Constant Mean Curvature Slices
in the Extended Schwarzschild Solution
and Collapse of the Lapse. Part II

Edward Malec
Niall Ó Murchadha


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Constant mean curvature slices in the extended Schwarzschild solution and collapse of
the lapse. Part II

Edward Malec*,** and Niall Ó Murchadha*,†
* ESI, A-1090 Wien, Boltzmanngasse 9, Austria.
† Institute of Physics, Jagiellonian University, 30-059 Cracow, Reymonta 4, Poland.
Institute of Physics, Jagiellonian University, 30-059 Cracow, Reymonta 4, Poland.
Institute of Physics, Jagiellonian University, 30-059 Cracow, Reymonta 4, Poland.

An explicit CMC Schwarzschildian line element is derived near the critical point of the foliation,
the lapse is shown to decay exponentially, and the coefficient in the exponent is calculated.

A. Introduction

This is a sequel of our previous work on the constant mean curvature (CMC) slices of the extended Schwarzschild
geometry. Here we get CMC foliations by solving Einstein equations in a particular gauge. A crucial role is played
by a condition (Eq. (5) below) imposed on the lapse. While this method is completely equivalent to the other more
geometric approach [see [1]], it seems to be more straightforward and technically simpler. We focus on the concise
derivation of the explicit CMC foliation near the critical point of the CMC foliation. The final result is identical to
the result derived in [1].

The constant mean curvature foliations have been recently investigated numerically in the simulation of a single
spherically-symmetric black hole [2]. We hope that our analytic results appear helpful in the verification of the
numerical schemes.

B. CMC slicing of the Schwarzschild spacetime

The notation is the same as in the preceding paper [1]. We define

\[(pR)^2 = 4 \left[ 1 - \frac{2m}{R} + \left( \frac{KR}{3} - \frac{C}{R^2} \right)^2 \right], \quad (1)\]

\[\gamma(R,t) = 1 + 8\delta R C \int_0^\infty dr \frac{1}{r^2 p^2}, \quad (2)\]

and

\[N = \gamma \frac{pR}{2}. \quad (3)\]

Here \(m\) is the mass, \(K\) (the trace of the extrinsic curvature) is a constant and \(C\) is a time-dependent parameter which
measures the transverse part of the extrinsic curvature.

The Schwarzschild line element, expressed in terms of coordinates adapted to the constant mean curvature foliation,
is given by [3]

\[ds^2 = -dt^2 \left( N^2 - \gamma^2 \left( \frac{KR}{3} - \frac{C}{R^2} \right)^2 \right) + 4N \frac{C}{r^2 R} dt dR + \frac{4}{(pR)^2} dR^2 + R^2 d\Omega^2. \quad (4)\]

The hypersurfaces of constant time are CMC slices, asymptotic to the CMC slices of Minkowskian geometry.
C. Elliptic slicing condition

A minimal surface is a locus of points defined by the condition \( p = 0 \). Choose a CMC Cauchy hypersurface \( \Sigma_C \) of the extended Schwarzschild manifold corresponding to a parameter \( C \) and let \( R_0 \) be an areal radius corresponding to a simple zero of \( p^2 \); that is \( p(R_0) = 0 \) but \( \partial_R p |_{R_0} \neq 0 \). Furthermore, assume that

\[
\frac{\partial_r N}{\sqrt{a}} |_{R_0} = 0
\]

at \( R_0 \). The condition (5) yields

\[
\dot{\alpha} C = \frac{1}{8I(R_0)}
\]

(6)

Here

\[
I(R_0) \equiv \int_{R_0}^r \frac{d\rho}{p^r} \frac{6C^2 + K^2 \rho^2}{2m + \frac{2KC^2}{3} + \frac{K^2 R_0^2}{9} - \frac{4C^2}{R_0^2}}
\]

(7)

The value of the lapse function \( N \) at the minimal surface, that is at the areal radius \( R_0 \), can be shown to be equal (using Eqs. (1 - 3)) to

\[
N = \frac{dC}{dt} m + \frac{KC^2}{3} + \frac{K^2 R_0^2}{9} - \frac{4C^2}{R_0^2}
\]

(8)

The lapse \( N \) is strictly positive at the minimal surface corresponding to a simple zero \( R_0 \). Eqs. (1 - 3) imply that \( N(R) > N(R_0) \) if \( R > R_0 \) and therefore the lapse exists on all of \( \Sigma_C \). Equation (6) dictates the rate of change of the parameter \( C \). It is clear that one can uniquely construct a foliation of a part of the extended Schwarzschild geometry by imposing the condition (5) at minimal surfaces on all slices to the future of a given one. The leaves of the resulting foliation connect two null infinities of the extended Schwarzschild spacetime. This gives us a curve \( R_0(t) \) of zeroes of the mean curvature \( p \). It is evident, just by inspecting the explicit solution presented above, that the line running along the locations of minimal surfaces \( R_0(t) \) can be arranged to be smooth. It can be chosen to coincide with the ‘vertical’ \( t = 0 \) axis in standard Schwarzschild coordinates.

This construction breaks down when \( R_0 \) ceases to be a simple zero of \( p^2 \), since expressions appearing in Eqs. (6) and (8) become unboundedly large. The goal of this paper is to show the asymptotic behaviour of the lapse at the critical minimal surface.

D. The evolution of \( C \) near critical point

Let \( C_* \) and \( R_* \) be degenerate, that is such that the zero of \( p^2 \) ceases to be simple. In this case both \( p \) and its derivative \( \partial_R p \) vanish; that means that

\[
1 - \frac{2m}{R_*} - \frac{2KC_*}{3R_*} + \frac{K^2 R_*^2}{9} + \frac{C^2}{R_*^2} = 0,
\]

\[
2m + \frac{2KC_*}{3} + \frac{K^2 R_*^2}{9} - \frac{4C^2}{R_*^2} = 0.
\]

(9)

One can easily show, if \( C_* \) and \( R_* \) are critical, then the sign of

\[
\beta \equiv -2C_* + \frac{2}{3}K R_*^2
\]

is the same as the sign of \( -C_* \).

There exists critical values of \( C_* \) that are positive \( (C^+_* ) \) or negative \( (C^-_* ) \). For definiteness we shall consider only the case when \( C(t = 0) > C^-_* \), therefore the only limiting case we consider is that with \( C \rightarrow C^+_* \). (That choice corresponds to a foliation formed by leaves connecting two null infinities which moves forward in time - see a discussion
in Sec IV in [1]). For simplicity we will drop the \( + \) suffix and \( C_* \) will mean a positive critical parameter. From the dynamical equation (6) follows that \( C \) can only increase.

Next, let us introduce the notation that

\[
\begin{align*}
\epsilon & \equiv C_* - C \\
R_0 & \equiv R_* + \delta.
\end{align*}
\] (11)

where both \( \delta \) and \( \epsilon \) are positive and small.

The equation \( p(R_0) = 0 \) yields a nonlinear algebraic equation whose truncation gives

\[
\delta^2 A + \epsilon \beta = 0. \tag{12}
\]

Here \( A \equiv 2R_*^2 + K^2 R_*^4 \). Eq. (12) is in fact the Lyapunov - Schmidt reduced equation constructed according to the standard rules [4]. Therefore in the vicinity of the critical point we have

\[
\delta = \sqrt{-\frac{\epsilon \beta}{A}} \tag{13}
\]

The function \( p \) can be expressed in a form

\[
\frac{p}{2} = \sqrt{1 - \frac{R_0}{r}} \left[ \frac{\kappa \delta}{R_0} + \frac{K^2}{9r} (r R_0 + r^2 - 2 R_*^3) - \frac{C^2}{R_*^3} \left( \frac{R_0}{r^2} + \frac{R_0^3}{r^3} + \frac{R_0^3}{r^3} - 3 \right) \right]^{1/2} \tag{14}
\]

The insertion of (11), (13) and (14) into the equation (7) and the change of the integration variable to \( y = \sqrt{1 - \frac{R_0}{r}} \), yield after a simple but tedious algebra

\[
I(R_0) \approx \sqrt{R_*} \int_0^1 \frac{dy}{\sqrt{y (\kappa \delta + y^2 F_2(\kappa \delta + y^2 F_3)^2)}}. \tag{15}
\]

Here the functions \( F_1, F_2 \) and \( F_3 \) are given by

\[
\begin{align*}
F_1(y) &= \frac{K^2 R_*^3}{3(1 - y^2)^4} + \frac{6C^2}{R_*^3}(1 - y^2)^2, \\
F_2(y) &= \frac{K^2 R_*^3}{9(1 - y^2)^2} + \frac{C^2}{R_*^3}(6 - 4y^2 + y^4), \\
F_3(y) &= (3 - 3y^2 + y^4) \left( \frac{2K^2 R_*^3}{9(1 - y^2)^2} + \frac{4C^2}{R_*^3} \right), \tag{16}
\end{align*}
\]

while \( \kappa \) reads

\[
\kappa = \frac{2K^2 R_*^2}{3} + \frac{12C^2}{R_*^4}. \tag{17}
\]

E. Limiting behaviour of the foliation.

The asymptotic behaviour of \( C \) will be dominated by the \( 1/\delta^2 \) part of \( I(R_0) \). As will be shown later, \( C \) tends exponentially to \( C_* \); the attenuation factor in the exponent depends only on the leading term of \( I(R_0) \). It is useful to define \( z = y/\sqrt{\kappa \delta} \). Then one obtains \( I(R_0) = \sqrt{\kappa \delta} \times I_d \), where

\[
I_d = \int_0^{1/\sqrt{\kappa \delta}} \frac{dz}{\sqrt{1 + \frac{z^2}{\kappa \delta} F_1(\sqrt{\kappa \delta} z)(1 + \frac{z^2}{\kappa \delta} F_2(\sqrt{\kappa \delta} z))^2}}. \tag{18}
\]

One can split the integral \( \int_0^{1/\sqrt{\kappa \delta}} \) into two parts: \( \int_0^{1/\sqrt{\kappa \delta}} = \int_0^{1/\sqrt{\kappa \delta}} + \int_{1/\sqrt{\kappa \delta}}^{1/\sqrt{\kappa \delta}} \). It is easy to check that the contribution coming from the second integral goes to zero as \( \delta \) approaches zero. Therefore \( F_1 \approx R_* \kappa/2, F_2 \approx R_* \kappa/2 \) and \( F_3 \approx R_* \kappa \). Thus the first integral (and also the integral \( I_d \)) is well approximated by
\[ I = \frac{k R_*}{2} \int_0^\infty \frac{dz}{\sqrt{1 + \frac{k R_*}{2} z^2 (1 + R_* k z^2)^2}} = \frac{\sqrt{k R_*}}{2} \int_0^\infty \frac{dz}{\sqrt{1 + \frac{k R_*}{2} z^2 (1 + z^2)^2}}. \]

(19)

The integral \( I_z = \int_0^\infty dz \frac{1}{\sqrt{1 + \frac{k R_*}{2} (1 + z^2)^2}} \) can be explicitly evaluated and gives \( \sqrt{2}/2 \).

In summary, near the critical point we have

\[ I(R_\theta) = \frac{\sqrt{k R_*}}{k^2 \beta^2} I \approx \frac{\sqrt{2}}{4} \frac{R_* A}{\kappa^3/2 |\beta|} \]

(20)

The insertion of (11) and (20) into (6) yields

\[ \partial_t \epsilon = -\Gamma \epsilon, \]

(21)

where

\[ \Gamma = \frac{|\beta| \kappa^3/2}{2 \sqrt{2} A R_*} \]

(22)

Eq. (21) immediately implies that \( \epsilon \) approaches 0 exponentially as

\[ \epsilon(t) = \epsilon_0 e^{-\Gamma t}, \]

(23)

where \( \epsilon_0 \) is an initial value of the parameter. Taking into account relations (11) and (13), one can conclude that the parameter \( C \) and the minimal radius \( R_\theta \) tend exponentially to their critical values \( C_* \) and \( R_* \), respectively.

Finally, collecting the above information and putting it into Eq. (8), we obtain the asymptotic behaviour of the lapse function near the critical point

\[ N = N_0 e^{-\Gamma t/2}. \]

(24)

This is exactly the same result as that obtained in [1]; in order to show equivalence, use expressions for the extrinsic curvatures (valid in the upper quadrant of the extended Schwarzschild geometry), which imply \( K = (2mR_* - 3m)/\sqrt{2mR_* - R_*^2} \) and \( C_* = (3mR_*^2 - R_*^4)/\sqrt{2mR_*^2 - R_*^4} \). In the case of maximal slicing \( (K = 0) \) the decay constant \( \Gamma/2 \) is equal to \( 4/(3\sqrt{2}) \), in agreement with the analytic derivation of [5] and close to the numerical result of [6]. The asymptotic behaviour of \( \gamma \) and \( p \) in a region close to the line \( R_\theta \) is given by

\[ \gamma = \gamma_0 \frac{e^{-\pi t}}{\sqrt{1 - \frac{R_*}{r_\theta}}}, \]

\[ \frac{pr}{2} = p_0 \sqrt{1 - \frac{R_*}{r_\theta}} e^{\pi t} \]

(25)

Here \( \gamma_0 \) and \( p_0 \) are initial values of \( \gamma \sqrt{1 - \frac{R_*}{r_\theta}} \) and \( pr/(2 \sqrt{1 - \frac{R_*}{r_\theta}}) \), respectively. The four constants \( (\epsilon_0, N_0, \gamma_0 \) and \( p_0) \) can be expressed in terms of one free parameter (say, \( \epsilon_0) \) and \( A, \beta, \kappa \). Equations (3), (15), (23) - (25) and \( C = C_* - \epsilon_0 e^{-\Gamma t} \) suffice to construct the metric (4) near the line \( R_\theta(t) \) of minimal surfaces.

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[3] M. Iriondo, E. Malec and N. Ø Murchadhha, gr-qc/9503030; notice that our $C$ is equal to $C_1/2$ therein. The published version, Phys. Rev. D54, 4792 (1996), contains i) a misprint in equation (52), which should read $\partial_t \left( R^2 (K' - K/3) \right) = 4\partial_t C_1$; ii) instead of $C_2(R, t)$ should be $C_2(t)$.

