Foliated CR Manifolds

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Abstract

We study foliations\(^1\) on CR manifolds and show the following\(^2\). 1) For a strictly pseudoconvex CR manifold \(M\), the relationship between a foliation \(\mathcal{F}\) on \(M\) and its pullback \(\pi^*\mathcal{F}\) on the total space \(C(M)\) of the canonical circle bundle of \(M\) is given, with emphasis on their interrelation with the Webster metric on \(M\) and the Fefferman metric on \(C(M)\), respectively. 2) With a tangentially CR foliation \(\mathcal{F}\) on a nondegenerate CR manifold \(M\), we associate the basic Kohn-Rossi cohomology of \((M, \mathcal{F})\) and prove that it gives the basis of the \(E_2\)-term of the spectral sequence naturally associated to \(\mathcal{F}\). 3) For a strictly pseudoconvex domain \(\Omega\) in a complex Euclidean space and a foliation \(\mathcal{F}\) defined by the level sets of the defining function of \(\Omega\) on a neighborhood \(U\) of \(\partial\Omega\), we give a new axiomatic description of the Graham-Lee connection, a linear connection on \(U\) which induces the Tanaka-Webster connection on each leaf of \(\mathcal{F}\). 4) For a foliation \(\mathcal{F}\) on a nondegenerate CR manifold \(M\), we build a pseudohermitian analogue to the theory of the second fundamental form of a foliation on a Riemannian manifold, and apply it to the flows obtained by integrating infinitesimal pseudohermitian transformations on \(M\).

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1 Introduction

Foliations on CR manifolds appear naturally in several contexts. For instance, if a CR manifold \((M, T_1, \theta(M))\) is Levi flat, then the maximally complex distribution \(H(M)\) of \(M\) is completely integrable so that \(M\) carries a foliation (the Levi foliation) by complex manifolds (cf. [16], [38]). Cf. Section 2 for notation and conventions. To see another example of this sort, let \(\Omega = \{\varphi < 0\} \subseteq C^{\alpha+1}\) be a strictly pseudoconvex domain with real analytic boundary \(M = \partial \Omega\). Let \(C(\Omega)\) be the algebra of functions on \(\Omega\) which admit a holomorphic extension to some neighborhood of \(\Omega\). Let \(\Sigma \subset M\) be a real analytic submanifold which is not \(C\)-tangent at any of its points. By a result in [8], if \(\Sigma\) is locally a maximum modulus set for \(C(\Omega)\) (cf., e.g., [14] for definitions), then \(L = T(\Sigma) \cap H(M)\) is completely integrable and gives rise to on \(\Sigma\) a \(C\)-tangent foliation \(F\) of codimension one. On the other hand, by a result in [5], if \(\Sigma\) is tangent to the characteristic direction \(T\) of a pseudohermitian structure \(\theta\) on \(M\), then \(\Sigma\) is a contact CR submanifold (in the sense of [43], and thus a CR manifold), \(F\) is a Riemannian foliation and the metric \(g_\Sigma\) induced on \(\Sigma\) by the Webster metric of \((M, \theta)\), where \(\theta = (i/2)(\overline{\theta} - \theta)\varphi\), is bundle-like (also, \(\Sigma\) is Levi flat and \(F\) is its Levi foliation).

Opposite to the Levi flat case, if \((M, T_{1,\theta}(M))\) is a nondegenerate CR manifold of hypersurface type whose pseudohermitian structure \(\theta\) is a contact form on \(M\), then the characteristic direction \(T\) of \((M, \theta)\) defines a flow on \(M\) (the contact flow, cf., e.g., [17]). Also, foliations by Riemann spheres appear (cf. [27]) on twistor spaces (nondegenerate 5-dimensional CR manifolds) of 3-dimensional conformal manifolds (a generalization of the example to \(n\) dimensions is due to [39]). A converse of this situation is known as well, namely if \(M\) is a nondegenerate CR manifold of CR dimension \(n = 2m\) carrying a foliation by compact complex manifolds of complex dimension \(\geq m\), then \(m = 1\), the leaves are \(CP^1\)'s, and \(M\) arises from a twistor construction (cf. [28]).

Furthermore, it should be noted that in [10] one considers foliations \(F\) on a CR manifold \(M\) such that, for any defining local submersion \(f : U \to U'\) (i.e., the leaves of \(F\) are the fibres of \(f\)), the local quotient manifold \(U'\) is a CR manifold, \(f\) is a CR map, and \(df : H(U) \to H(U')\) is surjective. Such \(F\) has a transverse CR structure (in the sense of [6]) and also a "tangential" CR structure (so that each leaf of \(F\) becomes a CR submanifold of \(M\)). While foliations with a transverse CR structure have been investigated (cf. [3] and [1]), a systematic treatment of foliations with tangential CR structure is still missing in the mathematical literature.

The purpose of the present paper is to study basic properties of foliations on CR manifolds, in particular, tangential CR foliations on nondegenerate CR manifolds, and prove the following as the first step.

If \(M\) is a strictly pseudoconvex CR manifold with a fixed contact form \(\theta\) whose corresponding Levi form \(G_\theta\) is positive definite, and \(F\) is a foliation on \(M\) which is tangent to the characteristic direction \(T\) of \(\theta\), then the pullback foliation
π⁎FM to the total space of the canonical circle bundle π : C(M) → M of M is nondegenerate with respect to the Fefferman metric Fθ on C(M). Furthermore, Fθ is bundle-like for π⁎F if and only if the Webster metric gθ of (M, θ) is bundle-like for F. For a transversally oriented codimension q foliation F on M, we show that if 1) F is tangent to T, 2) the transverse volume element of F in (M, gθ) is holonomy invariant, and 3) the mean curvature form κ of F in (M, gθ) is dπ⁎-exact, then the q-dimensional basic cohomology HqB(F) of F is nonvanishing. Thus we generalize a result in [23] (cf. also Corollary 9.22 in [41], p. 125) to the case of foliations on CR manifolds.

With any tangentially CR foliation F on M we associate a cohomology algebra H⁎B(F), the basic Kohn-Rossi cohomology of (M, F), which has the property that H⁎B(F) = CR*M (the space of CR functions on M) and that H⁎B(F) injects into the ordinary Kohn-Rossi cohomology group H⁎B(M) of M on (0, 1)-forms. We build a decreasing filtration {F∗Ω⁎B}r>0 of Ω⁎B(M) by JM-differential ideals, and show that if {E′r,F}r>0 is the corresponding spectral sequence, then F∗r ≅ H⁎B(F).

Given a smoothly bounded strictly pseudoconvex domain Ω = {φ < 0} ⊂ Cn+1 and a foliation F defined by the level sets of φ on a neighborhood U of ∂Ω, we give a new axiomatic description of the Graham-Lee connection, a linear connection ∇ on U which induces the Tanaka-Webster connection on each leaf of F, and then compute Faran’s invariants hφ and kφ (cf. [15]) in terms of the pseudohermitian torsion of the Graham-Lee connection and transverse curvature of φ, respectively. Also, for a foliation F on a nondegenerate CR manifold M we use the adapted connection determined by the Bott connection of F and the Tanaka-Webster connection (associated to a choice of contact form θ on M) to produce a pseudohermitian analogue to the theory of the second fundamental form of a foliation on a Riemannian manifold (cf. [41], p. 62).

The theory is applied to foliations which are tangent to the characteristic direction of θ and orthogonal to a semi- Levi foliation, and to flows obtained by integrating infinitesimal pseudohermitian transformations on a nondegenerate CR manifold.

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2 Foliations and the Fefferman metric

Given a foliation F on a strictly pseudoconvex CR manifold M with a contact form θ, whose corresponding real Levi form Gθ being positive definite, our main technique in this section is to consider the pullback foliation π⁎F of F on the total space C(M) of the canonical circle bundle over M.

This pullback foliation π⁎F enjoys many of the properties of the original foliation F. For instance, π⁎F is tangentially oriented ↔ F is tangentially oriented, the (canonical) transverse volume element of π⁎F is holonomy invariant ↔ the transverse volume element of F is holonomy invariant, π⁎F is har-
monic (with respect to the Fefferman metric $F_\theta$ on $C(M)$) $\iff$ $F$ is harmonic (with respect to the Webster metric $g_\theta$ on $M$), etc. Furthermore, $\pi^*F$ “lives” in the presence of a Lorentz metric (the Fefferman metric $F_\theta$). The resulting philosophy then is that one might get a better understanding of the geometry of a foliated (strictly pseudoconvex) CR manifold by establishing general theorems about foliated Lorentz manifolds.

2.1 CR and pseudohermitian geometry

We start by recalling a few notions of CR and pseudohermitian geometry, which are needed throughout the paper. Let $(M, T_{1,0}(M))$ be a CR manifold of type $(n, k)$, where $M$ is a real $(2n + k)$-dimensional $C^\infty$ manifold and $T_{1,0}(M)$ is its CR structure; that is, a complex rank $n$ subbundle of the complexified tangent bundle $T(M) \otimes \mathbb{C}$ of $M$ such that

$$T_{1,0}(M) \cap T_{0,1}(M) = \{0\},$$

i.e., $T_{1,0}(M)$ is totally complex, and

$$[\Gamma^\infty(\pi^*T_{1,0}(M)), \Gamma^\infty(\pi^*T_{1,0}(M))] \subseteq \Gamma^\infty(\pi^*T_{1,0}(M)),$$

i.e., $T_{1,0}(M)$ is involutive, or (formally) Frobenius integrable, where $T_{0,1}(M)$ stands for the complex conjugate of $T_{1,0}(M)$. The integers $n$ and $k$ are called the CR dimension and CR codimension of $(M, T_{1,0}(M))$, respectively. Clearly, if $k = 0$, then $(M, T_{1,0}(M))$ is a complex manifold of complex dimension $n$. We shall be mainly interested in CR manifolds of type $(n, 1)$, which are commonly referred to as CR manifolds of hypersurface type.

Let $(M, T_{1,0}(M))$ be a CR manifold of arbitrary but fixed type. Let

$$H(M) = \text{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$$

be the Levi, or maximally complex, distribution of $M$. It carries the complex structure $J : H(M) \to H(M)$ given by

$$J(Z + \overline{Z}) = i(Z - \overline{Z}), \quad Z \in T_{1,0}(M), \quad i = \sqrt{-1}.$$

The Levi form $L$ is then defined by

$$L(Z, \overline{W}) = i \pi([Z, \overline{W}]), \quad Z, W \in T_{1,0}(M),$$

where $\pi : T(M) \otimes \mathbb{C} \to \{T(M) \otimes \mathbb{C}\}/\{H(M) \otimes \mathbb{C}\}$ is the natural bundle map. A CR manifold with $L = 0$ is called Levi flat. Note that if $k = 0$, then $L = 0$ (i.e., a complex manifold is Levi flat). A CR manifold is called nondegenerate if $L$ is nondegenerate.

Assume now that $k = 1$ and $M$ is orientable. Let $\theta$ be a pseudohermitian structure on $M$, that is, a global nowhere zero $C^\infty$ section of $H(M)^{1,0} \subset T^*(M)$, the conormal bundle of $H(M)$ defined by $H(M)^{1,0} = \{\omega \in T^*_x(M) \mid \text{Ker}(\omega) \supseteq H(M)^{1,0}_x\}$, $x \in M$. Consider

$$G_\theta(X, Y) = d\theta(X, JY), \quad X, Y \in H(M),$$

(the real Levi form). It is also customary to consider the complex bilinear form

$$L_\theta(Z, \overline{W}) = -i d\theta(Z, \overline{W}), \quad Z, W \in T_{1,0}(M).$$
Then $L_\theta$ and the complex linear extension of $G_\theta$ to $H(M) \oplus C$ coincide on $T_{1,0}(M) \otimes T_{\overline{1},1}(M)$. Also, $L_\theta$ and $L$ coincide up to a bundle isomorphism $H(M)^{\perp} \cong T(M)/H(M)$.

A CR manifold $(M, T_{1,0}(M))$, of hypersurface type, is strictly pseudoconvex if $G_\theta$ is positive definite for some pseudohermitian structure $\theta$ on $M$.

When $(M, T_{1,0}(M))$ is nondegenerate (of hypersurface type), any pseudohermitian structure $\theta$ is a contact form on $M$ so that $\theta \wedge (d\theta)^n$ is a volume form on $M$. If this is the case, let $T$ be the characteristic direction of $(M, \theta$), that is, the unique tangent vector field on $M$, transverse to $H(M)$, determined by $\theta(T) = 1$ and $T \cdot d\theta = 0$. As usual, we extend $G_\theta$ to a degenerate metric $\tilde{G}_\theta = \pi_H^*G_\theta$ on $M$ given by $\tilde{G}_\theta(X, Y) = G_\theta(\pi_H(X), \pi_H(Y))$ for any $X, Y \in T(M)$, where $\pi_H : T(M) \to H(M)$ is the canonical projection associated to the direct sum decomposition $T(M) = H(M) \oplus RT$ [in particular, $\tilde{G}_\theta(T, T) = 0$]. The Webster metric of $(M, \theta)$ is then defined by
\[
g_\theta = \tilde{G}_\theta + \theta \otimes \theta.
\]

If $(2r, 2s)$ is the signature of $G_\theta$ ($r + s = n$), then $g_\theta$ is a semi-Riemannian metric on $M$ of signature $(2r + 1, 2s)$ [and if $M$ is strictly pseudoconvex with $G_\theta$ positive definite, then $g_\theta$ is a Riemannian metric on $M$]. For instance, let $H_n = \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group with the multiplication law $(z, t) \cdot (w, s) = (z + w, t + s + 2 \text{Im}(z, w))$, where $(z, w) = \delta_{ij} z^i \overline{w}^j$, and consider the Lewy operators
\[
L_\alpha = \frac{\partial}{\partial z^\alpha} - iz^\alpha \frac{\partial}{\partial t}, \quad 1 \leq \alpha \leq n.
\]

Then
\[
T_{1,0}(H_n)_x = \sum_{\alpha=1}^n C L_{\alpha, x}, \quad x \in H_n,
\]

where $L_{x, \alpha} = \overline{1}_{\alpha}$ is a CR structure on $H_n$ making $H_n$ into a strictly pseudoconvex CR manifold of CR dimension $n$ (and actually into a CR Lie group, that is, a Lie group and a CR manifold whose CR structure is left invariant).

A smooth map of CR manifolds $f : M \to M'$ is a CR map if $df_x (T_{1,0}(M)_x) \subset T_{1,0}(M')_{f(x)}$ for any $x \in M$. A CR isomorphism, or CR equivalence, is a CR map which is a $C^\infty$ diffeomorphism.

Any smooth real hypersurface $M$ in $\mathbb{C}^{n+1}$ is a CR manifold of CR dimension $n$, with the CR structure $T_{1,0}(M) = \{ T(M) \oplus C \} \cap T^{1,0}(\mathbb{C}^{n+1})$, where $T^{1,0}(\mathbb{C}^{n+1})$ is the span of $\{ \partial / \partial z_j | 1 \leq j \leq n + 1 \}$. In particular, the boundary $\partial \Omega_{n+1}$ of the Siegel domain
\[
\Omega_{n+1} = \left\{ (z, w) \in \mathbb{C}^{n+1} \times \mathbb{C} \mid \text{Im}(w) > \sum_{\alpha=1}^n |z^\alpha|^2 \right\}
\]
is a CR manifold which is CR isomorphic to the Heisenberg group (the CR isomorphism is given by $f(z, t) = (z, t + i |z|^2)$, $(z, t) \in H_n$).

For any nondegenerate CR manifold $M$ of hypersurface type, on which a contact form $\theta$ has been fixed, there is a unique linear connection $\nabla$ (the Tanaka-Webster connection of $(M, \theta)$, cf., e.g., [13]) such that 1) $H(M)$ is $\nabla$-parallel, 2) $\nabla g_\theta = 0$ and $\nabla J = 0$, 3) the torsion $T_\nabla$ of $\nabla$ is pure, that is, $T_\nabla(Z, W) = 0$ and $T_\nabla(Z, W) = i L_\theta(Z, W)T$ for any $Z, W \in T_{1,0}(M)$, and $\tau \circ J + J \circ \tau = 0$, where
\(\tau(X) = T\tau(T, X), \ X \in T(M)\). The vector valued 1-form \(\tau\) on \(M\) is called the pseudohermitian torsion of \(\nabla\) and satisfies \(g_\theta(\tau(X), Y) = g_\theta(X, \tau(Y))\) for any \(X, Y \in T(M)\), that is, \(\tau\) is self-adjoint with respect to \(g_\theta\).

2.2 The normal bundle

Generally, given a codimension \(q\) foliation \(F\) on a \(C^\infty\) manifold \(N\), we denote by \(T(\mathcal{F})\) the tangent bundle of \(\mathcal{F}\) and by \(\nu(\mathcal{F}) = T(N)/T(\mathcal{F})\) its normal (or transverse) bundle, and by \(\Pi : T(N) \to \nu(\mathcal{F})\) the natural bundle map.

Let \((M, T_{1, \theta}(M))\) be a strictly pseudoconvex CR manifold of CR dimension \(n\). Let \(\theta\) be a contact form on \(M\) such that \(G_\theta\) is positive definite. Let \(\mathcal{F}\) be a codimension \(q\) foliation of \(M\). Note that if \(2n \geq q\), then \(\theta\) is not basic. Indeed, if \(T(\mathcal{F}) \parallel \theta = 0\) then \(T(\mathcal{F}) \subseteq H(M)\), and if \(T(\mathcal{F}) \parallel d\theta = 0\), then for any \(X \in T(\mathcal{F})\)

\[0 = d\theta(X, JX) = G_\theta(X, X) \implies X = 0.\]

Hence \(\mathcal{F}\) is the foliation by points, that is, \(q = 2n + 1\). Let us extend \(G_\theta\) to the whole of \(T(M)\) as a degenerate metric \(\hat{G}_\theta\), by requesting that \(T\) is orthogonal to each \(V \in T(M)\), and let us consider

\[T(\mathcal{F})_0 = \{Y \in T(M) \mid \hat{G}_\theta(X, Y) = 0 \text{ for all } X \in T(\mathcal{F})\}.\]

We collect a few elementary facts in the following

**Proposition 1.** The tangent bundle \(T(\mathcal{F})\) is nondegenerate in \((T(M), \hat{G}_\theta)\) if and only if the characteristic direction \(T\) of \((M, \theta)\) is transverse to \(T(\mathcal{F})\). In general, let \(T(\mathcal{F})_{H(M)} = \pi_H (T(\mathcal{F}))\) be the projection of \(T(\mathcal{F})\) to \(H(M)\). Then we obtain

\[
T(\mathcal{F})_0 = [T(\mathcal{F})_{H(M)}]^\perp \oplus RT,
\]

where the orthogonal complement \([T(\mathcal{F})_{H(M)}]^\perp\) of \(T(\mathcal{F})_{H(M)}\) is taken in \((H(M), G_\theta)\). If \(T\) is tangent to \(\mathcal{F}\), then the following hold:

1. \(T(\mathcal{F})_{H(M)} = H(M) \cap T(\mathcal{F})\).
2. The natural bundle map \(\sigma_\theta : \nu(\mathcal{F}) \to T(\mathcal{F})_0\) is a bundle monomorphism and corestricts to a bundle isomorphism

\[
\nu(\mathcal{F}) \cong [T(\mathcal{F})_{H(M)}]^\perp.
\]

3. \(H_\theta(r, s) = \hat{G}_\theta(\sigma_\theta(r), \sigma_\theta(s)), \ r, s \in \nu(\mathcal{F})\), is a Riemannian metric on the normal bundle \(\nu(\mathcal{F}) \to M\).

**Proof.** Let us prove the first statement in Proposition 1. Assume that \(T\) is transverse to \(T(\mathcal{F})\). Let \(X \in T(\mathcal{F})\) such that \(\hat{G}_\theta(X, Y) = 0\) for any \(Y \in T(\mathcal{F})\). Then

\[0 = \hat{G}_\theta(X, X) = G_\theta(\pi_H(X), \pi_H(X)) = ||\pi_H(X)||^2,
\]

so that \(\pi_H(X) = 0\). Thus \(T(\mathcal{F}) \ni X = \theta(X)T\), which yields \(\theta(X) = 0\), i.e., \(X = 0\).

Vice versa, assume that \(T(\mathcal{F})\) is nondegenerate in \((T(M), \hat{G}_\theta)\). The proof is done by contradiction. If \(T_x \in T(\mathcal{F})_x\) for some \(x \in M\), then \(\hat{G}_\theta,x(v, T_x) = 0\) for
any $r \in T_x(M) \supset T(\mathcal{F})_x$, which yields $T_x = 0$ by the nondegeneracy of $T(\mathcal{F})_x$ in $(T_x(M), \tilde{G}_\theta(x))$, a contradiction.

To prove the second statement in Proposition 1, let $T(\mathcal{F})|_{H(M)}$ be the projection of $T(\mathcal{F})$ to $H(M)$, namely,

$$T(\mathcal{F})|_{H(M)} = \{ X - \theta(X)T \mid X \in T(\mathcal{F}) \}.$$

Since

$$[T(\mathcal{F})|_{H(M)}]^\perp \cap RT \subseteq H(M) \cap RT = \{ 0 \},$$

the sum in (2.1) is direct. To prove (2.1), first note that $T \in T(\mathcal{F})|_0$. Next, if $Z \in [T(\mathcal{F})|_{H(M)}]^\perp$, then $\tilde{G}_\theta(Z, Y) = 0$ for any $Y \in T(\mathcal{F})|_{H(M)}$, which is written as $Y = X - \theta(X)T$ with $X \in T(\mathcal{F})$. Thus

$$0 = \tilde{G}_\theta(Z, Y) = \tilde{G}_\theta(Z, X),$$

and hence $Z \in T(\mathcal{F})|_0$. To check the opposite inclusion, let $Z \in T(\mathcal{F})|_0 \subseteq H(M) \oplus RT$. Then $Z = Y + fT$ for some $Y \in H(M)$ and $f \in C^\infty(M)$. Since $\tilde{G}_\theta(Z, X) = 0$ for any $X \in T(\mathcal{F})$, it follows that

$$G_\theta(Y, X - \theta(X)T) = \tilde{G}_\theta(Z, X) = 0,$$

which implies $Y \in [T(\mathcal{F})|_{H(M)}]^\perp$.

Consider the bundle map $\sigma_0 : \nu(\mathcal{F}) \to T(\mathcal{F})|_0$ defined by

$$\sigma_0(\Pi(Y)) = (Y - \theta(Y)T)^\perp, \quad Y \in T(M),$$

where $(Y - \theta(Y)T)^\perp$ is the $[T(\mathcal{F})|_{H(M)}]^\perp$-component of $Y - \theta(Y)T$ in $H(M)$.

To see that $\sigma_0(\Pi(Y))$ is well-defined, suppose that $\Pi(Y) = \Pi(Z)$. Then $Y - Z \in T(\mathcal{F})$, and hence $Y - Z - \theta(Y - Z)T \in T(\mathcal{F})|_{H(M)}$ so that $(Y - Z - \theta(Y - Z)T)^\perp = 0$.

Assume now that $T \in T(\mathcal{F})$. The proof of (1) is immediate. To check that $\sigma_0$ is a bundle monomorphism, let $\sigma_0(\Pi(Y)) = 0$, that is,

$$Y - \theta(Y)T \in T(\mathcal{F})|_{H(M)} = H(M) \cap T(\mathcal{F}) \subseteq T(\mathcal{F}).$$

Thus, by $T \in T(\mathcal{F})$,

$$0 = \Pi(Y - \theta(Y)T) = \Pi(Y).$$

The isomorphism claimed in (2) of Proposition 1 follows by a dimension argument. Indeed, since $\dim_R \nu(\mathcal{F}) = q$, the fact that $H(M) + T(\mathcal{F}) \supseteq H(M) + RT = T(M)$ implies that

$$2n + 1 = \dim_R \nu(\mathcal{F}) = \dim_R H(M) + \dim_R T(\mathcal{F}) - \dim_R \{ H(M), T(\mathcal{F}) \},$$

and hence $\dim_R [T(\mathcal{F})|_{H(M)}]^\perp = 2n - q$ for any $x \in M$.

Let us now prove (3) of Proposition 1. As the image of $\sigma_0$ lies in $H(M)$, $H_\theta(r, r) = |\sigma_0(r)|^2 \geq 0$ and $= 0$ if and only if $Y - \theta(Y)T \in T(\mathcal{F})|_{H(M)}$ for each $Y \in T(M)$ such that $\Pi(Y) = r$. Therefore, $H_\theta(r, r) = 0$ if and only if $r \in R \Pi(T)$. In particular, if $T \in T(\mathcal{F})$, then $H_\theta$ is a Riemannian metric in $\nu(\mathcal{F})$. Proposition 1 is proved.
Remark 1. As the Webster metric $g_\theta$ is a Riemannian metric on $M$, one may consider as well the normal bundle

$$T(F)^\perp = \{ Y \in T(M) \mid g_\theta(Y, X) = 0 \text{ for all } X \in T(F) \}$$

with the corresponding bundle isomorphism $\sigma : \nu(F) \to T(F)^\perp$ given by $\sigma(\Pi(Y)) = Y^\perp$, where $Y^\perp$ is the $T(F)^\perp$-component of $Y \in T(M) = T(F) \oplus T(F)^\perp$, and the metric induced by $g_\theta$ on $\nu(F)$ via $\sigma$. However, when $T \in T(F)$, it follows that $T(F)^\perp = [T(F)_{H(M)}]^\perp$. Consequently, $\sigma = \sigma_0$, and the metric on $\nu(F)$ induced by $g_\theta$ is precisely $H_\theta$. Indeed, let $Y \in T(F)^\perp$, namely $g_\theta(Y, X) = 0$ for any $X \in T(F)$. Since $g_\theta(Y, T) = 0$ in particular, $Y \in H(M)$. Therefore

$$\tilde{G}_\theta(Y, X - \theta(Y)T) = \tilde{G}_\theta(Y, X) = g_\theta(Y, X) - \theta(Y) \theta(X) = 0$$

for any $X \in T(F)$, which shows that $T(F)^\perp \subset [T(F)_{H(M)}]^\perp$. The opposite inclusion may be proved in a similar manner. Also, it is immediate to see

$$\sigma(\Pi(Y)) = \sigma(\Pi(Y - \theta(Y)T)) = (Y - \theta(Y)T)^\perp$$

for any $Y \in T(M)$.

2.3 The Fefferman metric

The first statement in Proposition 1 shows that, under the natural assumption that $T$ be tangent to the leaves of $F$, $T(F)$ is degenerate in $(T(M), \tilde{G}_\theta)$. However, the pullback of $F$ to the total space of the principal $S^1$-bundle

$$C(M) = \{ K(M) \setminus \{ \text{zero section} \} \} / R_+ \to M$$

turns out to be nondegenerate in $(C(M), F_\theta)$, where $F_\theta$ is the Fefferman metric of $(M, \theta)$. Here $K(M) = \Lambda_{n+1}^n(M)$ is the canonical line bundle over $M$. To be more precise, a complex $p$-form $\omega$ on $M$ is said to be a $(p, 0)$-form, or a form of type $(p, 0)$ if $T_{\nu, 1}(M) \mid \omega = 0$, and $\Lambda^{p, 0}(M) \to M$ denotes the bundle of the $(p, 0)$-forms on $M$.

We proceed by recalling a few notions regarding the Fefferman metric (cf., e.g., [20]). Consider the 1-form $\eta$ on $C(M)$ given by

$$\eta = \frac{1}{n+2} \left\{ d\gamma + \pi^* \left( i \omega^\alpha_{\alpha} - \frac{i}{2} h^{\alpha\beta} d\alpha_{\beta} - \frac{R}{4(n+1)} \theta \right) \right\},$$

where $\gamma$ is the (local) fibre coordinate on $C(M)$, $\pi : C(M) \to M$ is the projection, $h_{\alpha\beta}$ are the (local) components of the Levi form with respect to a (local) frame $(T_\alpha)$ of $T_{\nu, 1}(M)$, i.e., $h_{\alpha\beta} = L_\nu(T_\alpha, \overline{T_\beta})$, $\omega^\alpha_{\alpha}$ are the connection 1-forms of the Tanaka-Webster connection $\nabla$ of $(M, \theta)$, that is,

$$\nabla T_\beta = \omega^\alpha_{\alpha} T_\alpha,$$
and $R = h^{\alpha \beta} R_{\alpha \beta}$ is the pseudohermitian scalar curvature (again cf. [29]). Also, $R_{\alpha \beta}$ is the pseudohermitian Ricci tensor. The Fefferman metric $F_\theta$ of $(M, \theta)$ is the Lorentz metric on $C(M)$ given by

$$F_\theta = \pi^* \hat{G}_\theta + 2(\pi^* \theta) \odot \eta,$$

where $\odot$ denotes the symmetric tensor product. Note that $\eta$ is a connection 1-form on the principal $S^1$-bundle $\pi : C(M) \to M$ (cf. also [18], p. 855). Let then

$$\beta_z = \{ d_z \pi : \text{Ker}(\eta_z) \to T_{\pi(x)}(M) \}^{-1}, \quad z \in C(M)_x, \; x \in M,$$

be the horizontal lift with respect to $\eta$. For a tangent vector field $X$ on $M$ we adopt the notation $X^\dagger = \beta(X)$. Let $S = \partial / \partial \gamma$ be the tangent vector filed to the $S^1$-action. Then $T^\dagger - S$ is timelike, and hence $(C(M), F_\theta)$ is time-oriented by $T^\dagger - S$, namely $(C(M), F_\theta)$ is a space-time (cf., e.g., [7], p. 17). Moreover, if $M$ is compact, then $(C(M), F_\theta)$ is not chronological (cf. Proposition 2.6 in [7], p. 23).

Let $\mathcal{F}$ be a foliation of $M$ and $\pi^* \mathcal{F}$ the pullback of $\mathcal{F}$ to $C(M)$, that is,

$$T(\pi^* \mathcal{F})_z = (d_z \pi)^{-1} T_\pi(\mathcal{F})_{\pi(z)}, \quad z \in C(M).$$

The leaves of $\pi^* \mathcal{F}$ are connected components of the inverse images (via $\pi$) of the leaves of $\mathcal{F}$. We may state the following

**Proposition 2.** Let $\mathcal{F}$ be a foliation on the strictly pseudoconvex CR manifold $M$, carrying the contact form $\theta$ (with $G_\theta$ positive definite). Let $T(\mathcal{F})^\dagger$ be the horizontal lift (with respect to $\eta$) of $T(\mathcal{F})$, that is, $T(\mathcal{F})^\dagger = \beta_z(T(\mathcal{F})_{\pi(z)}), \; z \in C(M)$. Then for the tangent bundle $T(\pi^* \mathcal{F})$ of the pullback foliation $\pi^* \mathcal{F}$ on $C(M)$ we obtain

$$T(\pi^* \mathcal{F}) = T(\mathcal{F})^\dagger \oplus \text{Ker}(d\pi).$$

Let $T$ be the characteristic direction of $(M, \theta)$. If $T$ is tangent to $\mathcal{F}$, then the following hold:

1. $T(\pi^* \mathcal{F})$ is nondegenerate in $(T(C(M)), F_\theta)$ and each leaf $L_\xi$ of $\pi^* \mathcal{F}$ is a Lorentz manifold with the induced metric $\iota^* F_\theta$, where $\iota : L_\xi \hookrightarrow C(M)$ denotes the inclusion.
2. The metric $h_\theta$ induced by $F_\theta$ on $\nu(\pi^* \mathcal{F}) = T(C(M))/T(\pi^* \mathcal{F})$, the normal bundle of $\pi^* \mathcal{F}$, is positive definite.
3. The Fefferman metric $F_\theta$ is bundle-like for $(C(M), \pi^* \mathcal{F})$ if and only if the Webster metric $g_\theta$ is bundle-like for $(M, \mathcal{F})$.

Here, by slightly generalizing the definition in, e.g., [32], p. 79, given a semi-Riemannian manifold $(N, g)$ (i.e., $g$ is nondegenerate, of constant index) and a foliation $\mathcal{F}$ on $N$, we call $g$ a bundle-like metric for $(N, \mathcal{F})$ if 1) $T(\mathcal{F})$ is nondegenerate in $(N, g)$ and 2) the metric $h$ induced by $g$ on $\nu(\mathcal{F})$ is holonomy invariant, that is, $\mathcal{L}_X h = 0$ for any $X \in T(\mathcal{F})$, where $\mathcal{L}_X$ stands for the Lie differentiation with respect to $X$. 

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Proof. Let us first prove (2.2) in Proposition 2. Since \( \eta \) is a connection 1-form, it follows that

\[
T(\mathcal{F})^\dagger \cap \text{Ker}\,(d\pi) \subseteq \text{Ker}\,(\eta) \cap \text{Ker}\,(d\pi) = \{0\}.
\]

Therefore the sum in (2.2) is direct. The inclusion "\( \supseteq \)" holds by the very definition of \( \pi^*\mathcal{F} \). Vice versa, if

\[
V \in T(\pi^*\mathcal{F}) \subseteq T(C(M)) = \text{Ker}\,(\eta) \oplus \text{Ker}(d\pi),
\]

then \( V = X^\dagger + fS \) for some \( X \in T(M) \) and \( f \in C^\infty(C(M)) \), where \( X^\dagger = \beta(X) \) is the horizontal lift of \( X \) with respect to \( \eta \). Also, \( V \in T(\pi^*\mathcal{F}) \) yields that \( X = d\pi(V) \in T(\mathcal{F}) \). Hence \( X^\dagger \in T(\mathcal{F})^\dagger \); that is, \( V \in T(\mathcal{F})^\dagger + \text{Ker}(d\pi) \). The identity (2.2) is thus proved.

Assume now that \( T \in T(\mathcal{F}) \). To see (1), consider \( V \in T(\pi^*\mathcal{F}) \) such that

\[
F_\theta(V, W) = 0 \quad \text{for any} \quad W \in T(\pi^*\mathcal{F}).
\]

Then we have

\[
(\pi^*\hat{G}_\theta)(V, W) + (\pi^*\theta)(V)\eta(W) + (\pi^*\theta)(W)\eta(V) = 0,
\]

which implies, by taking the decomposition \( V = V_H + V_V \in \text{Ker}(\eta) \oplus \text{Ker}(d\pi) \),

\[
(\pi^*\hat{G}_\theta)(d\pi(V_H), d\pi(W_H)) + \theta(d\pi(V_H))\eta(W_V) + \theta(d\pi(W_H))\eta(V_V) = 0.
\]

for any \( W \in T(\pi^*\mathcal{F}) \). If \( W = S \in \text{Ker}(d\pi) \subset T(\pi^*\mathcal{F}) \), then \( W_H = 0 \). Since

\[
\eta = \{d\gamma + \pi^*\eta_0\}/(n + 2)
\]

for some 1-form \( \eta_0 \) on \( M \), which is determined in terms of \( \theta \), and \( d\gamma(S) = 1 \), it follows that \( \eta(W_V) = 1/(n + 2) \). Then from (2.3) we see that \( \theta(d\pi(V_H)) = 0 \), which implies

\[
d\pi(V_H) \in H(M),
\]

with the corresponding simpler form of (2.3) as

\[
\hat{G}_\theta(d\pi(V_H), d\pi(W_H)) + \theta(d\pi(W_H))\eta(V_V) = 0.
\]

If \( W = V \), then it follows from (2.4) that \( \|d\pi(V_H)\|^2 = 0 \), that is, \( d\pi(V_H) = 0 \) and hence \( V_H \in \text{Ker}(d\pi) \cap \text{Ker}(\eta) = \{0\} \). Substituting \( V_H = 0 \) into (2.5), we then see

\[
\theta(d\pi(W_H))\eta(V_V) = 0
\]

for any \( W \in T(\pi^*\mathcal{F}) \). Setting \( W = T^\dagger \in T(\mathcal{F})^\dagger \subset T(\pi^*\mathcal{F}) \) in (2.6) then yields that \( \eta(V_V) = 0 \), that is, \( V_V = 0 \). Hence we conclude that \( V = 0 \), that is, \( T(\pi^*\mathcal{F}) \) is nondegenerate in \( (T(C(M)), g_\theta) \).

Note now that \( F_\theta(S, S) = 0 \), and hence \( F_\theta \) is indefinite on \( T(\pi^*\mathcal{F}) \). Since \( F_\theta \) is nondegenerate on \( T(\pi^*\mathcal{F}) \), there \( F_\theta \) must have signature \( (2n - 1 - q, 1) \). Yet \( F_\theta \) is a Lorentz metric, therefore \( F_\theta \) is positive definite on \( T(\pi^*\mathcal{F})^\perp \). Consequently, the metric \( h_\theta(r, s) = F_\theta(\rho(r), \rho(s)), r, s \in \nu(\pi^*\mathcal{F}) \), induced by \( F_\theta \) on the normal bundle \( \nu(\pi^*\mathcal{F}) \) of \( \pi^*\mathcal{F} \) is positive-definite, where \( \rho : \nu(\pi^*\mathcal{F}) \to T(\pi^*\mathcal{F})^\perp \) is the natural isomorphism. This proves (2).

To prove (3), note first that \( \mathcal{L}_{\hat{X}} h_\theta = 0 \) if and only if

\[
\hat{X}(F_\theta(V, W)) = F_\theta([\hat{X}, V], W) + F_\theta(V, [\hat{X}, W])
\]

for any \( \hat{X} \in T(\pi^*\mathcal{F}) \) and \( V, W \in T(\pi^*\mathcal{F})^\perp \). We now need the following
Lemma 1. \( T(\pi^*\mathcal{F})^\perp \subseteq \text{Ker}(\eta) \) and consequently

\[ (2.8) \quad \text{Ker}(\eta) = T(\mathcal{F})^\dagger \oplus T(\pi^*\mathcal{F})^\perp. \]

Moreover, \( d\pi\left( T(\pi^*\mathcal{F})^\perp \right) \subseteq H(M) \).

Proof of Lemma 1. For any

\[ V \in T(\pi^*\mathcal{F})^\perp \subset T(C(M)) = \text{Ker}(\eta) \oplus \text{Ker}(d\pi), \]

one has the decomposition \( V = V_H + fS, \ V_H \in \text{Ker}(\eta) \). On the other hand, since

\[ F_\theta(S, T^\dagger) = (\pi^*\theta)(T^\dagger)\eta(S) = \theta(d\pi(T^\dagger))/n + 2 = 1/(n + 2), \]

we have

\[ F_\theta(V, T^\dagger) = f/(n + 2) + F_\theta(V_H, T^\dagger). \]

As \( T \in T(\mathcal{F}) \), it follows that

\[ T^\dagger \in T(\mathcal{F})^\dagger \subset T(\pi^*\mathcal{F}), \]

so that \( T^\dagger \) is orthogonal to \( V \). Hence we obtain

\[ f = -(n + 2)F_\theta(V_H, T^\dagger) = -(n + 2)(\pi^*\tilde{G}_\theta)(V_H, T) \]

\[ = -(n + 2)\tilde{G}_\theta(d\pi(V_H), T) = 0, \]

that is, \( V \in \text{Ker}(\eta) \). Then the identity (2.8) follows, by (2.2), from the facts

\[ T(C(M)) = \{T(\mathcal{F})^\dagger \oplus \text{Ker}(d\pi)\} \oplus T(\pi^*\mathcal{F})^\perp, \]

\[ T(\mathcal{F})^\dagger \oplus T(\pi^*\mathcal{F})^\perp \subseteq \text{Ker}(\eta). \]

To prove the last statement in Lemma 1, let

\[ V \in T(\pi^*\mathcal{F})^\perp \subset T(C(M)) = H(M)^\dagger \oplus RT^\dagger, \]

that is, \( V = Y^\dagger + fT^\dagger \) for some \( Y \in H(M) \). Since \( S \in \text{Ker}(d\pi) \subset T(\pi^*\mathcal{F}) \), \( S \) and \( V \) are orthogonal. Thus we have

\[ 0 = F_\theta(V, S) = F_\theta(Y^\dagger, S) + f/(n + 2) \]

\[ = (\pi^*\theta)(Y^\dagger)\eta(S) + f/(n + 2). \]

Hence \( \theta(Y) = 0 \) yields \( f = 0 \). Lemma 1 is proved. \[ \square \]

By Lemma 1, (2.7) holds if and only if it is satisfied for vector fields \( V, W \) of the form \( V = Y^\dagger, W = Z^\dagger \) for some \( Y, Z \in H(M) \). Also, (2.7) is identically satisfied when \( \tilde{X} \in \text{Ker}(d\pi) \). Indeed, if this is the case, then (by a result in [25], Vol. I, p. 78) one has \([\tilde{X}, Y^\dagger] = [\tilde{X}, Z^\dagger] = 0\). Hence

\[ \tilde{X}(F_\theta(Y^\dagger, Z^\dagger)) = \tilde{X}(G_\theta(Y, Z) \circ \pi) = 0, \]

since \( d\pi(X) = 0 \).

Assume from now on that \( \tilde{X} \in T(\mathcal{F})^\dagger \), that is, \( \tilde{X} = X^\dagger \) for some \( X \in T(\mathcal{F}) \). By Proposition 1.3 in [25], Vol. I, p. 65, \([X, Y]^\dagger\) is the Ker(\eta)-component of
\[ [X^\dagger, Y^\dagger]. \] Then it follows from \( \theta(Y) = \theta(Z) = 0 \) that the identity (2.7) is equivalent to

\begin{equation}
X(\hat{G}_\theta(Y, Z)) = \hat{G}_\theta([X, Y], Z) + \hat{G}_\theta(Y, [X, Z])
\end{equation}

for any \( X \in T(F) \) and \( Y, Z \in H(M) \) such that \( Y^\dagger, Z^\dagger \in T(\pi^*F)^\perp \). Finally, note that for each \( V = \pi Y \in T(\pi^*F)^\perp \) with \( Y \in H(M) \) one has

\[ 0 = F_\theta(X^\dagger, Y) = \hat{G}_\theta(X, Y) \circ \pi \]

for any \( X \in T(F) \), and hence

\[ Y \in H(M) \cap T(F)^\perp = [T(F)H(M)]^\perp. \]

Therefore, \( \mathcal{L}_X h_\theta = 0 \) if and only if (2.9) holds for any \( X \in T(F) \) and \( Y, Z \in [T(F)H(M)]^\perp \), that is, if and only if \( \mathcal{L}_X F_\theta = 0 \). This completes the proof of Proposition 2. \( \square \)

### 2.4 Foliated Lorentz manifolds

Let \( N \) be a \( C^\infty \) manifold and \( F \) a codimension \( q \) foliation of \( N \). A differential \( p \)-form \( \omega \) on \( N \) is called basic if

\[ X|\omega = 0, \quad \mathcal{L}_X \omega = 0 \]

for all \( X \in T(F) \). Note that the exterior derivative \( d \) preserves basic forms, since

\[ X|d\omega = \mathcal{L}_X \omega - d(X|\omega) = 0, \quad \mathcal{L}_X d\omega = d\mathcal{L}_X \omega = 0. \]

Hence, denoting by \( \Omega^p_B(F) \) the set of basic \( p \)-forms, we obtain the basic complex of \( F \) (cf. [41], p. 119)

\[ \Omega^0_B(F) \xrightarrow{d_B} \Omega^1_B(F) \xrightarrow{d_B} \cdots \xrightarrow{d_B} \Omega^p_B(F) \xrightarrow{d_B} 0, \]

where \( d_B = d|\Omega_B \), and the corresponding basic cohomology of \( F \)

\[ H^j_B(F) = H^j(\Omega^*_B(F), d_B), \quad 0 \leq j \leq q. \]

Also, we consider the following multiplicative filtration of the de Rham complex \( \Omega^*(N) \) (a decreasing filtration by differential ideals, cf. [41], p. 120)

\[ F^r \Omega^m = \{ \omega \in \Omega^m(N) \mid X_1 \cdots X_{m-r+1} | \omega = 0 \text{ for } X_1, \cdots, X_{m-r+1} \in T(F) \}. \]

Let us now consider a codimension \( q \) foliation \( F \) on an \( n \)-dimensional connected Lorentz manifold \((N, g)\) such that \( T(F) \) is nondegenerate in \((T(N), g)\). The second fundamental form \( \alpha \) of \( F \) in \((N, g)\) is defined by

\[ \alpha: T(F) \otimes T(F) \to \nu(F), \quad \alpha(X, Y) = \Pi(\nabla_X Y), \quad X, Y \in T(N), \]

where \( \nabla^N \) is the Levi-Civita connection of \((N, g)\). As in the Riemannian case, since \( \nabla^N \) is torsion-free, the involutivity of \( T(F) \) implies that \( \alpha \) is symmetric. Next, by mere linear algebra (cf., e.g., [34], p. 49), \( T(N) = T(F) \oplus T(F)^\perp \) and we have a bundle isomorphism

\[ \sigma: Q = \nu(F) \to T(F)^\perp, \quad \sigma(s) = \text{the } T(F)\text{-component of } Y_s, \quad s \in \nu(F), \]

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where $Y_z \in T(N)$ with $\Pi(Y_z) = s$. Let us consider the induced metric

$$g_Q(r, s) = g(\sigma(r), \sigma(s)), \quad r, s \in \nu(F).$$

Set $\text{ind}(F) = -1$ if each leaf $L$ of $F$ is Lorentz, and $\text{ind}(F) = 1$ if each leaf $L$ of $F$ is Riemannian, with the induced metric $g_L = \iota^* g$, where $\iota : L \hookrightarrow N$, respectively. We remark here that if $g$ is a bundle-like metric for $F$, then no other possibility occurs. Indeed, let $\nabla$ be the connection in $Q \to N$ given by $\nabla_X s = \Pi[X, \sigma(s)]$ if $X \in \Gamma^\infty(P)$ and $\nabla_X s = \Pi X \sigma(s)$ if $X \in \Gamma^\infty(P^\perp)$ for any $X \in \mathfrak{X}(N)$ and $s \in \Gamma^\infty(Q)$. A verbatim repetition of the proof of Theorem 5.11 in [41], p. 53, shows that $g$ is bundle-like for $F$ if and only if $\nabla g_Q = 0$. Let $x, y \in N$ and $\alpha(t)$ a piecewise smooth curve joining $x$ and $y$. If $g$ is bundle-like a standard argument based on $\nabla$-parallel translation along $\alpha$ shows that $\text{ind}(g_Q)_x = \text{ind}(g_Q)_y$, where $\text{ind}(g_Q)$ is the index of $g_Q$ (in the sense of [34], p. 54), i.e., $(Q, g_Q)$ is a semi-Riemannian bundle. Let $g_P$ be the leafwise induced metric, i.e. if $x \in N$ and $L \in N/F$ is the leaf through $x$ then $g_P_x := (\iota^*g)_x = g_{L_x}$. Clearly $\text{ind}(g) = \text{ind}(g_P) + \text{ind}(g_Q)$ at each point of $N$, and hence $g_P$ has constant index.

For any $Z \in T(F)^\perp$, the Weingarten map $W(Z) : T(F) \to T(F)$ of $F$ is given by

$$g_P(W(Z)(X), X') = g_Q(\alpha(X, X'), \sigma^{-1}(Z)), \quad X, X' \in T(F).$$

Then $W(Z)$ is self-adjoint. The mean curvature form of $F$ in $(N, g)$ is the 1-form $\kappa \in \Omega^1(N)$ defined by

$$\kappa(Z) = \text{trace } W(Z), \quad Z \in \Gamma^\infty(P^\perp),$$

$$X | \kappa = 0, \quad X \in \Gamma^\infty(P).$$

Assume from now on that $g$ is bundle-like for $F$. Also, we assume that $F$ is tangentially oriented, that is, $F$ is equipped with a principal $GL^+(p, \mathbb{R})$-subbundle $B \to N$ of the principal $GL(p, \mathbb{R})$-bundle $L(P) \to N$, where $p = n-q$ and $L(P)_x$ is the set of $\mathbb{R}$-linear isomorphisms $u : \mathbb{R}^p \to P_x$, $x \in N$. Let $\{E_1, \ldots, E_p\}$ be a local $g_P$-orthonormal frame of $P$, adapted to $B$, defined on an open set $U \subset N$, satisfying $g_P(E_i, E_j) = \delta_{ij}$ with $\delta_{ij} = 1$ (thus $\text{ind}(F) = \varepsilon_1 \cdots \varepsilon_p$). The characteristic form of $F$ is a $p$-form $\chi_F \in \Omega^p(N)$ defined by

$$\chi_F(Y_1, \ldots, Y_p) = \det \left( g(Y_i, E_j) \right), \quad Y_1, \ldots, Y_p \in \Gamma(T(N)).$$

Note that $P^\perp | \chi_F = 0$. The Lorentian analogue of Rummler’s formula (cf., e.g., [41], p. 68) still holds, namely,

$$(2.10) \quad Z | d\chi_F + \kappa(Z)\chi_F = 0 \quad \text{along } P$$

for any $Z \in \Gamma^\infty(P^\perp)$. Indeed, as $\chi_F(E_1, \ldots, E_p) = \text{ind}(F)$,

$$(\mathcal{L}_Z \chi_F)(E_1, \ldots, E_p) = - \sum_{i=1}^p \chi_F(E_1, \ldots, \pi^+([Z, E_i]), \ldots, E_p)$$

$$= - \sum_{i=1}^p \varepsilon_i \text{ind}(F) g([Z, E_i], E_i),$$
where $\mathcal{L}_Z$ is the Lie differentiation and $\pi^\perp : T(N) \to P$ is the natural bundle morphism. On the other hand, by the definition of $\kappa$, we obtain

$$\kappa(Z) = \sum_{i=1}^p \epsilon_i g_p W(Z)(E_i), \quad E_i = \sum_{i=1}^p \epsilon_i g([Z, E_i], E_i),$$

and thus (2.10) is proved.

Assume further that $\mathcal{F}$ is transversally oriented (i.e., $P^\perp$ is oriented), and let $\nu$ be the characteristic form of $P^\perp$ defined in a completely analogous manner with $\chi_\mathcal{F}$. Let $\mu = d\operatorname{vol}(g)$ be the Lorentz volume form on $N$. Assume also that $N$ is oriented, and let $\{E_A \mid 1 \leq A \leq n\}$ be an oriented local orthonormal frame, satisfying $g(E_A, E_B) = \epsilon_A \delta_{AB}$, of $T(N)$ such that $\{E_i \mid 1 \leq i \leq p\}$ and $\{E_\alpha \mid p + 1 \leq \alpha \leq n\}$ are frames in $P$ and $P^\perp$, respectively, and denote its dual coframe by $\{\omega_A \mid 1 \leq A \leq n\}$. Then for any $\alpha \in \Omega^r(N)$

$$(\ast \alpha)(E_{A_1}, \ldots, E_{A_{n-r}}) \cdot \mu = \epsilon_{A_1} \cdots \epsilon_{A_{n-r}} \omega_{A_1} \wedge \cdots \wedge \omega_{A_{n-r}},$$

where $\ast : \Omega^r(N) \to \Omega^{n-r}(N)$ is the Hodge operator. In particular, for $\nu$ we have

$$(\ast \nu)(E_1, \ldots, E_p) \cdot \mu = \operatorname{ind}(\mathcal{F}) \nu \wedge \omega_1 \wedge \cdots \wedge \omega_p. \tag{2.11}$$

Also, a calculation based on the identity

$$\nu(Y_{p+1}, \ldots, Y_n) = \det(g(Y_{\alpha}, E_{\beta}))$$

leads to

$$\nu = \epsilon_{p+1} \cdots \epsilon_n \omega_{p+1} \wedge \cdots \wedge \omega_n.$$

Hence $\nu$ is proportional to the transverse volume element $\omega_{p+1} \wedge \cdots \wedge \omega_n$ of $\mathcal{F}$, and (2.11) may be written as

$$(\ast \nu)(E_1, \ldots, E_p) \cdot \mu = (-1)^{pq+1} \omega_1 \wedge \cdots \wedge \omega_n.$$

Since $\ast \nu, \chi_\mathcal{F} \in \Omega^p(N)$ and $P^\perp \mid \ast \nu = 0, P^\perp \mid \chi_\mathcal{F} = 0$, there is a function $f \in C^\infty(N)$ such that $\ast \nu = f \chi_\mathcal{F}$. A calculation shows that $f = (-1)^{pq+1} \operatorname{ind}(\mathcal{F})$ so that

$$(\ast \nu) = (-1)^{pq+1} \operatorname{ind}(\mathcal{F}) \chi_\mathcal{F}. \tag{2.12}$$

As a corollary of (2.12), we have

$$\nu \wedge \chi_\mathcal{F} = (-1)^{pq} \operatorname{ind}(\mathcal{F}) \mu. \tag{2.13}$$

At this point we may prove the following

**Proposition 3.** Let $\mathcal{F}$ be a transversally oriented foliation on a compact orientable Lorentz manifold $(N, g)$. Assume that the transverse volume element $\nu$ of $\mathcal{F}$ is holonomy invariant; hence $\nu \in \Omega^p_B(\mathcal{F})$ and $\operatorname{div} \nu = 0$. If $\mathcal{F}$ is harmonic (i.e., $\kappa = 0$), then $[\nu] \neq 0$ in $H^p_B(\mathcal{F})$.

The proof is a verbatim repetition of the proof of Theorem 9.21 in [41], p. 124 (and Proposition 3 is the Lorentzian analogue of a result by Kamber and Tondeur [23]). Indeed, Rummell’s formula (2.10) yields (when $\kappa = 0$)

$$d\chi_\mathcal{F} \in H^2 \Omega^{p+1},$$
and the assumption that \( \nu = d_B \alpha \) for some \( \alpha \in \Omega_B^{n-1}(\mathcal{F}) \) leads, by (2.13), to

\[
d(\alpha \wedge \chi_\mathcal{F}) = (-1)^q \text{ind}(\mathcal{F}) \mu,
\]
and then, by Green’s lemma, to a contradiction.

We may also establish the following

**Proposition 4.** Let \( \mathcal{F} \) be a foliation on a strictly pseudoconvex CR manifold \( M \), and assume that \( \mathcal{F} \) is tangent to the characteristic direction \( T \) of \((M, 0) \), for some contact form \( \theta \) on \( M \). Then the following hold:

1. \( \mathcal{F} \) is transversally oriented if and only if \( \pi^* \mathcal{F} \) is transversally oriented and, if this is the case, the transverse volume element \( \nu \) of \( \mathcal{F} \) in \((M, g_\theta)\) is holonomy invariant if and only if the transverse volume element \( \nu \) of \( \pi^* \mathcal{F} \) in \((C(M), F_\theta)\) is holonomy invariant.

2. \( \mathcal{F} \) is harmonic in \((M, g_\theta)\) if and only if \( \pi^* \mathcal{F} \) is harmonic in \((C(M), F_\theta)\).

Proposition 3 together with Proposition 4 then shows that for any transversally oriented codimension \( q \) foliation \( \mathcal{F} \) on a compact strictly pseudoconvex CR manifold \( M \), if 1) \( \mathcal{F} \) is tangent to the characteristic direction \( T \) of \((M, 0) \), 2) the transverse volume element \( \nu \) of \( \mathcal{F} \) in \((M, g_\theta)\) is holonomy invariant, and 3) \( \mathcal{F} \) is harmonic in \((M, g_\theta)\), then \( \nu \neq 0 \) in \( H^2_B(\mathcal{F}) \). Indeed, if \( M \) is compact, then so is \( C(M) \) and, given a local coordinate system \((U, x^A)\) on \( M \), \( (\pi^{-1}(U), u^A = x^A \circ \pi, u^{2n+2} = \gamma) \) yields a local coordinate on \( C(M) \). Hence an orientation of \( M \) induces that of \( C(M) \).

This is only illustrative of our ideas as to the use of the Fefferman metric. [The preceding statement also follows by directly applying the aforementioned result of Kamber and Tondeur (Theorem 9.21 in [23], p. 124) to \( \mathcal{F} \) on \((M, g_\theta)\).]

We may further exploit the relationship between pseudohermitian geometry and conformal Lorentzian geometry to prove the following

**Corollary 1.** Let \( \mathcal{F} \) be a transversally oriented codimension \( q \) foliation on a compact strictly pseudoconvex CR manifold \( M \), which is tangent to the characteristic direction \( T \) of \((M, 0) \) for a fixed contact form \( \theta \). Assume that the transverse volume element \( \nu \) of \( \mathcal{F} \) in \((M, g_\theta)\) is holonomy invariant and that the mean curvature form \( \kappa \) of \( \mathcal{F} \) in \((M, g_\theta)\) is closed (i.e., \( d\kappa = 0 \)). If \( [\kappa] = 0 \) in \( H^1_B(\mathcal{F}) \), then \( H^2_B(\mathcal{F}) \neq 0 \).

**Proof.** We shall need the following

**Lemma 2.** Let \( \mathcal{F} \) be a transversally oriented codimension \( q \) foliation on an \( n \)-dimensional Lorentz manifold \((N, g)\), and assume that \( T(\mathcal{F}) \) is nondegenerate in \((T(N), g)\). Then \( \mathcal{F} \) is harmonic in \((N, e^{2u}g)\), with \( u \in C^\infty(N) \), if and only if

\[
du(Z) = p^{-1} \kappa(Z), \quad Z \in T(\mathcal{F})^\perp,
\]
where \( p = n - q \) and \( \kappa \) is the mean curvature form of \( \mathcal{F} \) in \((N, g)\). Furthermore, the following are equivalent:

1. \( u \) is a basic function, i.e., \( u \in \Omega_B^0(\mathcal{F}) \).
(2) The transverse volume element $\tilde{\nu}$ of $\mathcal{F}$ in $(N,\hat{g} = e^{2u}g)$ is holonomy invariant if and only if the transverse volume element $\nu$ of $\mathcal{F}$ in $(N,g)$ is holonomy invariant.

The statement (1) in Lemma 2, that is, if $du = p^{-1}\kappa$, then the leaves of $\mathcal{F}$ are minimal in $(N, e^{2u}g)$, was first discovered in [24] for the case of a Riemannian metric $g$ (and our argument below follows closely the proof of Proposition 12.6 in [41], p. 151). The relationship between the Levi-Civita connections $\nabla^g$ and $\nabla^{\hat{g}}$ of Lorentz metrics $\hat{g} = e^{2u}g$ and $g$, respectively, is given by

$$\nabla^{\hat{g}} = \nabla^g + (du) \otimes I + I \otimes (du) - g \otimes \text{grad}_g u,$$

which implies that the second fundamental forms $\hat{\alpha}$ and $\alpha$ of $\mathcal{F}$ in $(N,\hat{g})$ and $(N,g)$, respectively, are related by

$$\hat{\alpha} = \alpha - g \otimes \Pi(\text{grad}_g u).$$

Hence, for the corresponding Weingarten maps, we have

$$\hat{W}(Z) = W(Z) - du(Z) I, \quad Z \in \Gamma^\infty(P^\perp),$$

where $I$ denotes the identity transformation. Consequently, the corresponding mean curvature forms satisfy

$$\hat{\kappa}(Z) = \kappa(Z) - p \ Z(u),$$

where $p = n - q$. Therefore $\hat{\kappa} = 0$ if and only if $u$ satisfies (2.14).

The statement (2) in Lemma 2 follows from the formula

$$\mathcal{L}_X \tilde{\nu} = e^{3u} \{ q \ du(X) \nu + \mathcal{L}_X \nu \}, \quad X \in \Gamma^\infty(P).$$

This completes the proof of Lemma 2.

Let us go back to the proof of Corollary 1. Since $[\kappa] = 0$ in $H^1_B(\mathcal{F})$ by assumption, there is a basic function $v \in \Omega^0_B(\mathcal{F})$ such that $\kappa = dv$. Set $u = (p + 1)^{-1} v$, where $p = 2n + 1 - q$ and $\dim M = 2n + 1$. Then it is immediate from (2.17) in the proof of Proposition 4 that

$$d(u \circ \pi)(Z^\perp) = (p + 1)^{-1} \hat{\kappa}(Z^\perp), \quad Z \in T(\mathcal{F})^\perp.$$

Hence it follows from Lemma 2 that $\pi^* \mathcal{F}$ is harmonic in $(C(M), e^{2u*}F^\delta)$, from which we see that $\pi^* \mathcal{F}$ is harmonic in $(C(M), F^\delta)$, since, by a result of Lee [29], the Fefferman metric changes conformally $F^\delta = e^{3u*}F^\delta$ under a transformation $\hat{\delta} = e^{2u}\delta$.

Now note that, by Proposition 4, the transverse volume element $\tilde{\nu}$ of $\pi^* \mathcal{F}$ in $(C(M), F^\delta)$ is holonomy invariant. Since $u \in \Omega^0_B(\mathcal{F})$, it follows that $u \circ \pi \in \Omega^0_B(\pi^* \mathcal{F})$ so that, again by Lemma 2, the transverse volume element $\tilde{\nu}$ of $\pi^* \mathcal{F}$ in $(C(M), F^\delta)$ is holonomy invariant. In consequence, by Proposition 3, we may conclude that $0 \neq H^3_B(\pi^* \mathcal{F}) \cong H^3_B(\mathcal{F})$.

Proof of Proposition 4. First we note that

$$\nu(\mathcal{F}) \approx T(\mathcal{F})^\perp \xrightarrow{\hat{\delta}} [T(\mathcal{F})^\perp]^\perp = T(\pi^* \mathcal{F})^\perp \approx \nu(\pi^* \mathcal{F}),$$

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from which it is immediate that $\mathcal{F}$ is transversally oriented if and only if so is $\pi^\ast \mathcal{F}$. We only need to justify here the equality in the sequence. To this end, let $\tilde{X} \in T(\pi^\ast \mathcal{F})$ and write $\tilde{X} = X^\top + V$ for some $X \in P = T(\mathcal{F})$ and $V \in \mathcal{V} = \text{Ker}(d\pi)$. Then for any $Y \in P^\perp$ we have

$$F_\theta(\tilde{X}, Y^\top) = (\pi^\ast \tilde{G}_\theta)(\tilde{X}, Y^\top) + (\pi^\ast \theta)(Y^\top)\eta(\tilde{X})$$

$$= \tilde{G}_\theta(X, Y) + \theta(Y)\eta(V) = \tilde{G}_\theta(\pi_H(X) + \theta(X)T, Y),$$

since $P^\perp \subset H(M)$. Therefore we see

$$F_\theta(\tilde{X}, Y^\top) = G_\theta(\pi_H(X), Y) = g_\theta(\pi_H(X) + \theta(X)T, Y)$$

$$= g_\theta(X, Y) = 0,$$

which shows $[P^\perp]^\top \subset T(\pi^\ast \mathcal{F})^\perp$ and hence the desired equality holds, for both bundles have rank $q$.

At this point we may relate the Weingarten maps of $\mathcal{F}$ and $\pi^\ast \mathcal{F}$. Let $\nabla^C(M)$ be the Levi-Civita connection of $(C(M), F_\theta)$. Given $Z \in P^\perp$, the Weingarten map $\tilde{W}(Z^\top): T(\pi^\ast \mathcal{F}) \to T(\pi^\ast \mathcal{F})$ of $\pi^\ast \mathcal{F}$ is given by

$$F_\theta(\tilde{W}(Z^\top)(\tilde{X}), \tilde{X}') = F_\theta(\nabla^{C(M)}_\tilde{X}, \tilde{X}')$$

for any $\tilde{X} = X^\top + V$ and $\tilde{X}' = X'^\top + V'$, where $X, X' \in P$ and $V, V' \in \mathcal{V}$. As $\pi: C(M) \to M$ is a principal $S^1$-bundle, the projection $\pi$ is a submersion. Recall, however, that for the vector field $S = \partial/\partial \gamma$ tangent to the $S^1$-action, $F_\theta(S, S) = 0$ and hence $S$ is null, or isotropic, so that $\pi$ is not a semi-Riemannian submersion (according to the terminology adopted in [34], p. 212). Nevertheless, we may relate $\nabla^C(M)$ to $\nabla^M$, in the spirit of [35]. Another difficulty is that $\text{Ker}(\eta)$ and $\mathcal{V}$ are not orthogonal (with respect to the Fefferman metric $F_\theta$), yet $H(M)^{\perp} \perp \mathcal{V}$ does hold. Noting that $[Y^\top, V] = 0$ for any $Y \in T(M)$, a calculation then leads to

$$2F_\theta(\tilde{W}(Z^\top)(\tilde{X}), \tilde{X}') = -Z(\tilde{G}_\theta(X, X')) - \tilde{G}_\theta(X, [X', Z]) - \tilde{G}_\theta(X', [X, Z])$$

$$+ \theta(X)\Omega(X^\top, Z^\top) + \theta(X')\Omega(X^\top, Z^\top)$$

$$+ d\theta(X, Z)\eta(V^\top) + d\theta(X', Z)\eta(V),$$

where $\Omega = D\eta$ is the curvature 2-form of $\eta$. Let $H(\mathcal{F})$ be the $g_\theta$-orthogonal complement of $R T$ in $P$. In particular, for any $X, X' \in H(\mathcal{F})$

$$2F_\theta(\tilde{W}(Z^\top)(\tilde{X}), \tilde{X}') = -Z(g_\theta(X, X')) - g_\theta(X, [X', Z]) - g_\theta(X', [X, Z])$$

$$+ d\theta(X, Z)\eta(V^\top) + d\theta(X', Z)\eta(V),$$

or (by exploiting the explicit expression of $\nabla^M$, cf., e.g., [25], p. 160)

$$2F_\theta(\tilde{W}(Z^\top)(\tilde{X}), \tilde{X}') = 2g_\theta(W(Z)(X), X')$$

$$+ d\theta(X, Z)\eta(V^\top) + d\theta(X', Z)\eta(V).$$

Similarly, we also have

$$F_\theta(\tilde{W}(Z^\top)(T^\top), T^\top) = \Omega(T^\top, Z^\top).$$
Next, we may calculate \( \tilde{\kappa}(Z^\dagger) = \text{trace } \tilde{W}(Z^\dagger) \). Let \( \{E_1, \ldots, E_{p-1}, T\} \) be a \( g_\theta \)-orthonormal frame of \( T(\mathcal{F}) \). Then

\[
\left\{ E_1^\dagger, \ldots, E_{p-1}^\dagger, T^\dagger + \frac{n+2}{2}S, T^\dagger - \frac{n+2}{2}S \right\}
\]

is an \( \mathcal{F}_0 \)-orthonormal frame of \( T(\pi^* \mathcal{F}) \). Note that \( T^\dagger - (n+2)/2S \) is timelike and \( \text{ind}(\pi^* \mathcal{F}) = -1 \), in particular. Since \( \theta(W(Z)(T)) = 0 \), a straightforward calculation based on (2.15) and (2.16) now leads to

\[
(2.17) \quad \tilde{\kappa}(Z^\dagger) = \kappa(Z) \circ \pi, \quad Z \in P^\perp.
\]

In particular, \( \mathcal{F} \) is harmonic in \( (M, g_\theta) \) if and only if \( \pi^* \mathcal{F} \) is harmonic in \( (C(M), F_\theta) \). Finally, if \( \tilde{\nu} \in \Omega^q(C(M)) \) is given by

\[
\tilde{\nu}(\tilde{Y}_1, \ldots, \tilde{Y}_q) = \text{det } \left( F_\theta(\tilde{Y}_\alpha, E_\alpha^\dagger) \right)
\]

for some oriented \( g_\theta \)-orthonormal frame \( \{E_\alpha \mid 1 \leq \alpha \leq q\} \) of \( P^\perp \), then \( \tilde{\nu} = \pi^* \nu \) and by a simple calculation we see that

\[
\mathcal{L}_{\tilde{X}} \tilde{\nu} = \pi^*(\mathcal{L}_X \nu)
\]

for any \( \tilde{X} = X^\dagger + V \) with \( X \in P \) and \( V \in \mathcal{V} \). Proposition 4 is now proved. \( \square \)

3 Tangentially CR foliations

Let \( (M, T_{1,0}(M)) \) be a CR manifold and \( \mathcal{F} \) a foliation on \( M \). We say that \( \mathcal{F} \) is a (tangentially) CR foliation if each leaf \( L \) of \( \mathcal{F} \) is a CR submanifold of \( M \), that is, \( L \) is a CR manifold and the inclusion \( \iota: L \hookrightarrow M \) is a CR map, i.e. \( d\iota_x(T_{1,0}(L)_x) \subseteq T_{1,0}(M)_x \) for each \( x \in L \).

A typical example of a CR foliation is illustrated by the following

**Example 1 (A CR foliation by level sets).** Let \( C^{n+1}_x \) be the \((n + 1)\)-dimensional complex Euclidean space with complex coordinates \((z^1, \ldots, z^n, w)\), \( w = u + iv \), and \( \alpha : R \to R \) a smooth function such that \( \alpha(0) = 0 \) and \( \alpha'(t) < 0 \) for any \( t \in R \). Define \( f : C^{n+1} \to R \) by

\[
f(z^1, \ldots, z^n, w) = \alpha \left( \left| z^1 \right|^2 + \cdots + \left| z^n \right|^2 - v \right) e^u.
\]

Then \( f \) is a smooth submersion so that it defines a foliation \( \mathcal{F} \) on \( C^{n+1} \) by the level sets of \( f \).

Note that

\[
f^{-1}(c) = \begin{cases} 
\{(z, w) \in \Omega_{n+1} \mid u = \log(c/\alpha(\rho))\} & \text{if } c > 0, \\
\partial \Omega_{n+1} & \text{if } c = 0, \\
\{(z, w) \in C^{n+1} \setminus \overline{\Omega_{n+1}} \mid u = \log(c/\alpha(\rho))\} & \text{if } c < 0,
\end{cases}
\]

where \( \rho = \sum_{n=1}^n |z^n|^2 - v \). Thus \( \mathcal{F} \) is a CR foliation on \( C^{n+1} \), one of whose leaves is the Heisenberg group \( H_n \approx \partial \Omega_{n+1} \).

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Now, let $\mathcal{F}$ be a CR foliation. Let $H(\mathcal{F}) \to M$ denote the subbundle of $T(\mathcal{F})$ whose portion over a leaf $L$ of $\mathcal{F}$ coincides with the Levi distribution $H(L)$ of $L$. Similarly, let $T^1_{1,0}(\mathcal{F}) \to M$ denote the complex subbundle of $T(\mathcal{F}) \otimes \mathbb{C}$ whose portion over a leaf $L$ of $\mathcal{F}$ coincides with the CR structure $T^1_{1,0}(L)$ of $L$.

Assume from now on that $M$ is a nondegenerate CR manifold (of hypersurface type), and fix a contact form $\theta$ on $M$ and the corresponding characteristic direction $T$ of $(M, \theta)$. It should be remarked that if $M$ is strictly pseudoconvex, then $\theta$ is holonomy invariant if and only if $H(\mathcal{F}) = 0$. Indeed, if $X \in H(\mathcal{F}) \subseteq H(M)$, then $\theta(X) = 0$. Hence,

$$0 = L_X \theta = X\lbrack d\theta \rbrack \implies G_\theta(X, X) = 0 \implies X = 0.$$

Recall that a $(0, s)$-form on $M$ is a complex $s$-form $\omega$ on $M$ such that $T^1_{1,0}(M) \lbrack \omega \rbrack = 0$ and $T \lbrack \omega \rbrack = 0$. Let $\Lambda^{0,s}(M) \to M$ be the bundle of $(0, s)$-forms on $M$ and set $\Omega^{0,s}(M) = \Gamma^\infty(\Lambda^{0,s}(M))$. We recall the tangential Cauchy-Riemann operator $\overline{\partial}_M$, which is the first order differential operator

$$\overline{\partial}_M : \Omega^{0,s}(M) \to \Omega^{0,s+1}(M)$$

defined as follows. If $\omega$ is a $(0, s)$-form, then $\overline{\partial}_M \omega$ is the unique $(0, s + 1)$-form which coincides with $d\omega$ on $T^1_{1,0}(M) \otimes \cdots \otimes T^1_{1,0}(M)$ ($s + 1$ terms). A smooth function $f : M \to \mathbb{C}$ is called a CR function if it satisfies the tangential Cauchy-Riemann equation

$$\overline{\partial}_M f = 0.$$

The space of CR functions on $M$ is denoted by $CR^\infty(M)$.

### 3.1 The basic tangentially Cauchy-Riemann complex

We say that $\omega \in \Omega^{0,s}(M)$ is a basic $(0, s)$-form if it satisfies

$$\overline{\partial}_M \omega = 0 \quad \text{for any } Z \in T^1_{1,0}(\mathcal{F}).$$

Let $\Omega^{0,s}_B(\mathcal{F})$ denote the space of all basic $(0, s)$-forms on $(M, \mathcal{F})$. Since $\overline{\partial}_M = 0$, we see easily that

$$\overline{\partial}_M \Omega^{0,s}_B(\mathcal{F}) \subseteq \Omega^{0,s+1}_B(\mathcal{F}).$$

Let $CR^\infty(\mathcal{F})$ be the space of smooth functions $f : M \to \mathbb{C}$ whose restriction $f|_L$ to each leaf $L$ of $\mathcal{F}$ is a CR function on $L$, namely $f|_L \in CR^\infty(L)$. Note that $CR^\infty(M) \subseteq CR^\infty(\mathcal{F})$. Moreover, we have

$$\Omega^{0,s}_B(\mathcal{F}) = CR^\infty(\mathcal{F}).$$

Let $\overline{\partial}_B$ be the restriction of $\overline{\partial}_M$ to $\Omega^{0,s}_B(\mathcal{F})$. Then we obtain a complex

$$H^{0,s}_B(\mathcal{F}) = H^s(\Omega^{0,s}_B(\mathcal{F}), \overline{\partial}_B) = \frac{\ker \{ \overline{\partial}_B : \Omega^{0,s}_B(\mathcal{F}) \to \Omega^{0,s+1}_B(\mathcal{F}) \}}{\overline{\partial}_B \Omega^{0,s-1}_B(\mathcal{F})},$$

which is called the basic tangentially Cauchy-Riemann complex of $(M, \mathcal{F})$. Here we suppose that $\dim M = 2N + 1$ and $\mathcal{F}$ has codimension $q = 2k$. For the remainder of this section, we set $n = N - k$ and assume $n \geq 1$. The cohomology of the complex (3.1) given by
where \( 0 \leq s \leq k \), is called the basic Kohn-Rossi cohomology of \((M, \mathcal{F})\). In particular, we obtain
\[
H^0_B(\mathcal{F}) = \text{Ker} \{ \partial_B | \Omega^0_B(\mathcal{F}) \to \Omega^1_B(\mathcal{F}) \} = \{ f \in C^\infty(\mathcal{F}) | \partial_B f = 0 \}
= \mathcal{C}^\infty(\mathcal{F}) \cap \mathcal{C}^\infty(M) = \mathcal{C}^\infty(M).
\]

Let
\[
H^{0, s}_B(M) = H^s(\Omega^{0, *}_B(M), \partial_B)
\]
be the ordinary Kohn-Rossi cohomology of the CR manifold \( M \). For any CR foliation \( \mathcal{F} \) on a nondegenerate CR manifold, there exists a natural injection of \( H^{0,1}_B(\mathcal{F}) \) into the Kohn-Rossi cohomology group \( H^{1,1}(M) \), namely the map
\[
(3.2) \quad H^{1,1}_B(\mathcal{F}) \hookrightarrow H^{1,1}(M), \quad [\omega] \mapsto [\omega]_{H^{1,1}(M)}
\]
is a monomorphism. Here \( \omega \in \Omega^{0,1}_B(\mathcal{F}) \) with \( \partial_B \omega = 0 \). Indeed, if \( \omega, \omega' \in \text{Ker} \{ \partial_B : \Omega^{0,1}_B(\mathcal{F}) \to \Omega^{1,2}_B(\mathcal{F}) \} \) lie in the same Kohn-Rossi cohomology class, then \( \omega' - \omega = \partial_M f \) for some smooth function \( f : M \to \mathbb{C} \). Then
\[
0 = \partial_M \omega' = \partial_M \omega + \partial_M f 
\]
for any \( Z \in T_{1,0}(\mathcal{F}) \), and hence \( f \in \Omega^{0,0}_B(\mathcal{F}) \). Thus \( \omega' - \omega = \partial_B f \), that is, \([\omega] = [\omega']\).

**Remark 2.** When \( M \) has CR codimension \( 0 \), that is, \( M \) is a complex manifold, \( \Omega^{0, s}_B(M) \) is the space of all \((0, s)\)-forms, which are locally spanned by monomials containing \( s \) anti-holomorphic differentials \( d\bar{z}^\alpha \), with respect to local complex coordinates \( z^\alpha \) on \( M \). Note that \( H^{0, s}_B(M) \) is then the Dolbeault cohomology, and given a foliation \( \mathcal{F} \) on \( M \) by CR submanifolds, \((3.2)\) still holds.

**Example 1** (continued). Set \( \beta(t) = \alpha'(t)/(\alpha(t) - i\alpha'(t)) \). Then \( T_{1,1}(\mathcal{F}) \) is spanned by
\[
(3.3) \quad \partial_{\bar{z}^\alpha} = \frac{\partial}{\partial z^\alpha} - 2\beta(\bar{z}) z^\alpha \frac{\partial}{\partial \bar{z}^\alpha}, \quad 1 \leq \alpha \leq n,
\]
where we set \( z^\alpha = z^0 \). Note that \( \beta(0) = i \), and hence along the leaf \( \partial \Omega_{n+1} \) of \( \mathcal{F} \) the vector fields \((3.3)\) correspond, under the CR isomorphism \( \partial \Omega_{n+1} \cong \mathcal{H}_n \), to the Levi operators.

We remark that \( H^{0, s}_B(\mathcal{F}) = 0 \) for \( s \in \{1, 2\} \). Indeed, first it follows from \((3.2)\) that \( H^{0,1}_B(\mathcal{F}) \hookrightarrow H^{1,1}(\mathbb{C}^{n+1}) = 0 \). On the other hand, if we define a \((0, 1)\)-form \( \Theta \in \Omega^{0,1}(\mathbb{C}^{n+1}) \) by
\[
\Theta = \partial \bar{\omega} + 2\beta(\bar{\rho}) z^\alpha d\bar{z}^\alpha,
\]
then we obtain that
\[
\Omega^1_B(\mathcal{F}) = \{ \lambda \Theta | \lambda \in \mathcal{C}^\infty(\mathbb{C}^{n+1}) \}, \quad \partial_B \Omega^1_B(\mathcal{F}) = \{ \partial f/\partial \bar{\omega} | \Theta | f \in \mathcal{C}^\infty(\mathcal{F}) \},
\]
and \( \partial_B \omega = 0 \) for any \( \omega \in \Omega^1_B(\mathcal{F}) \) as seen below. Thus the meaning of the fact \( H^{0,1}_B(\mathcal{F}) = 0 \) is that the system
\[
\frac{\partial f}{\partial \bar{\omega}} = 2\beta(\bar{\rho}) z^\alpha \lambda, \quad \frac{\partial f}{\partial \bar{\omega}} = \lambda
\]

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admits a solution \( f \in C^{\infty}(\mathbb{C}^{n+1}) \), provided that \( \lambda \) satisfies the compatibility relations 
\[
\overline{\partial}_M \lambda (\overline{\partial}_M \lambda) + i \beta(\rho) \bar{z}_\alpha \lambda = 0.
\]

To compute \( H_B^{2,0}(\mathcal{F}) \), let \( \omega = \omega_{\alpha\beta} \, d\bar{z}_\alpha \wedge d\bar{z}_\beta + \omega_\alpha \, d\bar{z}_\alpha \wedge d\bar{\omega} \) be a basic \((0, 2)\)-form. Then we see that the condition
\[
0 = \overline{\partial}_M \omega = 2(\omega_{\alpha\beta} + \beta(\rho) \bar{z}_\alpha \omega_\beta) d\bar{z}_\beta + \omega_\alpha d\bar{\omega}
\]
yields that \( \Omega_B^{2,0}(\mathcal{F}) = \{0\} \).

Similar to the above, let \( \Omega^{p,0}(M) \) denote the space of \((p, 0)\)-forms \( \omega \) such that \( T \mid \omega = 0 \), and consider the first order differential operator
\[
\partial_M : \Omega^{p,0}(M) \to \Omega^{p+1,0}(M)
\]
defined as follows. If \( \omega \in \Omega^{p,0}(M) \), then \( \partial_M \omega \) is the unique element of \( \Omega^{p+1,0}(M) \) which coincides with \( \omega \) on \( T_{1,0}(M) \cap \cdots \cap T_{1,0}(M) \) \((p+1)\) terms. Then \( \partial_M^2 = 0 \) in all degrees and one may consider the cohomology groups
\[
H^{p,0}(M) = H^p(\Omega^{p,0}(M), \partial_M).
\]

Moreover, if \( \mathcal{F} \) is a CR foliation on \( M \), then one may define the space of basic \((p, 0)\)-forms \( \Omega_B^{p,0}(\mathcal{F}) \), consisting of all elements \( \omega \in \Omega^{p,0}(M) \) satisfying
\[
T_{1,0}(\mathcal{F}) \mid \omega = 0, \quad T_{1,0}(\mathcal{F}) \mid \partial_M \omega = 0,
\]
and the corresponding cohomology group
\[
H_B^{p,0}(\mathcal{F}) = H^p(\Omega_B^{p,0}(\mathcal{F}), \partial_B),
\]
where \( \partial_B \) denotes the restriction of \( \partial_M \) to \( \Omega_B^{p,0}(\mathcal{F}) \). Then one sees that complex conjugation gives isomorphisms
\[
H^{p,0}(M) \approx H^{0,p}(M), \quad H_B^{p,0}(\mathcal{F}) \approx H_B^{0,p}(\mathcal{F}).
\]

**Example 2 (The contact flow).** Let \((M, T_{1,0})\) be a nondegenerate CR manifold of hypersurface type, and \( \theta \) a contact form on \( M \). Let \( T \) be the characteristic direction of \((M, \theta)\), and denote by \( \mathcal{F} \) the flow defined by \( T \) (cf., e.g., [41], p. 132). Following [17], p. 160, let us consider the space \( U_\nu^r \) of all horizontal \( r \)-forms on \( M \), where an \( r \)-form \( \omega \) on \( M \) is called *horizontal* if \( T \mid \omega = 0 \) and \( \mathcal{L}_T \omega = 0 \). Thus \( U_\nu^r \) is nothing but \( \Omega_B^r(\mathcal{F}) \). Employing Kohn’s solution (cf. [26]) to the Neumann problem for the \( \partial_M \) operator on a compact strictly pseudoconvex CR manifold, Gigante established the following

**Theorem 1 (Gigante [17]).** Let \( M \) be a compact strictly pseudoconvex CR manifold and \( \theta \) a contact form on \( M \). Let \( T \) be the characteristic direction of \((M, \theta)\), and \( \mathcal{F} \) the flow defined by \( T \). If the Tanaka-Webster connection of \((M, \theta)\) has vanishing pseudohermitian torsion \( (\tau = 0) \) and strictly positive definite pseudohermitian Ricci curvature, then \( H_B^1(\mathcal{F}) = 0 \).

We may give a short proof of Theorem 1, based on a result of Lee [30], as well as on our previous considerations. Indeed, \( H_B^1(\mathcal{F}) = H^{0,1}(M) \oplus H^{1,0}(M) \). Furthermore, by a result in [30], if \( R_{\alpha\beta\xi\eta} \xi^\alpha \eta^\beta > 0 \) for any \( \xi = (\xi^1, \cdots, \xi^n) \), then \( H^{0,1}(M) = 0 \) (note that the assumption \( \tau = 0 \) was removed).
3.2 The filtration $\{F^r \Omega^\bullet\}_{r \geq 0}$

We define a multiplicative filtration of the Cauchy-Riemann complex by setting

$$F^r \Omega^{0,m} = \{\omega \in \Omega^{0,m}(M) \mid \tilde{Z}_1 \cdots \tilde{Z}_{m-r+1} \omega = 0 \text{ for } Z_1, \cdots, Z_{m-r+1} \in T_{1,0}(\mathcal{F})\}.$$ 

Note that we have

$$\Omega^{0,m}(M) = F^0 \Omega^{0,m} \supseteq F^1 \Omega^{0,m} \supseteq \cdots \supseteq F^m \Omega^{0,m} \supseteq F^{m+1} \Omega^{0,m} = \{0\}$$

for any $0 \leq m \leq N$. Also, the following diagrams are commutative

$$
\begin{array}{ccc}
\Omega^{0,m}(M) & \supseteq & F^r \Omega^{0,m} \\
\downarrow \partial_M & & \downarrow \partial_M \\
\Omega^{0,m+1}(M) & \supseteq & F^r \Omega^{0,m+1} \\
\end{array}
$$

Indeed, let

$$\omega \in F^r \Omega^{0,m} \subset \Omega^{0,m}(M) \subset \Omega^{0,m+1}(M).$$

Then, since $T_{1,0}(\mathcal{F})$ is involutive, it follows that

$$\tilde{Z}_1 \cdots \tilde{Z}_{m-r+2} \partial_M \omega = 0$$

for any $Z_j \in T_{1,0}(\mathcal{F})$. Thus we have

$$\partial_M F^r \Omega^{0,m} \subset F^r \Omega^{0,m+1}.$$

Now, setting

$$F^r \Omega^{0,\bullet} = \bigoplus_{m=0}^{N} F^r \Omega^{0,m},$$

we obtain the following

**Proposition 5.** Let $\mathcal{F}$ be a CR foliation on the nondegenerate CR manifold $M$. Then $\{F^r \Omega^{0,\bullet}\}_{r \geq 0}$ is a decreasing filtration of $\Omega^{0,\bullet}(M)$ by differential ideals. Also, $\dim_{\mathbb{C}} T_{1,0}(\mathcal{F})_x = n$, $x \in M$, implies that $F^r \Omega^{0,n+r} = \Omega^{0,n+r}(M)$ and $\dim_{\mathbb{C}} T_{1,0}(M)_x / T_{1,0}(\mathcal{F})_x \cong k$, $x \in M$, yields that

$$F^{k+1} \Omega^{0,m} = \{0\}.\quad (3.4)$$

**Proof.** Since we have seen that $\partial_M F^r \Omega^{0,\bullet} \subset F^r \Omega^{0,\bullet}$, it remains to check that

$$\Omega^{0,\bullet}(M) \wedge F^r \Omega^{0,\bullet} \subset F^r \Omega^{0,\bullet}.$$ 

To this end, let $\omega = \omega_1 + \cdots + \omega_N$ with $\omega_m \in F^r \Omega^{0,m}$. Then we have

$$\alpha \wedge \omega_m \in F^r \Omega^{0,m+s}$$

for any $\alpha \in \Omega^{0,s}(M)$. Indeed, it is easy to see that

$$\tilde{Z}_1 \cdots \tilde{Z}_{m+s-r+1} (\alpha \wedge \omega_m) = 0$$

on the complement of $\mathcal{F}$.
for any $Z_j \in T_{1,0}(\mathcal{F})$, because at most $s$ of the $Z_j$'s enter $\alpha$, so that there are enough $Z_j$'s left to kill $\omega_m$. Hence we obtain

$$\Omega^{\alpha,i}(M) \wedge F^r \Omega^{\alpha,m} \subseteq F^r \Omega^{\alpha,m+s},$$

from which the desired inclusion follows.

To prove (3.4), we need some local considerations. Let $\{T_1, \ldots, T_N\}$ be a local frame of $T_{1,0}(M)$ such that $\{T_1, \ldots, T_n\}$ is a local frame of $T_{1,0}(\mathcal{F})$. Let $\{\theta^1, \ldots, \theta^n\}$ be a local dual frame determined by

$$\theta^i(T_j) = \delta^i_j, \quad \theta^i(T) = 0, \quad \theta^i(T) = 0.$$

Each $\omega \in F^r \Omega^{\alpha,m}$ is then locally a sum of monomials of the form

$$\bar{\omega}_{\alpha_1} \wedge \cdots \wedge \bar{\omega}_{\alpha_p} \wedge \bar{\omega}_{j_1} \wedge \cdots \wedge \bar{\omega}_{j_q},$$

with $C^\infty(M)$-coefficients, where $0 \leq p \leq m - r$ and $q = m - p$. If $r = k + 1$, then $0 \leq p \leq m - k - 1$ so that $q \geq k + 1$. Hence $\bar{\omega}_{\alpha_1} \wedge \cdots \wedge \bar{\omega}_{\alpha_p} = 0$, and (3.4) is proved.

For a given CR foliation $\mathcal{F}$ on the nondegenerate CR manifold $M$ with a fixed contact form $\theta$, we set $\langle Z, W \rangle = L_\theta(Z, W)$ and define

$$T_{1,0}(\mathcal{F})^\perp = \{ Z \in T_{1,0}(M) \mid \langle Z, W \rangle = 0 \text{ for any } W \in T_{1,0}(\mathcal{F}) \}.$$

An argument of mere algebra then shows that $T_{1,0}(\mathcal{F})$ is nondegenerate in $(T_{1,0}(M), \langle , \rangle)$ and $T_{1,0}(\mathcal{F}) \cap T_{1,0}(\mathcal{F})^\perp = T_{1,0}(M)$.

**Proposition 6.** Let $\mathcal{F}$ be a CR foliation on the nondegenerate CR manifold $M$. Let $\{E_{r,s}^{i,\alpha}\}_{i \geq 0}$ be the spectral sequence associated with the filtered differential space $(\Omega^{\alpha}(-), \partial_M, \{F^r \Omega^{\alpha}(-)\}_{r \geq 0})$. Then we have the following isomorphisms of linear spaces:

$$E_{0,s}^r \approx \text{Hom}(\Lambda^s T_{0,1}(\mathcal{F}), \Lambda^s [T_{0,1}(\mathcal{F})^\perp]^*),$$

$$E_{1,0}^r \approx \Omega_{B}^{\partial, r}(\mathcal{F}), \quad E_{2,0}^r \approx H_{B}^{0, r}(\mathcal{F}),$$

where $T_{0,1}(\mathcal{F})^\perp = \overline{T_{1,0}(\mathcal{F})}^\perp \subset T_{0,1}(M)$.

**Proof.** We set

$$Z_r^{r,m} = \{ \omega \in F^r \Omega^{\alpha,m} \mid \partial_M \omega \in F^{r+i} \Omega^{\alpha,m+1} \},$$

$$D_r^{r,m} = (F^r \Omega^{\alpha,m}) \cap \partial_M (F^{r-1} \Omega^{\alpha,m-1}),$$

and

$$E_r^{\alpha,m} = \frac{Z_r^{r,m}}{Z_{r-1}^{r+1,m} + D_r^{r,m}}.$$

(Cf., e.g., [22], Vol. III, p. 21.) Also, we set $E_{r,s}^{i,\alpha} = E_{i}^{\alpha, r+s}$. Then

$$E_{0,s}^r = \frac{F^r \Omega^{\alpha,r+s}}{F^{r+1} \Omega^{\alpha,r+s}} \approx \text{Hom}(\Lambda^s T_{0,1}(\mathcal{F}), \Lambda^s [T_{0,1}(\mathcal{F})^\perp]^*).$$

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With these understood, we now define

\[ Z_i^\sigma = \bigoplus_{m=0}^N Z_i^\sigma_m, \quad E_i^\sigma = \bigoplus_{m=0}^N E_i^\sigma_\Omega^{0,m}. \]

Then \( \overline{\partial}_M Z_i^\sigma \subset Z_i^{\sigma+1} \) and \( \overline{\partial}_M \text{Ker}(\pi_i^\sigma) \subset \text{Ker}(\pi_i^{\sigma+1}) \), where \( \pi_i^\sigma : Z_i^\sigma \to E_i^\sigma \) is the natural projection, and hence \( \overline{\partial}_M \) induces differentials \( d_i^\sigma : E_i^\sigma \to E_i^{\sigma+1} \). The resulting differential \( d_i^{0,s} : E_i^{0,s} \to E_i^{0,s+1} \) corresponds, under the isomorphism (3.5), to

\[
(\overline{\partial}_C \hat{\omega})(Z_1, \ldots, Z_{s+1}) = A_{s+1} \left[ (Z_1, \ldots, Z_{s+1}) \mapsto \overline{Z}_1 \right] \overline{\partial}_M \left( \hat{\omega}(\overline{Z}_1, \ldots, Z_{s+1}) \right)
\]

for any \( Z_j \in T_{1,0}(\mathcal{F}), \ 1 \leq j \leq s+1 \). Here \( A_{s+1} \) is the alternation map (cf., e.g., [25], Vol. I, p. 28) and, for any \((0, r+s)\)-form \( \omega \) which is locally (cf. the discussion preceding Proposition 6) a sum of monomials of the form \( \alpha \wedge \beta \) with \( \alpha \in \Lambda^r T_{0,1}(\mathcal{F})^* \) and \( \beta \in \Lambda^s [T_{0,1}(\mathcal{F})^+]^* \), we set

\[
\hat{\omega}(Z_1, \ldots, Z_s) = Z_1 \ldots Z_s \omega, \quad Z_j \in T_{1,0}(\mathcal{F}).
\]

Note that, as the notation suggests, \( \overline{\partial}_C \hat{\omega} \) is a CR analogue of the Chevalley-Eilenberg differential in [41], p. 122. Then we have

\[ E_i^{0,s} \approx H^s \left( \text{Hom} \left( \Lambda^\bullet T_{0,1}(\mathcal{F}), \Lambda^r [T_{0,1}(\mathcal{F})^+]^* \right), \overline{\partial}_C \hat{\omega} \right), \]

and hence

\[ E_i^{0,0} \approx \Omega^0_{\partial \Omega}(\mathcal{F}). \]

(3.6) \( E_i^{0,0} \) induces, on the right-hand side of (3.6), the differential \( \overline{\partial}_B \), it follows that \( E_i^{0,0} \approx H^0_{\partial \Omega}(\mathcal{F}). \)

\[ \square \]

3.3 The Graham-Lee connection

Let \( \Omega = \{ \varphi < 0 \} \subset \mathbb{C}^{n+1} \) be a smoothly bounded strictly pseudoconvex domain and \( \Delta_\varphi \) denote the Laplacian of the Kähler metric on \( \Omega \) whose Kähler 2-form is \( (i/2)\partial \overline{\partial} \log(-1/\varphi) \). Then, according to a result by Graham and Lee [20], if \( u \) is a local solution to \( \Delta_\varphi u = 0 \) which is smooth up to a portion of \( \partial \Omega \), then the boundary value \( f \) of \( u \) must satisfy \( \mathcal{C}_\varphi f = 0 \), where \( \mathcal{C}_\varphi \) is a differential operator on \( \partial \Omega \) of order \( 2n + 2 \), which was first studied by Graham [19] on the Siegel domain and the unit ball. In order to compute \( \mathcal{C}_\varphi \), one needs to understand the interrelation between the tangential pseudohermitian geometry of the leaves of the foliation \( \mathcal{F} \) defined by level sets of \( \varphi \), and the geometry of the ambient complex space.

One key instrument in this respect turns out to be a canonical connection on a one-sided neighborhood of \( \partial \Omega \) (the Graham-Lee connection, cf. Theorem 2 below) which induces the Tanaka-Webster connection on each leaf of \( \mathcal{F} \). We give a new axiomatic description of this connection and a hint on how one may recover Faran’s results (cf. [15]) in this setting, namely in the presence of a fixed defining function for the foliation. Actually, we merely look at Faran’s third order invariants \( h^a_T \) and \( k^a \), whereas the problem of recovering Faran’s result on whether a given real hypersurface may be a leaf of a Ricci flat foliation (cf. [15], p. 403) is left open.
To be more precise, let \( \Omega \subset \mathbb{C}^{n+1} \) be a strictly pseudoconvex domain. Let \( V \subset \mathbb{C}^{n+1} \) be an open set, and \( \varphi : V \to \mathbb{R} \) a smooth defining function for \( \Omega \) such that \( \Omega = \{ x \in V \mid \varphi(x) < 0 \} \) and \( \partial \Omega = \{ x \in V \mid \varphi(x) = 0 \} \), satisfying \( d\varphi(x) \neq 0 \) for any \( x \in \partial \Omega \). For a sufficiently small one-sided neighborhood \( U \) of the boundary \( \partial \Omega \), we set \( M_\epsilon = \{ x \in U \mid \varphi(x) = \epsilon \} \) so that \( M_0 = \partial \Omega \).

Consider now the foliation \( \mathcal{F} \) on \( U \) whose leaves are the level sets \( M_\epsilon \) of \( \varphi \), where \( \epsilon \in \varphi(U) \). Since \( \Omega \) is strictly pseudoconvex, the restriction of the real \((1,1)\)-form \( i \partial \bar{\partial} \varphi \) to \( T_{1,0}(\mathcal{F}) \) is definite, and by replacing \( \varphi \) by \( -\varphi \) if necessary, we may assume that it is positive definite. Note that there exists a uniquely defined complex vector field \( \xi \) of type \((1,0)\) on \( U \) which is orthogonal to \( T_{1,0}(\mathcal{F}) \) with respect to \( \partial \bar{\partial} \varphi \) and for which \( \partial \varphi(\xi) = 1 \) (cf. Lee and Melrose [31], p. 163).

Let us then define a function \( r : U \to \mathbb{R} \) by setting \( r = \partial \bar{\partial} \varphi (\xi, \bar{\xi}) \), so that \( \xi \) and \( r \) are characterized by

\[
\xi \cdot \partial \bar{\partial} \varphi = r \bar{\partial} \varphi, \quad \partial \varphi(\xi) = 1. \tag{3.7}
\]

Let \( \{W_1, \ldots, W_n\} \) be a local frame of \( T_{1,0}(\mathcal{F}) \). Then \( \{W_n, \xi\} \) is a local frame of \( T_{1,0}(U) \). Let \( \theta^a \) be the (local) complex 1-forms of type \((1,0)\) on \( U \) determined by \( \theta^a(W_0) = \delta^a_0 \) and \( \theta^a(\xi) = 0 \). Then \( \{\theta^a, \partial \varphi\} \) is a local frame of \( T^{1,0}(U)^* \), and, as a consequence of the first of the formulae (3.7),

\[
\partial \bar{\partial} \varphi = h_{\alpha \beta} \theta^\alpha \wedge \bar{\theta}^\beta + r \partial \varphi \wedge \bar{\partial} \varphi, \tag{3.8}
\]

for some positive definite Hermitian matrix of functions \( h_{\alpha \beta} \). It follows from (3.8) that \( r \) is positive if and only if \( \varphi \) is strictly plurisubharmonic, and \( r = 0 \) if and only if \( \varphi \) satisfies the homogeneous complex Monge-Ampère equation \( \det(\partial \bar{\partial} \varphi) = 0 \) (cf. [20]). We call \( r \) the transverse curvature of \( \varphi \).

Consider the real 1-form \( \theta = (i/2)(\bar{\partial} \varphi - \partial \varphi) \) on \( U \). Its exterior derivative is then given by

\[
d\theta = i h_{\alpha \beta} \theta^\alpha \wedge \bar{\theta}^\beta + r d\varphi \wedge \theta. \tag{3.9}
\]

Thus the Levi form \( L_\theta \) of \( \theta \), that is, the restriction of \(-i d\theta\) to \( T_{1,0}(\mathcal{F}) \otimes T_{0,1}(\mathcal{F}) \) is given by

\[
L_\theta(Z, \nabla) = h_{\alpha \beta} Z^\alpha \nabla^\beta, \tag{3.10}
\]

where \( Z = Z^a W_a, V = V^\beta W_\beta \in T_{1,0}(\mathcal{F}) \).

Let \( j_\epsilon : M_\epsilon \to U \) be the inclusion. Then \( \theta_\epsilon = j_\epsilon^* \theta \) is a pseudohermitian structure on \((M_\epsilon, T_{1,0}(M_\epsilon))\). If we write \( \xi = (1/2)(N - iT) \), with \( N \) and \( T \) real, then we have \( (d\varphi)(N) = 2, \theta(N) = 0 \), and the restriction \( T_\theta \) of \( T \) to \( M_\epsilon \) is tangent to \( M_\epsilon \). Also, (3.7) shows that \( T_\epsilon \) is the characteristic direction of \((M_\epsilon, \theta_\epsilon)\). Among the linear connections on \( U \) which restrict to the Tanaka-Webster connection \( \nabla^\epsilon \) on each leaf \( M_\epsilon \) of \( \mathcal{F} \), we single out a canonical one (cf. also Proposition 1.1 in [26], p. 701) in the following manner.

Let \( \nu(\mathcal{F}) = T(U)/T(\mathcal{F}) \) and \( \Pi : T(U) \to \nu(\mathcal{F}) \) be the projection. Given a linear connection \( \nabla \) on \( U \), we consider the bundle map

\[
a : T(\mathcal{F}) \otimes T(\mathcal{F}) \to \nu(\mathcal{F}), \quad a(X, Y) = \Pi(\nabla_X Y),
\]

where \( X, Y \in T(\mathcal{F}) \). Let \( T\nabla \) be the torsion of \( \nabla \) and set \( \tau(X) = T\nabla(T, X) \) for
any $X \in T(U)$. We say $T_T$ is pure if

\begin{align}
(3.10) & \quad T_T(Z, W) = 0, \\
(3.11) & \quad T_T(Z, \overline{W}) = i L_\theta(Z, \overline{W})T, \\
(3.12) & \quad T_T(N, Z) = \tau Z + i\tau(Z)
\end{align}

for any $Z, W \in T_{1,0}(\mathcal{F})$ and

\begin{equation}
(3.13) \quad \tau \circ J + J \circ \tau = 0.
\end{equation}

Here $J$ is the restriction of $J_0$ (the complex structure of $U$) to $H(\mathcal{F})$.

Now, we may state the following

**Theorem 2.** Let $\Omega \subset C^{n+1}$ be a strictly pseudoconvex domain and $\varphi$ a smooth defining function for $\Omega$. Let $\mathcal{F}$ be the foliation by level sets of $\varphi$ on a one-sided neighborhood $U$ of $\partial \Omega$. Then there exists a unique linear connection $\nabla$ on $U$ such that

1. $T_{1,0}(\mathcal{F})$, $N$, $T$ and $L_\theta$ are parallel with respect to $\nabla$, and
2. the torsion $T_T$ of $\nabla$ is pure.

Consequently, one has $\alpha = 0$.

The canonical connection $\nabla$ furnished by Theorem 2 is referred to as the *Graham-Lee connection* of $(U, \varphi)$. Let $\pi_+ : T(U) \otimes C \to T_{1,0}(\mathcal{F})$, respectively $\pi_- : T(U) \otimes C \to T_{0,1}(\mathcal{F})$, be the canonical projections associated with the direct sum decomposition:

\begin{equation}
(3.14) \quad T(U) \otimes C = T_{1,0}(\mathcal{F}) \oplus T_{0,1}(\mathcal{F}) \oplus CT \oplus CN.
\end{equation}

**Proof.** Let us first prove the uniqueness statement in Theorem 2. The identity (3.11) may be written as

\[ [Z, \overline{Y}] = \nabla_Z \overline{Y} - \nabla_{\overline{Y}} Z - iL_\theta(Z, \overline{Y})T. \]

Hence, by (1), we have

\begin{equation}
(3.15) \quad \nabla_{\overline{Y}} Z = \pi_+(\overline{Y}, Z)
\end{equation}

for any $Y, Z \in \Gamma^\infty(T_{1,0}(\mathcal{F}))$. Next, $\nabla L_\theta = 0$ may be written as

\[ X(L_\theta(Y, \overline{Z})) = L_\theta(\nabla_X Y, \overline{Z}) + L_\theta(Y, \nabla_X \overline{Z}), \]

from which we obtain

\begin{equation}
(3.16) \quad L_\theta(\nabla_X Y, \overline{Z}) = X(L_\theta(Y, \overline{Z})) - L_\theta(Y, \pi_-(\overline{X}, \overline{Z}))
\end{equation}

for any $X, Y, Z \in \Gamma^\infty(T_{1,0}(\mathcal{F}))$.

We now compute $\nabla T_X$. To this end, set

\[ K_TX = -(1/2)J(\mathcal{L}_TJ)(X) \]

for $X \in H(\mathcal{F})$. Then, since $T_{1,0}(\mathcal{F})$ is $\nabla$-parallel and $\nabla$ is a real differential operator, it follows that $T_{0,1}(\mathcal{F})$ is also $\nabla$-parallel. Hence $H(\mathcal{F})$ is $\nabla$-parallel and $\nabla J = 0$. Consequently, it follows from (3.13) that

\[ \tau(X) = K_TX \]
for any $X \in H(\mathcal{F})$ (actually, it holds for any $X \in T(\mathcal{F})$ if one extends $J$ to $T(\mathcal{F})$ by $JT = 0$). Moreover, it is verified that

$$(3.17) \quad \nabla T X = \tau(X) + \mathcal{L}_T X$$

for any $X \in \Gamma^\infty(T_{1,0}(\mathcal{F}))$. Finally, by (3.12) and $\nabla N = 0$, we have

$$(3.18) \quad \nabla N Z = rZ + i\tau(Z) + [N, Z]$$

for any $Z \in \Gamma^\infty(T_{1,0}(\mathcal{F}))$. From (3.15) through (3.18) and $\nabla T = \nabla N = 0$, it follows that $\nabla$ is uniquely determined.

To establish the existence statement in Theorem 2, let

$$\nabla : \Gamma^\infty(T(U) \odot C) \times \Gamma^\infty(T(U) \odot C) \to \Gamma^\infty(T(U) \odot C)$$

be given by the following identities:

$$\nabla X Y = \pi_+([X, Y]), \quad \nabla X Y = \nabla X Y,$$

$$\nabla X Y = U_{XY}, \quad \nabla X Y = \nabla X Y,$$

$$(3.19) \quad \nabla T X = \mathcal{L}_T X + K T X, \quad \nabla T X = \nabla T X,$$

$$\nabla N X = rX + iK T X + \mathcal{L}_N X, \quad \nabla N X = \nabla N X,$$

where $X, Y \in \Gamma^\infty(T_{1,0}(\mathcal{F}))$ and $U_{XY} \in \Gamma^\infty(T_{1,0}(\mathcal{F}))$ is given by

$$L_\theta(U_{XY}, \overline{Z}) = X(L_\theta(Y, \overline{Z})) - L_\theta(Y, \pi_-(\overline{[X, Z]}))$$

for $Z \in \Gamma^\infty(T_{1,0}(\mathcal{F}))$. Then it is immediate that $\nabla$ extends to define a linear connection on $U$.

Hence it suffices to check that $\nabla$ obeys the axioms (1) and (2). To see this, let $\Pi^\perp : T(U) \to T(\mathcal{F})$ be the canonical projection associated with the direct sum decomposition $T(U) = T(\mathcal{F}) \oplus \mathbb{R} N$. Set $\nabla^\perp_X Y = \Pi^\perp(\nabla_X Y)$ for any $X, Y \in \Gamma^\infty(T(\mathcal{F}))$. Then it is verified that $\nabla^\perp$ restricts on each leaf $M_\theta$ of $\mathcal{F}$ to a linear connection on $M_\theta$, and we have

$$\nabla X Y = \nabla^\perp X Y + \sigma(\alpha(X, Y))$$

for any $X, Y \in \Gamma^\infty(T(\mathcal{F}))$, where $\sigma : \nu(\mathcal{F}) \to \mathbb{R} N$ denotes the natural bundle isomorphism. Then it follows that $\alpha = 0$, since by the very definition (3.19) of $\nabla$, the $N$-component of $\nabla_X Y$ vanishes for all $X, Y \in \Gamma^\infty(T(\mathcal{F}))$. Furthermore, it follows from (3.19) again that $\nabla$ restricts on each leaf $M_\theta$ of $\mathcal{F}$ to the Tanaka-Webster connection of $M_\theta$. Hence $T_{1,0}(\mathcal{F})$ and $L_\theta$ are parallel with respect to $\nabla$ and the identities (3.10), (3.11) and (3.13) are satisfied. Up to now, we see that $\nabla$ obeys axiom (1). Finally, note that (3.12) follows from $\nabla_N X = rX + iK T X + \mathcal{L}_N X$ and $\nabla N = 0$. Hence $\nabla$ obeys (2) as well. The proof of Theorem 2 is now complete.

**Remark 3.** The purity axiom (3.12) is natural in the following sense. Note that we may write $T_\nabla(N, Z)$ in the form

$$T_\nabla(N, Z) = A^\alpha W_\alpha + A^\beta \overline{W_\alpha} + B T + C N$$


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with unknown functions \( A^\alpha, A^\overline{\alpha}, B \) and \( C \) to be determined, where \( \{W_\alpha\} \) is a local frame of \( T_{1,\beta}(\mathcal{F}) \). Then the condition that the linear connection \( \nabla \) we look for restricts to the Tanaka-Webster connection \( \nabla^t \) on each leaf \( \mathcal{M}_\beta \) of \( \mathcal{F} \) together with the requirement \( d^2 \theta = 0 \) for the exterior derivative of (3.9) and the integrability of the complex structure on \( U \) implies that

\[
A^\alpha = r Z^\alpha, \quad A^\overline{\alpha} = i r^\overline{\alpha}(Z), \quad B = C = 0,
\]

where \( Z = Z^\alpha W_\alpha \in T_{1,\beta}(\mathcal{F}) \) and \( \tau = \tau^\alpha \otimes W_\alpha + \tau^{\overline{\alpha}} \otimes \overline{W}_\alpha \) (cf. [26], p. 703). Hence we require that \( T^\tau(N, Z) = r Z + i \tau(Z) \).

Faran [15] determined a complete system of local invariants under biholomorphic mappings of foliations of \( U \) by nondegenerate real hypersurfaces. His study is imitative of the work of Chern and Moser [9], and indeed the local invariants of a foliation by real hypersurfaces turn out to be similar to the local invariants of a single real hypersurface. There is, however, a remarkable difference between them, since one of these invariants is the intrinsic normal direction \( N = 2 \text{Re}(\xi) \).

The flow along \( N \) gives a foliate map (that is, a map sending leaves to leaves) whose restriction to each leaf of \( \mathcal{F} \) is a contact transformation, yet in general not a CR diffeomorphism. Indeed, by (3.9) we have

\[
(\mathcal{L}_N \theta)(W_\beta) = N(\theta(W_\beta)) - \theta([N, W_\beta]) = d\theta(N, W_\beta)
\]

\[
= i h_{\alpha\overline{\alpha}} \theta^\alpha \wedge \overline{\theta}^\overline{\alpha}(N, W_\alpha) + r d\varphi \wedge \theta(N, W_\alpha) = 0,
\]

\[
(\mathcal{L}_N \theta)(T) = N(\theta(T)) - \theta([N, T]) = d\theta(N, T)
\]

\[
= r d\varphi \wedge \theta(N, T) = rd\varphi(N) \theta(T) = 2r,
\]

\[
(\mathcal{L}_N \theta)(N) = d\theta(N, N) = 0.
\]

Summing up, we have

\[
\mathcal{L}_N \theta = 2r \theta.
\]

Faran [15] has built the third order invariants \( h^\alpha_\beta \) and \( k^\alpha \), that is, invariants which may be calculated at a point by using only the 3-jet of \( \varphi \) at that point, and gave their geometric interpretation. Precisely, it turns out that \( h^\alpha_\beta \) measures the failure of the flow on \( N \) from being a CR map, while \( k^\alpha = 0 \) if and only if there is a defining function \( \varphi \) of \( \Omega \) so that \( d([\partial \varphi / \partial Z_\gamma]) = 0 \), namely \( \varphi \) has vanishing transverse curvature. As observed in [26], Graham and Lee’s setting bears the same relationship to Faran’s setting as does Webster’s (cf. [42]) to that of Chern and Moser [9]. We need the following structure equation (cf. [26]):

\[
\begin{align*}
(3.20) \quad d\theta^\alpha &= \theta^\beta \wedge \varphi^\alpha_\beta - i \partial \varphi \wedge \tau^\alpha + i r\theta^\alpha \wedge \theta \alpha + \frac{1}{2} r d\varphi \wedge \theta^\alpha.
\end{align*}
\]

Here \( \tau^\alpha = A^\alpha_\beta \overline{\tau}^\beta \) and \( A^\alpha_\beta \) is given by \( \tau([W_\beta]) = A^\alpha_\beta W_\alpha \). Also, \( r^\alpha = h^\alpha_\overline{\beta} \tau^\overline{\beta} \) and \( r_\overline{\beta} = d\tau(W_\beta) = \overline{W}_\beta \sigma \). Finally, \( \varphi^\alpha_\beta \) are given by \( \nabla W_\beta = \varphi^\alpha_\beta \otimes W_\alpha \), where \( \nabla \) is the Graham-Lee connection. Using (3.20), one may derive

\[
(\mathcal{L}_N \theta^\alpha)(W_\beta) = r \delta^\alpha_\beta - \varphi^\alpha_\beta(N), \quad (\mathcal{L}_N \theta^\alpha)(W_\overline{\beta}) = -i A^\alpha_\overline{\beta},
\]

\[
(\mathcal{L}_N \theta^\alpha)(T) = 2r \phi^\alpha, \quad (\mathcal{L}_N \theta^\alpha)(N) = 0.
\]

Summing up, one has

\[
\mathcal{L}_N \theta^\alpha = (r \delta^\alpha_\beta - \varphi^\alpha_\beta(N)) \theta^\beta - i \tau^\alpha + 2i r \phi^\alpha.
\]
In particular,

\[ \mathcal{L}_N \theta^\alpha \equiv -i \tau^\alpha \mod \theta, \theta^\alpha, \]
\[ \mathcal{L}_N \bar{\theta}^\alpha \equiv 2i r^\alpha \bar{\theta} \mod \theta^\alpha, \bar{\theta}^\alpha. \]

A comparison with (2.4) and (2.5) in [15], p. 401, shows that Faran’s third order invariants \( h^\alpha \tau \) and \( k^\alpha \) are essentially \( A^\alpha_\tau \) and \( r^\alpha \), respectively. Hence, the flow along \( N \) is a CR map (when restricted to a leaf of \( \mathcal{F} \)) if and only if \( r^\alpha = 0 \) (that is, each leaf of \( \mathcal{F} \) has vanishing pseudohermitian torsion).

## 4 Foliations and the Tanaka-Webster connection

We adopt the following terminology. If \((M, T_{1,0}(M))\) is a CR manifold and \( \mathcal{F} \) is a foliation on \( M \), then \( \mathcal{F} \) is called a semi-Levi foliation if \( T(\mathcal{F}) \subset H(M) \), where \( H(M) \) is the Levi distribution of \( M \). A semi-Levi foliation is a Levi foliation if \( JT(\mathcal{F}) = T(\mathcal{F}) \) with respect to the complex structure \( J : H(M) \to H(M) \).

Note that if \( \mathcal{F} \) is a semi-Levi foliation of codimension one, then \( \mathcal{F} \) is a Levi foliation and \((M, T_{1,0}(M))\) is Levi flat, that is, \( L = 0 \). Let \((M, \mathcal{F})\) be a foliated CR manifold of hypersurface type, and \( \theta \in \Omega^1(M) \) a pseudohermitian structure on \( M \). If \( \theta \in \Omega^1(M) \), then \( \mathcal{F} \) is a semi-Levi foliation and (the Levi form of) \( M \) is degenerate.

Generally, let \((N, \mathcal{F})\) be a foliated manifold. Then \( \mathcal{F} \) is called a semi-Riemannian foliation if there is a holonomy invariant semi-Riemannian bundle metric \( g_Q \) on the normal bundle \( Q = \nu(\mathcal{F}) \). Note that for any semi-Riemannian foliation \((\mathcal{F}, g_Q)\) on \( N \) there is a bundle-like semi-Riemannian metric \( h \) on \( N \) which induces \( g_Q \) on \( Q \).

Our aim in this section is to study foliations on non-degenerate CR manifolds, on which a contact form \( \theta \) has been fixed. Recall that with any foliation \( \mathcal{F} \) of a Riemannian manifold \( M \) one has a natural connection on the normal bundle \( \nu(\mathcal{F}) \), induced by both the Bott (partial) connection of \( \mathcal{F} \) and the Levi-Civita connection of the given Riemannian metric on \( M \) (cf., e.g., (5.3) in [41], p. 48). In the spirit of pseudohermitian geometry (cf. [42]), when \( M \) is a CR manifold, we replace the Riemannian connection by the Tanaka-Webster connection of \( \theta \) and investigate the resulting theory of the “second fundamental form” of \( \mathcal{F} \) in \( M \) (cf. also [12] and [4] where similar ideas lead to a study of the geometry of the second fundamental form of a CR immersion).

### 4.1 The second fundamental form

Let \((M, T_{1,0}(M))\) be a non-degenerate CR manifold (of hypersurface type) and \( \theta \) a contact form on \( M \). Let \( g_\theta \) and \( \nabla \) be the Webster metric and the Tanaka-Webster connection of \((M, \theta)\), respectively. Let \( \mathcal{F} \) be a foliation on \( M \) such that the tangent bundle \( P = T(\mathcal{F}) \) is non-degenerate in \((T(M), g_\theta)\). We denote by \( P^\perp \) the orthogonal complement of \( P \) in \( T(M) \) with respect to \( g_\theta \), and by \( g_\theta \) the bundle metric induced by \( g_\theta \) on the normal bundle \( Q = \nu(\mathcal{F}) \). Let \( D \) be the connection in \( Q \) defined by

\[
D_X s = \begin{cases} \nabla_X s, & \text{if } X \in \Gamma^\infty(P), \\ \Pi(\nabla_X \sigma(s)), & \text{if } X \in \Gamma^\infty(P^\perp), \end{cases}
\]
where $s \in \Gamma^\infty(Q)$. $\nabla$ is the Bott connection of $(M, \mathcal{F})$, $\Pi : T(M) \to Q$ is the natural bundle map and $\sigma : Q \to P^\perp$ is the natural bundle isomorphism, respectively. Then it is immediate (cf. [41]) to see the following

**Proposition 7.** Let $M$ be a nondegenerate CR manifold and $\theta$ a fixed contact form on $M$. Let $\mathcal{F}$ be a foliation on $M$ such that $T(\mathcal{F})$ is nondegenerate in $(T(M), g_\theta)$. Then $D$ is an adapted connection in $Q$ and its torsion $T_D$ satisfies

1. $P \mid T_D = 0$ for $P = T(\mathcal{F})$, and
2. $T_D(Z, Z') = \Pi(T_\nabla(Z, Z'))$ for any $Z, Z' \in \Gamma^\infty(P^\perp)$.

Moreover, $\mathcal{F}$ is semi-Riemannian and $g_\theta$ is bundle-like if and only if $g_Q$ is parallel with respect to $D$.

A pseudohermitian analogue $a : P \oplus P \to Q$ of the second fundamental form (of a foliation on a Riemannian manifold) is given by

$$a(X, X') = \Pi(\nabla_X X')$$

for any $X, X' \in \Gamma^\infty(P)$. Also, if $Z \in \Gamma^\infty(P^\perp)$, we consider the Weingarten map $W(Z) : P \to P$ given by

$$g_\theta(W(Z)(X), X') = g_Q(a(X, X'), \sigma^{-1}(Z)).$$

It should be noted that in general $a$ is not symmetric and $W(Z)$ is not self-adjoint with respect to $g_\theta$, since the Tanaka-Webster connection $\nabla$ has nontrivial torsion. Next, define $a^\perp : P^\perp \oplus P^\perp \to P$ by setting

$$a^\perp(Z, Z') = \frac{1}{2} \Pi^\perp(\nabla_Z Z' + \nabla_{Z'} Z)$$

for any $Z, Z' \in \Gamma^\infty(P^\perp)$, where $\Pi^\perp : T(M) \to P$ is the natural bundle map. Also, let $\kappa \in \Omega^1(M)$ be defined by

$$P \mid \kappa = 0, \quad \kappa(Z) = \text{trace } W(Z)$$

for any $Z \in \Gamma^\infty(P^\perp)$. A pseudohermitian analogue $t \in \Gamma^\infty(P^\perp)$ of the mean curvature vector (of a foliation on a Riemannian manifold) is then given by

$$g_\theta(t, Z) = \kappa(Z)$$

for any $Z \in \Gamma^\infty(P^\perp)$.

Let $g_P$ be the bundle metric induced by $g_\theta$ on $P$. Since $\nabla g_\theta = 0$, we have

$$\langle L_Z g_P(X, X') \rangle = -2g_Q(a(X, X'), \sigma^{-1}(Z)) + g_\theta(T_\nabla(X, X'), Z)$$

$$+ g_\theta(T_\nabla(Z, X), X') + g_\theta(T_\nabla(Z, Z'), X)$$

for any $X, X' \in \Gamma^\infty(P)$ and $Z \in \Gamma^\infty(P^\perp)$. Also, by a similar calculation, we have

$$\langle L_X g_Q(\sigma^{-1}(Z), \sigma^{-1}(Z')) \rangle = -2g_\theta(a^\perp(Z, Z'), X)$$

$$+ g_\theta(T_\nabla(X, Z), Z') + g_\theta(T_\nabla(X, Z'), Z)$$

for any $X \in \Gamma^\infty(P)$ and $Z, Z' \in \Gamma^\infty(P^\perp)$.
Let us now extend the complex structure $J$ to the whole $T(M)$ by setting

$$JT = 0.$$ 

This furnishes a bundle morphism $J : T(M) \rightarrow T(M)$ satisfying

$$J^2 = -I + \theta \otimes T,$$

$I$ being the identity transformation of $T(M)$, and

$$g_\theta (JX, JY) = g_\theta (X, Y) - \theta(X)\theta(Y)$$

for any $X, Y \in T(M)$. Finally, we remark that the real expression of the purity axioms of the torsion $T_{\tau}$ yield the identity

\[(4.3) \quad T_{\tau} = \theta \wedge \tau + d\theta \otimes T,\]

where $T$ is the characteristic direction of $(M, \theta)$ (cf. [12], p. 174).

We apply these notions and formulas to foliations $\mathcal{F}$ all of whose leaves are tangent to $T$. If this is the case, that is, if $T \in \Gamma^\infty (P)$, let $H(\mathcal{F})$ be the orthogonal complement (with respect to $g_\theta$) of $RT$ in $P$. Then $H(M) = H(\mathcal{F}) \oplus P^\perp$. Note that the flow determined by $T$ is a subfoliation of $\mathcal{F}$, that is, each leaf of $\mathcal{F}$ is foliated by real curves which are the maximal integral curves of $T$. The study of the corresponding exotic characteristic classes (cf. [11]) is an open problem. We now obtain the following

**Theorem 3.** Let $M$ be a nondegenerate CR manifold and $\theta$ a contact form on $M$ with vanishing pseudohermitian torsion $(\tau = 0)$. Let $\mathcal{F}$ be a foliation of $M$ such that $T(\mathcal{F})$ is nondegenerate in $(T(M), g_\theta)$. Assume that $T$ is tangent to the leaves of $\mathcal{F}$. Then the following hold:

1. $D$ is torsion-free.
2. $\alpha$ is symmetric and $W(Z)$ is self-adjoint for any $Z \in \Gamma^\infty (P^\perp)$.
3. The induced metric $g_\mathcal{F}$ along the leaves is invariant under the flows of vector fields orthogonal to the foliation if and only if $\alpha = 0$ and $H(\mathcal{F})$ is $J$-invariant.
4. $\mathcal{F}$ is semi-Riemannian and $g_\theta$ is bundle-like if and only if $\alpha^\perp = 0$.

**Proof.** The fact that $T_P = 0$ follows from $\tau = 0$ together with Proposition 7 and the purity axiom (4.3), for $\Pi(T) = 0$.

To see (2) through (4), let us drop the assumption $\tau = 0$ for the moment. First note that, since $\nabla T = 0$, one has

\[(4.4) \quad \alpha (X, T) = 0\]

for any $X \in \Gamma^\infty (P)$. Moreover,

\[(4.5) \quad \alpha (T, X) = \Pi(\tau (X))\]

as a consequence of (4.4) and of

$$\alpha (X, X') = \alpha (X', X) + \Pi(T_{\tau} (X, X'))$$

for any $X, X' \in \Gamma^\infty (P)$. Then it is known from (4.4) that $W(Z)$ is $H(\mathcal{F})$-valued and from (4.5) together with the self-adjointness of $\tau$ with respect to $g_\theta$ that

\[(4.6) \quad W(Z)(T) = \Pi^\perp (|\tau(Z)|)\]

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for any $Z \in \Gamma^\infty(P^\perp)$. Finally, we note that
\[
\alpha(X, X') = \alpha(X', X),
\]
\[
g_\theta(W(Z)X, X') = g_\theta(X, W(Z)X')
\]
for any $X, X' \in \Gamma^\infty(H(F))$, since $\Pi(T_TO(X, X')) = 0$. Then (2) follows from the following more general statement that 1) $\alpha$ is symmetric if and only if $\tau(P) \subseteq P$, and 2) $W(Z)$ is self-adjoint (with respect to $g_\theta$) if and only if $\tau(P^\perp) \subseteq P^\perp$, provided that $T \in \Gamma^\infty(P)$.

Using the purity axiom (4.3) and noting that $\tau$ is self-adjoint with respect to $g_\theta$, one may compute the torsion terms in (4.1) to obtain
\[
(L_{GP})(X, T) = -g_\theta((J + 2\tau)(X), Z)
\]
for any $X \in \Gamma^\infty(P)$, and
\[
(L_{GP})(X, X') = -2g_\theta(\alpha(X, X'), \sigma^{-1}(Z))
\]
for any $X, X' \in \Gamma^\infty(H(F))$. Then (3) is a corollary of a more general statement that $L_{GP} = 0$ along the leaves if and only if $\alpha = 0$ on $H(F) \otimes H(F)$ and $J + \tau$ is a bundle endomorphism of $H(F)$.

Finally, note that (4.2) furnishes, by a similar computation, that
\[
(L_T g_\theta)(\sigma^{-1}(Z), \sigma^{-1}(Z')) = -26(\alpha^{-1}(Z, Z')) + 2g_\theta(\tau(Z), Z'),
\]
\[
(L_X g_\theta)(\sigma^{-1}(Z), \sigma^{-1}(Z')) = -2g_\theta(\alpha^{-1}(Z, Z'), X)
\]
for any $X \in H(F)$ and $Z, Z' \in \Gamma^\infty(P^\perp)$. Then $L_X g_\theta = 0$ for any $X \in P$ if and only if $\alpha^{-1} = A \otimes T$, where $A \in \Gamma^\infty((P^\perp)^* \otimes (P^\perp)^*)$ is given by $A(Z, Z') = g_\theta(\alpha^{-1}(Z, Z'))$. Hence (4) holds when $T = 0$. \qed

4.2 The characteristic form

Let $(M, T_{1, 0}(M))$ be a nondegenerate CR-manifold and $\theta$ a contact form on $M$. If $E \to M$ is a vector bundle over $M$ with standard fibre $R^k$, then we denote by $L(E) \to M$ the principal $GL(k, R)$-bundle of frames (in the fibres) of $E$. Let $F$ be a foliation on $M$ such that the tangent bundle $P = T(F)$ is nondegenerate in $(T(M), g_\theta)$. Let $g_P$ be the bundle metric induced on $P$ by the Webster metric $g_\theta$ and $(\nu, p - \nu)$ the signature of $g_P$, where $p = \text{dim}_R P_x$, $x \in M$. Denote by $O(P) \to M$ the principal $O(\nu, p - \nu)$-subbundle of $L(P) \to M$ determined by $g_P$. From now on, we assume that $F$ is tangentially oriented, namely the structure group $O(\nu, p - \nu)$ reduces to $SO(\nu, p - \nu)$. Let then $\chi_F \in \Omega^p(M)$ be the characteristic form of $(M, F)$ defined by
\[
\chi_F(v_1, \cdots, v_p) = \det \langle g_\theta(v_i, u(e_j)) \rangle
\]
for any $v_i \in T_x(M)$ and some frame $u : R^p \to T_x(M)$ adapted to the “tangential” $SO(\nu, p - \nu)$-structure. Clearly, the definition of $\chi_F(v_1, \cdots, v_p)$ is independent of the choice of adapted frames at $x$. Also, $P^\perp \mid \chi_F = 0$, where $P^\perp$ is the orthogonal complement of $P$ in $(M, g_\theta)$. We shall need the following
Lemma 3. Let $M$ be a nondegenerate CR manifold with a fixed contact form $\theta$, and $\mathcal{F}$ a tangentially oriented foliation of $M$ whose tangent bundle $P$ is nondegenerate in $(T(M), g_\theta)$. Then

$$(4.7) \quad \mathcal{L}_Z \chi_\mathcal{F} |_P = \{-\kappa(Z) + \theta (\text{trace}(\tau_P) Z + \Pi^\perp((J - \tau)(Z)))\} \chi_\mathcal{F} |_P$$

for any $Z \in \Gamma^\infty(P^\perp)$.

Here $\tau_P : P \to P$ is given by $\tau_P(X) = \Pi^\perp(\tau(X))$ for any $X \in P$. The identity (4.7) is the pseudohermitian analogue of a formula in [40] (cf. also (6.17) in [41], p. 66) and will be referred to as the pseudohermitian Rummler formula.

Proof. Let $\{E_1, \cdots, E_p\}$ be an oriented local orthonormal frame of $P$ so that $g_\theta(E_i, E_j) = \epsilon_i \delta_{ij}$, where $\epsilon_1 = \cdots = \epsilon_p = -1$ and $\epsilon_{p+1} = \cdots = \epsilon_p = 1$. Then it is immediate from

$$(\mathcal{L}_Z \chi_\mathcal{F})(E_1, \cdots, E_p) = -\sum_{i=1}^p \chi_\mathcal{F}(E_1, \cdots, \Pi^\perp([Z, E_i]), \cdots, E_p)$$

and

$$\Pi^\perp([Z, E_i]) = \sum_{j=1}^p \epsilon_j g_\theta([Z, E_i], E_j) E_j$$

that

$$(4.8) \quad \mathcal{L}_Z \chi_\mathcal{F} |_P = -\sum_{i=1}^p \epsilon_i g_\theta([Z, E_i], E_i) \chi_\mathcal{F} |_P.$$ 

On the other hand, we have

$$\kappa(Z) = \sum_{i=1}^p \epsilon_i g_\theta(W(Z) E_i, E_i) = \sum_{i=1}^p \epsilon_i g_\theta(\Pi(\nabla E_i, E_i), \sigma^{-1}(Z))$$

$$= -\sum_{i=1}^p \epsilon_i g_\theta(E_i, \nabla E_i Z) = \sum_{i=1}^p \epsilon_i g_\theta([Z, E_i] + T_\nabla(Z, E_i, E_i),$$

since $2g_\theta(E_i, \nabla Z E_i) = Z(\epsilon_i) = 0$. Finally, again by making use of the purity axiom (4.3), we obtain

$$\kappa(Z) = \sum_{i=1}^p \epsilon_i g_\theta([Z, E_i], E_i) + \theta (\text{trace}(\tau_P) Z + \Pi^\perp((J - \tau)(Z)))$$

and hence (4.8) yields (4.7). \hfill \Box

By the pseudohermitian Rummler formula, the $p$-form

$$\eta = Z \mid d\chi_\mathcal{F} + \{\kappa(Z) - \theta (\text{trace}(\tau_P) Z + \Pi^\perp((J - \tau)(Z)))\} \chi_\mathcal{F}$$

vanishes along the leaves of $\mathcal{F}$. As an immediate application, we may look at the case of a foliation tangent to the characteristic direction of $(M, \theta)$ and orthogonal to a semi- Levi foliation. Then we obtain
Proposition 8. Let $\mathcal{F}$ be tangent to $T$. If $P^\perp$ is involutive, then the following statements are equivalent:

1. $\kappa = 0$.
2. $L_Z \chi_{\mathcal{F}} = 0$ for any $Z \in \Gamma^\infty(P^\perp)$.
3. $d\chi_{\mathcal{F}} = 0$.

The proof of this proposition mimics closely that of Theorem 6.23 in [41], p. 69, and is therefore omitted.

In contrast with the case of foliations on Riemannian manifolds, it should be remarked that a foliation $\mathcal{F}$ with $\kappa = 0$ is not necessarily harmonic. To see a geometric interpretation of this condition, let $\beta$ denote the second fundamental form of (each leaf of) $\mathcal{F}$ in $(M, g_\theta)$ (cf., e.g., (6.1) in [41], p. 62). Note that the Levi-Civita connection $\nabla^\theta$ of $(M, g_\theta)$ is related to the Tanaka-Webster connection $\nabla$ as follows (cf. [12], p. 174):

\[(4.9)\quad \nabla^\theta = \nabla + \left( \frac{1}{2} \Omega_\theta - A \right) \odot T + \tau \odot \theta + \theta \odot J,\]

where $\Omega_\theta$ and $A$ are given respectively by

$$\Omega_\theta (X, Y) = g_\theta(X, JY), \quad A(X, Y) = g_\theta(\tau(X), Y)$$

for $X, Y \in T(M)$ and $\odot$ denotes the symmetric tensor product. It then follows from (4.9) that $\beta$ is related to $\alpha$ as

\[(4.10)\quad \beta(X, X') = \alpha(X, X') + \{(1/2) \Omega_\theta (X, X') - A(X, X')\} \Pi(T)
+ \theta(X') \Pi(\tau(X)) + (1/2) \{\theta(X) \Pi(JX') + \theta(X') \Pi(JX)\}\]

for any $X, X' \in P$. For a given $Z \in P^\perp$, let $\alpha[Z] : P \to P$ be the Weingarten map associated to $\beta$ (cf., e.g., (6.3) in [41], p. 62). Then, by (4.10), we have

\[(4.11)\quad \alpha(Z)(X) = W(Z)(X) - \theta(Z) \Pi^\perp ((1/2) J + \tau) (X) + \alpha(X, Z) \Pi^\perp(T)
- (1/2) \theta(X) \Pi^\perp(JZ) + (1/2) g_\theta(JX, Z) \Pi^\perp(T).\]

Let $\ell \in \Omega^1(M)$ be the mean curvature of $\mathcal{F}$ in $(M, g_\theta)$ (cf. e.g. (6.13) in [41], p. 65). Taking traces in (4.11), we then obtain

\[(4.12)\quad \ell(Z) = \kappa(Z) - \theta (\text{trace} \tau_\theta) Z + \Pi^\perp((J - \tau)(Z))\]

for any $Z \in P^\perp$.

As an immediate application, we may observe the following. If $T$ is tangent to $\mathcal{F}$, then $\mathcal{F}$ is harmonic if and only if $\kappa = 0$. Or, assume that $(M, \theta)$ has vanishing pseudohermitian torsion (e.g., $M$ is an odd dimensional sphere, the Heisenberg group, or the pseudoconvex locus of a pseudo-Siegel domain, cf. [4], p. 84-85). Then $\mathcal{F}$ is harmonic if and only if $\kappa = \theta(\Pi^\perp \circ J)$. 

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4.3 Flows

Let $(M, T_{1,0}(M))$ be a nondegenerate CR manifold and $\theta$ a contact form on $M$. Let $\epsilon \in \{\pm 1\}$ and let $X$ be a tangent vector field on $M$ so that $g_\theta(X, X) = \epsilon$ everywhere on $M$. Let $\mathcal{F}$ be the flow determined by $X$, that is, the foliation whose leaves are the integral curves of $X$. Let $\chi_{\mathcal{F}}$ be defined by $\chi_{\mathcal{F}}(Y) = g_\theta(Y, X)$ for any $Y \in T(M)$. Then $\chi_{\mathcal{F}} \in \Omega^1(M)$ is the characteristic form of $\mathcal{F}$ on $(M, g_\theta)$.

Now, note that the pseudohermitian analogue of the mean curvature form $\kappa \in \Omega^1(M)$ is given by $\kappa(Z) = \epsilon g_\theta(\nabla_X X, Z)$. Since $2g_\theta(\nabla_X X, X) = X(\epsilon) = 0$, it follows that $\nabla_X X \in \Gamma^\infty(P^\perp)$ and the mean curvature vector $t \in \Gamma^\infty(P^\perp)$ is given by

(4.13) \[ t = \epsilon \nabla_X X. \]

We first prove the following

**Theorem 4.** Let $(M, T_{1,0}(M))$ be a nondegenerate CR manifold with a fixed contact form $\theta$, and $T$ the characteristic direction of $(M, \theta)$. Let $\mathcal{F}$ be the flow on $M$ defined by $T$. Then the following hold:

1. $\mathcal{F}$ is totally geodesic in $(M, g_\theta)$.
2. The orbits of $T$ are autoparallel curves of $\nabla$.
3. $\mathcal{L}_T \chi_{\mathcal{F}} = 0$.
4. $g_\theta$ is invariant under flows of vector fields lying in the Levi distribution of $M$.
5. $d\chi_{\mathcal{F}} \in F^2\Omega^2(M)$.

Theorem 4 follows from the following

**Proposition 9.** Let $\epsilon \in \{\pm 1\}$ and $X \in T(M)$ so that $g_\theta(X, X) = \epsilon$. Let $\mathcal{F}$ be the flow on $M$ defined by $X$. Then the following statements are equivalent:

1. $\kappa = 0$.
2. The orbits of $X$ are autoparallel curves of $\nabla$.
3. The characteristic form $\chi_{\mathcal{F}}$ satisfies

$$\mathcal{L}_X \chi_{\mathcal{F}} = \theta(X)\left(JX + \tau(X)\right) - A(X, X)\theta,$$

where $b$ denotes lowering of indices by $g_\theta$, that is, $X(b) = g_\theta(X, Y)$ for $X, Y \in T(M)$.
4. For any $Z \in P^\perp$ the Lie derivative $\mathcal{L}_Z g_\theta$ satisfies

$$(\mathcal{L}_Z g_\theta)(X, X) = 2A(X, X)\theta(Z) - 2\theta(X)g_\theta((J + \tau)(X), Z).$$

Proof. The equivalence of (1) and (2) follows from (4.13). Moreover, again by (4.13), we have

$$\kappa(Z) = -\epsilon g_\theta(X, \nabla_X Z),$$

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and hence
\begin{equation}
(\mathcal{L}_{X} \chi_{r})Z = \epsilon \kappa(Z) + g_{\theta}(T_{\tau}(X, Z), X)
\end{equation}
for any \( Z \in T(M) \). Also, by the purity axiom (4.3), we have
\begin{equation}
g_{\theta}(T_{\tau}(X, Z), X) = -A(X, X)\theta(Z) + \theta(X)g_{\theta}((J + \tau)(X), Z)
\end{equation}
for any \( Z \in T(M) \). Then (4.14) together with (4.15) yields the equivalence of (1) and (2). On the other hand, it follows from (4.1) and (4.15) that
\begin{equation}
(\mathcal{L}_{Z} \gamma_{\theta})(X, X) = -2\epsilon \kappa(Z) + 2A(X, X)\theta(Z) - 2\theta(X)g_{\theta}((J + \tau)(X), Z),
\end{equation}
which implies the equivalence of (1) and (4).
\[ \square \]

All that remains to be checked is (5) in Theorem 4. This follows from
\[ F^{\theta} \Omega^{2}(M) = \{ \omega \in \Omega^{2}(M) \mid X \omega = 0 \} \]
and the fact that \( \kappa = 0 \) if and only if
\[ X \omega = -A(X, X)\theta + \theta(X)(JX + \tau(X)). \]
This completes the proof of Theorem 4.
\[ \square \]

Next, we restrict our attention to flows defined by infinitesimal pseudohermitian transformations. A \( C^{\infty} \) diffeomorphism \( f : M \to M \) is called a pseudohermitian transformation of \((M, \theta)\) if 1) \( f \) is a CR map and 2) \( f^{*}\theta = \theta \). Let \( \text{Psh}(M, \theta) \) denote the group of all pseudohermitian transformations of \((M, \theta)\), which has been studied by Webster (cf. Theorem 1.2 in [42]) and Musso (cf. Theorem 4.10 in [33]). Let \( U(M, \theta) \to M \) be the principal \( U(r, s) \)-
subbundle of \( L(T(M)) \to M \) consisting of all linear frames of the form \( u = (x, \{ X_{\alpha}, J_{x}X_{\alpha}, T(x) \}) \) with
\[ g_{\theta, x}(X_{i}, X_{j}) = \epsilon \delta_{ij}, \quad 1 \leq i, j \leq 2n, \]
where \( X_{\alpha+n} = J_{x}X_{\alpha}, \ X_{\alpha} \in H(M)_{x}, \ 1 \leq \alpha \leq n \). Then we have

**Proposition 10.** A \( C^{\infty} \) diffeomorphism \( f \) of \( M \) is a pseudohermitian transformation of \((M, \theta)\) if and only if the induced transformation of \( L(T(M)) \) maps \( U(M, \theta) \) into itself. Any fibre preserving transformation of \( U(M, \theta) \) which leaves the canonical form of \( U(M, \theta) \) invariant is induced by a pseudohermitian transformation \( f \in \text{Psh}(M, \theta) \).

The proof of Proposition 10 follows from the fact that \( f \in \text{Psh}(M, \theta) \) if and only if \( f \) is a \( C^{\infty} \) diffeomorphism, \( f^{*}\theta = \theta \), and \( f^{*}\theta^{\alpha} = U_{\beta}^{\alpha}\theta^{\beta} \) for any local frame \( \{ \theta^{\alpha} \} \) of \( T_{1,0}(M)^{*} \) and some (locally defined) \( C^{\infty} \) functions \( U_{\beta}^{\alpha} \) on \( M \) (it mimics the proof of Proposition 3.1 in [25], p. 236).

A tangent vector field \( X \) on \( M \) is said to be an infinitesimal pseudohermitian transformation of \((M, \theta)\) if its local 1-parameter group of local transformations consists of local pseudohermitian transformations of \((M, \theta)\). Let \( \mathfrak{i}(M, \theta) \) be the set of all infinitesimal pseudohermitian transformations of \((M, \theta)\). By analogy with Proposition 3.2 in [25], p. 237, we also have
Proposition 11. For a vector field $X$ tangent to a nondegenerate CR manifold $M$, the following statements are equivalent:

1. $X \in \mathfrak{i}(M, \theta)$.
2. The natural lift of $X$ to $L(T(M))$ is tangent to $U(M, \theta)$ at each point of $U(M, \theta)$.
3. $\mathcal{L}_X \theta = 0$ and $\mathcal{L}_X \theta^\alpha = V_\alpha \theta^\beta$ for any local frame $\{\theta^\alpha\}$ of $T_1,\bar{\theta}(M)^*$ and some local smooth functions $V_\alpha^\beta$ on $M$.

It is verified from (3) in Proposition 11 that $\mathfrak{i}(M, \theta)$ is a Lie algebra, as $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$. Since each pseudohermitian transformation of $(M, \theta)$ preserves the Webster metric $g_\theta$,

\begin{equation}
\text{Psh}(M, \theta) \subseteq \text{Iso}(M, g_\theta).
\end{equation}

Let $\epsilon \in \{\pm 1\}$ and $X \in \mathfrak{i}(M, \theta)$ such that $g_\theta(X, X) = \epsilon$. Let $\mathcal{F}$ be the flow determined by $X$. Then it follows from (4.16) that $[X, Z] \in P^\perp$ for any $Z \in P^\perp$.

Consequently, we have

\begin{equation}
\mathcal{L}_X \chi_{\mathcal{F}} = 0.
\end{equation}

Assume, in particular, that $M$ is compact. Then, by virtue of Theorem 1.2 in [42], p. 31, Psh$(M, \theta)$ is compact. Let $G$ be the closure in Psh$(M, \theta)$ of the 1-parameter group of transformations obtained by integrating $X$. Then $G$ is compact and abelian, and hence is a torus. Let $\Omega^k(M)^G$ denote the space of $G$-invariant $k$-forms on $M$. As a corollary of (4.17), there is a short exact sequence

\begin{equation}
0 \rightarrow \Omega^k_B(\mathcal{F}) \rightarrow \Omega^k(M)^G \overset{X|}{\rightarrow} \Omega^{k-1}_B(\mathcal{F}) \rightarrow 0,
\end{equation}

from which one may conclude (as in [41], p. 139) that $H^k_B(\mathcal{F})$ are finite dimensional for $0 \leq k \leq 2n$, and zero for $k > 2n$.

Note that, if $\tau = 0$, then it is known by Proposition 2.2 in [42], p. 33 that $T \in \mathfrak{i}(M, \theta)$. If $X = T$, then the connecting homomorphism

$$
\Delta : H^{k-1}_B(\mathcal{F}) \rightarrow H^{k+1}_B(\mathcal{F}),
$$

in the long exact cohomology sequence associated with (4.18), is given by $\Delta[\omega] = [(d\theta) \wedge \omega]$ for any $[\omega] \in H^{k-1}_B(\mathcal{F})$. As a corollary of Theorem 1 (and of Theorem 10.13 in [41], p. 139), the map $(i_T)_* : H^1(M, \mathfrak{R}) \rightarrow H^1_B(\mathcal{F})$ is injective.

References


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