Distance Sets Corresponding to Convex Bodies

Mihail N. Kolountzakis

April 3, 2003

Supported by the Austrian Federal Ministry of Education, Science and Culture  
Available via http://www.esi.ac.at
Distance sets corresponding to convex bodies

Mihail N. Kolountzakis

March 2003

Abstract

Suppose that $K \subseteq \mathbb{R}^d$ is a 0-symmetric convex body which defines the usual norm

$$\|x\|_K = \sup \{ t \geq 0 : x \notin tK \}$$

on $\mathbb{R}^d$. Let also $A \subseteq \mathbb{R}^d$ be a measurable set of positive upper density $\rho$. We show that if the body $K$ is not a polytope, or if it is a polytope with many faces (depending on $\rho$), then the distance set

$$D_K(A) = \{ \|x - y\|_K : x, y \in A \}$$

contains all points $t \geq t_0$ for some positive number $t_0$. This was proved by Katznelson and Weiss, by Falconer and Marstrand and by Bourgain in the case where $K$ is the Euclidean ball in any dimension. As corollaries we obtain (a) an extension to any dimension of a theorem of Iosevich and Laba regarding distance sets with respect to convex bodies of well-distributed sets in the plane, and also (b) a new proof of a theorem of Iosevich, Katz and Tao about the nonexistence of Fourier spectra for smooth convex bodies.

1 Introduction

Suppose that $K \subseteq \mathbb{R}^d$ is a 0-symmetric convex body which defines the usual norm

$$\|x\|_K = \sup \{ t \geq 0 : x \notin tK \}$$

on $\mathbb{R}^d$. Define the $K$-distance set of $A \subseteq \mathbb{R}^d$ as the set of $K$-distances that show up in $A$:

$$D_K(A) = \{ \|x - y\|_K : x, y \in A \}.$$

Katznelson and Weiss [3], Falconer and Marstrand [2] (in the plane) and Bourgain [1] have proved that if $B$ is the Euclidean ball and $A$ has positive upper density, i.e., if there is $\rho > 0$ such that there are arbitrarily large cubes $Q$ in which $A$ has a fraction at least $\rho$ of their measure: $|A \cap Q| \geq \rho |Q|$, then $D_B(A)$ contains all numbers $t \geq t_0$, for some $t_0$. In this paper we study the following question: for which other convex bodies $K$ in place of $B$ is this true? We obtain that if this property (of eventually all numbers showing up in $D_K(A)$) fails for some set $A$ with positive upper density then the body $K$ is necessarily a polytope, and with a number of faces that is bounded above by a number that is inversely proportional to the density of $A$.

*Supported in part by European Commission HPRN Network HARP (Harmonic Analysis and Related Problems), Contract Number: HPRN-CT-2001-00273 - HARP.
To state our results more precisely let us call a measure $\mu \in M(\mathbb{R}^d)$ $\epsilon$-good if for some $R > 0$ we have $|\hat{\mu}(x)| \leq \epsilon$ for $|x| \geq R$. In what follows the Fourier Transform of a measure is defined by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \langle \xi, x \rangle} d\mu(x).$$

**Theorem 1** Suppose that $A \subseteq \mathbb{R}^d$ has upper density equal to $\epsilon > 0$ and that the 0-symmetric convex body $K$ affords $C_d\epsilon$-good probability measures supported on its boundary (the constant $C_d$ depends on the dimension only). Then $D_K(A)$ contains all positive real numbers beyond a point.

This is complemented by the following result.

**Theorem 2** Suppose $K$ is a 0-symmetric convex body. Then

(a) If $K$ is a polytope with $N$ non-parallel face directions then it does not afford $\epsilon$-good probability measures on its boundary if $\epsilon < 1/(\sqrt{2N})$.

(b) If $K$ is a polytope with $N$ non-parallel face directions then it affords $(\frac{1}{N} + \delta)$-good probability measures on its boundary for all $\delta > 0$.

If $K$ is not a polytope then it affords $\epsilon$-good probability measures on its boundary for all $\epsilon > 0$.

Theorems 1 and 2 allow us to prove the following result, a weaker version of which was the motivation for this work and was proved in dimension 2 by losevich and Laba [5]. A set $\Lambda \subseteq \mathbb{R}^d$ is called well-distributed if there is a constant $r > 0$ such that every cube of side $r$ contains at least one point of $\Lambda$. And a set $D \subseteq \mathbb{R}$ is called separated if there exists $\epsilon > 0$ such that $|x - y| \geq \epsilon$ for any two distinct $x, y \in D$.

**Corollary 1** Suppose that $\Lambda$ is a well-distributed subset of $\mathbb{R}^d$ and $K$ is a 0-symmetric convex body. If $D_K(\Lambda) \cap \mathbb{R}^+$ has infinitely many gaps of length at least $\epsilon > 0$ then $K$ is a polytope.

**Remark 1** It was proved in [5] that if $\Lambda$ is a well-distributed set in the plane and $D_K(\Lambda)$ is separated then $K$ is a polygon.

**Remark 2** By taking $K = (-1/2, 1/2)^d$ and $\Lambda = \mathbb{Z}^d$ we see that there are indeed polytopes and well-distributed sets for which $D_K(\Lambda)$ is separated. It is not clear which polytopes can play this role.

**Proof of Corollary 1.** Suppose that the intervals $(x_k, x_k + \epsilon)$, $x_k \to +\infty$, do not intersect $D_K(\Lambda)$. Let $A$ be the subset of $\mathbb{R}^d$ that arises if we put a copy of the body $(\epsilon/10)K$ centered at each point of $\Lambda$. By the fact that $\Lambda$ is well-distributed we obtain that $A$ has positive upper density and for any two points $x, y \in A$, we can write $x = \lambda + x_1$, $y = \mu + y_1$, with $\lambda, \mu \in \Lambda$, $x_1, y_1 \in (\epsilon/10)K$. It follows that $x - y = (\lambda - \mu) + (x_1 - y_1)$ and

$$\frac{1}{5} \epsilon \geq \|x_1 - y_1\|_K \geq |\|x - y\|_K - \|\lambda - \mu\|_K|,$$

which implies that $D_K(A)$ does not intersect the intervals $(x_k + \epsilon/5, x_k + 4\epsilon/5)$, $k = 1, 2, \ldots$, hence $D_K(A)$ cannot contain a half-line. From Theorems 1 and 2 it follows that $K$ is a polytope, and
From Corollary 1 one can easily show that smooth convex bodies do not have Fourier spectra, a fact first proved by losevich, Katz and Tao [4], who used a different approach. A bounded open domain \( \Omega \subseteq \mathbb{R}^d \) is said to have the set \( \Lambda \subseteq \mathbb{R}^d \) as a Fourier spectrum if the collection of exponentials

\[
E(\Lambda) = \{ \exp(2\pi i \langle \lambda, x \rangle) : \lambda \in \Lambda \}
\]

is an orthogonal basis for \( L^2(\Omega) \). It is easy to see from the orthogonality that any two distinct points \( \lambda \) and \( \mu \) of a Fourier spectrum must satisfy

\[
\hat{\chi}_\Omega(\lambda - \mu) = 0,
\]

and that the set \( \Lambda \) is necessarily a well-distributed set.

**Corollary 2** (losevich, Katz and Tao [4])

*If K is a smooth, 0-symmetric convex body it does not have a Fourier spectrum.*

**Proof of Corollary 2.** Suppose \( \Lambda \) is a Fourier spectrum of \( K \). It is a well known fact (see, for example, [4]) that if \( \xi \) is a zero of \( \hat{\chi}_K \) and \( \xi \to \infty \) then

\[
\|\xi\|_{K^*} = \left( \frac{\pi}{2} + \frac{d\pi}{4} \right) + k\pi + o(1), \quad (\xi \to \infty),
\]

where \( K^* \) is the dual body (which is also smooth), \( d \) is the dimension and \( k \) is an integer.

Let \( R > 0 \) be such that any zero \( \xi \) of \( \hat{\chi}_K \), outside a cube of side \( R \) centered at the origin, satisfies

\[
\|\xi\|_{K^*} = \left( \frac{\pi}{2} + \frac{d\pi}{4} \right) + k\pi + \theta, \quad (k \in \mathbb{Z}, |\theta| < \pi/10).
\]

We also take \( R \) to be large enough so as to be certain that we find at least one \( \Lambda \)-point in any cube of side \( R \). (We can do this since \( \Lambda \) is well-distributed.)

Let now the set \( \Lambda' \) arise by keeping only one point of \( \Lambda \) in each cube of the type \( n + (-R/2, R/2)^d \), with \( n \in \mathbb{Z}^d \) having all its coordinates even. We keep nothing outside these cubes. It follows that \( \Lambda' \) is also a well distributed set and that for any two distinct points \( \lambda \) and \( \mu \) of \( \Lambda' \), \( \mu \) is not contained in the cube of side \( R \) centered at \( \lambda \). From (1) we obtain that for any two distinct points \( \lambda, \mu \in \Lambda' \) we have

\[
\|\lambda - \mu\|_{K^*} = k\pi + \theta, \quad (k \in \mathbb{Z}, |\theta| \leq \pi/5).
\]

This means that the set of \( K^* \) distances \( D_{K^*}(\Lambda') \) has infinitely many gaps of length at least \( 3\pi/5 \), so by Corollary 1 \( K^* \) should be a polytope, which is a contradiction.

\[\square\]

## 2 Proofs of the main theorems

**Proof of Theorem 1.**

*Notation:* \( B_1(0) \) is the unit ball in \( \mathbb{R}^d \) and \( \omega_d \) is its volume. In what follows the dimension \( d > 1 \) is fixed and constants may depend on it.
As in [1], whose method we follow, it suffices to prove the following theorem.

**Theorem A** Suppose \( \epsilon, R > 0 \) and let \( K \) be a 0-symmetric convex body contained in \( B_1(0) \) and \( \sigma \) be a probability measure on \( \partial K \) such that

\[
|\hat{\sigma}(\xi)| \leq \eta(\epsilon) = \frac{\omega_d}{80 \cdot 4^d \pi^d} \epsilon, \quad \text{if } |\xi| \geq R.
\]  

(2)

Then there is \( J = J(\epsilon, R) \) such that if \( A \subseteq B_1(0) \) is a set of measure \( \epsilon \) and \( 0 < t_j < 1, \ j \geq 1 \), is a sequence with \( t_{j+1} < \frac{1}{2} t_j \) then there is \( j \leq J \) such that \( t_j \in D_K(A) \).

**Theorem A** implies **Theorem 1**: Suppose that \( K \) is a 0-symmetric convex body, and \( \sigma \in M(\partial K) \) is a probability measure on its surface which satisfies \( |\hat{\sigma}(\xi)| \leq \eta(\epsilon) \) if \( |\xi| \geq R \), and let \( A \subseteq \mathbb{R}^d \) be a measurable set with upper density larger than \( \epsilon > 0 \). Suppose also that Theorem 1 fails and there is a sequence \( 0 < x_1 < x_2 < \cdots \) tending to infinity all of whose elements are not in \( D_K(A) \).

Renaming, we pass to lacunary subsequence of \( x_j \) such that \( x_{j+1} > 2x_j \) with \( J = J(C_d \epsilon, R) \) terms. Let \( Q \) be a cube of side-length greater than \( 10 x_J \) for which \( |Q \cap A| \geq \epsilon |Q| \). Scaling everything down to \( B_1(0) \) and renaming the scaled \( x_j \) as \( t_j \) (and reversing their order so as to have a decreasing sequence) we now have a set of measure \( \geq C \epsilon \) contained in \( B_1(0) \) and a sequence \( t_1 > t_2 > \cdots > t_J \), with \( t_{j+1} < \frac{1}{2} t_j \).

By Theorem A there exists a \( j \leq J \) such that \( t_j \) appears as a \( K \)-distance in the scaled-down set \( A \), which implies that the corresponding \( x_{j'} \) appears as a \( K \)-distance in the set \( A \), a contradiction.

\( \square \)

**Proof of Theorem A.** It is enough to show that there is \( j \leq J(\epsilon, R) \) such that, for \( t = t_j \),

\[
\int \int f(x)f(x+ty)dxd\sigma(y) > 0,
\]

where \( f \) is the indicator function of \( A \). The integral on the left may be rewritten as a constant \( C(t) \) times

\[
\int \left| \hat{f}(\xi) \right|^2 \hat{\sigma}(t\xi) \ d\xi.
\]  

(3)

The integral in (3) is broken into three parts

\[ I_1 = \int_{|\xi| \leq \frac{\delta}{t}}, \quad I_2 = \int_{\frac{\delta}{t} < |\xi| < \frac{\delta}{4t}}, \quad I_3 = \int_{\frac{\delta}{4t} \leq |\xi|} \cdot \]

where \( \delta \) will be determined later.

We shall now need three lemmas to control the quantities \( I_1, I_2, \) and \( I_3 \).

**Lemma 1** Let \( K \) be a 0-symmetric convex body contained in the unit ball \( B_1(0) \). Let also \( \sigma \) be a probability measure on \( \partial K \) and \( A \subseteq B_1(0) \) be a measurable set with indicator function \( f \). Then, writing

\[
I_1(t, \delta) = \int_{|\xi| \leq \frac{\delta}{t}} \left| \hat{f}(\xi) \right|^2 \hat{\sigma}(t\xi) \ d\xi,
\]

we have

\[
I_1(t, \delta) \geq \frac{\omega_d}{8 \cdot 4^d \pi^d} \left| A \right|^2, \quad \text{if } t \leq 4 \pi \delta.
\]  

(4)
Proof of Lemma 1. Since \( f \) and \( \sigma \) are supported in \( B_1(0) \) it follows that \( |\nabla_x f| \leq 2\pi |A| \) and \( |\nabla_x \hat{\sigma}| \leq 2\pi \), for all directions \( u \in S^{d-1} \). Hence

\[
\hat{f}(\xi) \geq \frac{|A|}{2}, \quad \text{if } |\xi| \leq \frac{1}{4\pi},
\]

and

\[
\hat{\sigma}(\xi) \geq \frac{1}{2}, \quad \text{if } |\xi| \leq \frac{1}{4\pi}.
\]

So

\[
I_1(t, \delta) \geq \int_{|\xi| \leq \frac{1}{4\pi}} |\hat{f}(\xi)|^2 \hat{\sigma}(t\xi) \, d\xi
\]

\[
\geq \int_{|\xi| \leq \frac{1}{4\pi}} \left( \frac{|A|}{2} \right)^2 \frac{1}{2} \, d\xi
\]

\[
\geq \frac{1}{8} |A|^2 \left\{ \left\{ |\xi| \leq \frac{1}{4\pi} \right\} \right\}
\]

\[
\geq \frac{\omega_d}{8 \cdot 4^d \pi^d} |A|^2.
\]

\[\Box\]

**Lemma 2** Suppose that \( \sigma \) is any measure for which \( |\hat{\sigma}(x)| \leq \eta \) provided \( |x| \geq R \) and write

\[
I_3(t, \delta) = \int_{|\xi| \geq \frac{1}{4\pi}} |g(\xi)|^2 \hat{\sigma}(t\xi) \, d\xi,
\]

where \( g \in L^2(\mathbb{R}^d) \). Then

\[
|I_3(t, \delta)| \leq \eta \int |g|^2, \quad (\text{if } \delta \leq \frac{1}{R}).
\]  

(5)

**Proof of Lemma 2.**

\[
|I_3(t, \delta)| \leq \eta \int_{|\xi| \geq \frac{1}{4\pi}} |g(\xi)|^2 \, d\xi
\]

\[
\leq \eta \int |g|^2.
\]

\[\Box\]

**Lemma 3** If \( t, \delta > 0 \), \( \delta < 1 \), \( \sigma \) is a finite measure with total variation at most 1, \( f \in L^2(\mathbb{R}^d) \) with \( \epsilon = \int |f|^2 \),

\[
I_2(t, \delta) = \int_{|\xi| < \frac{1}{4\pi}} \left| \hat{f}(\xi) \right|^2 \hat{\sigma}(t\xi) \, d\xi,
\]

and \( \{t_j\}_{j=1}^\infty \) is a sequence with

\[
0 < t_j < 1 \quad \text{and} \quad t_{j+1} < \frac{1}{2} t_j
\]

5
\[ j \leq \frac{2}{\log 2} \theta^{-1} \frac{1}{\epsilon} \log \frac{1}{\delta} \]

such that
\[ |I_2(t, \delta)| \leq \theta \epsilon^2. \tag{6} \]

**Proof of Lemma 3.** Note first that \( \log \frac{1}{t_{j+1}} - \log \frac{1}{t_j} \geq \log 2 \). For \( x \in \mathbb{R} \) let \( N(x) \) be the number of intervals \( \left( \frac{x}{t_j}, \frac{1}{t_j} \right) \) to which \( x \) belongs. It follows that
\[
N(x) \leq \frac{2}{\log 2} \log \frac{1}{\delta},
\] as \( x \in \left( \frac{x}{t_j}, \frac{1}{t_j} \right) \) is equivalent to
\[
\log \frac{1}{t_j} - \log \frac{1}{\delta} < \log x < \log \frac{1}{t_j} + \log \frac{1}{\delta}.
\]

For any positive integer \( J \) we have thus
\[
\sum_{j=1}^{J} |I_2(t_j, \delta)| \leq \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \hat{\sigma}(t \xi) N(|\xi|) \, d\xi \leq \frac{2}{\log 2} \log \frac{1}{\delta} \cdot \epsilon,
\]
by \( \int |\hat{f}(\xi)|^2 \, d\xi = \epsilon \) and \( |\hat{\sigma}(\xi)| \leq 1 \). Hence, if we let
\[
J = \frac{2}{\log 2} \theta^{-1} \frac{1}{\epsilon} \log \frac{1}{\delta},
\]
we obtain that there is \( j \leq J \) for which
\[ |I_2(t_j, \delta)| \leq \theta \epsilon^2. \]
\[ \square \]

**Proof of Theorem A (continued).** Let \( \delta = 1/R \) and \( t \leq 4\pi \delta \). By Lemma 1 we get
\[ I_1(t, \delta) \geq \frac{\omega_d}{8 \cdot 4^d \pi^d} \epsilon^2. \tag{7} \]

By Lemma 2, applied to \( g = \hat{f} \) we get
\[ |I_3(t, \delta)| \leq \eta \epsilon = \frac{\omega_d}{80 \cdot 4^d \pi^d} \epsilon^2. \tag{8} \]

Inequalities (7) and (8) hold for all \( t \leq 4\pi /R \).

Define \( j_0 = j_0(R) \) by \( t_{j_0} \leq 4\pi /R \) (clearly \( j_0 \leq C \log R \), as \( t_1 \leq 1 \)) and \( \theta = \frac{\omega_d}{80 \cdot 4^d \pi^d} \), and apply Lemma 3 to the sequence \( t_{j_0}, t_{j_0+1}, \ldots \). It follows that there is \( j \) with
\[ j \leq j_0 + 10 \theta^{-1} \frac{1}{\epsilon} \log \frac{1}{\delta} \leq C \frac{\log R}{\epsilon} =: J(\epsilon, R) \]
such that
\[ I_2(t_j, \delta) \leq \theta \varepsilon^2. \]  
(9)

Putting together (7), (8) and (9) we obtain for this \( j \)
\[
I(t_j) \geq I_1(t_j, \delta) - |I_2(t_j, \delta)| - I_3(t_j, \delta) \\
\geq \frac{\omega_d}{8 \cdot 4^d \pi^d} \varepsilon^2 - \frac{\omega_d}{80 \cdot 4^d \pi^d} \varepsilon^2 - \frac{\omega_d}{8 \cdot 4^d \pi^d} \varepsilon^2 \\
\geq \frac{\omega_d}{40 \cdot 4^d \pi^d} \varepsilon^2 \\
> 0,
\]
which shows that \( t_j \in D_K(A) \), as we had to show. The proof of Theorem A and therefore of Theorem 1 is complete.

\[ \square \]

We denote by \( S^{d-1} \) the surface of the unit ball in \( \mathbb{R}^d \) and whenever \( \Omega \) is a hypersurface in \( \mathbb{R}^d \) we denote by \( \sigma_\Omega \) its surface measure. Also, if \( x, y \in S^{d-1} \), by \( \text{dist}(x, y) \) we understand their geodesic distance in \( S^{d-1} \), in other words the angle formed by \( x \), \( 0 \) and \( y \).

**Lemma 4** Let \( \Theta \subseteq S^{d-1} \) satisfy \( \Theta = -\Theta \), and \( \delta > 0 \). Suppose also that \( K \subseteq \mathbb{R}^d \) is a convex body and \( D \subseteq \partial K \) is a measurable subset of the boundary on which almost every point has a normal vector \( \xi \in \Theta \). Let also \( \sigma_D \) be the surface measure of \( \partial K \) restricted to \( D \). Then
\[ \lim_{t \to \infty} \sigma_D(t \eta) = 0, \]  
(10)
uniformly for
\[ \eta \in \mathcal{N} = \left\{ x \in S^{d-1} : \text{dist}(x, \Theta) \geq \delta \right\}.
\]

**Proof.** If the set \( D \) is contained in a hyperplane then \( \Theta \) may be taken to be a the set \( \{ \pm \theta_0 \} \). If, in addition, it is a rectangle, one gets the validity of (10) by direct calculation of \( \sigma_D \), which tends to 0 except in the normal direction \( \theta_0 \) and does uniformly so if one keeps away from the normal direction by any fixed angle. If \( D \) is not a rectangle in its hyperplane one gets (10) by approximating the set \( D \) with a union of finitely many disjoint rectangles. Further, if \( D \) is polytopal, then it consists of a finite number of flat pieces \( D_i \) and we can represent \( \sigma_D \) as the sum of \( \sigma_{D_i} \), the surface measure restricted to \( D_i \). Since each of the \( \sigma_{D_i}(t \eta) \) goes to 0 uniformly in \( \mathcal{N} \) so does their sum \( \sigma_D(t \eta) \).

To any measure \( \mu \in M(\mathbb{R}^d) \) and \( \eta \in S^{d-1} \) we associate the projection measure on the line of \( \eta \), \( \mu^\eta \in M(\mathbb{R}) \), defined by
\[ \mu^\eta(A) = \mu(A \eta + \eta^\perp), \quad (A \text{ a Borel subset of } \mathbb{R}), \]  
(11)
where \( \eta^\perp \) is the hyperplane orthogonal to \( \eta \). By Fubini’s theorem we see easily that
\[ \widehat{\mu^\eta}(t) = \hat{\mu}(t \eta), \quad t \in \mathbb{R}. \]  
(12)

We now use the fact that for all \( \epsilon, \delta > 0 \) there exists a polytopal approximation \( P \) of \( D \) such that

- all normals to \( P \) are at distance \( \leq \delta/2 \) from \( \Theta \),
for all \( \eta \in \mathcal{N} \) we have that the measures \( \sigma_D^\beta \) and \( \sigma_P^\beta \) are in fact \( L^1 \) functions and
\[
\| \sigma_D^\beta - \sigma_P^\beta \|_{L^1(\mathbb{R})} \leq \epsilon, \quad (\eta \in \mathcal{N}).
\]
(13)

To prove this one first shows this under the assumption that the normal vector is a continuous function on \( D \) and then uses the well known theorem (see, for example, [6, p. 23]) which says that we can throw away a part of the surface of arbitrarily small measure so that the normal is continuous on what remains.

From (13) and from (12) it follows that \( |\widehat{\sigma_D(t\eta)} - \widehat{\sigma_P(t\eta)}| \leq \epsilon \) for all \( t \in \mathbb{R}, \eta \in \mathcal{N} \). Since \( \epsilon \) is arbitrary Lemma 4 follows for the general convex surface piece \( D \) from the fact that it holds for polytopal surfaces.

\( \square \)

**Proof of Theorem 2.** (\( a \)) Suppose that \( K \) is a symmetric polytope with \( N \) non-parallel faces and that \( \mu \) is a probability measure on \( \partial K \) which is \( \epsilon \)-good. We will show that \( \epsilon \geq \frac{1}{\sqrt{2N}} \).

The distinct direction vectors \( \pm \theta_1, \ldots, \pm \theta_N \) are all the normals that appear on the faces of \( K \). Since all faces are partitioned into those which are normal to \( \theta_1 \), normal to \( \theta_2 \), and so on, it follows that at least one of these pairs of faces has total \( \mu \)-measure at least \( 1/N \), say the pair of faces normal to \( \theta_1 \).

The projection measure (see (11)) \( \mu^{\theta_1} \) has then one or two nonnegative point masses \( c_1 \) and \( c_2 \) of total mass at least \( 1/N \). But, if \( c_i, i \in I \), are all the point masses of the finite measure \( \mu^{\theta_1} \), Wiener's Theorem tells us that
\[
\sum_{i \in I} |c_i|^2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\mu^{\theta_1}(t)|^2 dt.
\]
From (12) and the assumption that \( \mu \) is \( \epsilon \)-good it follows that the right hand side above is \( \leq \epsilon^2 \), from which it follows that
\[
\epsilon^2 \geq c_1^2 + c_2^2 \geq \frac{1}{2N^2},
\]
since \( c_1 + c_2 \geq 1/N \). This is the claimed inequality.

(\( b \)) For the remaining part of Theorem 2 we must show that whenever \( K \) is not a polytope or is a polytope with at least \( N \) non-parallel faces, we can find, for any \( \delta > 0 \), a probability measure on \( \partial K \) which is \( (\frac{1}{N} + \delta) \)-good. For this we recall that almost all points \( x \) on \( \partial K \) have a unique tangent hyperplane \( T_x \), whose outward normal unit vector we denote by \( n(x) \). This map \( x \to n(x) \) is called the Gauss map and, through it, a measure is defined on \( S^{d-1} \), called the area measure \( S_K \):
\[
S_K(A) = \sigma_{\partial K} \{ x \in \partial K : n(x) \in A \}, \quad \text{for any Borel set } A \subseteq S^{d-1}.
\]
It is well known that the 0-symmetric convex body is a polytope with \( N \) pairs of opposite faces if and only if the measure \( S_K \) is a symmetric measure supported on \( N \) pairs of opposite points on the sphere \( S^{d-1} \). Therefore the support of the area measure of \( K \) contains at least \( 2N \) points \( \pm \theta_1, \ldots, \pm \theta_N \). For half of these points, the points \( \theta_1, \ldots, \theta_N \), we choose an open neighborhood \( N_i \subseteq S^{d-1}, i = 1, \ldots, n \), around each of them so that all these neighborhoods are disjoint, and we call \( \delta_0 > 0 \) the minimum geodesic distance between any two of them. Let then \( D_i = n^{-1}(N_i) \subseteq \partial K \) be the parts of the boundary almost all points of which get mapped into \( N_i \) via the Gauss map \( n(x) \). This implies that all points in \( D_i \) have a normal in \( N_i \), and the \( D_i \) all have positive surface measure.
Define now the probability measure $\tilde{\mu} \in M(\partial \Omega)$ to be an appropriate multiple of surface measure on each $D_i$, so as to have total mass $1/N$ on each $D_i$. Call $\tilde{\mu}_i$ the measure $\mu$ restricted to $D_i$. From Lemma 4 it follows that $\tilde{\mu}_i(t\eta) \to 0$ as $t \to \infty$ uniformly for all $\eta$ which are distance at least $\delta_0/10$ from $N_i$. And for all $x$ we have trivially $|\tilde{\mu}_i(x)| \leq 1/N$.

Let now $\delta > 0$ and choose $R > 0$ large enough so that for all $i = 1, \ldots, N$ we have $|\tilde{\mu}_i(t\eta)| \leq \delta/(N-1)$ if $|t| > R$ and $\eta$ has distance more than $\delta_0/10$ from $N_i$. If now $x \in \mathbb{R}^d$ is an arbitrary vector with $|x| > R$ the vector $\eta = \frac{1}{|x|} x$ can have distance at most $\delta_0/10$ from at most one neighborhood $N_i$, say from $N_1$. It follows that

$$\begin{align*}
|\tilde{\mu}(x)| & \leq |\tilde{\mu}_1(x)| + \cdots + |\tilde{\mu}_N(x)| \\
& \leq \frac{1}{N} + (N - 1) \frac{\delta}{N - 1} \\
& = \frac{1}{N} + \delta.
\end{align*}$$

\[ \square \]

Bibliography


Department of Mathematics, University of Crete, Knossos Ave., 714 09 Iraklio, Greece.
E-mail: kolount@member.ams.org

9