A Convergence Theorem for the Standard Luria–Delbrück Distribution

M. Möhle

A CONVERGENCE THEOREM FOR THE
STANDARD LURIA-DELBRÜCK DISTRIBUTION

M. Möhle\textsuperscript{1} Mathematisches Institut, Eberhard Karls Universität Tübingen,
Auf der Morgenstelle 10, 72076 Tübingen, Germany

Abstract
A new scaling for the standard discrete Luria-Delbrück distribution is provided which leads to a weak convergence result as the parameter of the distribution tends to infinity. We show that the stable limiting probability measure has the Fourier transform \( t \mapsto \exp(-\pi t^2/2 - it \log |t|) \). For the corresponding density an integral representation is derived, which differs from that found in a closely related paper of Kepler and Oprea [4]. In addition, we indicate how the approach is connected to more general compound Poisson distributions.

Keywords: Compound Poisson distribution; Continuous Luria-Delbrück distribution; Fourier transform; Stable distribution; Weak convergence

AMS 2000 Mathematics Subject Classification: Primary 60F05; 92D15
Secondary 60E10; 92D10

1 Introduction

One way to define the standard Luria-Delbrück (LD) distribution with parameter \( m > 0 \) is via its generating function

\[
\sum_{n=0}^{\infty} p_n s^n := (1 - s)e^{m(1-s)/s}.
\]  

The probabilities \( p_n = p_n(m) \) depend on \( m \). Models in population biology which lead to the standard LD distribution are for example described in Kemp [3] or Lea and Coulson [6]. The name of this distribution goes back to the original work of Salvador Luria and Max Delbrück [8]. The LD distribution is of interest in biological applications, as it provides a basis for procedures which estimate mutation rates. Unfortunately, the probabilities \( p_n, n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\} \) of the LD distribution are not simple to compute, in particular for large parameter \( m \). Ma. et al [9] found the recursion \( p_0 := e^{-m} \) and

\[
p_n = \frac{m}{n} \sum_{i=1}^{n} \frac{p_{n-i}}{i+1} \quad \text{for } n \in \mathbb{N} := \{1, 2, \ldots\}.
\]

Based on this recursion properties of the LD distribution can be derived. For example, it was shown that for each fixed parameter \( m \) the asymptotics \( \lim_{n \to \infty} n(n+1)p_n(m) = m \) holds. For more details on the tail behavior we refer to Kemp [3] and Prodinger [10].

\textsuperscript{1}E-mail address: martin.moehle@uni-tuebingen.de
In contrast to the situation where \( m \) is fixed and \( n \) is large, we are interested in the asymptotic behavior of the LD distribution for large parameter \( m \). An important question is whether or not the LD distribution, properly scaled, converges to some limiting distribution as the parameter \( m \) tends to infinity. The scaling is certainly not simple to find, as the moments of the LD distribution do not exist. The paper of Kepler and Oprea [4] is closely related to this question and already indicates that there might be a positive answer to this problem.

Our main convergence theorem (Theorem 4.1) provides the appropriate scaling and characterizes the limiting distribution via its Fourier transform. Due to the fact that the LD distribution is a special compound Poisson distribution, we start with a few results (Section 2) on convergence of them. In Section 3 more details on the LD distribution are presented which are needed in order to verify the main convergence theorem presented in Section 4. The paper finishes with a discussion in Section 5.

2 Convergence of compound Poisson distributions

Compound Poisson distributed random variables are (by definition) of the form

\[
Y_m := \sum_{k=1}^{K} X_k,
\]

where \((X_k)_{k \in \mathbb{N}}\) is a sequence of iid random variables, and \(K\) is a Poisson random variable with parameter \( m > 0 \), which is independent of the sequence \((X_k)_{k \in \mathbb{N}}\).

The Fourier transform \( \varphi_m : IR \rightarrow \mathbb{C}^\ast \) of \( Y_m \) is \( \varphi_m(t) := \mathbb{E}(e^{itY_m}) = e^{m(\varphi(t)-1)} \),

where \( \varphi \) denotes the Fourier transform of \( X := X_1 \). Compound Poisson distributions appear in many situations and they have been the object of intensive research. For example, 'compound Poisson approximation' and 'large deviation principles' are two of these fields. In this paper we are simply interested in the behavior of \( Y_m \) for large parameter \( m \). The following convergence theorem (Theorem 2.1) is the analog of the classical law of large numbers and the classical central limit theorem. It states in particular, that \( Y_m \) is asymptotically normal, provided that \( X \) has a finite and non-vanishing second moment. Although the result is well known from the literature, we present a simple proof based on Fourier analysis, as this technique will be used again later in this paper.

**Theorem 2.1**

a) If \( \mathbb{E}(|X|) < \infty \), then \( Y_m / m \) converges in probability to \( \mathbb{E}(X) \).

b) If \( 0 < \mathbb{E}(X^2) < \infty \), then \( Y_m \) is asymptotically normal, i.e.

\[
\frac{Y_m - m \mathbb{E}(X)}{\sqrt{m \mathbb{E}(X^2)}}
\]

converges in distribution to the standard normal distribution.
Remark. Due to the fact that $Y_m$ is a random sum of iid variables, the scaling variance $\text{Var}(Y_m) = m \text{E}(X^2)$ in part b) of the theorem is larger than the value $m \text{Var}(X)$, which one would expect from the classical central limit theorem for a deterministic sum of iid variables.

Proof.

a) Fix $t \in \mathbb{R}$ and consider the Fourier transform of $Y_m/m$. Obviously $\text{E}(e^{itY_m/m}) = e^{m(\varphi(x) - 1)}$ with $x = x(m) := t/m$. From $\dot{x} := dx/dm = -t/m^2$ and L’Hospital’s rule conclude that

$$m(\varphi(x) - 1) = \frac{\varphi(x) - 1}{m} \sim \frac{\dot{x}\varphi(x)}{m^2} = t\varphi(x) \sim t\varphi'(0) = it\text{E}(X).$$

Thus $\text{E}(e^{itY_m/m})$ converges to $e^{it\text{E}(X)}$ as $m$ tends to infinity for each $t \in \mathbb{R}$. This point-wise convergence of the Fourier transforms is equivalent to the convergence of $Y_m/m$ in distribution to $\text{E}(X)$. As the limiting random variable is constant this is equivalent to the convergence in probability.

b) Fix $t \in \mathbb{R}$ and define $\mu_m := m \text{E}(X)$ and $\sigma_m^2 := m \text{E}(X^2)$ for convenience. Obviously

$$\text{E}(e^{it(Y_m-\mu_m)/\sigma_m}) = \exp\left(\frac{\mu_m}{\sigma_m}it + m\left(\frac{t}{\sigma_m}\varphi(\frac{t}{\sigma_m}) - 1\right)\right).$$

Taylor expansion of $\varphi$ together with $\varphi'(0) = i\text{E}(X)$ and $\varphi''(0) = -\text{E}(X^2)$ yields

$$-\frac{\mu_m}{\sigma_m}it + m\left(\frac{t}{\sigma_m}\varphi\left(\frac{t}{\sigma_m}\right) - 1\right)$$

$$= -\frac{\mu_m}{\sigma_m}it + m\left(\varphi'(0)\frac{t}{\sigma_m} + \varphi''(0)\frac{t^2}{2\sigma_m^2} + o(\sigma_m^{-2})\right)$$

$$= it\frac{\text{E}(X)}{\sigma_m} - \frac{\mu_m}{\sigma_m}t^2m\text{E}(X^2) = \frac{t^2}{2} + o(1).$$

Thus $\text{E}(e^{it(Y_m-\mu_m)/\sigma_m})$ converges to $e^{-t^2/2}$ as $m$ tends to infinity. This point-wise convergence of the Fourier transforms implies the convergence in distribution.

If the first moment of $X$ does not exist, then the behavior of $Y_m$ for large $m$ can be quite different. The following lemma illustrates this.

Lemma 2.2 Assume that $\text{E}(|X|) = \infty$. If there exists a constant $c \in \mathbb{R}$ such that the Fourier transform $\varphi$ of $X$ satisfies the condition

$$\lim_{t \to 0} t\varphi''(t) = -ic,$$  \hfill (3)

then the scaled random variable $Z_m := Y_m/(m \log m)$ converges as $m$ tends to infinity in probability to $c$, i.e. $\lim_{m \to \infty} P(|Z_m - c| \geq \varepsilon) = 0$ for all $\varepsilon > 0$. 

3
**Proof.** It is sufficient to verify the point-wise convergence of the corresponding Fourier transforms. Thus for \( t \in \mathbb{R} \) consider \( E(e^{itX_m}) = \exp(m(\varphi(x) - 1)) \) with \( x = x(m) := t/(m \log m) \). Note that \( m \) tends to infinity if and only if \( x \) tends to zero. Moreover
\[
\dot{x} := \frac{dx}{dm} = -\frac{t(1 + \log m)}{(m \log m)^2} \sim -\frac{t}{m^2 \log m} = -\frac{x}{m}.
\]
L'Hospital's rule implies that
\[
m(\varphi(x) - 1) = \frac{1 - \varphi(x)}{-\frac{x}{m}} \sim \frac{-\dot{x}\varphi'(x)}{-\frac{x}{m^2}} = -m^2 \frac{\dot{x}\varphi'(x)}{m} \sim \frac{t\varphi'(x)}{m}.
\]
Applying L'Hospital's rule once more yields
\[
m(\varphi(x) - 1) \sim \frac{t\dot{x}\varphi''(x)}{m} \sim -tx\varphi''(x) \sim itc,
\]
where in the last step the condition (3) has been used. \( \Box \)

In the following section we focus on a special random variable \( X \) which satisfies the condition (3) in Lemma 2.2 with \( c = 1 \). The corresponding compound Poisson distribution is the LD distribution.

### 3 The Luria-Delbrück distribution

The LD distribution with parameter \( m > 0 \) is the distribution of a special compound Poisson distributed random variable \( Y_m \) of the form (2), where the random variable \( X := X_1 \) takes the integer values \( k \in \mathbb{N} \) with probability
\[
a_k := P(X = k) := \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.
\]
In order to see this not that \( X \) has the generating function
\[
h(s) := \sum_{k=1}^{\infty} a_k s^k = \sum_{k=1}^{\infty} \frac{s^k}{k(k+1)} = 1 + \frac{(1-s)\log(1-s)}{s}.
\]
From \( E(s^{Y_m}) = e^{m(h(s)-1)} \) it follows that \( Y_m \) has the generating function (1). For our purpose it is necessary to study the Fourier transform \( \varphi \) of \( X \). Obviously
\[
\varphi(t) := E(e^{itX}) = \sum_{k=1}^{\infty} a_k e^{itk}
\]
\[
= \sum_{k=1}^{\infty} \frac{e^{itk}}{k(k+1)} = \sum_{k=1}^{\infty} \frac{\cos(tk)}{k(k+1)} + i \sum_{k=1}^{\infty} \frac{\sin(tk)}{k(k+1)}, \quad t \in \mathbb{R}.
\]
Using the principal branch of the complex logarithm log \( z := \log(|z|) + i \arg z \), where the log on the right hand side of the equation denotes the usual real valued logarithm and \(-\pi < \arg z < \pi\), it follows that
\[
\varphi(t) = 1 + \frac{(1-e^{it}) \log(1-e^{it})}{e^{it}}
\]
for all \( t \in \mathbb{R} \) with the convention \( 0 \log 0 := 0 \) for \( t = 0 \). After some algebraic manipulation the derivatives turn out to be
\[
\varphi'(t) = -i \left( 1 + \frac{\log(1-e^{it})}{e^{it}} \right)
\]
and
\[
\varphi''(t) = -\frac{1}{1-e^{it}} - \frac{\log(1-e^{it})}{e^{it}} = -\frac{e^{it} + \log(1-e^{it})}{e^{it}},
\]
where \( t \in \mathbb{R} \setminus \{0\} \). Obviously \( \varphi' \) and \( \varphi'' \) have singularities at \( t = 0 \). L'Hospital's rule implies that
\[
\lim_{t \searrow 0} \frac{te^{it}}{1-e^{it}} = i \quad \text{and} \quad \lim_{t \searrow 0} t \log(1-e^{it}) = 0.
\]
Thus the second derivative of the Fourier transform of \( X \) fulfills the condition \( \lim_{t \searrow 0} t \varphi''(t) = -i \) and hence Lemma 2.2 is applicable with \( c = 1 \). Therefore \( Y_m/(m \log m) \) converges in probability to one. In the following section a much stronger convergence result (Theorem 4.1) is derived.

4 Convergence to the continuous Luria-Delbrück distribution

As before let \( Y_m \) be a (discrete) L D distributed random variable with parameter \( m > 0 \). We are now able to present our main convergence theorem.

**Theorem 4.1 (Asymptotics of the L D distribution)**

The normalized random variable
\[
\frac{Y_m - m \log m}{m} = \frac{Y_m}{m} - \log m
\]
converges as \( m \) tends to infinity in distribution to a stable limiting probability measure \( Q \) on \((\mathbb{R}, \mathcal{B})\) uniquely determined via its Fourier transform
\[
\varphi_Q(t) := \int_{\mathbb{R}} e^{itx}Q(dx) = (-it)^{-it} = \exp\left(-\frac{t^2}{2}|t|-it \log |t|\right), \quad t \in \mathbb{R}. \quad (4)
\]

**Remarks.** Define \( \mu_m := m \log m \) and \( \sigma_m := m \) for convenience. The theorem ensures that
\[
\lim_{m \to \infty} \sup_{x \in \mathbb{R}} |P(Y_m \leq \sigma_m x + \mu_m) - F(x)| = 0,
\]
where $F$ denotes the distribution function of $Q$. Thus uniformly in $k$ the approximation

$$P(Y_m \leq k) \approx F(k/m - \log m)$$

holds for large $m$. The probability measure $Q$ is hence very useful to approximate the discrete LD distribution. It is shown in the following corollary that $Q$ has a density with respect to the Lebesgue measure. Therefore it is natural to call $Q$ the standard continuous Luria-Delbrück distribution.

Due to the fact that the moments of the (discrete) LD distribution do not exist, the normalizing sequences $\mu_m = m \log m$ and $\sigma_m = m$ differ quite dramatically from the “usual” sequences ($\mu_m = m E(X)$ and $\sigma_m = \sqrt{m}$) known from Theorem 2.1(b). The main difficulty in finding the theorem was to discover the correct normalization. After the scaling was found, the proof (see below) became straightforward via a simple Fourier analysis argument.

From

$$Z_m := \frac{Y_m}{m \log m} = \frac{1}{\log m} \frac{Y_m - m \log m}{m} + 1,$$

we conclude by an application of Slutsky’s theorem, that the convergence in Theorem 4.1 is stronger than the convergence $Z_m \to 1$ in probability.

**Proof.** (of Theorem 4.1) Fix $t \in \mathbb{R}$ and define $z := e^{it} / m$ for convenience. The Fourier transform of $Y_m / m - \log m$ is

$$E(e^{it(Y_m/m - \log m)}) = e^{-it \log m} E(e^{itY_m/m}) = e^{-it \log m} m \varphi(t \mid m \log m - 1) = e^{-it \log m + m \varphi(t \mid m \log m - 1)} = e^{-it \log m + m(1 - z) / z \log(1 - z)}.

Note that $z = 1 + O(1/m)$ and $m(1 - z) = -it + O(1/m)$. Thus

$$-it \log m + m(1 - z) / z \log(1 - z) \sim -it \log m + m(1 - z) \log(1 - z) \sim -it \log m - it \log(1 - z) = -it \log |m(1 - z)| \sim -it \log(-it).

Obviously $\arg(-it) = \pm \pi / 2$ and hence

$$-it \log(-it) = -it \log |t| + i \arg(-it) = -it \log |t| \pm \frac{\pi}{2} |t|.

Thus the Fourier transform of $Y_m / m - \log m$ converges point-wise as $m$ tends to infinity to the continuous function $t \mapsto e^{-at \log(-it)} = e^{-\frac{\pi}{2} a |t| \log |t|}$. The continuity theorem for Fourier transforms ensures that this function is a Fourier transform of a certain probability measure $Q$ on $(\mathbb{R}, \mathcal{B})$ and that $Y_m / m - \log m$ converges in distribution to $Q$. Standard results for characteristic functions (see for example [7, Theorem 5.7.3]) ensures that $Q$ is a stable distribution with exponent $\alpha = 1$. 

6
Remark. (Laplace transform of the standard continuous LD distribution)
A similar argument as in the previous proof shows that the continuous Luria Delbrück distribution $Q$ has the Laplace transform

$$\psi_Q(\lambda) := \int_{\mathbb{R}} e^{-\lambda x} Q(dx) = e^{\lambda \log \lambda} = \lambda^\lambda, \quad \lambda \geq 0,$$

in agreement with the formal substitution $\lambda \leftrightarrow -it$ in the Fourier transform (4).

**Corollary 4.2** (Density of the standard continuous LD distribution)
The continuous LD distribution is absolutely continuous with density

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\lambda x} \varphi_Q(t) \lambda(dt), \quad x \in \mathbb{R}. \tag{5}$$

In particular, $f(x) \leq 2/\pi^2 \approx 0.2026$ for all $x \in \mathbb{R}$.

Proof. Obviously $|\varphi_Q(t)| = \exp(-\frac{\pi}{2} t^2)$ and hence $\int_{\mathbb{R}} |\varphi_Q(t)| \lambda(dt) = 4/\pi < \infty$. Therefore the Fourier inversion theorem ensures that

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\lambda x} \varphi_Q(t) \lambda(dt), \quad x \in \mathbb{R},$$

is a density of $Q$. Plugging in $\varphi_Q$, splitting up the integral into the two integrals over the negative and positive part of the real axis, and substituting $t \leftrightarrow -t$ in the second integral, yields

$$f(x) = \frac{1}{2\pi} \int_0^\infty e^{-\lambda x} e^{\frac{\pi}{2} t^2} e^{i t \log t} dt + \frac{1}{2\pi} \int_0^\infty e^{\lambda x} e^{\frac{\pi}{2} t^2} e^{-i t \log t} dt$$

$$= \frac{1}{2\pi} \int_0^\infty e^{-\frac{\pi}{2} t^2} (e^{-i(x t + t \log t)} + e^{i(x t + t \log t)}) dt$$

$$= \frac{1}{\pi} \int_0^\infty e^{-\frac{\pi}{2} t^2} \cos(xt + t \log t) dt.$$

The rest follows from $f(x) \leq \pi^{-1} \int_0^\infty e^{-\frac{\pi}{2} t^2} dt = 2/\pi^2$. \hfill \Box

**Definition 4.3** Let $a \in \mathbb{R}$, $b > 0$ and let $F$ denote the distribution function of the standard continuous LD distribution. The probability measure which corresponds to the linear transformed distribution function $F_{a,b}(x) := F((x-a)/b)$ is called the continuous LD distribution with parameters $a$ and $b$. We denote this probability measure by $Q_{a,b}$.

Remark. From Theorem 4.1 it follows that the probability measure $Q_{a,b}$ has the Fourier transform

$$\varphi_{a,b}(t) = \exp \left( -\frac{\pi |t|}{b} - \frac{t}{b} \left( a + \log \frac{|t|}{b} \right) \right).$$
As the standard continuous LD distribution is stable, it follows that for all parameters \( a_1, a_2 \in IR \) and \( b_1, b_2 > 0 \) there exists two other parameters \( a \in IR \) and \( b > 0 \) such that the convolution property

\[
Q_{a_1, b_1} \ast Q_{a_2, b_2} = Q_{a, b}
\]

holds. The following corollary determines \( a \) and \( b \) explicitly.

**Corollary 4.4** Let \( a_1, a_2 \in IR \) and \( b_1, b_2 > 0 \). The continuous LD distribution satisfies the convolution property (6) with

\[
b := \left( \frac{1}{b_1} + \frac{1}{b_2} \right)^{-1} = \frac{b_1 b_2}{b_1 + b_2}
\]

and

\[
a := b \left( \frac{a_1}{b_1} + \frac{a_2}{b_2} - \frac{\log b_1}{b_1} - \frac{\log b_2}{b_2} \right) + \log b = \frac{a_1 b_2 + a_2 b_1 - b_1 \log b_2 - b_2 \log b_1 + \log b}{b_1 + b_2}.
\]

**Proof.** This follows from the equivalent equation \( \varphi_{a_1, b_1}(t) \varphi_{a_2, b_2}(t) = \varphi_{a, b}(t), \ t \in IR \) for the corresponding Fourier transforms, which can be verified easily. □

## 5 Final remarks, discussion and open problems

The density \( f \) is well known from the literature. Kepler and Oprea [4, p. 46, eqn. (32)] discovered \( f \) using a different approach starting with a class of discrete LD distributions with characteristic functions originally derived by Bartlett [2]. They showed that

\[
f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-x^2 - t \log t} \sin(\pi t) \, dt, \quad x \in IR.
\]

This result follows also from standard theory on characteristic functions, for example from Eqn. (5.8.15) in Lukacs [7] with \( \beta = 1 \) and \( \gamma = 0 \).

Both integral representations (5) and (7) can be integrated numerically. We prefer (5), as the integrand \( e^{-\frac{x^2}{2}} \cos(x t + t \log t) \) is bounded by the decreasing function \( t \mapsto e^{-\frac{x^2}{2}} \) no matter how \( x \) is chosen. Applying Eqn. (5.8.16) in Lukacs [7] shows that \( f \) has the Taylor expansion \( f(x) = 1/\pi \sum_{k=0}^{\infty} a_k x^k, \ x \in IR \) with coefficients

\[
a_k := \frac{(-1)^k}{k!} \int_0^{\infty} t^k \sin(2t) e^{-\frac{t^2}{2} \log t} \, dt.
\]

The novelty in the present paper is the convergence theorem (Theorem 4.1) which connects directly the discrete LD distribution with the continuous LD distribution. The theorem in particular presents the normalizing sequences \( \mu_m = m \log m \) and \( \sigma_m = m \). Moreover, our approach leads to closed forms
for the Fourier transform and the Laplace transform of the continuous LD distribution. The convergence result also differs from the limiting results of Angerer [1]. He considers versions of the standard LD distribution having finite moments which are more appropriate for biological applications, but which lead to discrete limiting distributions.

The density \( f \) together with Theorem 4.1 is quite useful to derive approximative results for the discrete LD distribution with parameter \( m \). For example, the median and the mode are approximately given by

\[
\lfloor m \log m - x_{\text{median}} \rfloor \quad \text{and} \quad \lfloor m \log m - x_{\text{mode}} \rfloor,
\]

where \( x_{\text{median}} \) and \( x_{\text{mode}} \) denote the median and the mode of \( f \). Note that the mode of \( f \) is uniquely determined as all stable distributions are unimodal (Theorem of Ibragimov and Czeern). Numerical analysis shows that \( x_{\text{median}} \approx 1.35 \) and \( x_{\text{mode}} \approx -0.23 \). These values reflect the skewness of the continuous LD distribution.

Theorem 4.1 might be also helpful to estimate the parameter \( m \) of the discrete LD distribution and to provide approximative confidence intervals for \( m \), but we do not want to go into detail here. In this context we also refer to [4, 5].

Numerical computation supports the conjecture that 'local' convergence of the discrete LD distribution to the continuous LD distribution holds, i.e.

\[
\lim_{m \to \infty} \sup_{x \in \mathbb{R}} |m P(Y_m = [mx + m\log m]) - f(x)| = 0.
\]

Unfortunately, this stronger type of convergence could not be verified rigorously.

Acknowledgement. The author is grateful to the Erwin Schrödinger International Institute for Mathematical Physics (ESI) in Vienna for support during a stay in winter 2002/2003, where part of this work was done. In particular, the author thanks Wolfgang Angerer for his talk related to the Luria-Delbrück distribution during a workshop at the ESI in February 2003.

References


