilon (\eta \psi^\varepsilon, F_\varepsilon)_\mathcal{S} = 0$. Indeed, by (3.18), we may assume that $F_0 \in \mathcal{S}$, and then the required fact follows from (3.16). Thus, $\lim_{\varepsilon \to 0} \varepsilon \Theta_{11}(\varepsilon) = 0$. Finally, relations (3.5), (3.8) and (3.16) yield that $\lim_{\varepsilon \to 0} \varepsilon \Theta_{12}(\varepsilon) = 0$. This implies (3.17). □

3.4. Proof of Theorem 3.1 (continuation). On the basis of the obvious identity
\[ \langle \psi^\varepsilon, p_\varepsilon \rangle_{\mathbb{C}^n} = \langle g^\varepsilon \psi^\varepsilon, b(D)\mathbf{u}_\varepsilon \rangle_{\mathbb{C}^n}, \]
one can calculate the limit in (3.14) by another method. Namely,
\begin{equation}
(\eta \psi^\varepsilon, p_\varepsilon)_\mathcal{S} = (\eta g^\varepsilon \psi^\varepsilon, b(D)\mathbf{u}_\varepsilon)_\mathcal{S},
\end{equation}
\begin{equation}
= (\eta g^\varepsilon \psi^\varepsilon, b(D)\mathbf{u}_\varepsilon)_\mathcal{S} + (\eta g^\varepsilon \psi^\varepsilon, b(D)(\mathbf{u}_\varepsilon - \mathbf{u}))_\mathcal{S} = : \Theta_3(\varepsilon) + \Theta_4(\varepsilon).
\end{equation}
From (3.13) it follows that
\begin{equation}
\lim_{\varepsilon \to 0} \Theta_3(\varepsilon) = \langle g^0 C, \eta^+ b(D)\mathbf{u} \rangle_{\mathcal{S}} = (\eta C, g^0 b(D)\mathbf{u})_{\mathcal{S}}.
\end{equation}
Here $\eta^+$ is the function, complex conjugate to the function $\eta$. Below we shall show that
\begin{equation}
\lim_{\varepsilon \to 0} \Theta_4(\varepsilon) = 0.
\end{equation}
Relations (3.19)-(3.21) imply that
\begin{equation}
\lim_{\varepsilon \to 0} (\eta \psi^\varepsilon, p_\varepsilon)_\mathcal{S} = (\eta C, g^0 b(D)\mathbf{u})_{\mathcal{S}}.
\end{equation}
Comparing the right-hand sides of (3.14) and (3.22) for any $\eta \in C^\infty_0(\mathbb{R}^d)$, $C \in \mathbb{C}^n$, we conclude that
\begin{equation}
p = g^0 b(D)\mathbf{u}.
\end{equation}
In order to prove that $\mathbf{u} = \mathbf{u}_0$ (and then $p = p_0$), we pass to the limit as $\varepsilon_j \to 0$ in (3.6). For this, it suffices to consider $z \in C^\infty_0(\mathbb{R}^d, \mathbb{C}^n)$. Using (3.11), (3.23), we have:
\begin{equation}
(g^0 b(D)\mathbf{u}_\varepsilon, b(D)\mathbf{z})_{\mathcal{S}} = (p_\varepsilon, b(D)\mathbf{z})_{\mathcal{S}} \to (p, b(D)\mathbf{z})_{\mathcal{S}} = (g^0 b(D)\mathbf{u}, b(D)\mathbf{z})_{\mathcal{S}}.
\end{equation}
Condition (3.4) implies that

\begin{equation}
(\mathbf{F}_\varepsilon, \mathbf{z})_\Theta \longrightarrow (\mathbf{F}_0, \mathbf{z})_\Theta, \quad \mathbf{z} \in \mathcal{C}_{0}^\infty(\mathbb{R}^d; \mathbb{C}^n).
\end{equation}

Let \( \mathcal{K}_0 \) be a ball such that \( \mathcal{K}_0 \supset \text{supp} \mathbf{z} \). From (3.10) it is seen that, as \( \varepsilon_j \to 0, \mathbf{u}_{\varepsilon_j} \) converges to \( \mathbf{u} \) in \( L_2(\mathcal{K}_0; \mathbb{C}^n) \). Combining this with (0.1), we arrive at

\begin{equation}
(\mathcal{Q}'\mathbf{u}_\varepsilon, \mathbf{z})_\Theta \longrightarrow (\mathcal{Q}\mathbf{u}, \mathbf{z})_\Theta, \quad \mathbf{z} \in \mathcal{C}_{0}^\infty(\mathbb{R}^d; \mathbb{C}^n).
\end{equation}

As a result, by (3.24)-(3.26), passage to the limit in (3.6) implies (3.7) with \( \mathbf{u}_\varepsilon \) replaced by \( \mathbf{u} \). Thus, \( \mathbf{u} = \mathbf{u}_\varepsilon, \mathbf{p} = \mathbf{p}_0 \), which is equivalent to the conclusions of Theorem.

3.5. **Proof of Theorem 3.1 (end).** It remains to check (3.21). We represent \( \Theta_4(\varepsilon) \) as

\[ \Theta_4(\varepsilon) = \int_\mathbb{R}^d \langle g\psi, b(D)\varepsilon \rangle \mathcal{C}_m \, d\mathbf{x} = 0, \]

which proves (3.21).

The scale transformation allows us to rewrite (3.27) as

\begin{equation}
\int_\mathbb{K} \langle g\psi, b(D)\mathbf{z} \rangle_{\mathcal{C}^m} \, d\mathbf{x} = 0,
\end{equation}

where \( \mathbb{K} \) is some ball and \( \mathbf{z} \) is some function such that \( \mathbf{z} \in \mathcal{S}_1 \) and \( \text{supp} \mathbf{z} \subset \mathbb{K} \). (Since now \( \varepsilon \) is fixed, we do not reflect dependence of \( \mathbb{K} \) and \( \mathbf{z} \) on \( \varepsilon \) in the notation.) Now, we choose the appropriate "partition of unity" \( \{\tilde{\zeta}_l\}_{l} \). Namely, let \( \tilde{\zeta} \in \mathcal{C}_{0}^\infty(\mathbb{R}^d) \) be such that \( \sum_l \tilde{\zeta}(\mathbf{x}) = 1 \) for \( \mathbf{x} \in \mathbb{K} \) and \( \text{supp} \tilde{\zeta} \subset \Omega + \mathbf{x}_l^0 \) for each \( l \) and some \( \mathbf{x}_l^0 \in \mathbb{R}^d \). Then

\begin{equation}
\int_\mathbb{K} \langle g\psi, b(D)\tilde{\zeta}\mathbf{z} \rangle_{\mathcal{C}^m} \, d\mathbf{x} = \int_{\Omega + \mathbf{x}_l^0} \langle g\psi, b(D)\mathbf{w}_l \rangle_{\mathcal{C}^m} \, d\mathbf{x},
\end{equation}

where \( \mathbf{w}_l = \tilde{\zeta}\mathbf{z} \). We extend the function \( \mathbf{w}_l \) from \( \Omega + \mathbf{x}_l^0 \) to the \( \Gamma \)-periodic function in \( \mathbb{R}^d \). Then

\begin{equation}
\int_\mathbb{K} \langle g\psi, b(D)\tilde{\zeta}\mathbf{z} \rangle_{\mathcal{C}^m} \, d\mathbf{x} = \int_{\Omega} \langle g\psi, b(D)\mathbf{w}_l \rangle_{\mathcal{C}^m} \, d\mathbf{x} = 0.
\end{equation}

The latter equality in (3.29) directly follows from (3.1.3), since \( \mathbf{w}_l|_{\Omega} \in \bar{H}^1(\Omega; \mathbb{C}^n) \). Summing equalities (3.29), we arrive at (3.28). \( \square \)
4. Convergence of solutions and flows (part 2)

Here we deduce some consequences from Theorem 3.1, and also analyze the question about conditions under which convergence of solutions or convergence of flows turns out to be strong.

4.1. The following theorem is a direct consequence of Theorem 3.1.

**Theorem 4.1.** Let \( u_\varepsilon \) be solution of the equation (3.1), and let \( u_0 \) be solution of the equation (3.2) with \( F \in \mathcal{G}^{-1} \). Then the following is true.

1°. As \( \varepsilon \to 0 \), the solutions \( u_\varepsilon \) converge to \( u_0 \) weakly in \( \mathcal{G}^1 \).

2°. As \( \varepsilon \to 0 \), the flows \( p_\varepsilon \) converge to the flow \( p_0 \) weakly in \( \mathcal{G}_* \).

4.2. The case where \( f \neq 1_n \). Now we consider the equation

\[
A \varepsilon (g, f) u_\varepsilon + Q^2 u_\varepsilon = F, \quad F \in \mathcal{G},
\]

more general than (3.1). We call attention to the fact that now we assume \( F \in \mathcal{G} \), in order to avoid encumbering of formulation. We put \( u_\varepsilon = f^* u_\varepsilon \), and denote

\[
F_\varepsilon = (f^*)^{-1} F = ((f^*)^{-1})^T F, \quad Q = (f^*)^{-1} Q f^{-1}.
\]

Then (4.1) turns into the following equation for \( u_\varepsilon \):

\[
\hat{A} \varepsilon (g) u_\varepsilon + Q^2 u_\varepsilon = F_\varepsilon.
\]

We apply Theorem 3.1 to this equation. Namely, we put

\[
F_\varepsilon = (f^*)^{-1} F = (f^*)^{-1} F,
\]

and consider the equation

\[
\hat{A} (g^0) u_0 + Q u_0 = F_0.
\]

Equations (4.3), (4.5) satisfy conditions of Theorem 3.1. Indeed, by (0.2),

\[
(w, \mathcal{G}) \lim_{\varepsilon \to 0} F_\varepsilon = F_0.
\]

Let \( K \subset \mathbb{R}^d \) be a ball. Since the natural embedding \( \mathcal{G} \subset H^{-1}(K; \mathbb{C}^n) \) is compact,

\[
(w, H^{-1}(K; \mathbb{C}^n)) \lim_{\varepsilon \to 0} F_\varepsilon = F_0.
\]

Thus, (3.4) is satisfied. Obviously, other conditions of Theorem 3.1 are also satisfied. Hence, we arrive at the following theorem.

**Theorem 4.2.** Let \( \tilde{u}_\varepsilon \) be solution of the equation (4.1), and let \( Q \) and \( F_\varepsilon \) be as in (4.2) and (4.4). Let \( u_0 \) be solution of the equation (4.5). We put \( p_\varepsilon = g^0 b(D) f^* u_\varepsilon \),

\[
p_0 = g^0 b(D) u_0.
\]

Then

\[
(w, \mathcal{G}^1) \lim_{\varepsilon \to 0} f^* \tilde{u}_\varepsilon = u_0,
\]

\[
(w, \mathcal{G}_*) \lim_{\varepsilon \to 0} \tilde{p}_\varepsilon = p_0.
\]

**Remark 4.3.** Theorem 2.9 can be proved on the basis of Theorem 4.2. We shall not dwell on this.
4.3. About strong convergence. It is interesting to distinguish the cases where, in one of the conclusions of Theorem 4.1, the weak limit turns into the strong one.

**Theorem 4.4.** Under the assumptions of Theorem 4.1, let \( \mathbf{F} \in \mathcal{S} \). Then the following is true.

1°. \((\mathfrak{G}^1)\)-convergence of the solutions \( \mathbf{u}_\varepsilon \) of equations (3.1) to the solution \( \mathbf{u}_0 \) of equation (3.2) is equivalent to the condition

\[
(4.7) \quad (\mathcal{F} - \mathcal{F}_0) \mathbf{b}(\mathbf{D}) \mathbf{u}_0 = 0.
\]

2°. \((\mathfrak{G}_e)\)-convergence of the flows \( \mathbf{p}_\varepsilon = \mathcal{F}_\varepsilon \mathbf{b}(\mathbf{D}) \mathbf{u}_\varepsilon \) to the flow \( \mathbf{p}_0 = \mathcal{F}_0 \mathbf{b}(\mathbf{D}) \mathbf{u}_0 \) is equivalent to the condition

\[
(4.8) \quad (\mathcal{F}_0 - \mathcal{F}_\varepsilon) \mathbf{b}(\mathbf{D}) \mathbf{u}_0 = 0.
\]

**Proof** of both statements is similar. To be definite, let us prove the statement 2°. We have:

\[
J_\varepsilon := ((\mathcal{F}_\varepsilon)^{-1} (\mathbf{p}_\varepsilon - \mathbf{p}_0), (\mathbf{p}_\varepsilon - \mathbf{p}_0))_\mathcal{S},
\]

\[
= (\mathcal{F}_\varepsilon \mathbf{b}(\mathbf{D}) \mathbf{u}_\varepsilon, \mathbf{b}(\mathbf{D}) \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) - (\mathcal{F}_0 \mathbf{b}(\mathbf{D}) \mathbf{u}_0, \mathbf{b}(\mathbf{D}) \mathbf{u}_0, \mathbf{u}_0),
\]

\[
+ ((\mathcal{F}_\varepsilon)^{-1} \mathcal{F}_0 \mathbf{b}(\mathbf{D}) \mathbf{u}_0, \mathcal{F}_0 \mathbf{b}(\mathbf{D}) \mathbf{u}_0, \mathbf{u}_0, \mathbf{u}_0) - (\mathbf{b}(\mathbf{D}) \mathbf{u}_\varepsilon, \mathbf{b}(\mathbf{D}) \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon).
\]

From equations (3.1), (3.2) it follows that

\[
J_\varepsilon^{(1)} - J_\varepsilon^{(2)} = ((\mathcal{F} - \mathcal{F}_\varepsilon) \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)_\mathcal{S},
\]

\[
= ((\mathcal{Q} - \mathcal{Q}_\varepsilon) \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)_\mathcal{S} + ((\mathcal{Q}^e \mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)_\mathcal{S} + ((\mathcal{Q}_\varepsilon - \mathcal{Q}^e) \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon)_\mathcal{S}.
\]

According to Theorem 2.4, \((\mathfrak{G})\)-lim \( \mathbf{u}_\varepsilon = \mathbf{u}_0 \). Combining this with (0.2), we deduce that

\[
\lim_{\varepsilon \to 0} (J_\varepsilon^{(1)} - J_\varepsilon^{(2)}) = 0.
\]

Next, by (0.2),

\[
\lim_{\varepsilon \to 0} J_\varepsilon^{(3)} = ((\mathcal{F}_\varepsilon)^{-1} \mathcal{F}_0 \mathbf{b}(\mathbf{D}) \mathbf{u}_\varepsilon, \mathbf{b}(\mathbf{D}) \mathbf{u}_\varepsilon)_\mathcal{S}.
\]

Finally, by Theorem 4.1, 1°,

\[
(\mathfrak{u}, \mathfrak{G}_e)\)-lim \( \mathbf{b}(\mathbf{D}) \mathbf{u}_\varepsilon = \mathbf{b}(\mathbf{D}) \mathbf{u}_0,
\]

whence

\[
\lim_{\varepsilon \to 0} J_\varepsilon^{(4)} = (\mathbf{b}(\mathbf{D}) \mathbf{u}_\varepsilon, \mathbf{b}(\mathbf{D}) \mathbf{u}_\varepsilon)_\mathcal{S}.
\]

Relations (4.9)-(4.11) and (4.13) imply that

\[
\lim_{\varepsilon \to 0} J_\varepsilon = ((\mathcal{F}_0 - \mathcal{F}_\varepsilon) \mathbf{b}(\mathbf{D}) \mathbf{u}_0, (\mathcal{F}_\varepsilon)^{-1} \mathbf{b}(\mathbf{D}) \mathbf{u}_\varepsilon)_\mathcal{S}.
\]
Hence, if (4.8) is satisfied, then $J_z \to 0$ as $\varepsilon \to 0$. Whence,
\begin{equation}
(\mathfrak{G}_*) \lim_{\varepsilon \to 0} p_\varepsilon = p_0.
\end{equation}

Conversely, suppose that, for a given $F$, (4.14) is fulfilled. Then, in the equality
$b(D)u_\varepsilon = (g^\varepsilon)^{-1}(p_\varepsilon - p_0) + (g^\varepsilon)^{-1}p_0$, the first summand tends to zero in $\mathfrak{G}_*$. By
(0.2), the second summand $\mathfrak{G}_*$-weakly converges to $(g^\varepsilon)^{-1}p_0$. Thus,
\begin{equation}
(w, \mathfrak{G}_*) \lim_{\varepsilon \to 0} b(D)u_\varepsilon = (g^\varepsilon)^{-1}g^\varepsilon b(D)u_0.
\end{equation}
Combining this with (4.12), we arrive at (4.8). \qed

Relations (4.7) or (4.8) are valid for any $F \in \mathfrak{G}$, if $g^\varepsilon = \mathcal{F}$ or $g^\varepsilon = \mathcal{G}$ respectively. Moreover (cf. (3.8), (3.9)), the estimates
\[ |u_\varepsilon|_{\mathfrak{G}_*} \leq c_\varepsilon + |F|_{\mathfrak{G}_*}^{-1}, \quad |p_\varepsilon|_{\mathfrak{G}_*} \leq a_1 \varepsilon^{1/2} c_\varepsilon \|g\|_{L_\infty} |F|_{\mathfrak{G}_*}^{-1}, \]
allow us to extend the statements about strong convergence to any $F \in \mathfrak{G}_*$. Thus, we have proved the following theorem.

**Theorem 4.5.** Under the assumptions of Theorem 4.1, the following is true.

1°. If $g^\varepsilon = \mathcal{F}$, then, for any $F \in \mathfrak{G}_*$,
\begin{equation}
(\mathfrak{G}_*) \lim_{\varepsilon \to 0} u_\varepsilon = u_0.
\end{equation}

2°. If $g^\varepsilon = \mathcal{G}$, then, for any $F \in \mathfrak{G}_*$,
\begin{equation}
(\mathfrak{G}_*) \lim_{\varepsilon \to 0} p_\varepsilon = p_0.
\end{equation}
In particular, (4.15) is a fortiori valid, if $m = n$.

In contrast to Theorem 4.4, now the sufficient conditions for strong convergence are not necessary.

**4.4.** The latter theorem does not satisfy all needs of applications. Let $\mathcal{R}$ be one more $\Gamma$-periodic matrix-valued function such that $\mathcal{R}(x) > 0$ and $\mathcal{R} + \mathcal{R}^{-1} \in L_\infty$.

Consider the equations
\begin{align}
(4.16) & \quad (\hat{A}_z(g) + \mathcal{Q}^z)u_\varepsilon = \mathcal{R}^z F, \quad F \in \mathfrak{G}, \\
(4.17) & \quad (\hat{A}_z(g^\varepsilon) + \mathcal{Q}^z)u_0 = \mathcal{R}F, \quad F \in \mathfrak{G}.
\end{align}

First of all, we distinguish the following proposition.

**Proposition 4.6.** We have
\begin{equation}
(\mathfrak{G}_*) \lim_{\varepsilon \to 0} u_\varepsilon = u_0.
\end{equation}

**Proof.** It suffices to prove (4.18) for $F \in C_0^\infty(\mathbb{R}^d, \mathfrak{G}^z)$. Let $u_\varepsilon^\gamma$ be the solution of equation (4.17) with the right-hand side replaced by $\mathcal{R}^z F$. By Theorem 2.4, we have $|u_\varepsilon - u_\varepsilon^\gamma|_{\mathfrak{G}_*} \to 0$ as $\varepsilon \to 0$. From (0.1) it follows that
\begin{equation}
(w, \mathfrak{G}_*) \lim_{\varepsilon \to 0} (\mathcal{R}^z - \mathcal{R})F = 0.
\end{equation}
Let $\eta \in C_0^\infty(\mathbb{R}^d)$ be such that $\eta F = F$. Then
\[ u_\varepsilon^\gamma - u_0 = ((\hat{A}_z(g^\varepsilon) + \mathcal{Q}^z)^{-1} \eta) (\mathcal{R}^z - \mathcal{R})F \xrightarrow{\mathfrak{G}_*} 0. \]
This follows from (4.19) and from the presence of the compact operator (in $\mathfrak{G}$) applied to $(\mathcal{R}^z - \mathcal{R})F$. \qed
Theorem 4.7. Let \( u_\varepsilon, u_0 \) be solutions of the equations (4.16) and (4.17) respectively. Then the following is true.

1°. If \( g^0 = \f, \) then 
\[
\lim_{\varepsilon \to 0} (\mathcal{G}^1) u_\varepsilon = u_0.
\]

2°. If \( g^0 = q, \) then, for the flows (1.3), 
\[
\lim_{\varepsilon \to 0} (\mathcal{G}_*) p_\varepsilon = p_0.
\]

Proof slightly differs from that of Theorem 4.4. So, we omit the proof. Note only that we can again assume that \( F \in C_0^\infty (\mathbb{R}^d; \mathbb{C}^n), \) and, in the estimates, take (4.18) and (4.19) into account. \( \square \)

Now we apply Theorem 4.7 to the case where \( f \neq 1_n. \) Consider the equation (4.1):

\[
\mathcal{A}_\varepsilon (g, f) \tilde{u}_\varepsilon + \mathcal{Q} \tilde{u}_\varepsilon = F, \quad F \in \mathcal{G}.
\]

We put \( u_\varepsilon = f^* \tilde{u}_\varepsilon \) and denote

\[
Q = (f^*)^{-1} \Omega f^{-1}, \quad R = (f^*)^{-1}.
\]

Then equation (4.20) turns into equation (4.16). Applying Theorem 4.7 to the latter equation, we arrive at the following theorem.

Theorem 4.8. Let \( \tilde{u}_\varepsilon \) be solution of the equation (4.20), and let \( u_0 \) be solution of the equation (4.17), where \( Q \) and \( R \) are defined by (4.21). Then the following is true.

1°. If \( g^0 = \f, \) then 
\[
(\mathcal{G}^1) \lim_{\varepsilon \to 0} f^* \tilde{u}_\varepsilon = u_0.
\]

2°. If \( g^0 = q, \) then, for the flows \( \bar{p}_\varepsilon = g^* b(D) f^* \tilde{u}_\varepsilon, \) 
\[
(\mathcal{G}_*) \lim_{\varepsilon \to 0} \bar{p}_\varepsilon = p_0 = g^0 b(D) u_0.
\]

Comments on Chapter 4

Here we collect various remarks and comments concerning theorems about the homogenization. This material supplements what was said in §1.

1. The general method of Chapter 1 concerns the operators admitting an appropriate factorization. The linear dependence of the factors on the parameter corresponds to the study of second order DO’s. Both assumptions, from one side, allow us to advance far in analysis, from the other side, they somewhat restrict the circle of applications. However, as it is seen from Chapters 5–7, many important periodic operators of mathematical physics belong to the classes of DO’s of the form \( \mathcal{A}(g) \) and \( \mathcal{A}(g, f), \) which were distinguished in Chapters 2, 3.
2. In the case of operators of the form $\hat{A}_s(g)$, we deal with the real homogenization: among the operators which are threshold equivalent to a given one, there exists an operator with constant coefficients. This is directly related to the fact that, for the operator family $\hat{A}(k; g)$, $k = t\theta$, the corresponding kernel $\hat{\eta}$ consists of constants, and calculating of projections onto $\hat{\eta}$ is reduced to ordinary averaging. The class of operators $A(g)$ (with fixed $b(D)$) is closed with respect to homogenization as the resolvent convergence in the operator norm. This is well illustrated by Theorems 2.1 and 2.4.

A different situation arises with a wider class of DO’s $\hat{A}(g, f)$. A good approximation for $(\lambda \hat{A}(g, f) + I)^{-1}$ is given by another operator family, also having the oscillating factors. The choice of such family is not unique. In this respect, a good result is given by Theorem 2.5, while the result of Theorem 2.3 is less convenient.

The class of generalized resolvents (2.15) is already closed with respect to homogenization as the weak limit procedure (cf. Theorem 2.9). This class contains three functional parameters: $g, f, \Omega$. Each of them is "homogenized" by its own rule.

3. Usually, the matrix $b(D)$ is responsible for the type of the physical process, while $g, f$ are responsible for its parameters. However, the choice of $b(D)$ is not always unique. Thus, e.g., (cf. Subsection 5.2.3 below) the isotropic operator of elasticity theory with constant shear modulus (the Hill body) can be described not only in the universal form, but also via the matrix $b(D)$ of lower size. This fact yields useful conclusions.

4. Theorem 2.7 was deduced from Theorem 2.4 with the help of interpolation. Similarly, one could apply interpolation, relying on Theorem 3.2.3. More particularly, we could supplement the estimate (3.2.11) by the estimate of the $(G^{-s} \to G^s)$-norm of the difference of resolvents for $0 < s < 1$.

5. Apparently, in the homogenization theory, the results like that of Theorems from §2 have not been distinguished before. These theorems are the most essential results of the paper both in content and in the method of the proof. The corresponding estimates are of precise order, and the constants in estimates are well controlled. Theorems 2.1 and 2.3 have been published by the authors before in the paper [BSu1]. Other theorems of §2 are published for the first time. As for other possible approaches to obtaining such results, see comments on Chapter 5.

6. The material of §3, 4 is closer to the usual formulations and technique of the homogenization theory. For instance, the conclusion of Theorem 4.1 looks quite traditional, and in a number of special cases leads to the known results (cf. [BeLP, Sa, ZhKO]). Herein, representation (3.1.14) for the (main) effective matrix $\hat{g}^0$ is viewed as initial one. As well as predecessors, in the proof of the (main) Theorem 3.1, we use the periodic solution of the equation on the cell, but only for $k = 0$. Thus, we do not use much of the Floquet decomposition.

Note that we do not use (at least, explicitly) the method of the proof which is usually called (cf. [BeLP, ZhKO]) the compensated compactness.

New useful observations are related to Theorems 4.5, 4.7 and 4.8. The corresponding results will be applied in Subsections 5.2.3, 6.2.3, 6.3.3; see also Remarks 5.1.1, 6.1.9 and Propositions 7.2.4, 7.3.8.
Chapter 5. Applications of general scheme. The case where \( f = 1_n \)

Here we consider applications of the results of Chapters 3 and 4 to specific periodic operators of mathematical physics in the case where \( f = 1_n \). We discuss specific properties of threshold characteristics and the homogenization problems. The main examples are the acoustic operator, the Schrödinger operator and operator of elasticity theory.

\[ \text{§ 1. The periodic acoustic and Schrödinger operators} \]

1.1. The acoustic operator. In \( L_2(\mathbb{R}^d) \), \( d \geq 1 \), we consider the operator

\[
\hat{A}(g) = D^* g(x)D = \nabla^* g(x) \nabla,
\]

describing a periodic acoustic medium. Here \( g(x) \) is a \( \Gamma \)-periodic \( (d \times d) \)-matrix-valued function with real-valued entries such that

\[
g(x) > 0, \quad g + g^{-1} \in L_\infty.
\]

Now \( n = 1 \), \( m = d \geq 1 \), \( b(x) = \xi \), \( \mathcal{H} = L^1(\Omega) \), \( \mathcal{H}_* = L_2(\Omega; \mathbb{C}^d) \). Obviously, condition (2.1.2) is satisfied. One can realize representation (cf. (2.1.10))

\[
g(x) = h(x)^* \gamma h(x),
\]

e. g., putting \( h = g^{1/2} \). However, the specific choice of representation (1.2) is not essential. The kernels (3.1.1) and (2.3.5) are given by the relations

\[
\hat{\mathcal{H}} = \{ u \in \mathcal{H} : u = \text{const} \}, \quad d \geq 1,
\]

\[
\hat{\mathcal{H}}_* = \{ q \in \mathcal{H}_*: h^* q \in \tilde{H}^1(\Omega; \mathbb{C}^d), \quad \text{div} h^* q = 0 \}, \quad n_* = \infty, \quad d \geq 2,
\]

\[
\hat{\mathcal{H}}_* = \{ q \in \mathcal{H}_*: h^* q = \text{const} \}, \quad \mathcal{H}_* = \mathcal{H}, \quad n_* = 1, \quad d = 1.
\]

According to (3.1.5) and (3.1.8), we describe the operators \( \hat{R}(\theta) : \hat{\mathcal{H}} \to \hat{\mathcal{H}}_* \) and \( \hat{S}(\theta) : \hat{\mathcal{H}} \to \hat{\mathcal{H}} \). Let \( r = r(\theta) \in \tilde{H}^1(\Omega) \) be a (weak) solution of the equation

\[ D^* g(D r + \theta) = 0. \]

We put \( u(\theta) = h(D r(\theta) + \theta) \in \hat{\mathcal{H}}_* \). Then \( \hat{R}(\theta)|c = \gamma u(\theta) \), \( c \in \hat{\mathcal{H}} \), and \( \hat{S}(\theta) \) is reduced to multiplication by the number

\[
\gamma(\theta) = |\Omega|^{-1} \| u(\theta) \|_{\mathcal{H}_*}^2, \quad \theta \in \mathbb{S}^{d-1}.
\]

Now, we construct the effective matrix \( g^0 \) for operator (1.1). We introduce the functions \( v_j \in \tilde{H}^1(\Omega), v_j = v(e_j) \), where \( \{ e_j \} \) is a fixed basis in \( \mathbb{R}^d \). Then (cf. (3.1.12))

\[
g^0 = |\Omega|^{-1} \{ (u_{j,l}, u_l)_{\mathcal{H}_*} \}, \quad j,l = 1, \ldots, d, \quad u_j = h(D v_j + e_j).
\]

According to Subsection 3.1.3, now the effective matrix \( g^0 \) with real-valued entries is unique. Therefore, it is fair to call \( g^0 \) the matrix of effective medium.

Formula (3.1.8) shows that

\[
\hat{S}(\theta) = \theta^* g^0 \theta = \langle g^0 \theta, \theta \rangle_{\mathcal{H}_*}.
\]
It is easily seen that expressions (1.6)-(1.8) do not depend on the factorization (1.2).

We put

\[ \hat{\gamma}(k) = t^2 \hat{\gamma}(|\theta|) = \langle g^0, k \rangle_{\mathbb{R}^d}, \quad k = t\theta, \quad t = |k|. \] 

Then the effective DO \( \hat{\mathcal{A}}^0 = \hat{\mathcal{A}}(g^0) \) can be written as

\[ \hat{\mathcal{A}}(g^0) = D^* g^0 D = \hat{\gamma}(D). \]

Note that, for \( d = 1 \), we have \( m = n (= 1) \), whence (cf. Theorem 3.1.5),

\[ g^0 = \bar{g} \quad \text{for} \quad d = 1. \]

By (1.8), if \( d = 1 \), the number \( \hat{\gamma}(\theta) \) does not depend on \( \theta \), which now has only two values \( \theta = \pm 1 \).

Theorem 3.2.3 can be directly applied to operator (1.1). This gives the estimate for the \( \mathcal{O}(\theta) \)-norm of the difference of resolvents

\[ (A(g) + \varepsilon^2 I)^{-1} - (\hat{A}(g^0) + \varepsilon^2 I)^{-1} \quad \text{for small} \quad \varepsilon > 0; \quad \text{now} \quad \mathcal{O} = L_2(\mathbb{R}^d). \]

One can also apply interpolation, as it was done in the proof of Theorem 4.2.7. Then we obtain the estimates for the \( \mathcal{O} \rightarrow \mathcal{O}^\ast \)-norm of the difference of resolvents with \( s \in (0, 1) \). Such estimates may be of interest for the study of the negative discrete spectrum for periodic operator \( \hat{\mathcal{A}}(g) \) perturbed by a potential decaying as \( |x| \to \infty \) (potential of the "adixture" type).

To the resolvent of the operator \( \hat{\mathcal{A}}_\varepsilon(g) = \hat{\mathcal{A}}(g^\varepsilon) \), we can apply results about the homogenization. Namely, Theorems 4.2.1 and 4.2.7 (the latter theorem with \( Q = 1 \)) are applicable. Theorems 4.4.1, 4.4.4, 4.4.5 (again with \( Q = 1 \)) are also applicable.

**Remark 1.1.** Let us dwell on application of Theorem 4.4.5 in more detail. In Subsection 3.1.4, we have already mentioned the following known (cf. [ZhKO]) fact. The relation \( g^0 = \bar{g} \) for the acoustic operator is equivalent to the fact that the columns of the matrix \( g \) are solenoidal; the relation \( g^0 = \bar{g} \) is equivalent to the fact that the columns of the matrix \( g^{-1} \) are potential. Respectively, in the first case the \( \mathcal{O} \)-convergence of the solutions \( u_\varepsilon \) to \( u_0 \) is strong, and in the second case the \( \mathcal{O} \)-convergence of the flows \( p_\varepsilon \) to \( p_0 \) is strong.

### 1.2. The Schrödinger operator

Now, we consider the operator

\[ \mathcal{H}(g) = \hat{\mathcal{A}}(g) + Q(x), \]

where \( Q(x) > 0 \) is a \( \Gamma \)-periodic function such that \( Q + Q^{-1} \in L_\infty \). The operator \( \mathcal{H}(g) \) is a periodic Schrödinger operator (with metric \( g \)). If \( g = 1_d \), the operator (1.11) is an ordinary Schrödinger operator for the quantum particle in the external electric field corresponding to the periodic potential \( Q \). To the operator

\[ \mathcal{H}_\varepsilon(g) = \hat{\mathcal{A}}_\varepsilon(g) + Q^\varepsilon(x), \]

Theorems 4.2.4 and 4.2.7 about homogenization are applicable. Now, it is convenient to interpret them as the estimates for the norm of the difference of inverse operators for the operator (1.12) and for the "effective Schrödinger operator"

\[ \mathcal{H}^0 = \hat{\mathcal{A}}(g^0) + Q. \]
Theorems 4.4.1, 4.4.4 and 4.4.5 are also applicable, and the analogue of Remark 1.1 is true. The operator (1.13) can be interpreted as the operator of the free particle in the space with homogeneous metric $g^0$. The constant $Q$ is responsible for the shift in the energy scale (the "renormalization"). Thus, for rapidly oscillating periodic metric and electric field, the quantum particle behaves like a free one. The quoted theorems give qualitative and quantitative characteristics of the passage of the particle into the free particle, as $\varepsilon \to 0$. Below, in §6.1, we shall consider another statement of the homogenization problem for the Schrödinger operator.

1.3. Other examples, which we proceed to, will be useful in §6.2 for the study of the two-dimensional Pauli operator. These examples themselves are also interesting.

Let $d = 2$, $n = m = 1$, and let $\omega(\mathbf{x})$ be a $\Gamma$-periodic function such that $\omega(\mathbf{x}) > 0$ and $\omega + \omega^{-1} \in L_\infty$. We consider a pair of operators

$$\hat{\mathcal{B}}_\pm(\omega^2) = \partial_\omega \omega^2(\mathbf{x}) \partial_\pm, \quad \hat{\mathcal{B}}^\pm(\omega^2) = \partial_\pm \omega^2(\mathbf{x}) \partial_\pm, \quad \omega_\pm = D_1 \pm iD_2.$$  

The operator $\hat{\mathcal{B}}^\pm(\omega^2)$ is of the form $\hat{\mathcal{A}}(g)$ with $g = \omega^2$ and $b(\xi) = \xi^1 \pm i\xi^2$. Since $m = n = 1$, the number $g_\omega^0$ is defined by the formula

$$g_\omega^0 = g_\omega^0 = g^0 = (\omega^2) = \left( |\Omega|^{-1} \int_\Omega (\omega(\mathbf{x}))^{-2} \, d\mathbf{x} \right)^{-1}.$$  

We calculate the germ $\hat{\mathcal{S}}(\omega^2) = \hat{\mathcal{S}}(\omega^0)$ of the operator $\hat{\mathcal{B}}^\pm(\omega^2)$. By (3.1.9), we have:

$$\hat{\mathcal{S}}(\omega^2) = (\theta^1 \mp i\theta^2) g^0 (\theta^1 \pm i\theta^2) = g^0.$$  

Thus, the germs $\hat{\mathcal{S}}_\pm$ do not depend on $\theta$, coincide with each other ($\hat{\mathcal{S}}_+ = \hat{\mathcal{S}}_-$), and are reduced to multiplication by the constant $\gamma = g^0$ defined by (1.15). According to (3.1.10), the effective DO $\hat{\mathcal{B}}^\pm(\omega^2)$ is given by the formula

$$\hat{\mathcal{B}}_\pm^0 = \partial_\omega g^0 \partial_\pm = -g^0 \Delta.$$  

Thus, the effective operators coincide with each other:

$$\hat{\mathcal{B}}_\pm^0 = \hat{\mathcal{B}}_\pm^0 =: \hat{\mathcal{B}}_\pm^0 = -g^0 \Delta.$$  

Expression (1.16) gives the reason to call the constant $(g^0)^{-1}$ the effective mass for the operators $\hat{\mathcal{B}}^\pm(\omega^2)$ at the lower edge of the spectrum (i. e., for $\lambda = 0$).

One can treat operators (1.14) as the two-dimensional (complex) analogues of the one-dimensional acoustic operator. The fact that $\hat{\mathcal{S}}(\omega^2)$ for any $\omega(\mathbf{x})$ does not depend on $\theta$, attests some hidden symmetry.

Note that the operators (1.14) can be also written in more traditional form

$$\hat{\mathcal{B}}^\pm(\omega^2) = D^* \omega^2 g^0 D, \quad \omega^0 = \begin{pmatrix} 1 & \pm i \\ -\mp i & 1 \end{pmatrix},$$  

but the matrices $g^0$ have one-dimensional kernels.
Now, consider the "vector" variant of the previous example, where \( d = n = m = 2 \). Let \( \omega_{\pm}(x) \) be two (real-valued) \( \Gamma \)-periodic functions such that \( \omega_{\pm}(x) > 0 \) and \( \omega_{\pm} + \omega_{\pm}^{-1} \in L_{\infty} \). We put
\[
(1.17) \quad f_0 = \text{diag}(\omega_+, \omega_-).
\]
In \( \Phi = L_2(\mathbb{R}^2; \mathbb{C}^2) \), we consider the operators
\[
(1.18) \quad \mathcal{E} = \begin{pmatrix} 0 & \partial_- \\ \partial_+ & 0 \end{pmatrix}, \quad \mathbf{\hat{B}}_x = \mathcal{E} f_0 \mathcal{E}.
\]
Then
\[
(1.19) \quad \mathbf{\hat{B}}_x = \text{diag}(\partial_- \omega_+^2, \partial_+ \omega_-^2, \partial_- \omega_+^2, \partial_+ \omega_-^2) = \text{diag}(\mathbf{\hat{B}}_x(\omega_+^2), \mathbf{\hat{B}}_x(\omega_-^2)).
\]
The operator \( \mathbf{\hat{B}}_x \) is of the form \( \mathbf{\hat{A}}(g_x) \) with \( g_x = f_0 \).
\[
(1.20) \quad b_x(x) = \begin{pmatrix} 0 & \xi^1 - i\xi^2 \\ \xi^1 + i\xi^2 & 0 \end{pmatrix}.
\]
The effective matrix is given by the formula (cf. (1.15)):
\[
g_x^0 = g_x = \text{diag}\{g_x^0, g_x^0\}, \quad g_x^0 = (\omega_x^2).
\]
The germ \( \hat{S}(\theta; g_x) = b_x(\theta) g_x^0 b_x(\theta) \) again does not depend on \( \theta \) and is reduced to multiplication by the constant diagonal matrix
\[
\hat{S}(\theta; g_x) = \text{diag}\{g_x^0, g_x^0\}.
\]
Finally, now the effective \( \text{DO} \hat{B}_x^0 \) has the form
\[
(1.21) \quad \hat{B}_x^0 = \text{diag}\{g_x^0, g_x^0\}(-\Delta).
\]
As a concluding example, we consider the analogue of \( \hat{B}_x \), introducing the metric. Namely, let \( \bar{g}(x) \) be a \( \Gamma \)-periodic \( (2 \times 2) \)-matrix-valued function with real-valued entries such that \( \bar{g}(x) > 0 \) and \( \bar{g} + \bar{g}^{-1} \in L_{\infty} \). Consider the operator
\[
(1.22) \quad \hat{B}_x^0 = \mathcal{E} g \mathcal{E}, \quad g = f_0 \bar{g} f_0.
\]
(Of course, we could include the factors \( f_0 \) in \( \bar{g} \), but in this form the operator (1.22) would be less convenient for application in §6.2.) The operator (1.22) is of the form \( \mathbf{\hat{A}}(g) \) with \( b(\xi) = b_x(\xi) \) (cf. (1.20)). The effective matrix is again unique (cf. Subsection 3.1.3) and can be calculated explicitly:
\[
(1.23) \quad g^0 = g = \left( \Omega^{-1} \int_{\Omega} f_0^{-1} \bar{g}^{-1} f_0^{-1} d\Omega \right)^{-1}.
\]
Now, the germ \( \hat{S}_x(\theta; g) = b_x(\theta) g b_x(\theta) \) depends on \( \theta \), but, obviously,
\[
\text{Tr} \hat{S}_x(\theta; g) = \text{Tr} g, \quad \det \hat{S}_x(\theta; g) = \det g.
\]
Hence, the eigenvalues \( \gamma_1, \gamma_2 \) of the germ \( \hat{S}_x(\theta; g) \) do not depend on \( \theta \) and coincide with the eigenvalues of the matrix (1.23). The effective operator for \( \text{DO} \) (1.22) is given by the general formula (3.1.10):
\[
\hat{B}_x^0 = \mathcal{E} g^0 \mathcal{E}.
\]
We shall not comment on applications of general results to the homogenization for examples of the present subsection. Note only that, since \( m = n (= 2) \), Theorem 4.4.5 ensures the strong convergence of the flows for the operator \( \mathcal{E} f_0^0 g f_0 \mathcal{E} \).
§2. The operator of elasticity theory on $\mathbb{R}^d$, $d \geq 2$

2.1. Preliminaries. To represent the operator of elasticity theory in the form $\hat{A} = b(D) \cdot \eta(x) b(D)$, we need some agreements. Let $\zeta$ be an orthogonal second rank tensor in $\mathbb{R}^d$. In a fixed orthonormal basis in $\mathbb{R}^d$, it can be represented by the matrix $\zeta = \{\zeta_{ij}\}$. In the linear space of tensors, we introduce the norm $|\zeta|^2 = \sum_{i,j} |\zeta_{ij}|^2$; this norm does not depend on the choice of the orthonormal basis in $\mathbb{R}^d$. We shall consider symmetric tensors which will be identified with vectors $\zeta \in \mathbb{C}^n$, $2m = d(d + 1)$, by the following rule. The vector $\zeta$, is formed by all components $\zeta_{ij}$, $j < l$, and the pairs $(j, l)$ are put in order in some fixed way. Let $\chi$ be an $(m \times m)$-matrix, $\chi = \text{diag}\{\chi_{(j,l)}\}$, where $\chi_{(j,l)} = 1$ for $j = l$ and $\chi_{(j,l)} = 2$ for $j < l$. Then

$$|\zeta|^2 = \langle \chi \zeta, \zeta \rangle_{\mathbb{C}^{2m}}.$$ 

For the displacement vector $u \in H^1(\mathbb{R}^d, \mathbb{C}^d)$, we introduce the tensors

$$\nabla u = \left\{ \frac{\partial u_i}{\partial x_j} \right\}, \quad \epsilon(u) = \frac{1}{2} \left\{ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right\}, \quad r(u) = \frac{1}{2} \left\{ \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right\}.$$

The deformation tensor $\epsilon(u)$ is symmetric. Let $\epsilon_s(u)$ be the vector corresponding to the tensor $\epsilon(u)$, according to the described rule. The relation

$$b(D)u = -i \epsilon_s(u)$$

uniquely defines the $(m \times d)$-matrix $b(D)$. For instance, with appropriate putting in order, we have

$$b(\xi) = \begin{pmatrix} \xi^1 & 0 \\ \frac{1}{\sqrt{3}} \xi^2 & \frac{1}{\sqrt{3}} \xi^3 \\ 0 & \frac{1}{\sqrt{3}} \xi^3 \end{pmatrix}, \quad d = 2; \quad b(\xi) = \begin{pmatrix} \xi^1 & 0 & 0 \\ \frac{1}{\sqrt{3}} \xi^2 & \frac{1}{\sqrt{3}} \xi^1 & 0 \\ 0 & \xi^2 & 0 \\ 0 & 0 & \xi^3 \\ \frac{1}{\sqrt{3}} \xi^3 & 0 & \frac{1}{\sqrt{3}} \xi^1 \end{pmatrix}, \quad d = 3.$$

Now $n = d \geq 2, 2m = d(d + 1), \theta = L_2(\mathbb{R}^d, \mathbb{C}^d), \Theta_s = L_2(\mathbb{R}^d, \mathbb{C}^n)$. It is easily seen that, for $b(D)$ from (2.2), rank $b(\xi)$ = $d$ for $\xi \neq 0$, i. e., condition (2.1.2) is satisfied. Now, let $g(x)$ be a 1-periodic $(m \times m)$-matrix-valued function with real-valued entries such that $g(x) > 0$ and $g + g^{-1} \in L_\infty$, and let

$$\sigma_s(u) = g(x) \epsilon_s(u).$$

The corresponding tensor $\sigma(u)$ is called the stress tensor, and relation (2.3) expresses the Hooke law about proportionality of the stress and the deformation. The matrix $g$ gives an "economic" description of the Hooke tensor which relates the tensors $\sigma(u)$ and $\epsilon(u)$. The matrix $g$ characterizes the parameters of elastic (in general, anisotropic) medium filling $\mathbb{R}^d$. The quadratic form

$$\langle \sigma(u), \epsilon_s(u) \rangle = \frac{1}{2} \int_{\mathbb{R}^d} \langle \sigma_s(u), \epsilon_s(u) \rangle_{\mathbb{C}^{2m}} \, dx = \frac{1}{2} \langle \sigma(u), \epsilon(u) \rangle_{\Theta_s},$$

where
gives the energy of elastic deformations. The operator $\mathcal{W} = \mathcal{W}(g)$ of elasticity theory is generated in the space $\mathfrak{G}$ by the quadratic form (2.4). Now, it is clear that $2\mathcal{W}(g) = \hat{A}(g)$, where $g$ is the "Hooke matrix" from (2.3), and $b(D)$ is defined by (2.2).

In the case of an isotropic medium, expression for the form (2.4) simplifies significantly and depends only on two functional* Lame parameters* $\lambda(x), \mu(x)$:

$$ w[u,u] = \int_{\mathbb{R}^d} \left( \mu(x) \left| \varepsilon(u) \right|^2 + \frac{\lambda(x)}{2} |\text{div} u|^2 \right) dx $$

$$ = \int_{\mathbb{R}^d} \left( \mu(x)(\chi_+(u),\varepsilon_+(u))_{ce} + \frac{\lambda(x)}{2} |\text{div} u|^2 \right) dx. $$

The parameter $\mu$ is the *shear modulus*. Often, instead of $\lambda(x)$, another parameter $K(x)$ is introduced; $K(x)$ is called the *modulus of volume compression*. We shall also use one more modulus $\beta(x)$. Here are the relations:

$$(2.5) \quad K(x) = \lambda(x) + \frac{2\mu(x)}{d}, \quad \beta(x) = \mu(x) + \frac{\lambda(x)}{2}. $$

The modulus $\lambda(x)$ may be negative. In the isotropic case, conditions ensuring the positive definiteness of the matrix $g(x)$ are the following: $\mu(x) \geq \mu_0 > 0$, $K(x) \geq K_0 > 0$. We write down the "isotropic" matrices $g$ for $d = 2$ and $d = 3$:

$$ g_{\mu,K}(x) = \begin{pmatrix} K+\mu & 0 & K-\mu \\ 0 & 4\mu & 0 \\ K-\mu & 0 & K+\mu \end{pmatrix}, \quad \text{for } d = 2, $$

$$ g_{\mu,K}(x) = \frac{1}{3} \begin{pmatrix} 3K+4\mu & 0 & 3K-2\mu & 0 & 3K-2\mu \\ 0 & 12\mu & 0 & 0 & 0 \\ 3K-2\mu & 0 & 3K+4\mu & 0 & 3K-2\mu \\ 0 & 0 & 0 & 12\mu & 0 \\ 3K-2\mu & 0 & 3K-2\mu & 0 & 3K+4\mu \end{pmatrix}, \quad \text{for } d = 3. $$

To the operator $\mathcal{W}(g)$, all the results of general scheme, in particular, those valid only for the case where $f = 1_n$, are applicable. In particular, the upper and lower estimates (3.1.21) are valid. Note that, in the isotropic case,

$$ \overline{g_{\mu,K}} = \overline{g_{\mu,K}}, \quad g_{\mu,K} = \overline{g_{\mu,K}}. $$

This follows from the fact that the isotropic Hooke tensor is linear with respect to $\mu$ and $K$, and the inverse tensor (see [ZhKO], Ch. 12, §1) is linear with respect to $\mu^{-1}$ and $K^{-1}$.

### 2.2. About homogenization for the operator $\mathcal{W}$

According to (2.3), the flows $p_{e} = g^{e} b(D) u_{e} = -i g^{e} \varepsilon_{e} (u_{e})$ corresponding to the operator $2\mathcal{W}(g) = \hat{A}(g^{e})$ have the meaning of the stress. The effective matrix $g^{e}$ and the effective DO $\mathcal{W}(g^{e})$ are constructed according to general rules. Though now the effective matrix $g^{e}$ is
not unique, its choice is determined by the desire to observe the behavior of the stress tensor as \( \varepsilon \to 0 \). By Theorem 4.4.1, the solutions \( u_\varepsilon \) of the equation

\[
(2.6) \quad 2W(\varepsilon u_\varepsilon + u_\varepsilon) = F, \quad F \in \mathcal{G}^{-1},
\]

\( \mathcal{G} \)-weakly converge to the solution \( u_0 \) of the equation

\[
(2.7) \quad 2W(u_0) + u_0 = F, \quad F \in \mathcal{G}^{-1},
\]

and the stresses (the flows) \( g^\varepsilon \epsilon_*(u_\varepsilon) \) \( \mathcal{G} \)-weakly converge to the limit flow \( g^0 \epsilon_*(u_0) \), which corresponds to the "limit" stress tensor. For the solutions of equations (2.6), (2.7), these (known) facts are essentially supplemented by applying Theorems 4.2.1 and 4.2.7.

To more general equation

\[
(W(\varepsilon) + Q^\varepsilon)u_\varepsilon = F,
\]

also Theorems 4.2.4, 4.2.7 and 4.4.1 are applicable.

In the isotropic case, no simplification or strengthening occurs; in general, the effective medium is anisotropic.

2.3. The Hill body. In mechanics (see, e. g., [ZhKO]), the isotropic medium with \( \mu(x) = \mu_0 = \text{const} \) is called the Hill body. In this case, more economic description of the operator \( W \) is possible, and also the homogenization results can be strengthened. We start with the identity

\[
\int_{\mathbb{R}^d} \|\epsilon(u)\|^2 dx = \int_{\mathbb{R}^d} \|r(u)\|^2 + |\text{div} u|^2 dx, \quad u \in \mathcal{G}^1;
\]

here we use the notation (2.1). Then, for \( \mu(x) = \mu_0 \), the form of energy can be represented (using the notation (2.5)) as

\[
(2.8) \quad w[u, u] = \mu_0 \int_{\mathbb{R}^d} \|r(u)\|^2 dx + \int_{\mathbb{R}^d} \beta|\text{div} u|^2 dx, \quad u \in \mathcal{G}^1.
\]

We write the form (2.8) as \( (g_\lambda b_\lambda(D)u, b_\lambda(D)u)_{\mathcal{G}_\lambda^*} \), with another \( m = m_\lambda \), \( g = g_\lambda \), \( b = b_\lambda \), than in Subsection 2.1. Namely, let \( m_\lambda = 1 + \frac{d(d-1)}{2} \). Let us describe the symbol of \( (m \times d) \)-matrix \( b_\lambda \). The first line of \( b_\lambda(\xi) \) is equal to \( (\xi^1, \xi^2, \ldots, \xi^d) \).

Other lines correspond to (different) pairs of indices \( (j, l) \), \( 1 \leq j < l \leq d \). The element standing in the \( (j, l) \)-th line and \( j \)-th column is \( \xi^j \), and that in the \( (j, l) \)-th line and \( l \)-th column is \( -\xi^j \); all other elements of the \( (j, l) \)-th line are equal to zero. The order of \( (j, l) \)-th lines does not matter. Finally, \( g_\lambda(x) \) is defined by the formula

\[
g_\lambda(x) = \text{diag}\{\beta(x), \mu_0/2, \mu_0/2, \ldots, \mu_0/2\}.
\]

It is easily seen that \( w[u, u] = (g_\lambda b_\lambda(D)u, b_\lambda(D)u)_{\mathcal{G}_\lambda^*} \), \( \mathcal{G}_\lambda^* = L_2(\mathbb{R}^d; \mathbb{C}^m) \), or, equivalently,

\[
W = (b_\lambda(D))^* g_\lambda(x) b_\lambda(D).
\]

Now we show that

\[
(2.9) \quad g_\lambda^0 = g_\lambda = \text{diag}\{\beta, \mu_0/2, \mu_0/2, \ldots, \mu_0/2\}.
\]
For this, we apply Proposition 3.1.7. We need to check condition of this Proposition only for the first column col(\(\beta_1(x)^{-1}, 0, \ldots, 0\)) of the matrix \(g_{\alpha}(x)^{-1}\). Let \(\vartheta \in H^2(\Omega)\) be solution of the equation \(\Delta \vartheta = i(\beta(x)^{-1} - \beta^{-1})\) and let \(v = \nabla \vartheta \in H^1(\Omega; \mathbb{C}^d)\). Then \((b_{\alpha}(D)\nu)^j = -i \text{div} v = \beta^{-1} - \beta^{-1}\). Obviously, relations \((b_{\alpha}(D)v)^j = 0, j > 1\), are equivalent to \(r(v) = r(\nabla \vartheta) = 0\). This yields (2.9). Note that, if \(\mu(x) = \mu_0\), there is no analogue of the equality \(g^0_{\alpha} = g_{\alpha}\) for the representation of \(\mathcal{V}_r\), used in Subsection 2.1. However, the "ordinary" effective matrix \(g^0\) in the case of the Hill body also corresponds to the isotropic medium with parameters \(\mu_0\). 

From (2.9) and Theorem 4.4.5 it follows that, in the homogenization problem for the Hill body, the flows converge strongly. Namely, let \(u_\varepsilon\) be the solution of the equation

\[
(b_{\alpha}(D))^\varepsilon g_{\alpha}^\varepsilon b_{\alpha}(D)u_\varepsilon + u_\varepsilon = F, \quad F \in \mathcal{G}^{-1},
\]

and let \(u_0\) be the solution of the equation

\[
(b_{\alpha}(D))^0 g_{\alpha}^0 b_{\alpha}(D)u_0 + u_0 = F.
\]

Then

\[
(\mathcal{G}^{-1})\lim_{\varepsilon \to 0} g_{\alpha}^\varepsilon b_{\alpha}(D)u_\varepsilon = g_{\alpha}^0 b_{\alpha}(D)u_0.
\]

Clearly, (2.10) is equivalent to the pair of relations

\[
(L_2(\mathbb{R}^d))\lim_{\varepsilon \to 0} \beta^\varepsilon \text{div} u_\varepsilon = \text{div} u_0,
\]

\[
(L_2(\mathbb{R}^d))\lim_{\varepsilon \to 0} r(u_\varepsilon) = r(u_0).
\]

However, this is less than the strong convergence of the full stress tensor.

**Comments on Chapter 5**

1. For all three main examples (the acoustic operator, the Schrödinger operator and the operator of elasticity theory), application of Theorems 4.2.1, 4.2.4, 4.2.7 to the homogenization problem yields new results (part of these results has been published by the authors in [BSu2]). In the opposite, application of Theorem 4.4.1 to these cases gives well known statements. Relations (2.11) and (2.12) about the strong convergence for the Hill body have not been mentioned before.

2. Examples of Subsection 1.3 have illustrative meaning; they also prepare the study of the two-dimensional periodic Pauli operator in § 6.2. In many respects, the operators \(\mathcal{B}_{\pm}(\omega^2)\) (cf. (1.14)) are analogous to the acoustic operator with \(d = 1\). In particular, formulas (1.15) have the same form as (1.10).

3. For the acoustic and Schrödinger operators, we have \(n = 1\). Then the single eigenvalue \(\gamma(k)\) of the operator \(\hat{A}(k, \rho)\) acting in \(L_2(\Omega)\) (and corresponding to DO (1.1)), is analytic near the point \(k = 0\). The same is true for the corresponding eigenfunction. On this basis, one can prove the estimate of the type

\[
|||\hat{A}(\rho) - I|||_0 \leq (\text{const})\varepsilon
\]

in more direct way. For this, one should apply the Floquet decomposition, but does not need to use general results of Chapters 1 and 3. Similar methods were applied before for the study of the Schrödinger operator perturbed by potentials of the admixture type (cf. [B1.2, BlaSu1]). However, using this method, it is much more difficult to control the constant in (3).
4. In the paper [Zh1] (cf. also [ZhKo, §II.6]) V. V. Zhikov, using the Floquet theory, obtained substantial estimates for the fundamental solution of the parabolic equation corresponding to DO (1.1). When discussing the results of [BSu2], V. V. Zhikov [Zh2] communicated to the authors that the estimate of the form \( (*) \) can be deduced from the estimates for the fundamental solution in comparatively easy way. For this, on the basis of [Zh1], one needs some smoothness assumptions on \( g \), and it is difficult to control the constants in estimates.

Chapter 6. Applications of general scheme. The case where \( f \neq 1 \)

In this case, the results of general scheme are less perfect, although even these results allow us to obtain a number of essential advances. In particular, this concerns the homogenization problems. Our main examples are the Schrödinger operator and the two-dimensional Pauli operator. Both operators can be represented as \( A(g, f) \), by appropriate factorization. Finally, we consider the homogenization problem for one first order DO.

§1. The periodic Schrödinger operator

1.1. Preliminaries. Factorization. In the space \( L^2(\mathbb{R}^d) \), \( d \geq 1 \), we consider the periodic Schrödinger operator (with metric) of the form

\[
\mathcal{H} = \mathbf{D}^* g(\mathbf{x}) \mathbf{D} + p(|\mathbf{x}|), \quad \mathbf{x} \in \mathbb{R}^d,
\]

where a \((d \times d)\)-matrix \( g(\mathbf{x}) > 0 \) with real-valued entries and a real-valued potential \( p(|\mathbf{x}|) \) are \( \Gamma \)-periodic and such that

\[
\text{(1.2)} \quad \bar{g} + g^{-1} \in L_\infty,
\]

\[
\text{(1.3)} \quad p \in L_s(\Omega), \quad 2s > d \text{ for } d \geq 2; \quad s = 1 \text{ for } d = 1.
\]

The precise definition of the selfadjoint operator \( \mathcal{H} \) in \( \mathcal{G} = L_2(\mathbb{R}^d) \) is given via the quadratic form

\[
\mathfrak{h}[u, u] = \int_{\mathbb{R}^d} (\langle \bar{g} \mathbf{D} u, \mathbf{D} u \rangle_{C^0} + p|u|^2) \, d\mathbf{x}, \quad u \in \mathcal{G}^1 = H^1(\mathbb{R}^d).
\]

Under conditions (1.2) and (1.3), this form is lower semi-bounded and closed in \( \mathcal{G} \). By adding an appropriate constant to \( p \), we can assume that the point \( \lambda = 0 \) is the lower edge of the spectrum of \( \mathcal{H} \). Suppose that the latter condition is satisfied.

Consider the homogeneous equation \((\mathbf{D}^* \bar{g} \mathbf{D} + p)\omega = 0\), understood in the sense of the following identity for \( \omega \in \bar{H}^1(\Omega)\):

\[
\int_\Omega (\langle \bar{g} \mathbf{D} \omega, \mathbf{D} \omega \rangle_{C^0} + p \omega \zeta^+ \zeta^-) \, d\mathbf{x} = 0, \quad \zeta \in \bar{H}^1(\Omega).
\]

Under conditions (1.2), (1.3), this equation has a positive \( \Gamma \)-periodic solution \( \omega \in H^1_{loc}(\mathbb{R}^d) \cap \text{Lip} \tau \) (for some \( \tau > 0 \)). Besides, the function \( \omega \) is a multiplier in the classes \( \mathcal{G}^1 \) and \( \bar{H}^1(\Omega) \). By substitution \( u = \omega v \), the form (1.4) turns into

\[
\mathfrak{h}[u, u] = \int_{\mathbb{R}^d} \omega^2 \langle \bar{g} \mathbf{D} v, \mathbf{D} v \rangle_{C^0} \, d\mathbf{x}, \quad u = \omega v, \quad v \in \mathcal{G}^1.
\]
This means that the operator (1.1) is represented as a product
\begin{equation}
\mathcal{H} = \omega^{-1} \mathbf{D}^* \omega \nabla g \mathbf{D} \omega^{-1}.
\end{equation}

Thus, the operator $\mathcal{H}$ takes the form
\begin{equation}
\mathcal{H} = \mathcal{A}(g, f), \quad g = \omega^2 \tilde{g}, \quad f = \omega^{-1},
\end{equation}
with $n = 1, m = d \geq 1, \beta(\xi) = \xi$.

**Remark 1.1.** Expression (1.6) can be taken as the *definition* of the operator $\mathcal{H}$, if $\omega$ is a measurable $T$-periodic function such that
\[ \omega(x) > 0, \quad \omega + \omega^{-1} \in L_\infty. \]

One can return to the form (1.1), using the formula $p = -\omega^{-1}(\mathbf{D}^* \tilde{g} \mathbf{D} \omega)$. The corresponding potential $p$ may be strongly singular.

**Remark 1.2.** Let $E_j(k)$ be the band functions of the periodic operator (1.1). Representation (1.7) directly implies that $E_j(k)$ has non-degenerate minimum (equal to zero) at $k = 0$ and that $\min E_j(k) > 0$. These properties follow from the existence of a positive periodic solution of the equation (1.5). In fact, any other way of establishing these properties of the band functions, is equivalent to the proof of the existence of a positive solution $\omega$.

1.2. *Relation to the acoustic operator.* The operator $\hat{A}(g)$ corresponding to the operator (1.7) with $f = 1$, coincides with the acoustic operator (5.1.1). Thus, we are under conditions of general scheme of Subsection 1.1.5 with $M = M^* = \omega^{-1}$. According to (5.1.3), $\mathcal{H} = L_2(\Omega)$ and
\begin{equation}
\mathfrak{M} = \{ u \in \mathfrak{H} : u = cu, \quad c \in \mathbb{C} \},
\end{equation}
while $\mathfrak{M}_c = \hat{\mathfrak{N}}_c$ is defined by (5.1.4), (5.1.5). By definition, the effective matrix $g^\theta$ for the operator (1.7) is the same as for the acoustic operator. For the latter, since $n = 1$, the germ $\hat{S}(\theta)$ acting in $\mathfrak{H}$ is reduced to multiplication by the number (see (5.1.8))
\begin{equation}
\hat{\gamma}(\theta) = \langle g^\theta \theta, \theta \rangle_{\mathbb{R}^d}.
\end{equation}

For calculating the germ $S(\theta)$ of the operator $\mathcal{H}$, we use formula (1.1.26). Now $M = \omega^{-1}$, and one can put (cf. (1.8)) $\xi = \omega$. Then (1.1.26) shows that $S(\theta)$ acts as multiplication of the elements of the kernel (1.8) by the number
\begin{equation}
\gamma(\theta) = \hat{\gamma}(\theta) \Omega \| \omega \|^2, \quad \theta \in S^{d-1}.
\end{equation}

We normalize the choice of solution $\omega$ by the condition
\begin{equation}
|\Omega|^{-1} \int_{\Omega} \omega^2 dx = 1.
\end{equation}

Then (1.10) implies the following useful proposition.
Proposition 1.3. Suppose that a positive solution \( \omega \) of the equation (1.5) is normalized by the condition (1.11), and let the matrix \( g \) be defined by (1.7). Then we have

\[
\gamma(\theta) = \tilde{\gamma}(\theta) = \langle g^0 \theta, \theta \rangle_{\mathbb{R}^d}, \quad \theta \in S^{d-1}.
\]

Remark 1.4. In the quantum mechanics, the tensor inverse to the tensor \( \gamma \) is called the tensor of effective masses. Thus, (1.12) means that, under condition (1.11) and for \( g = \bar{g} \omega^2 \), the tensors of effective masses for operators (1.1) and (5.1.1) coincide with each other.

According to accepted definitions, under condition (1.11) and for \( g = \bar{g} \omega^2 \), the effective DO for the operator \( H \) can be written as

\[
(\omega(x))^{-1} \tilde{\gamma}(D)(\omega(x))^{-1},
\]

where the tensor \( \tilde{\gamma} \) (see (1.9)) is related to the corresponding acoustic operator.

We distinguish the case where \( d = 1 \). Then \( m = n = 1 \) and (cf. (5.1.10)) \( \tilde{\gamma} = g^0 = \bar{g} \). Combining this with (1.10), we obtain

\[
\gamma = |\Omega|^2 ||\omega||^{-2}_{\mathbb{R}^d} ||(\bar{g}\omega^{-1})^{-1} - \omega^{-1}||^{-2}_{\mathbb{R}^d}, \quad d = 1.
\]

In particular,

\[
\gamma = |\Omega|^2 ||\omega||^{-2}_{\mathbb{R}^d} ||\omega^{-1}||^{-2}_{\mathbb{R}^d} \text{ for } d = 1, \bar{g} = 1.
\]

The latter formula is well known in the quantum mechanics as the formula for the effective mass \( \gamma^{-1} \) on the left edge of the spectrum.

1.3. Homogenization. We consider the family of operators \( H_{\varepsilon} \), which, according to (1.6), (1.7), is defined by the formulas

\[
H_{\varepsilon} = H_{\varepsilon}(\bar{g}, \omega) = (\omega^\varepsilon)^{-1} D^* g^\varepsilon D(\omega^\varepsilon)^{-1}, \quad g^\varepsilon = (\omega^2 \bar{g})^\varepsilon, \quad \varepsilon > 0.
\]

Expression (1.14) can be also rewritten in the initial terms:

\[
H_{\varepsilon} = D^* (\bar{g})^\varepsilon D + \varepsilon^{-2} p^\varepsilon.
\]

The family (1.15) differs from the family (5.1.12) by the fact that now the potential \( p \) is, a fortiori, non-signdefinite. Besides, there is a coefficient \( \varepsilon^{-2} \) before \( p^\varepsilon \), which "equalizes" the roles of both summands in (1.15). In the homogenization of the family (5.1.12), the potential term less influence the result.

To the operator \( H_{\varepsilon} \), all the results of general nature for the family \( A_{\varepsilon}(g, f) \) with variable \( f \) are applicable. We formulate some consequences of the general results. Herewith, we shall not write down the constants in estimates explicitly, though it can be easily done: general formulas for the constants may only simplify. All constants in estimates in this subsection will be denoted by \( C \) without indices.

First of all, we apply Theorem 4.2.5.
\textbf{Theorem 1.5.} Let $g = \bar{g}\omega^2$ and suppose that (1.11) is satisfied. Denote by $\hat{H}(g^0)$ the operator $D^*g^0D$. Then

\begin{equation}
\| (H_{\varepsilon}(\bar{g}, \omega) + I)^{-1} - \omega^\varepsilon (\hat{H}(g^0) + I)^{-1} \omega^\varepsilon \|_{\mathcal{S} \to \mathcal{S}} \leq C\varepsilon, \quad 0 < \varepsilon \leq 1.
\end{equation}

\textit{Proof} is reduced to referring to (3.3.2) and the normalization condition (1.11). As a result, we obtain $\overline{Q} = 1$. \hfill $\Box$

This estimate can be supplemented by the interpolation estimate from Theorem 4.2.8. The following theorem is true.

\textbf{Theorem 1.6.} Under the assumptions of Theorem 1.5, we have

\begin{equation}
\| (\omega^\varepsilon)^{-1} (H_{\varepsilon}(\bar{g}, \omega) + I)^{-1} - (\hat{H}(g^0) + I)^{-1} \omega^\varepsilon \|_{\mathcal{S} \to \mathcal{S}} \leq C\varepsilon^{1-s}, \quad 0 < s < 1, \quad 0 < \varepsilon \leq 1.
\end{equation}

Next, Theorem 4.2.9 (with $\mathcal{Q} = 1$) implies the following result.

\textbf{Theorem 1.7.} Under the assumptions of Theorem 1.5, we have

\begin{equation}
(w, \mathcal{S} \to \mathcal{S}) \cdot \lim_{\varepsilon \to 0} (H_{\varepsilon}(\bar{g}, \omega) + I)^{-1} = (\mathcal{S}^2)(\hat{H}(g^0) + I)^{-1}.
\end{equation}

Finally, we mention the following consequence of Theorem 4.4.2 (again with $\mathcal{Q} = 1$). Namely, let $\tilde{u}_\varepsilon$ be the solution of the equation

$$(H_{\varepsilon}(\bar{g}, \omega) + I)\tilde{u}_\varepsilon = F, \quad F \in \mathcal{S}.$$

Suppose that (1.11) is satisfied and $g = \bar{g}\omega^2$. Let $u_0$ be the solution of the equation

$$(\hat{H}(g^0) + I)u_0 = \mathcal{S}F.$$

\textbf{Theorem 1.8.} Under the above assumptions,

\begin{align}
(\omega^\varepsilon)^{-1} \cdot \lim_{\varepsilon \to 0} \tilde{u}_\varepsilon &= u_0, \\
(w, \mathcal{S_*} \cdot \lim_{\varepsilon \to 0} g^\varepsilon D((\omega^\varepsilon)^{-1} \tilde{u}_\varepsilon) &= g^0D u_0.
\end{align}

By Theorem 4.4.8 and Propositions 3.1.6, 3.1.7, for the Schrödinger operator (1.1), the following analogue of Remark 5.1.1 is valid.

\textbf{Remark 1.9.} If the columns of the matrix $g = \omega^2 \bar{g}$ are solenoidal, convergence in (1.17) is strong. If the columns of the matrix $g^{-1} = \omega^{-2} \bar{g}^{-1}$ are potential, convergence in (1.18) is weak.

§2. The two-dimensional periodic Pauli operator

\textbf{2.1. Definition and factorization for the Pauli operator.} Let the magnetic potential $A = \{A_1, A_2\}$ be a $\Gamma$-periodic $\mathbb{R}^2$-valued function on $\mathbb{R}^2$ such that

\begin{equation}
A \in L_p(\Omega; \mathbb{C}^2), \quad \rho > 2.
\end{equation}

Recall the standard notation for the Pauli matrices

\begin{equation}
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{equation}
In $\mathfrak{G} = L_2(\mathbb{R}^2; \mathbb{C}^2)$, consider the operator

$$
(2.2) \quad \mathcal{D} := (D_1 - A_1)\sigma_1 + (D_2 - A_2)\sigma_2, \quad \text{Dom} \mathcal{D} = \mathfrak{G}^1 = H^1(\mathbb{R}^2; \mathbb{C}^2).
$$

By definition, the Pauli operator $\mathcal{P}$ is the square of the operator $\mathcal{D}$:

$$
(2.3) \quad \mathcal{P} := \mathcal{D}^2 = \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix}.
$$

The precise definition of the operator $\mathcal{P}$ is given via the closed quadratic form

$$
(2.4) \quad \|\mathcal{D}u\|^2_{\mathfrak{G}} = \mathcal{U}(u), \quad u \in \text{Dom} \mathcal{D},
$$

in $\mathfrak{G}$. If the potential $\mathbf{A}$ is sufficiently smooth, the blocks $P_{\pm}$ of the operator (2.3) can be written as

$$
(2.5) \quad P_{\pm} = (\mathcal{D} - \mathbf{A})^2 \pm (\partial_1 A_2 - \partial_2 A_1).
$$

The expression $\partial_1 A_2 - \partial_2 A_1$ is the strength of the magnetic field.

We rely on the known factorization (see [BSu1,2] for details) for operators (2.2), (2.3). By the gauge transformation, the potential $\mathbf{A}$ can be subject to the conditions

$$
(2.6) \quad \text{div } \mathbf{A} = 0, \quad \int_{\Omega} \mathbf{A} \cdot d\mathbf{x} = 0,
$$

without violating (2.1). Under conditions (2.1), (2.6), there exists a (unique) real-valued $\Gamma$-periodic function $\varphi$ such that

$$
(2.7) \quad \nabla \varphi = \{A_2, -A_1\}, \quad \nabla \varphi \equiv 0.
$$

From (2.1), (2.7) it follows that

$$
(2.8) \quad \varphi \in \mathring{W}_\rho^1(\Omega) \subset \text{Lip } \tau, \quad \tau = 1 - 2\rho^{-1}.
$$

Here $\mathring{W}_\rho^1(\Omega)$ is the subspace of the Sobolev space $W_\rho^1(\Omega)$ formed by the functions whose $\Gamma$-periodic extension belongs to $W_\rho^{1,\text{loc}}(\mathbb{R}^2)$.

We introduce the notation (cf. (5.1.14), (5.1.17), (5.1.18))

$$
(2.9) \quad \omega_{\pm} := \exp(\pm \varphi), \quad f_0 = \begin{pmatrix} \omega_+ & 0 \\ 0 & \omega_- \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} 0 & \partial_- \\ \partial_+ & 0 \end{pmatrix}.
$$

By (2.8), $\omega_{\pm} \in \mathring{W}_\rho^1(\Omega)$ and

$$
(2.10) \quad \omega_-(\mathbf{x})\omega_+(\mathbf{x}) = 1, \quad \mathbf{x} \in \mathbb{R}^2.
$$

The operators (2.2), (2.3) can be written as

$$
(2.11) \quad \mathcal{D} = f_0 \mathcal{E} f_0, \\
(2.12) \quad \mathcal{P} = f_0 \mathcal{E} f_0^2 \mathcal{E} f_0.
$$
The blocks (2.5) of the operator (2.3) admit the representations
\begin{equation}
P_+ = Y^*Y, \quad P_- = YY^*, \quad \text{where } Y := \omega_+ \Delta \omega_-
\end{equation}

It is convenient to take (2.11)-(2.13) as definitions of the operators $\mathcal{D}$, $\mathcal{P}$, $P_\pm$, assuming that $\omega_\pm$ are arbitrary $1$-periodic functions such that
$$\omega_\pm(x) > 0, \quad \omega_+, \omega_- \in L_\infty,$$
and (2.10) is satisfied. Precisely, the operator $\mathcal{D}$ is given by the expression (2.11) on the domain
\begin{equation}
\text{Dom} \mathcal{D} = \{ u \in \mathcal{G}: \int u \in \mathcal{G}^1 \}.
\end{equation}
The operator $\mathcal{P}$ corresponds to the quadratic form (2.4) defined on the domain (2.14). The blocks $P_\pm$ are defined via the quadratic forms
$$\| \omega_\pm \Delta \omega_\pm u \|^2_{L_2(\mathbb{R}^2)}, \quad \omega_\pm u \in H^1(\mathbb{R}^2).$$

Herewith, the magnetic field strength in (2.5) loses immediate meaning.

Clearly, Ker $Y = \text{Ker} Y^* = \{0\}$. Therefore, (2.13) implies that the operator $P_+$ is unitarily equivalent to $P_-$. Obviously, the study of the operator $\mathcal{P}$ reduces to the study of the blocks $P_\pm$.

2.2. Effective masses and DO. The operators $P_\pm$ and $\mathcal{P}$ are included in the general scheme.

We start with the operators $P_\pm$. Now $d = 2$, $m = n = 1$, $\mathcal{G} = \mathcal{G} = L_2(\mathbb{R}^2)$, $\mathfrak{g} = \mathfrak{g} = L_2(\Omega)$. The corresponding kernels are given by the relations
$$\mathfrak{M}_\pm = \{ u \in \mathfrak{g}: u = \omega_\pm, \, c \in \mathbb{C} \}, \quad (\mathfrak{M}_\pm)_{\pm} = \mathfrak{M}_\mp, \quad n = n_\mp = 1.$$
The operators $P_\pm$ can be represented as $\mathcal{A}(g, f)$. Namely, the roles of $g$, $f$ and $b(\xi)$ are played by $\omega^2_\pm$, $\omega_\mp$ and $\xi^1 \mp i \xi^2$ respectively. Then $P_\pm = \mathcal{A}(\omega^2_\pm, \omega_\mp)$, and the role of the corresponding operator $\mathcal{A}(g)$ is played by the operator (5.1.14) $\hat{\mathcal{B}}_\pm(\omega^2_\pm)$.

Hence (cf. (5.1.15)), the effective matrix $g^0_\pm$ is equal to the number
\begin{equation}
g^0_\pm = \left( |\Omega|^{-1} \int_\Omega \omega^2_\pm \, dx \right)^{-1}.
\end{equation}

Correspondingly (cf. (5.1.16)), the effective DO is given by the formula
\begin{equation}
P^0_\pm = -g^0_\pm \omega_\pm(x) \Delta \omega_\pm(x).
\end{equation}

For calculating the germ $S_\pm(\Theta)$ of the operator $P_\pm$, we use (3.1.30). By this formula, $S_\pm(\Theta)$ reduces to multiplication by the number $\gamma_\pm(\Theta) = \hat{\gamma}_\pm(\Theta)|\Omega|\|\omega_\pm\|^2_{\mathfrak{g}}$. Now, by the relation $\hat{\gamma}_\pm(\Theta) = g^0_\pm$ from Subsection 5.1.3, applied to $\hat{\mathcal{B}}_\pm(\omega^2_\pm)$, we have $\hat{\gamma}_\pm(\Theta) = g^0_\pm$. By (2.10), (2.15), we obtain
\begin{equation}
\gamma_\pm(\Theta) = |\Omega|^{\frac{1}{2}} \| \omega_- \|_{\mathfrak{g}}^{-2} \| \omega_+ \|_{\mathfrak{g}}^{-2} =: \gamma.
\end{equation}
Thus, the numbers \( \gamma_{\pm}(\Theta) \) do not depend on \( \Theta \) and coincide with each other. Formulas (2.17) are the rigorous analogues of the formula (1.13) for the one-dimensional Schrödinger operator. Herewith, the effective masses for both Pauli operators \( P_{\pm} \) coincide with each other and do not depend on \( \Theta \).

We briefly dwell on the operator \( \mathcal{P} \). Here \( m = n = d = 2, \mathcal{G} = \mathcal{G}_{\varepsilon} = L_2(\mathbb{R}^2; \mathbb{C}) \). The operator \( \mathcal{P} \) is of the form \( \mathcal{A}(g, f) \), where \( g = f_0^1, f = f_0 \) and \( b(\xi) \) is defined by (5.1.20). The role of the corresponding operator \( \widetilde{\mathcal{A}}(g) \) is played by the operator \( \widetilde{B}_x \) defined by (5.1.19). According to (3.1.29) and (5.1.21), the effective DO \( \mathcal{P}^0 \) for the operator \( \mathcal{P} \) is represented as

\[
\mathcal{P}^0 = f_0(x)\widetilde{B}_x f_0(x) = f_0(x) \begin{pmatrix} -\frac{g_{\pm}}{\Delta} & 0 \\ 0 & -g_{\pm} \Delta \end{pmatrix} f_0(x) = \begin{pmatrix} P_{\pm}^0 & 0 \\ 0 & P_{\pm}^0 \end{pmatrix}.
\]

Thus, \( \mathcal{P}^0 \) is expressed via the operators (2.16).

2.3. **About homogenization.** It is more convenient to start with the vector Pauli operator. As usual, we define the operators \( \mathcal{P}_{\varepsilon} \) and \( \mathcal{P}_{\varepsilon}^0 \), replacing \( f_0(x) \) by \( f_0(x) = f_0(\varepsilon^{-1}x) \). Theorems 4.2.3, 4.2.6, 4.2.9, 4.4.2, 4.4.8 are applicable. Herewith, it is convenient to change the notation a little bit. Let \( \tilde{\omega}_\pm \) be a positive function from the kernel \( \mathcal{K}_\pm \) such that \( \| \tilde{\omega}_\pm \|^2_{L_2(\Omega)} = \| \Omega \|. \) We put \( f = \text{diag}[\tilde{\omega}_-, \tilde{\omega}_+] \). Let \( \gamma \) be the number (2.17). Then Theorem 4.2.6 implies that

\[
(2.18) \quad ||(\mathcal{P}_{\varepsilon} + I)^{-1} - \tilde{f} (-\gamma \Delta + I)^{-1} \tilde{f} ||_{\mathcal{G} \to \mathcal{G}} \leq C\varepsilon,
\]

and the constant \( C \) can be easily decoded. Theorem 4.2.9 shows that the limit

\[
(2.19) \quad (w, \mathcal{G} \to \mathcal{G}) \lim_{\varepsilon \to 0} (\mathcal{P}_{\varepsilon} + I)^{-1} = (f)(-\gamma \Delta + I)^{-1}(f)
\]

exists.

Now, let \( \tilde{u}_c \) be the solution of the equation

\[
(2.20) \quad \mathcal{P}_{\varepsilon} \tilde{u}_c + \tilde{u}_c = F, \quad F \in \mathcal{G},
\]

and let \( u_0 \) be the solution of the equation

\[
(2.21) \quad -\gamma \Delta u_0 + u_0 = \text{diag} \left\{ \tilde{\omega}_- (\tilde{\omega}_-)^{-1} \tilde{\omega}_+ (\tilde{\omega}_+)^{-1} \right\} F.
\]

Then, by Theorems 4.4.2 and 4.4.8 (2*) (with \( \mathcal{Q} = 1_2 \)), we have

\[
(2.22) \quad (w, \mathcal{G}^1) \lim_{\varepsilon \to 0} f_0^2 \tilde{u}_c = u_0,
\]

\[
(2.23) \quad (\mathcal{G}^1) \lim_{\varepsilon \to 0} \tilde{p}_c = p_0.
\]

Here the flows \( \tilde{p}_c, p_0 \) are defined by the formulas

\[
(2.24) \quad \tilde{p}_c = (f_0^2)^2 \tilde{c} f_0^2 \tilde{u}_c = \begin{pmatrix} 0 & (\tilde{\omega}_+)^2 \partial_+ \tilde{\omega}_- \\ (\tilde{\omega}_-) \partial_+ \tilde{\omega}_+ & 0 \end{pmatrix} \tilde{u}_c,
\]

\[
(2.25) \quad p_0 = \begin{pmatrix} 0 & (\tilde{\omega}_-)^{-1} \partial_- \\ (\tilde{\omega}_+) \partial_- & 0 \end{pmatrix} u_0.
\]
The results on homogenization for \( \mathcal{P} \) directly imply the results for the blocks \( P_{\pm} \). According to (2.13), we put

\[
(2.26) \quad P_{\pm, \varepsilon} = Y_{\pm}^* Y_{\varepsilon}, \quad P_{\mp, \varepsilon} = Y_{\mp}^* Y_{\varepsilon}, \quad Y_{\varepsilon} = \omega_{\varepsilon}^2 \partial_{\omega_{\varepsilon}}.
\]

Then \( \mathcal{P} = \text{diag}(P_{\mp, \varepsilon}, P_{\pm, \varepsilon}) \), and (2.18) implies that

\[
(2.27\pm) \quad \| (P_{\pm, \varepsilon} + I)^{-1} - \omega_{\varepsilon}^2 (-\gamma \Delta + I)^{-1} \omega_{\varepsilon} \|_{L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)} \leq C \varepsilon.
\]

From (2.19) it follows that

\[
(2.28\pm) \quad (w, L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)) \text{-} \lim_{\varepsilon \to 0} (P_{\pm, \varepsilon} + I)^{-1} = (\omega_{\pm})^2 (-\gamma \Delta + I)^{-1} = (\omega_{\pm})^2 (\omega_{\pm})^{-1} (-\gamma \Delta + I)^{-1}.
\]

Now, let \( \overline{u}_0(\pm) \) be the solution of the equation

\[
P_{\pm, \varepsilon} \overline{u}_0(\pm) + \overline{u}_0(\pm) = F_\pm, \quad F_\pm \in L_2(\mathbb{R}^2),
\]

and let \( u_0(\pm) \) be the solution of the equation

\[
-\gamma \Delta u_0(\pm) + u_0(\pm) = \omega_{\pm}(\omega_{\pm})^{-1} F_\pm.
\]

Then (2.20)–(2.25) imply that

\[
(2.29\pm) \quad (w, H^1(\mathbb{R}^2)) \text{-} \lim_{\varepsilon \to 0} \omega_{\varepsilon}^2 \overline{u}_0(\pm) = u_0(\pm),
\]

\[
(2.30\pm) \quad (L_2(\mathbb{R}^2)) \text{-} \lim_{\varepsilon \to 0} p_0(\pm) = p_0(\pm),
\]

where the flows \( \overline{p}_0(\pm), p_0(\pm) \) are defined by the formulas

\[
(2.31\pm) \quad \overline{p}_0(\pm) = (\omega_{\varepsilon}^2 \partial_{\omega_{\varepsilon}} \overline{u}_0(\pm),
\]

\[
p_0(\pm) = (\omega_{\pm})^{-1} \partial_{\omega_{\pm}} u_0(\pm).
\]

2.4. The periodic Pauli operator with metric. Suppose that \( \overline{g}(x) \) is a \( \Gamma \)-periodic \( (2 \times 2) \)-matrix-valued function with real-valued entries such that \( \overline{g}(x) > 0 \) and \( \overline{g} + \overline{g}^{-1} \in L_\infty \). In \( L_2(\mathbb{R}^2; \mathbb{C}^2) \), we consider the operator

\[
(2.32) \quad \mathcal{P}(g) = f_0 E g E f_0, \quad g = f_0 \overline{g} f_0.
\]

If \( \overline{g} = 1_2 \), the operator (2.32) turns into the Pauli operator \( \mathcal{P} \). However, in contrast to the operator \( \mathcal{P} \), in general, the operator \( \mathcal{P}(g) \) does not split into blocks. We note that

\[
(2.33) \quad \mathcal{P}(g) = f_0 \hat{B}_x(g) f_0, \quad g = f_0 \overline{g} f_0,
\]

where the operator \( \hat{B}_x(g) \) is defined by (5.1.22).
The operator $\mathcal{P}(y)$ is included in general scheme with $m = n = d = 2$, $\mathcal{G} = \mathcal{G}_i = L_2(\mathbb{R}^2; \mathbb{C}^2)$. From \((2.33)\) it is clear that the operator \((2.32)\) can be written as $\mathcal{A}(y, f_0)$ with $b(\xi) = b_y(\xi)$ defined by \((5.1.20)\). Now the kernel $\mathfrak{H}$ has the form

$$\mathfrak{H} = \{u \in L_2(\Omega; \mathbb{C}^2) : f_0 u = c \in \mathbb{C}^2\}.$$  

The effective matrix $g^\circ$ is unique and is defined by \((5.1.23)\). The effective DO is given by the formula

$$\mathcal{P}(g^\circ) = f_0(\mathbf{x}) \mathcal{E} g^\circ \mathcal{E} f_0(\mathbf{x}).$$

Now, the germ $S(\theta)$ (acting in $\mathfrak{H}$) of the operator $\mathcal{P}(y)$ depends on $\theta$. However, one can check that its eigenvalues $\gamma_1(\theta)$, $\gamma_2(\theta)$ coincide with the eigenvalues of the constant matrix

$$\gamma^\square = \text{diag} \{\|\omega_+\|_{L^2(\Omega)}, \|\omega_-\|_{L^2(\Omega)}^{-1}\}.$$  

Thus, the eigenvalues of the germ do not depend on $\theta$. It is easy to check this using \((1.1.22)\); cf. [BSu2, Subsection 7.3] for details.

The homogenization problem for the operator $\mathcal{P}(y)$ can be studied similarly to the study of homogenization for the operator $\mathcal{P}_z$; cf. Subsection 2.3. We shall not dwell on this.

§3. The homogenization problem for the operator $\mathcal{D}$

3.1. Here we study the operator $\mathcal{D}$ introduced in Subsection 2.1. We discuss this operator using the scheme intended in Subsection 1.1.8, though now there is no need to use the abstract results directly. Note that the operator $\mathcal{D}$ may be interpreted as the two-dimensional Dirac operator with zero mass in the absence of electric field. However, for us, this operator is interesting as the model one. In its study, we clarify some factors, which are related also to the much more difficult Maxwell operator. The latter will be considered in Chapter 7.

3.2. We start with the definition \((2.11)\):

$$\mathcal{D} = f_0 \mathcal{E} f_0 = \begin{pmatrix} 0 & Y \\ Y^* & 0 \end{pmatrix},$$

where the operator $Y$ is defined by \((2.13)\). As usual, we define the operator

$$\mathcal{D}_\varepsilon = f_0 \mathcal{E} f_0^\varepsilon = \begin{pmatrix} 0 & Y_\varepsilon \\ Y_\varepsilon^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega_+^\varepsilon \partial_+ \omega_-^\varepsilon \\ \omega_-^\varepsilon \partial_+ \omega_+^\varepsilon & 0 \end{pmatrix}, \quad \varepsilon > 0.$$

Let $u_\varepsilon = \text{col}(u_\varepsilon^-, u_\varepsilon^+)\) be the solution of the equation

$$\mathcal{D}_\varepsilon - i\lambda u_\varepsilon = s = \text{col}(q, r), \quad s \in \mathcal{G} = L_2(\mathbb{R}^2; \mathbb{C}^2).$$

We write $u_\varepsilon$ as

$$u_\varepsilon = u_{q, \varepsilon} + u_{r, \varepsilon}, \quad u_{q, \varepsilon} = \text{col}(u_{q, \varepsilon}^-, u_{q, \varepsilon}^+), \quad u_{r, \varepsilon} = \text{col}(u_{r, \varepsilon}^-, u_{r, \varepsilon}^+),$$

where $u_{q, \varepsilon}$ is the solution of equation \((3.2)\) with $r = 0$, and $u_{r, \varepsilon}$ is the solution of \((3.2)\) with $q = 0$. 


Let \( \mathbf{u}_0 = \text{col}(u_0^(-), u_0^+) \) be the solution of the equation

\[
(D^0 - i\Lambda)\mathbf{u}_0 = \mathbf{s} = \text{col}(q, r), \quad \mathbf{s} \in \mathfrak{S},
\]

where

\[
D^0 = \begin{pmatrix} \omega_+ & 0 \\ \partial_+ & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} (\omega_+)^{-2} & 0 \\ 0 & (\omega_+)^{-2} \end{pmatrix}.
\]

Similarly to (3.3), we write \( \mathbf{u}_0 \) as

\[
\mathbf{u}_0 = \mathbf{u}_{q,0} + \mathbf{u}_{r,0}, \quad \mathbf{u}_{q,0} = \text{col}(u_{q,0}^(-), u_{q,0}^+), \quad \mathbf{u}_{r,0} = \text{col}(u_{r,0}^(-), u_{r,0}^+).
\]

Our goal is to prove the following theorem.

**Theorem 3.1.** 1°. Let \( \mathcal{D}_\varepsilon \) be the operator defined by (3.1). Suppose that the operator \( D^0 \) and the matrix \( \Lambda \) are as in (3.5). Then

\[
(w, \mathfrak{S} \rightarrow \mathfrak{S}) - \text{lim}_{\varepsilon \rightarrow 0} (\mathcal{D}_\varepsilon - iI)^{-1} = (D^0 - i\Lambda)^{-1}.
\]

2°. We have

\[
(w, H^1(\mathbb{R}^2)) - \text{lim}_{\varepsilon \rightarrow 0} \omega_+ u_{q,\varepsilon}^{(-)} = \omega_+ u_{q,0}^{(-)},
\]

\[
(w, H^1(\mathbb{R}^2)) - \text{lim}_{\varepsilon \rightarrow 0} \omega_- u_{q,\varepsilon}^{(+)} = \omega_- u_{q,0}^{(+)},
\]

\[
(L_2(\mathbb{R}^2)) - \text{lim}_{\varepsilon \rightarrow 0} \omega_+ u_{r,\varepsilon}^{(-)} = \omega_+ u_{r,0}^{(-)},
\]

\[
(L_2(\mathbb{R}^2)) - \text{lim}_{\varepsilon \rightarrow 0} \omega_- u_{r,\varepsilon}^{(+)} = \omega_- u_{r,0}^{(+)}.
\]

It is natural to call the operator \( D^0 \) the effective DO for \( D \). However, the right-hand side of (3.6) is a generalized resolvent. Relations (3.7)–(3.10) show that the results about the limit procedure are of different strength for different “blocks” of the solution \( \mathbf{u}_\varepsilon \). The combination of these results inevitably leads to the loss. However, we formulate the ”unit” result. Let \( f_0 \) be the matrix defined by (2.9). From (3.7)–(3.10) it is clear that \( f_0 \mathbf{u}_\varepsilon \) is represented as the sum of two summands. One of them converges weakly in \( \mathfrak{S}^1 = H^1(\mathbb{R}^2; \mathbb{C}^3) \), and another one converges strongly in \( \mathfrak{S} \). Then the sum converges strongly in \( \mathfrak{S}_{\text{loc}}^1 \). As a result, we obtain the following theorem.

**Theorem 3.2.** Under the assumptions of Theorem 3.1, we have

\[
(\mathfrak{S}_{\text{loc}}^1) - \text{lim}_{\varepsilon \rightarrow 0} f_0 \mathbf{u}_\varepsilon = f_0 \mathbf{u}_0.
\]

### 3.3. Proof of Theorem 3.1.

The summands \( \mathbf{u}_{q,\varepsilon} \) and \( \mathbf{u}_{r,\varepsilon} \) in (3.3) are considered similarly. To be definite, let us consider \( \mathbf{u}_{q,\varepsilon} \). From the equation (3.2) with \( r = 0 \), we have

\[
Y_\varepsilon u_{q,\varepsilon}^{(+)} - i u_{q,\varepsilon}^{(-)} = q, \quad Y_\varepsilon u_{q,\varepsilon}^{(-)} - i u_{q,\varepsilon}^{(+)} = 0.
\]
From (3.11) it follows that
\[(3.12) \quad u_{0,-}^{(-)} = i(Y_2 Y_2^* + I)^{-1} q = i(P_{-\varepsilon} + I)^{-1} q,\]
\[(3.13) \quad u_{0,+}^{(+)} = Y_2^* (P_{-\varepsilon} + I)^{-1} q,\]
where \(P_{-\varepsilon}\) is defined by (2.26). Relations (3.12) and (2.28) show that
\[(3.14) \quad (w, L_2(\mathbb{R}^2)) \lim_{\varepsilon \to 0} u_{0,-}^{(-)} = i(\overline{\omega}_-)^2 (-\gamma \Delta + I)^{-1} q.\]
By (3.12) and (2.29) (with \(F_\varepsilon = iq\)), we obtain that
\[(3.15) \quad (w, H^1(\mathbb{R}^2)) \lim_{\varepsilon \to 0} \omega_+^\varepsilon u_{0,-}^{(-)} = i(\overline{\omega}_-)^2 (-\gamma \Delta + I)^{-1} q.\]
Combining (3.13), (2.26) and (2.31) (with \(F_\varepsilon = q\)), we see that
\[u_{0,-}^{(-)} = \omega_+^\varepsilon \tilde{p}_{0,-}^{(-)}, \quad \tilde{p}_{0,-}^{(-)} = (\omega_+^\varepsilon)^2 \partial_+ \omega_+^\varepsilon (P_{-\varepsilon} + I)^{-1} q.\]
This and (2.30) imply the strong \(L_2(\mathbb{R}^2)\)-convergence for the product \(\omega_+^\varepsilon u_{0,+}^{(+)}\):
\[(3.16) \quad (L_2(\mathbb{R}^2)) \lim_{\varepsilon \to 0} \omega_+^\varepsilon u_{0,+}^{(+)} = p_0^{(--)} = \overline{\omega}_-\gamma \partial_+ (-\gamma \Delta + I)^{-1} q.\]
From (3.16), by Proposition 4.0.1, we obtain that
\[(3.17) \quad (w, L_2(\mathbb{R}^2)) \lim_{\varepsilon \to 0} u_{0,+}^{(+)} = (\overline{\omega}_-)(\overline{\omega}_+)\gamma \partial_+ (-\gamma \Delta + I)^{-1} q.\]
Similarly, using the properties of the operator \(P_{+\varepsilon}\), we arrive at the following facts about convergence for \(u_{0,-}^{(-)}\), \(u_{0,+}^{(+)}\):
\[(3.18) \quad (w, L_2(\mathbb{R}^2)) \lim_{\varepsilon \to 0} u_{0,+}^{(+)} = i(\overline{\omega}_+)^2 (-\gamma \Delta + I)^{-1} r,\]
\[(3.19) \quad (w, H^1(\mathbb{R}^2)) \lim_{\varepsilon \to 0} \omega_+^\varepsilon u_{0,+}^{(+)} = i(\overline{\omega}_+)^2 (-\gamma \Delta + I)^{-1} r,\]
\[(3.20) \quad (L_2(\mathbb{R}^2)) \lim_{\varepsilon \to 0} \omega_-^\varepsilon u_{0,+}^{(+)} = \overline{\omega}_+ \gamma \partial_- (-\gamma \Delta + I)^{-1} r,\]
\[(3.21) \quad (w, L_2(\mathbb{R}^2)) \lim_{\varepsilon \to 0} u_{0,+}^{(-)} = (\overline{\omega}_-)(\overline{\omega}_+)\gamma \partial_- (-\gamma \Delta + I)^{-1} r.\]
As a result, relations (3.3), (3.14), (3.17), (3.18), (3.21) show that
\[(3.22) \quad (w, \mathcal{G}) \lim_{\varepsilon \to 0} u_- = u_0 = \text{col}(u_{0,-}^{(-)}, u_{0,+}^{(+)}),\]
where
\[u_{0,-}^{(-)} = i(\overline{\omega}_-)^2 (-\gamma \Delta + I)^{-1} q + (\overline{\omega}_-)(\overline{\omega}_+)\gamma \partial_- (-\gamma \Delta + I)^{-1} r,\]
\[u_{0,+}^{(+)} = i(\overline{\omega}_+)^2 (-\gamma \Delta + I)^{-1} r + (\overline{\omega}_-)(\overline{\omega}_+)\gamma \partial_+ (-\gamma \Delta + I)^{-1} q.\]
It can be directly checked that \(u_0\) satisfies equation (3.4). Now, (3.22) implies (3.6).
Relations (3.7), (3.8), (3.9), (3.10) follow from (3.15), (3.19), (3.16), (3.20) respectively. \(\square\)

We note that, for \(u_{0,-}^{(-)}\) and \(u_{0,+}^{(+)}\), we can deduce some results on the basis of (2.27±). Let us write down the estimate for \(u_{0,-}^{(-)}\):
\[\|u_{0,-}^{(-)} - \overline{\omega}_+^{\varepsilon} (-\gamma \Delta + I)^{-1} \omega_+^{\varepsilon} q\|_{L_2(\mathbb{R}^2)} \leq C\|q\|_{L_2(\mathbb{R}^2)}.\]
Homogenization of the oscillating factors \(\omega_+^{\varepsilon}\) necessarily affects the quality of convergence (cf. (3.14)).
1. About factorization for the Schrödinger operator, see, e. g., [KiSi, BSu1]. Apparently, for the first time, Proposition 1.3 (formula (1.12)) was mentioned in [BSu1]. The results of §1 about the homogenization are new; at least, this concerns Theorems 1.5–1.7. Especially, we distinguish estimate (1.16).

2. The two-dimensional periodic Pauli operator was discussed by the authors in [BSu1,2]. In these papers the factorization formulas were given and it was shown that the effective masses for the two-dimensional Pauli operator can be calculated explicitly and do not depend on the direction of the quasimomentum, i. e., on the vector $\Theta$. All the results concerning homogenization for the Pauli operator are new. In this case, we benefited from the relation $m=n$.

3. For the first time, the Pauli operator with metric was introduced in [BSu1], although the variant suggested there was unlikely appropriate. Another variant considered in Subsection 2.4 was proposed in [BSu2]. From the mathematical point of view, its definition looks natural, although the authors cannot suggest the physical interpretation for this operator.

4. The operator $D$ from §3 has an illustrative meaning. It is instructive that, even for proving the existence of the weak limit (3.6), we have to use the specific properties of the case $m=n$.

Chapter 7. The periodic Maxwell operator

§1. Preliminary remarks

In this chapter, we apply general results of Chapters 3, 4 to the stationary periodic Maxwell operator, one of the most difficult cases important for applications. Mainly, we concentrate on the homogenization problems. The study of the Maxwell operator meets a number of complications. First, the Maxwell operator, written in terms of electric and magnetic field strengths, acts in the weighted $L_2$-spaces, which depend on the parameter $\varepsilon$. As a result, statement of the problem about the resolvent convergence loses direct meaning. Second, the Maxwell operator has "reasonable" properties only in the corresponding solenoidal spaces, which, moreover, depend on $\varepsilon$. In order to avoid these difficulties, we write the Maxwell operator in terms of displacements instead of strengths. Herewith, we are forced to sacrifice the selfadjointness of the Maxwell operator. Some results for the strengths can be obtained afterwards.

Next, it turns out that it is convenient to study the homogenization for the Maxwell operator, representing each field as a sum of two summands, like it was done in Subsection 1.1.8 and in §6.3 (for the operator $D$). Herewith, we obtain the results of different quality for different summands.

We are not able to include the Maxwell operator of general type in the scheme developed above, and are forced to assume that one of two coefficients (dielectric permittivity $\varepsilon$ or magnetic permeability $\mu$) is unit. This assumption is related to the following fact. Only in this case, the study can be based on the results about homogenization for an appropriate operator of the form $\hat{A}(g)$. Such operator is preliminarily considered in §2. The results about homogenization of the Maxwell operator are given in §3. In Comments on Chapter 7, we compare the obtained
results with the known ones. To be definite, assume that $\mu = 1$; this is preferentially from the physical point of view.

A separate paper will be devoted to the study of the homogenization problem for the periodic Maxwell operator in general case.

\section{The operator $L(\epsilon, \nu) = \rot \epsilon^{-1} \rot - \nabla \nu \div$}

\subsection{Definition of the operator.}

Let $\epsilon(x)$ be a $\Gamma$-periodic $(3 \times 3)$-matrix-valued function in $\mathbb{R}^3$ with real-valued entries and such that

\begin{equation}
\epsilon(x) > 0, \quad \epsilon + \epsilon^{-1} \in L_\infty.
\end{equation}

Let $\nu(x)$ be a real-valued $\Gamma$-periodic function in $\mathbb{R}^3$ such that

\begin{equation}
\nu(x) > 0, \quad \nu + \nu^{-1} \in L_\infty.
\end{equation}

In $\mathfrak{G} = L_2(\mathbb{R}^3; \mathbb{C}^3)$, we consider the operator $L(\epsilon, \nu)$, formally given by the expression

\begin{equation}
L(\epsilon, \nu) = \rot \left((\epsilon(x))^{-1} \rot - \nabla \nu \div x\right).
\end{equation}

The precise definition of the operator $L(\epsilon, \nu)$ as a selfadjoint operator in $\mathfrak{G}$ is given via the closed positive form

$$
\int_{\mathbb{R}^3} \left(\langle(\epsilon(x))^{-1} \rot u, \rot u \rangle + \nu(x) \div u^2\right) \, dx, \quad u \in \mathfrak{G}^1 = H^1(\mathbb{R}^3; \mathbb{C}^3).
$$

The operator $L(\epsilon, \nu)$ is of the form $\hat{A}(\gamma) = b(D)^* g(x) b(D)$ with $n = 3$, $m = 4$, $\mathfrak{G} = L_2(\mathbb{R}^3; \mathbb{C}^4)$,

$$
b(D) = \begin{pmatrix} -i \rot \\ -i \div \end{pmatrix}, \quad g(x) = \begin{pmatrix} (\epsilon(x))^{-1} & 0 \\ 0 & \nu(x) \end{pmatrix}.
$$

The corresponding symbol $b(\xi)$ has the form

\begin{equation}
b(\xi) = \begin{pmatrix} 0 & -\xi^3 & \xi^2 \\ \xi^3 & 0 & -\xi^1 \\ -\xi^2 & \xi^1 & 0 \end{pmatrix}.
\end{equation}

Obiously, rank $b(\xi) = 3$ for $\xi \neq 0$, i.e., condition (2.1.2) is satisfied. Now $\mathfrak{G} = L_2(\Omega; \mathbb{C}^3)$. The kernel $\hat{\mathfrak{G}}$ is defined according to (3.1.1):

\begin{equation}
\hat{\mathfrak{G}} = \{ \mathfrak{G} \in \mathfrak{G} : \mathfrak{G} = \mathfrak{C} \in \mathbb{C}^3 \}.
\end{equation}
2.2. The effective matrix \( g^\theta \) for the operator \( L(\epsilon, \nu) \) can be calculated according to general rules from §3.1. Let \( C \in \mathbb{C}^4 \) and let \( v \in \widetilde{H}^1(\Omega; \mathbb{C}^3) \) be solution of the equation

\[
b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})v + C) = 0,
\]

which now takes the form

\[
\text{rot } \epsilon(\mathbf{x})^{-1}(\text{rot } v + i\widetilde{C}) - \nabla \nu(\mathbf{x})(\text{div } v + iC^4) = 0,
\]

where \( C = \sum_{j=1}^{4} C^j e_j, \widetilde{C} = \sum_{j=1}^{3} C^j e_j \). In other words, \( v \in \widetilde{H}^1(\Omega; \mathbb{C}^3) \) satisfies identity of the form (3.1.3):

\[
(2.5) \quad \int_\Omega \left( \langle \epsilon(\mathbf{x})^{-1}(\text{rot } v + i\widetilde{C}), \text{rot } z \rangle_{C^3} + \nu(\mathbf{x})(\text{div } v + iC^4)(\text{div } z^+) \right) d\mathbf{x} = 0,
\]

\[
z \in \widetilde{H}^1(\Omega; \mathbb{C}^3).
\]

Representing \( z \) as the sum \( z = \overline{z} + \nabla \varphi \), where \( \text{div } \overline{z} = 0 \), we write (2.5) for \( z = \overline{z} \). Then the term with \( \text{div } \overline{z} \) is equal to zero. By the identity \( \text{rot } \overline{z} = -\text{rot } z \), we obtain that

\[
(2.6) \quad \int_\Omega \langle \epsilon(\mathbf{x})^{-1}(\text{rot } v + i\widetilde{C}), \text{rot } z \rangle_{C^3} d\mathbf{x} = 0, \quad z \in \widetilde{H}^1(\Omega; \mathbb{C}^3).
\]

From (2.6) it follows that

\[
(2.7) \quad \epsilon(\mathbf{x})^{-1}(\text{rot } v + i\widetilde{C}) = \nabla \Phi + i\mathbf{c}, \quad \Phi \in \widetilde{H}^1(\Omega), \quad \mathbf{c} \in \mathbb{C}^3,
\]

whence

\[
(2.8) \quad \text{rot } v + i\widetilde{C} = \epsilon(\mathbf{x})(\nabla \Phi + i\mathbf{c}).
\]

Relation (2.8) implies that

\[
\int_\Omega \langle \epsilon(\mathbf{x})(\nabla \Phi + i\mathbf{c}), \nabla \Psi \rangle_{C^3} d\mathbf{x} = 0, \quad \Psi \in \widetilde{H}^1(\Omega).
\]

Thus, \( \Phi \) is a \( \widetilde{H}^1(\Omega) \)-solution of the equation

\[
\text{div } \epsilon(\mathbf{x})(\nabla \Phi + i\mathbf{c}) = 0, \quad \mathbf{c} \in \mathbb{C}^3.
\]

Let \( \epsilon^\theta \) be the effective matrix for the acoustic operator \( -\text{div } \epsilon(\mathbf{x}) \nabla \). According to representation (3.1.14), we have

\[
(2.9) \quad \epsilon^\theta \mathbf{c} = |\Omega|^{-1} \int_\Omega \epsilon(\mathbf{x})(-i\nabla \Phi + \mathbf{c}) d\mathbf{x}, \quad \mathbf{c} \in \mathbb{C}^3.
\]

Integrating (2.8) and taking account of (2.9), we obtain that

\[
(2.10) \quad \widetilde{C} = \epsilon^\theta \mathbf{c}.
\]
From the other side, integrating (2.7), we arrive at

\[ (2.11) \quad i \epsilon |\Omega| = \int_{\Omega} \epsilon(x)^{-1} \text{rot} \nu + i \bar{C} \, d\mathbf{x}. \]

Next, (2.5) and (2.6) imply the identity

\[ \int_{\Omega} \nu(x)(\text{div} \nu + iC^4)(\text{div} \mathbf{z})^+ \, d\mathbf{x} = 0, \quad \mathbf{z} \in H^1(\Omega; \mathbb{C}), \]

which means that

\[ (2.12) \quad \nu(x)(\text{div} \nu + iC^4) = \alpha \in \mathbb{C}. \]

From (2.12) it is clear that

\[ iC^4|\Omega| = \alpha \int_{\Omega} \nu(x)^{-1} \, d\mathbf{x}, \]

i.e.,

\[ (2.13) \quad \alpha = iC^4 \nu. \]

Now we calculate \( g^0 \), starting with representation (3.1.14) (corresponding to the operator (2.2)). By (2.10)-(2.13), we have

\[ g^0 C = |\Omega|^{-1} \int_{\Omega} g(x)(b(D \nu + C) \, d\mathbf{x} = |\Omega|^{-1} \int_{\Omega} \left( \left( \frac{\epsilon(x)^{-1} - i \text{rot} \nu + \bar{C}}{\nu(x)(-i \text{div} \nu + C^4)} \right) \right) \, d\mathbf{x} \]

\[ = \left( \begin{array}{c} e \\ -i\alpha \end{array} \right) = \left( \begin{array}{c} (\epsilon^0)^{-1} - \bar{C} \\ 0 \end{array} \right) = \left( \begin{array}{c} 0 \\ \nu \end{array} \right) C. \]

Thus, the effective matrix \( g^0 \) of the operator \( \mathcal{L}(\epsilon, \nu) \) is expressed via the effective matrix \( \epsilon^0 \) for the operator \( -\text{div} \epsilon(x) \nabla \) and via \( \nu \) by the relation

\[ (2.14) \quad g^0 = \text{diag}\{\epsilon^0)^{-1}, \nu\}. \]

It is easy to check that the uniqueness condition (3.1.16) for the effective matrix with real-valued entries is satisfied. Indeed,

\[ b(\theta) \mathbf{y} = \text{col}(\theta \times \mathbf{y}, \theta \cdot \mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^3, \]

whence

\[ b(\theta)\mathbb{R}^3 = \{\text{col}(e, a) \in \mathbb{R}^4 : e \perp \theta, a \in \mathbb{R}\}, \]

\[ \text{clos}_0 b(\theta)\mathbb{R}^3 = \mathbb{R}^4. \]

The following operator is the effective DO for the operator \( \mathcal{L}(\epsilon, \nu) \)

\[ (2.15) \quad \mathcal{L}^0 = \mathcal{L}(\epsilon^0, \nu) = \text{rot}(\epsilon^0)^{-1} \text{rot} - \nabla \nu \cdot \text{div}. \]

The germ \( S_\epsilon(\theta) \) corresponding to the operator \( \mathcal{L}(\epsilon, \nu) \) acts in \( \mathfrak{N} \) as multiplication by the matrix

\[ S_\epsilon(\theta) = b(\theta)^* g^0 b(\theta), \quad \theta \in \mathbb{S}^2. \]

According to (2.3), (2.14), \( S_\epsilon(\theta) \) can be represented as the sum

\[ (2.16) \quad S_\epsilon(\theta) = b_r(\theta)^* (\epsilon^0)^{-1} b_r(\theta) + \nu b_d(\theta)^* b_d(\theta), \]

where

\[ (2.17) \quad b_r(\theta) = \begin{pmatrix} 0 & -\theta^3 & \theta^2 \\ \theta^3 & 0 & -\theta^1 \\ -\theta^2 & \theta^1 & 0 \end{pmatrix}, \quad b_d(\theta) = (\theta^1 \theta^2 \theta^3). \]
2.3. Splitting of the operator \( \mathcal{L}(\epsilon, \nu) \). We use the Weyl decomposition

\[
(2.18) \quad \mathfrak{g} = L^2(\mathbb{R}^3; \mathbb{C}^3) = J(\mathbb{R}^3) \oplus G(\mathbb{R}^3),
J(\mathbb{R}^3) = J = \{ w \in \mathfrak{g} : \text{div } w = 0 \},
G(\mathbb{R}^3) = \{ u = \nabla \varphi : \varphi \in H^1_{\text{loc}}(\mathbb{R}^3), \nabla \varphi \in \mathfrak{g} \}.
\]

Decomposition (2.18) reduces the operator \( \mathcal{L}(\epsilon, \nu) \):

\[
\mathcal{L}(\epsilon, \nu) = \mathcal{L}_J(\epsilon) \oplus \mathcal{L}_G(\nu).
\]

The operator \( \mathcal{L}_J(\epsilon) \) acting in the subspace \( J(\mathbb{R}^3) \) corresponds to the differential expression \( \text{rot } \epsilon(x)^{-1} \text{rot} \), and the operator \( \mathcal{L}_G(\nu) \) acting in \( G(\mathbb{R}^3) \) is defined by the expression \( -\nabla \nu(x) \text{div} \). Mainly, we are interested in the operator \( \mathcal{L}_J(\epsilon) \), the definition of which does not depend on \( \nu \).

Note that the germ (2.16) acting in the kernel (2.4) also splits in the "induced" orthogonal decomposition

\[
\mathcal{H} = J(\Theta) \oplus G(\Theta),
J(\Theta) = \{ c \in \mathbb{C}^3 : c \perp \Theta \},
G(\Theta) = \{ c = \lambda \Theta : \lambda \in \mathbb{C} \}.
\]

Here, \( S_{\mathcal{L}, J}(\Theta) \) (the part of \( S_{\mathcal{L}}(\Theta) \) in \( J(\Theta) \)) corresponds to the first summand in (2.16), and \( S_{\mathcal{L}, G}(\Theta) \) (the part of \( S_{\mathcal{L}}(\Theta) \) in \( G(\Theta) \)) corresponds to the second summand. In the subspace \( G(\Theta) \), the operator \( S_{\mathcal{L}}(\Theta) \) has a single eigenvalue \( \gamma_1(\Theta) = \frac{\epsilon}{\nu} \).

In the subspace \( J(\Theta) \), it has two eigenvalues \( \gamma_1(\Theta), \gamma_2(\Theta) \), which correspond to the algebraic problem

\[
b_{\gamma}(\Theta)^*(\epsilon)^{-1}b_{\gamma}(\Theta)c = \gamma c, \quad c \perp \Theta.
\]

In some sense, \( S_{\mathcal{L}, J}(\Theta) \) plays the role of the germ for the operator \( \mathcal{L}_J(\epsilon) \), and \( S_{\mathcal{L}, G}(\Theta) \) plays the role of the germ for \( \mathcal{L}_G(\nu) \). We shall not dwell on the corresponding analysis.

2.4. Homogenization. To the operator

\[
\mathcal{L}_\epsilon(\epsilon, \nu) = \mathcal{L}(\epsilon^\epsilon, \nu^\epsilon) = \text{rot } (\epsilon^\epsilon)^{-1} \text{rot } -\nabla \nu^\epsilon \text{div},
\]

general theorems about homogenization for the case where \( f = 1_n \) are applicable. As in §5.1, we shall not write down the constants in estimates explicitly, though it is easy to do this. In what follows, we denote constants in estimates by \( C \) without indices.

First of all, we apply Theorem 4.2.1. Take into account that decomposition (2.18) simultaneously reduces the operator \( \mathcal{L}_\epsilon(\epsilon, \nu) \) and the effective operator \( \mathcal{L}_\epsilon^0 \), defined by (2.15). Hence, the estimate for the difference of resolvents of the operators \( \mathcal{L}_\epsilon(\epsilon, \nu) \) and \( \mathcal{L}_\epsilon^0 \) directly implies the estimate for the difference of resolvents of the operators \( \mathcal{L}_{\mathcal{J}, \epsilon}(\epsilon) = \mathcal{L}_J(\epsilon^\epsilon) \) and \( \mathcal{L}_{\mathcal{J}^0} = \mathcal{L}_J(\epsilon^\epsilon) \), acting in the subspace \( J \). As a result, we obtain the following theorem.
**Theorem 2.1.** We have
\[
\|(L_\varepsilon(\varepsilon, \nu) + I)^{-1} - (L_0^\varepsilon + I)^{-1}\|_{\mathcal{G}^s \to \mathcal{G}} \leq C\varepsilon, \quad 0 < \varepsilon \leq 1,
\]
\[
\|(L_{J_\varepsilon}(\varepsilon) + I)^{-1} - (L_0^\varepsilon + I)^{-1}\|_{J^s \to J^s} \leq C\varepsilon, \quad 0 < \varepsilon \leq 1.
\]

Now we discuss application of the "interpolation" Theorem 4.2.7 (with \(Q = I\)). Let \(P_J\) be the orthoprojector of \(\mathcal{G}\) onto the subspace \(J\). In the Fourier representation, the operator \(P_J\) turns into multiplication by the symbol \((ib_r(\Theta))\), \(\Theta = \xi/|\xi|\), where \(b_r\) is defined by (2.17). In other words, \(P_J\) is a pseudo-differential operator of zero order, which acts continuously in the whole scale of spaces \(\mathcal{G}^s = H^s(\mathbb{R}^3, \mathbb{C})\), \(s \in \mathbb{R}\). Moreover, let \(P_J^s\) be the restriction of \(P_J\) onto \(\mathcal{G}^s\) for \(s > 0\) and the extension of \(P_J\) by continuity to \(\mathcal{G}^s\) for \(s < 0\). Then \(P_J^s\) is the orthoprojector in all spaces \(\mathcal{G}^s\), \(s \in \mathbb{R}\). We introduce the scale of spaces \(J^s = P_J^s\mathcal{G}^s\) with the norm induced by the norm in \(\mathcal{G}^s\). Then the following result is true.

**Theorem 2.2.** For \(0 \leq s < 1\), we have
\[
\|(L_\varepsilon(\varepsilon, \nu) + I)^{-1} - (L_0^\varepsilon + I)^{-1}\|_{\mathcal{G}^s \to \mathcal{G}^s} \leq C\varepsilon^{1-s}, \quad 0 < \varepsilon \leq 1,
\]
\[
\|(L_{J_\varepsilon}(\varepsilon) + I)^{-1} - (L_0^\varepsilon + I)^{-1}\|_{J^s \to J^s} \leq C\varepsilon^{1-s}, \quad 0 < \varepsilon \leq 1.
\]

Here (2.19) is a direct consequence of Theorem 4.2.7, and (2.20) follows from (2.19), by applying the corresponding orthoprojectors.

Now we apply Theorem 4.4.1 (again with \(Q = I\)). Let \(v_\varepsilon\) be the solution of the equation
\[
(L_\varepsilon(\varepsilon, \nu)v_\varepsilon + v_\varepsilon = F, \quad F \in \mathcal{G}^{-1},
\]
and let \(v_0\) be the solution of the equation
\[
L_0^\varepsilon v_0 + v_0 = F, \quad F \in \mathcal{G}^{-1}.
\]

Note that, for \(F \in J^{-1}\), we automatically have \(\text{div} v_\varepsilon = 0\), \(\text{div} v_0 = 0\). In this case \(v_\varepsilon\) coincides with the solution of the problem
\[
\text{rot}(\varepsilon^s(x))^{-1}\text{rot} v_\varepsilon + v_\varepsilon = F, \quad \text{div} v_\varepsilon = 0, \quad F \in J^{-1},
\]
and \(v_0\) coincides with the solution of the problem
\[
\text{rot}(\varepsilon^s)^{-1}\text{rot} v_0 + v_0 = F, \quad \text{div} v_0 = 0, \quad F \in J^{-1}.
\]

Theorem 4.4.1 implies the following result.

**Theorem 2.3.** 1°. Let \(v_\varepsilon\) be the solution of equation (2.21), and let \(v_0\) be the solution of (2.22). Then
\[
(w, \mathcal{G}^1)_{\varepsilon \to 0} v_\varepsilon = v_0,
\]
\[
(w, \mathcal{G}^1)_{\varepsilon \to 0} \varepsilon b(D)v_\varepsilon = \varepsilon b(D)v_0.
\]
2°. Let \( \mathbf{v}_\varepsilon \) be the solution of the problem (2.23), and \( \mathbf{v}_0 \) be the solution of the problem (2.24). Then

\[
(2.27) \quad (w, J^1)_{\varepsilon \to 0} \lim \mathbf{v}_\varepsilon = \mathbf{v}_0,
\]

\[
(2.28) \quad (w, \mathbf{\Theta})_{\varepsilon \to 0} \lim (\varepsilon^\varepsilon)^{-1} \text{rot} \mathbf{v}_\varepsilon = (\varepsilon^0)^{-1} \text{rot} \mathbf{v}_0.
\]

We distinguish the case where convergence of solutions or convergence of flows is strong. By (2.14), the case where \( \varepsilon^0 = \mathcal{G} \) corresponds to \( \varepsilon^\varepsilon = \varepsilon \varepsilon \), \( \nu = \text{const} \). The case where \( \varepsilon^0 = \mathcal{G} \) is equivalent to the relation \( \varepsilon^\varepsilon = \varepsilon \). Then Theorem 4.4.8 and Propositions 3.1.6, 3.1.7 imply the following proposition.

**Proposition 2.4.** 1°. If the columns of the matrix \( \varepsilon(\mathbf{x}) \) are solenoidal, then convergence in (2.26), (2.28) is strong.

2°. If the columns of the matrix \( \varepsilon(\mathbf{x})^{-1} \) are potential (up to a constant summand), then convergence in (2.27) is strong. If, besides, \( \nu(\mathbf{x}) = \text{const} \), convergence in (2.25) is also strong.

### §3. The homogenization for the periodic Maxwell system with \( \mu = 1 \)

#### 3.1. Statement of the problem

The results of §2 are applied to the homogenization problem for the periodic Maxwell system. (Herewith, it suffices to assume that \( \nu = 1 \) in the definition (2.2) for \( \mathcal{L}(\varepsilon, \nu) \).) We assume that the magnetic permeability is unit: \( \mu = 1 \). In what follows, \( \mathbf{u}, \mathbf{v} \) stand for the strengths of electric and magnetic fields correspondingly. The dielectric permittivity \( \varepsilon(\mathbf{x}) \) is a \( \Gamma \)-periodic \( (3 \times 3) \)-matrix-valued function with real-valued entries and subject to (2.1). Next, \( \mathbf{w} = \mathbf{c} \mathbf{u} \) is an electric displacement vector (a magnetic displacement vector is equal to \( \mathbf{v} \), since \( \mu = 1 \)). We denote \( \mathcal{G} = L_2(\mathbb{R}^3; \mathbb{C}^3) \).

The Maxwell operator \( \mathcal{M} = \mathcal{M}(\varepsilon) \), written in terms of the displacement vectors, acts in the space

\[
\mathcal{G} = J \oplus J, \quad J = J(\mathbb{R}^3),
\]

and is defined by the formula

\[
(3.1) \quad \mathcal{M}(\varepsilon) \text{col}(\mathbf{w}, \mathbf{v}) = \text{col}(i \text{rot} \mathbf{v}, -i \text{rot}(\varepsilon(\mathbf{x}))^{-1} \mathbf{w})
\]

on the domain

\[
(3.2) \quad \text{Dom} \mathcal{M}(\varepsilon) = \{ \text{col}(\mathbf{w}, \mathbf{v}) : \mathbf{w} \in J, \text{rot} \varepsilon^{-1} \mathbf{w} \in \mathcal{G}, \mathbf{v} \in J^1 \}.
\]

Conditions \( \mathbf{w} \in J, \mathbf{v} \in J^1 \) automatically add the equations

\[
\text{div} \mathbf{w} = \text{div} \mathbf{v} = 0
\]

to relations (3.1). The operator \( \mathcal{M}(\varepsilon) \) is closed in \( \mathcal{G} \), but non-selfadjoint.

We define the operator \( \mathcal{M}_\varepsilon(\varepsilon), \varepsilon > 0 \), acting in \( \mathcal{G} \), by the relation

\[
\mathcal{M}_\varepsilon(\varepsilon) = \mathcal{M}(\varepsilon^\varepsilon).
\]

The domain \( \text{Dom} \mathcal{M}_\varepsilon \) is given by the relations

\[
\mathbf{v} \in J^1, \quad \mathbf{w} \in J, \quad \text{rot}(\varepsilon^\varepsilon)^{-1} \mathbf{w} \in \mathcal{G},
\]
and, hence, it does not depend on $\varepsilon$.

Our goal is to study the behavior of the resolvent $(\mathcal{M}_\varepsilon - iI)^{-1}$ as $\varepsilon \to 0$. Consider the equation

\[(3.3) \quad (\mathcal{M}_\varepsilon - iI)\text{col}(w_\varepsilon, v_\varepsilon) = \text{col}(q, r), \quad q, r \in J^{-1}.\]

In detailed writing, (3.3) has the form

\[
\begin{align*}
\left\{ \begin{array}{l}
 i \text{rot} v_\varepsilon - iw_\varepsilon = q, \\
 -i \text{rot}(\varepsilon^b)^{-1}w_\varepsilon - iv_\varepsilon = r, \\
 \text{div} v_\varepsilon = 0, \\
 \text{div} w_\varepsilon = 0.
\end{array} \right.
\end{align*}
\]

(3.4)

It is convenient (cf. §6.3) to represent the solutions $w_\varepsilon$, $v_\varepsilon$ as the sums

\[
(3.5) \quad w_\varepsilon = w_\varepsilon^{(q)} + w_\varepsilon^{(r)},
\]

(3.6) \quad $v_\varepsilon = v_\varepsilon^{(q)} + v_\varepsilon^{(r)}$,

where $\text{col}(w_\varepsilon^{(q)}$, $v_\varepsilon^{(q)})$ is the solution of system (3.4) with $r = 0$, and $\text{col}(w_\varepsilon^{(r)}$, $v_\varepsilon^{(r)})$ is the solution of system (3.4) with $q = 0$.

3.2. The case where $q = 0$. The system (3.4) for $\text{col}(w_\varepsilon^{(r)}$, $v_\varepsilon^{(r)})$ takes the form

\[
\left\{ \begin{array}{l}
 w_\varepsilon^{(r)} = \text{rot} v_\varepsilon^{(r)}, \\
 \text{rot}(\varepsilon^b)^{-1}v_\varepsilon^{(r)} + v_\varepsilon^{(r)} = i\varepsilon r, \\
 \text{div} v_\varepsilon^{(r)} = 0.
\end{array} \right.
\]

(3.7)

Hence,

\[
(3.8) \quad v_\varepsilon^{(r)} = i(\mathcal{L}_{J,\varepsilon}(\varepsilon) + I)^{-1}r,
\]

where the operator $\mathcal{L}_{J,\varepsilon}(\varepsilon)$ is defined in Subsection 2.3. The displacement vector $w_\varepsilon^{(r)}$ can be found from the relation

\[
(3.9) \quad w_\varepsilon^{(r)} = \text{rot} v_\varepsilon^{(r)} = i \text{rot}(\mathcal{L}_{J,\varepsilon}(\varepsilon) + I)^{-1}r,
\]

while the strength $u_\varepsilon^{(r)}$ is expressed via the flow for $v_\varepsilon^{(r)}$:

\[
(3.10) \quad u_\varepsilon^{(r)} = (\varepsilon^b)^{-1}w_\varepsilon^{(r)} = (\varepsilon^b)^{-1}\text{rot} v_\varepsilon^{(r)}.
\]

Let $v_\varepsilon^{(r)} = i(\mathcal{L}_{J,\varepsilon}^{(b)} + I)^{-1}r$, i.e., $v_\varepsilon^{(r)}$ is the solution of the problem

\[
\text{rot}(\varepsilon^{(b)})^{-1}v_\varepsilon^{(r)} + v_\varepsilon^{(r)} = i\varepsilon r, \quad \text{div} v_\varepsilon^{(r)} = 0.
\]

We put

\[
\left\{ \begin{array}{l}
 w_\varepsilon^{(r)} = \text{rot} v_\varepsilon^{(r)}, \\
 u_\varepsilon^{(r)} = (\varepsilon^{(b)})^{-1}\text{rot} v_\varepsilon^{(r)}.
\end{array} \right.
\]

On the basis of (3.8)–(3.10), Theorems 2.1–2.3 imply the following result.
Theorem 3.1. For the solutions of system (3.7), the following is true.
1°. If \( r \in J \), then \( \mathbf{v}_z^{(r)} \) converges in \( \mathcal{G} \) to \( \mathbf{v}_0^{(r)} \), and
\[
\|\mathbf{v}_z^{(r)} - \mathbf{v}_0^{(r)}\|_{\mathcal{G}} \leq C\varepsilon\|\mathbf{r}\|_{\mathcal{G}}, \quad 0 < \varepsilon \leq 1.
\]
2°. If \( r \in J^{-s}, 0 \leq s < 1 \), then \( \mathbf{v}_z^{(r)} \) converges in \( \mathcal{G}^s \) to \( \mathbf{v}_0^{(r)} \), and
\[
\|\mathbf{v}_z^{(r)} - \mathbf{v}_0^{(r)}\|_{\mathcal{G}^s} \leq C\varepsilon^{1-s}\|\mathbf{r}\|_{\mathcal{G}^{-s}}, \quad 0 < \varepsilon \leq 1.
\]
3°. If \( r \in J^{-1} \), then \( \mathbf{v}_z^{(r)} \) converges to \( \mathbf{v}_0^{(r)} \) weakly in \( \mathcal{G}^1 \), as \( \varepsilon \to 0 \).
4°. If \( r \in J^{-1} \), then \( \mathbf{w}_z^{(r)} \) converges to \( \mathbf{w}_0^{(r)} \) weakly in \( \mathcal{G} \), and \( \mathbf{u}_z^{(r)} \) converges to \( \mathbf{u}_0^{(r)} \) weakly in \( \mathcal{G} \), as \( \varepsilon \to 0 \).

Note also that, as it follows from the second equation in (3.7),
\[
\text{rot } \mathbf{u}_z^{(r)} = i\mathbf{r} - \mathbf{v}_z^{(r)}.
\]
Similarly, \( \text{rot } \mathbf{u}_0^{(r)} = i\mathbf{r} - \mathbf{v}_0^{(r)} \). Hence,
\[
\|\text{rot } \mathbf{u}_z^{(r)} - \text{rot } \mathbf{u}_0^{(r)}\|_{\mathcal{G}} \leq C\varepsilon\|\mathbf{r}\|_{\mathcal{G}}, \quad r \in J, \quad 0 < \varepsilon \leq 1.
\]

Proposition 2.4 implies the following result.

Proposition 3.2. If the columns of the matrix \( e(x)^{-1} \) are potential, then \( \mathbf{v}_z^{(r)} \) converges to \( \mathbf{v}_0^{(r)} \) strongly in \( \mathcal{G}^1 \). If the columns of the matrix \( e(x) \) are solenoidal, then \( \mathbf{u}_z^{(r)} \) converges to \( \mathbf{u}_0^{(r)} \) strongly in \( \mathcal{G} \).

3.3. The case where \( r = 0 \). System (3.4) for \( \text{col}(\mathbf{w}_z^{(q)}, \mathbf{v}_z^{(q)}) \) has the form
\[
\begin{align*}
&i\mathbf{r} \text{ rot } \mathbf{v}_z^{(q)} - i\mathbf{w}_z^{(q)} = \mathbf{q}, \\
&\text{rot } (e^r)^{-1}\mathbf{w}_z^{(q)} + \mathbf{v}_z^{(q)} = 0, \\
&\text{div } \mathbf{v}_z^{(q)} = 0, \\
&\text{div } \mathbf{w}_z^{(q)} = 0.
\end{align*}
\]

(3.11)

In this Subsection, we restrict ourselves to considering the case of \( \mathbf{q} \in J \).

Lemma 3.3. Let \( \mathbf{q} \in J \). Then the solutions of system (3.11) satisfy the inequalities
\[
\|\mathbf{v}_z^{(q)}\|_{\mathcal{G}^1} \leq C\|\mathbf{q}\|_{\mathcal{G}},
\]
(3.12)
\[
\|\mathbf{w}_z^{(q)}\|_{\mathcal{G}} \leq C\|\mathbf{q}\|_{\mathcal{G}}.
\]
(3.13)

Whence, for \( \mathbf{u}_z^{(q)} = (e^r)^{-1}\mathbf{w}_z^{(q)} \), we have
\[
\|\mathbf{u}_z^{(q)}\|_{\mathcal{G}} \leq C\|\mathbf{q}\|_{\mathcal{G}}.
\]

Proof. We multiply the second equation in (3.11) by \( \mathbf{v}_z^{(q)} \) and integrate:
\[
((e^r)^{-1}\mathbf{w}_z^{(q)}, \text{rot } \mathbf{v}_z^{(q)})_{\mathcal{G}} + \|\mathbf{v}_z^{(q)}\|_{\mathcal{G}}^2 = 0.
\]
Substituting

$$w^{(q)}_x = \text{rot } v^{(q)}_x + iq$$

from the first equation, we obtain that

$$\left((\varepsilon^\varepsilon)^{-1} \text{rot } v^{(q)}_x, \text{rot } v^{(q)}_x\right)_\Theta + \|v^{(q)}_x\|^2_\Theta = -i((\varepsilon^\varepsilon)^{-1} q, \text{rot } v^{(q)}_x)_\Theta.$$

Combining this with the relation $\text{div } v^{(q)}_x = 0$, we arrive at (3.12). Estimate (3.13) follows from (3.12) and (3.14). \□

By Lemma 3.3, for the proof of weak $(\mathcal{G}^1)$-convergence of $v^{(q)}_x$ and weak $(\mathcal{G})$-convergence of $w^{(q)}_x$ and $u^{(q)}_x$, it suffices to assume that

$$q = \text{rot } F, \quad F \in J^1. \tag{3.15}$$

We put

$$\Phi^{(q)}_x = i(L_{J^1}(\varepsilon) + l)^{-1} F. \tag{3.16}$$

It can be easily checked that the functions

$$w^{(q)}_x = \text{rot } \Phi^{(q)}_x, \tag{3.17}$$

$$v^{(q)}_x = \Phi^{(q)}_x - iF, \tag{3.18}$$

satisfy the system (3.11) with $q$ defined by (3.15). Then, by (3.17),

$$u^{(q)}_x = (\varepsilon^\varepsilon)^{-1} \text{rot } \Phi^{(q)}_x, \tag{3.19}$$

i.e., $u^{(q)}_x$ is expressed via the flow for $\Phi^{(q)}_x$. Besides, second relation in (3.11) means that

$$\text{rot } u^{(q)}_x = -v^{(q)}_x. \tag{3.20}$$

Let

$$\Phi^{(q)}_0 = i(L^0_{J^1} + l)^{-1} F. \tag{3.21}$$

$$v^{(q)}_0 = \Phi^{(q)}_0 - iF, \tag{3.22}$$

$$w^{(q)}_0 = \text{rot } \Phi^{(q)}_0, \quad u^{(q)}_0 = (\varepsilon^0)^{-1} \text{rot } \Phi^{(q)}_0. \tag{3.23}$$

Theorem 2.3 and (3.16)–(3.23) imply the following theorem.

**Theorem 3.4.** Let $q \in J$. Then the following is true.

1. The fields $v^{(q)}_x$ converge to $v^{(q)}_0$ weakly in $\mathcal{G}^1$, as $\varepsilon \to 0$.
2. The fields $w^{(q)}_x$ converge to $w^{(q)}_0$ weakly in $\mathcal{G}$, as $\varepsilon \to 0$.
3. The fields $u^{(q)}_x$ converge to $u^{(q)}_0$ weakly in $\mathcal{G}$, and $\text{rot } u^{(q)}_x$ converges to $\text{rot } u^{(q)}_0$ weakly in $\mathcal{G}^1$, as $\varepsilon \to 0$.

Besides, Proposition 2.4 implies the following result.
Proposition 3.5. If the columns of the matrix \( \epsilon(x)^{-1} \) are potential, the fields \( \mathbf{v}_z^{(q)} \) converge to \( \mathbf{v}_0^{(q)} \) strongly in \( \mathcal{G}^1 \). If the columns of the matrix \( \epsilon(x) \) are solenoidal, the fields \( \mathbf{u}_z^{(q)} \) converge to \( \mathbf{u}_0^{(q)} \) strongly in \( \mathcal{G} \).

Now, we apply Theorems 2.1, 2.2 to functions (3.16), and take (3.18) into account. Then, for \( F \in J^1 \), \( q = \text{rot } F \), we have

\[
\left\| \mathbf{v}_z^{(q)} - \mathbf{v}_0^{(q)} \right\|_{\mathcal{G}} \leq C\varepsilon\|F\|_{\mathcal{G}}, \quad 0 < \varepsilon \leq 1,
\]

\[
\left\| \mathbf{v}_z^{(q)} - \mathbf{v}_0^{(q)} \right\|_{\mathcal{G}^s} \leq C\varepsilon^{1-s}\|F\|_{\mathcal{G}^s}, \quad 0 \leq s < 1, \quad 0 < \varepsilon \leq 1.
\]

Relations (3.24), (3.25) and estimate (3.12) yield the \((\mathcal{G}^s)\)-convergence of \( \mathbf{v}_z^{(q)} \) to \( \mathbf{v}_0^{(q)} \) for any \( q \in J \), but already without a "qualified" estimate. The latter is related to the fact that condition (3.15) does not imply the estimate of the norm \( \|F\|_{\mathcal{G}} \) in terms of \( \|q\|_{\mathcal{A}} \). Thus, we arrive at the following theorem.

Theorem 3.6. Let \( q \in J \). Then the fields \( \mathbf{v}_z^{(q)} \) converge to \( \mathbf{v}_0^{(q)} \) strongly in \( \mathcal{G}^s \), \( 0 \leq s < 1 \), as \( \varepsilon \to 0 \).

3.4. As we have seen, the quality of convergence for separate summands may be better than for the sums (3.5), (3.6). Still, let us formulate the summarizing result. Theorems 3.1, 3.4 and 3.6 lead to the following conclusion.

Theorem 3.7. Let \( \mathcal{M}^0 = \mathcal{M}(\epsilon^0) \) be the effective Maxwell operator defined according to (3.1), (3.2) with \( \epsilon(x) \) replaced by \( \epsilon^0 \). Let

\[
\text{col}(\mathbf{w}_z, \mathbf{v}_z) = (\mathcal{M}_z - iI)^{-1} \text{col}(q, \mathbf{r}), \quad \text{col}(\mathbf{w}_0, \mathbf{v}_0) = (\mathcal{M}_0 - iI)^{-1} \text{col}(q, \mathbf{r}),
\]

where \( q \in J, \mathbf{r} \in J \). Let

\[
\mathbf{u}_z = (\epsilon^z)^{-1} \mathbf{w}_z, \quad \mathbf{u}_0 = (\epsilon^0)^{-1} \mathbf{w}_0.
\]

Then the following is true.

1°. The fields \( \mathbf{v}_z \) converge to \( \mathbf{v}_0 \) strongly in \( \mathcal{G}^s \), \( 0 \leq s < 1 \), and weakly in \( \mathcal{G}^1 \), as \( \varepsilon \to 0 \).

2°. The fields \( \mathbf{w}_z \) converge to \( \mathbf{w}_0 \) weakly in \( \mathcal{G} \), as \( \varepsilon \to 0 \).

3°. The fields \( \mathbf{u}_z \) converge to \( \mathbf{u}_0 \) weakly in \( \mathcal{G} \), as \( \varepsilon \to 0 \). Besides, \( \text{rot } \mathbf{u}_z \) converges to \( \text{rot } \mathbf{u}_0 \) strongly in \( \mathcal{G} \), as \( \varepsilon \to 0 \).

Combining Propositions 3.2 and 3.5, we obtain the following proposition.

Proposition 3.8. If the columns of the matrix \( \epsilon(x)^{-1} \) are potential, the fields \( \mathbf{v}_z \) converge to \( \mathbf{v}_0 \) strongly in \( \mathcal{G}^1 \). If the columns of the matrix \( \epsilon(x) \) are solenoidal, the fields \( \mathbf{u}_z \) converge to \( \mathbf{u}_0 \) strongly in \( \mathcal{G} \).

Comments on Chapter 7

1°. The results of Theorems 2.1, 2.2 about homogenization as applied to the operator \( L_{J_x}(\epsilon) \) are new. Relation (2.27) from Theorem 2.3 is close to some results of [BeLP]. Let us dwell on this in more detail.

In [BeLP], the equation of the type (2.23) (but in a bounded domain and with appropriate boundary conditions) is called the equation of the "Maxwell type".
Condition $\text{div} \mathbf{v}_z = 0$ is not assumed, but, if $\text{div} \mathbf{F} = 0$, it holds automatically. The extension of the system to an elliptic one is not employed. Up to these small differences, our statement about the limit (2.27) repeats the result of [BeLP]. In [BeLP], convergence of the flows is not discussed. Therefore, our relation (2.28) is of certain interest: it is applied in \S 3 to the homogenization problem for the Maxwell system itself.

2. In [BeLP], application of the results to the stationary Maxwell system is not discussed. However, along with (2.23), in [BeLP] the more general system

$$(+) \quad \text{rot}(\varepsilon^e)^{-1} \text{rot} \mathbf{v}_z + \sigma^e \mathbf{v}_z = \mathbf{F}$$

is considered. It is proved that solutions $\mathbf{v}_z$ converge weakly in the space with the metric form $||\text{rot} \mathbf{v}||^2_{L^2} + ||\mathbf{v}||^2_{L^2}$. Here with, in the limit equation, $a$ is also homogenized by the acoustic rule. Condition $\text{div} \mathbf{F} = 0$ automatically implies the condition $\text{div} \sigma^e \mathbf{v}_z = 0$. Application to the stationary Maxwell system is not discussed.

Equation $(+)$ is not included in our scheme. It is impossible to apply Theorem 4.4.1, moreover, the homogenization rule for the lower term given by this theorem does not correspond to the acoustic rule.

3. Homogenization for the stationary and non-stationary Maxwell systems have been studied in [ZhKO]. In both cases, the solenoidal conditions were not included in the system. The weak $L^2$-convergence of solutions to the solution of the limit equation was established. Here with, both electric permittivity and magnetic permeability were homogenized according to the acoustic rule. The earlier and weaker results about the non-stationary Maxwell system can be found in [Sa, BeLP].

4. Recall that, in our approach to the homogenization problem for the Maxwell system, two simplifying assumptions have been made: 1) $\mu(\mathbf{x}) = 0$, and 2) solenoidal conditions are included in the system. Besides, representation of the fields as a sum of two summands was rather useful.

**Concluding remarks**

1. As compared with [BSu2], in the present paper the abstract basis enriched. Besides the fact that all constants in estimates are carefully controlled, new Theorems 1.5.8, 1.5.9 appear. As applied to the homogenization, this leads to advantageous Theorems 4.2.5, 4.2.6.

2. As it has been already mentioned, in the homogenization theory, it is more traditional to consider, e.g., the equations of the form $\mathcal{A}(\mathbf{g}, f) = \mathbf{F}$ in a fixed bounded domain $\mathcal{O}$ with appropriate boundary conditions. As compared with the case of the whole $\mathbb{R}^d$, there are differences. From one side, the compactness of the embedding of $H^1(\mathcal{O})$ into $L^2(\mathcal{O})$ may improve matters. From the other side, the effects near $\partial \mathcal{O}$ may hinder a good estimate for the difference $\mathbf{u}_z - \mathbf{u}_\parallel$ (cf. [ZhKO]).

3. The homogenization problems may be related not only to the lower edge of the spectrum, but also to the edges of internal gaps. Then the shift to the high energy area is necessary. Thus, the homogenization effects start to interact with high-energy ones. For the simplest model, these factors were clarified in the recent paper [B3].
4. During all the text, the authors tried to distinguish the cases where the weak convergence turns into the strong one. It is no need to explain that the weak limit is much weaker than the strong one. In this connection, mention that, e.g., in the problem considered in [B3], the weak limit always exists, but is equal to zero. In this case, those approximations of solutions that depend on \( \varepsilon \) themselves are informative.

5. In the homogenization theory, the specific methods are well developed. The idea to apply the analytic perturbation theory methods (on the basis of the Floquet-Bloch decomposition) stands here by itself. However, this very idea is not new at all. Some material of this kind can be found in [BelP, Ch. 4] and [ZhKO, Ch. 2]. Apparently, this way was not employed consistently and intensively enough. We think that there was a lack of the exactly defined notion of a threshold effect. Also, the class of operators admitting an appropriate factorization was not distinguished. Finally, the abstract operator theory basis of threshold effects was not analyzed. By these reasons, the present paper has almost no direct intersections with papers of other authors that used the Floquet-Bloch decomposition and the perturbation theory. Among such papers, we have already mentioned the remarkable paper [Zh]. Mention also the paper [Se], where the acoustic operator with \( g \in C^\infty \) and \( d \geq 3 \) was considered. By means of the perturbation theory, for the solution of the equation \( \mathbf{D}^*g(\varepsilon^{-1}\mathbf{x})\mathbf{D}u = F \in C_0^\infty(\mathbb{R}^d) \), the full asymptotic expansion in powers of \( \varepsilon \) was obtained. In [Cv], for the acoustic operator, formula (5.1.9) (in our notation) was proved, where the left-hand side was defined in terms of the perturbation theory, and in the right-hand side \( g^0 \) was the effective matrix arising in the classical homogenization theory. In all these papers \( n = 1 \), which simplifies considerations significantly.

6. The limits of applicability of the method. The number of examples could be substantially extended. Besides periodic operators in \( \mathbb{R}^d \), on the same abstract basis, we can study periodic problems in the domains of cylinder or layer type, etc. In such problems, the momentum dimension is less than the coordinate dimension. This leads to new phenomena in the study of the effective characteristics and to other technical difficulties. A substantive example of such type is studied in [Su].

Now, we mention the cases where our method is not applicable.

1) Consider the matrix Schrödinger operator \( H_p = -\Delta + p(\mathbf{x}) \) in \( L_2(\mathbb{R}^d; \mathbb{C}^n) \), where \( p \) is a periodic Hermitian \( (n \times n) \)-matrix-valued potential. Now we do not have the analogue of the factorization (6.1.6). Assume that \( \lambda = 0 \) is the lower edge of the spectrum of \( H_p \), and let \( H_p(\mathbf{k}) \) be the corresponding operator in \( L_2(\Omega; \mathbb{C}^n) \). Now, the point \( \mathbf{k} = 0 \) does not play a distinguished role. It may happen that \( \text{Ker} H_p = \{0\} \). As a result, the possibility to study the threshold properties efficiently is lost.

2) The periodic Pauli operator for \( d = 3 \) is defined by the formula \( D_3 = (D_1 - A_1)\sigma_1 + (D_2 - A_2)\sigma_2 + (D_3 - A_3)\sigma_3 \). For this operator, there is no analogue of the factorization (6.2.12). Moreover, the kernel \( \text{Ker} D_3(0) \) cannot be described in a reasonable way.

3) The magnetic periodic Schrödinger operator is defined by the formula \( \mathcal{M} = (\mathbf{D} - \mathbf{A}(\mathbf{x}))^*(\mathbf{D} - \mathbf{A}(\mathbf{x})) \), \( d \geq 2 \), where \( \text{div} \mathbf{A} = 0 \), \( \int_{\Omega} \mathbf{A} d\mathbf{x} = 0 \). Now the lower edge of the spectrum is a positive number. After shifting the lower edge to zero, the factorization is lost. The question, whether or not one can reconstruct the factorization, is not simple. If the magnetic potential is not too large, the answer is positive. In the opposite case, serious complications may occur even for \( d = 2 \).
The corresponding analysis was recently fulfilled in [Sh].

References


