Some Remarks about Pairs of Dynamical Systems and Strange Factor-Representations of Type $\text{II}_1$

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Vienna, Preprint ESI 1254 (2002)  
December 17, 2002

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at
SOME REMARKS ABOUT PAIRS OF DYNAMICAL SYSTEMS AND STRANGE FACTOR-REPRESENTATIONS OF TYPE II₁.

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ABSTRACT. We describe a construction of the factor-representations of type II₁ corresponding to some pair of the dynamical systems. This is a generalization of the classical von Neumann construction of the factors as the crossed-product and of the groupoid approach. We present the natural examples of factor-representations of this type for which so called coupling constant could be not equal to one and consequently there are no spatial traces. The simplest example of such situation gives a pair of discrete subgroups of Heisenberg group (see [2]), which provide also the new kind of factor-representations of type II₁ of rotation algebra. Another examples come from the theory of the lattices in the groups and from the theory of representations of infinite symmetric group.

1. GENERAL SCHEME.

Let X is locally-compact separable space with sigma-finite infinite continuous borel measure µ. Suppose two arbitrary countable groups G and H acts with homeomorphisms on the space X, and

1) the actions of G and H commute. It is convenient to consider the action of one group (G) as a left action and of the second one (H) as right action.

2) both actions preserve the measure µ.

3) both actions are totally disconnected which means that no orbits of the groups has limit points in X.

By condition 3) we can correctly define the measurable fundamental domains - the sets F_G ⊂ X and F_H ⊂ X for the action of G and H correspondingly, and then restrict the measure µ on the sets F_G, F_H. Also we can define the spaces of orbits of groups G and H i.e. G\X and X/H as the quotient topological spaces over actions. Using the isomorphism between F_G and G\X, (corresp. H and X/H) we can define quotient measures µ_G, (µ_H) on spaces quotient spaces (up to some positive multipliers).

Accordingly to the commutation condition 1) there is a natural action of the group G on the space X/H and of the group H on the space G\X. Equivalently, we can transfer these actions to the fundamental domains.

Because the measure µ is infinite we can multiply it on any positive constant preserving the invariance of the measure under actions of group. For example we

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Above we use only the fact that orbit partitions of the measure preserving groups are measurable partitions, moreover all our construction has pure measure-theoretical feature; in order to avoid the details which is not so important here I use rather language of topological dynamics than the language of ergodic theory. In the case when fundamental domain has infinite measure the space (X/H, µ) also has infinite invariant measure and we must use a fundamental domain for the correct definition of factor measure.
can choose this constant in such a way that the measure of the first fundamental domain $F_G$ (if it is finite) becomes equal to $\mu(F_G) = 1$, this normalization defines the value of measure of the second fundamental domain and if it is finite, as some constant $\mu(F_H) = \lambda < \infty$. If we exchange the roles of $G$ and $H$ then the constant will be equal to $\lambda^{-1}$. Evidently the ratio of the values of measure on the fundamental domain does not depend on the choice of the normalization.

**Definition.**
Suppose the measure of the both fundamental domains is finite. The ratio of measures $\lambda(G, H) = \frac{\mu(F_H)}{\mu(F_G)}$ will be called as coupling constant of pair $(G, H)$ in the context above. From definition it is clear that $\lambda(G, H) \cdot \lambda(H, G) = 1$ and these values do not depend on the normalization of measure $\mu$.

Let $\mathcal{C}_G$ (corresp. $\mathcal{C}_H$) is the algebra of all bounded continuous functions which are constant on the orbits of $G$ (corresp. $H$). It is clear from condition 3) that the algebra $\mathcal{C}_G$, (corresp. $\mathcal{C}_H$) is canonically isomorphic to the algebra $C(G\backslash X)$, (corresp. $C(X/H)$) of the bounded continuous functions on the space $G\backslash X$ ($X/H$).

From the condition 1) it is clear that the action of the group $G$ preserves algebra $\mathcal{C}_G$, and defines the group of automorphisms this algebra. The same is true for the action of the group $H$ which preserves algebra $\mathcal{C}_H$. Also from 2) we can conclude that the actions of $G$ on $X/H$ as well as action of $H$ on $G\backslash X$ are measure-preserving actions.

Now consider the complex Hilbert space of the square integrable functions $L^2(X, \mu)$ and the representations of all those objects; representations of the groups $G$ and $H$ with unitary operators by formulas $(f \in L^2(X, \mu))$

\[ [(U_g) f](x) = f(g^{-1}x), g \in G; \quad [(V_h) f](x) = f(xh), h \in H, \]

and representations of the algebras $\mathcal{C}_G, \mathcal{C}_H$ as algebras of multipliers by formulas:

\[ (M_\phi f)(x) = \phi(x) \cdot f(x), \phi \in \mathcal{C}_G, \mathcal{C}_H. \]

Consider von Neumann algebras $\mathcal{A}_G$ (corresp. $\mathcal{A}_H$) generated (= more exactly - weak closure of) by the set of all operators $\{U_g, g \in G\}$ and $\{M_\phi, \phi \in \mathcal{C}_H\}$ (corresp. $\{U_h, h \in H\}$ and $\{M_\phi, \phi \in \mathcal{C}_G\}$) in the Hilbert space $L^2(X, \mu)$.

Before formulation of the theorem we recall the classical notion of the coupling constant of the finite factor from the classical papers by Murray-von Neumann. Let $\mathcal{B}$ be a finite factor and $\mathcal{B}'$ its commutant in some Hilbert space $\mathcal{K}$. Let $tr(\cdot), tr'(\cdot)$ be the canonical normal traces correspondingly in the $W^*$-algebras $\mathcal{B}, \mathcal{B}'$ of the finite type. Then coupling constant of factor $\mathcal{B}$ is the number $\lambda(\mathcal{B}) = \frac{tr(P_h)}{tr'(P'_h)}$, where $h \in \mathcal{K}$ is an arbitrary unit vector, and $P_h$ and $P'_h$ are orthogonal projectors from the corresponding factors on the cyclic subspaces generated by factors with vector $h$. The theorem claims that the ratio does not depend on the choice of $h$ (see f.e.[1]). Coupling constant plays role of the measure of multiplicity in the theory of representations of factor type II, more important that it is complete spatial
invariant of the finite factors, which means that two algebraically isomorphic factors (as $W^*$-algebras) are spatially isomorphic iff their coupling constants are equal.

Recall also that the factor $B$ has a cyclic vector $f$ (⇒ cyclic subspace generated by $f$ is whole Hilbert space) iff $\lambda(B) \leq 1$,

factor has separating vector $h$ (⇒ $U h \neq 0$ for all $U \in B$) iff $\lambda(B) \geq 1$,

factor has bicyclic vector (⇐ separating and cyclic vector or equivalently cyclic for factor and its commutant) iff $\lambda(B) = \lambda(B') = 1$.

Trivially the following symmetry shows what is the connections between those properties for the factor and its commutant: $\lambda(B') \cdot \lambda(B) = 1$. All this facts have a direct application in representation theory and even for example to classical Fourier analysis (see below).

Remember that two representations $\rho_1$ and $\rho_2$ of type II of some algebra which are algebraically isomorphic (means there exists an algebraic isomorphism between factors which brings one representation to another) must be quasiequivalent (means each of two representations (say $\rho_2$) are spatially isomorphic to the subrepresentation of the direct sum $\oplus_k \rho_1$ for some multiplicity $k \in \mathbb{N}$). So the trace is unique invariant of the factor-representations of type II up to algebraic equivalence or quasiequivalence.

**Theorem 1.** Suppose conditions (1)-3) are satisfied, the measures $\mu_G$ and $\mu_H$ are finite, and the actions of the group $G \times H$ on the space $(X, \mu)$ with invariant sigma-finite measure $\mu$ is ergodic.

1. The $W^*$-algebra generated by both algebras $A_G$ and $A_H$ is the algebra $B(L^2)$ of all bounded operators. So the representation of the crossed product of the joint action of the group $G \times H$ on the space $(X, \mu)$ is irreducible representation.

2. Von Neumann algebra $A_G, A_H$ as subalgebras of algebra of all bounded operators $B(L^2)$ are factors of type II$_1$. Those algebras $A_G$ and $A_H$ are mutual commutants in $B(L^2)$.

3. Coupling constant $\lambda$ of factors $A_G$ and $A_H$ is equal to the coupling constant defined above as $\lambda(G, H) = \frac{\|\mu_H^*\|}{\|\mu_G^*\|} = \lambda(A_G)$.

**Proof**

First of all ergodicity of the action of $G \times H$ on $(X, \mu)$ implies the ergodicity of both actions of the group $G$ on $(X/H, \mu_H)$ and the groups $H$ on $(G \backslash X, \mu_G)$. The natural representations of both algebras in $L^2(X, \mu)$ can be considered as representation of the tensor product of two algebras

$$A_G \otimes A_H \equiv A_{G,H}$$

in the space $L^2(X, \mu)$. This representation of tensor product is irreducible. Indeed because of ergodicity the abelian subalgebra generated by multiplicants $M_\phi$ from both abelian subalgebras ($\phi \in C_G \vee C_H$) has as a weak closure the maximal abelian subalgebra of algebra $B(L^2)$; the action of the group $G \times H$ on $(X, \mu)$ is ergodic. But the restriction of the irreducible representation of the tensor product of two algebras (which of course commute) on each efficient is a factor representation it (see for example [1]), so the weak closure of algebras $A_G$ and $A_H$ are factors and are mutual commutants: $[A_G]^* = A_H$. Now define a type of factors.
Let us define the traces on our $A_G$ and $A_H$ as on von Neumann algebras. It is clear that each of them are isomorphic to standard $W^*$-crossed-product of commutative algebras with the action of groups, more exactly, we can construct regular representation of crossed-product using groupoid construction. Because of ergodicity of actions and measure preserving we obtain factors type $II_1$ (measure is finite). But this isomorphism is NOT spatial in generally, and consequently we can not say that the standard crossed product representations are isomorphic to our representations.

Now let us calculate a coupling constant for our factors. For this we must choose some unit vector $\chi \in L^2(X, \mu)$ and find the ratio of the traces of the factors $A_G$ and $A_H$ on the corresponding cyclic subspaces of vector $\chi$. Because the result does not depend on the choice of that vector we will set $\chi = \chi_{F_H}$, where $\chi_{F_H} \in C_G \subset A_G$ is a characteristic function of the fundamental domain $F_H$. Let us suppose that a normalization of the measure $\mu$ is chosen in such a way: $\mu(F_H) = 1$. From the theory of crossed product we know that the trace (under that normalization) of the projector $P_\chi$ corresponding to multiplicator on the characteristic function $\chi_{F_H}$ from maximal regular abelian subalgebra of factor is equal to the measure of that set.

So $\text{tr}(P_{\chi_{F_H}}) = \mu(F_H) = 1$. In the same way the value of the trace of the second factor $A_H$ on the projector onto cyclic subspace generated by the same characteristic function in the factor $A_H$ equal to the measure of the set $F_G$ but under another normalization: $\mu(F_G) = 1$. It means that the ratio of traces coupling constant of the factors in the sense von Neumann theory and defined in the beginning coupling constant sense the pair of dynamical systems are coincided: we can rewrite this in the form which does not depend on the normalization of the measure $\mu$.

$$\lambda(A_G) = \frac{\mu(F_H)}{\mu(F_G)}; \quad \lambda(A_H) = \frac{\mu(F_G)}{\mu(F_H)}$$

Remark. As was mentioned the value of coupling constant gives the criteria of the existence of bicyclic vector e.g. cyclic vector for factor and its commutant - this condition is $\lambda = 1$. In our terms it means that if the measures of fundamental domains are equal then such vector exists, but it is not evident how to find it if the fundamental domains are not coincided but have equal measure. The problem of finding of such vector reduced to the analysis of the structure of factor and orbit theory of dynamical systems.

The problem appeared - what kind of pairs of the dynamical systems $(Y, G, \phi)$ and $(Z, H, \psi)$ with countable groups $G$ and $H$ which act on the spaces with measure $(Y, \mu)$ and $(Z, \nu)$ correspondingly, allow the "joining" to a general scheme of the first paragraph. It means when there exists the space $(X, \mu)$ and action of the group $G$ and $H$ with imbedding of both systems as quotient with all conditions of our scheme.

More concrete Lie-theoretical aspect of this problem is the following: to describe the connected Lie groups in which there exist two lattices (=discrete subgroups with the arbitrary volume of fundamental domains) which commute and together generate topologically the whole group. This can not be for semisimple Lie groups, but it is possible for nilpotent Lie groups.

Below we will give some examples.

Here we describe well-known construction of von Neumann representation of dynamical systems with invariant measure as a partial case our scheme. Suppose we have a countable group $G$ which acts with homeomorphisms on the locally compact separable space $X_0$ with finite or infinite continuous measure $\mu_0$ (so we have Lebesgue space). Then we define the space $X = X_0 \times G$ and two actions $L$ and $R$ of the group $G$:

$$R_g(x, q) = (gx, qg) \quad L_g(x, q) = (x, g^{-1}q), \quad g, q \in G, x \in X_0, (x, q) \in X$$

It is easy to check that these two actions commute and satisfy to all other conditions of the general scheme of the first section. In those terms both groups are coincided: $G = H$ (more precisely the group $H$ is opposite to $G$; $H = G^\circ$ - see below), and have common fundamental domain $X_0 \equiv X_0 \times \{e\} \subset X$ ($e$ is the unit in the group $G$) which can be identified with the both spaces of the orbits. In the same time the actions of the group of course are different.

Both commutative algebras $C_G$ and $C_H$ can be identified with $C(X_0)$ but as subalgebras of $C(X)$ they are differ because of different actions. The left orbit of the point $(x, e)$, $x \in X_0$ are the set $\{(x, g): g \in G\}$, and the right orbit of such point are the sets $\{(gx, g): g \in G\}$. So the right action of $G$ induced on the space $X_0$ (as a fundamental domain of the second action) the initial action of the group: $x \mapsto gx$, and the left action is the action of the group which is opposite to group $G$ with the same orbits: element $g \in G$ acts as element $g^{-1}$). So, we can consider two von Neumann algebras $\mathcal{A}_G$ and $\mathcal{A}_H$ in the Hilbert space

$$L^2(X, \mu) = L^2(X_0, \mu_0) \otimes l^2(G)$$

and obtain the classical von Neumann crossed product construction of factor, generated by this dynamical system as a right action, and the commutant which is realized using the left action. The representation of both algebras are mutual commutants The construction above also gave an antiisomorphism of the factors.

As a representation of crossed-product algebra this representation is so called

regular or von Neumann representations, it can be realized also in the framework of groupoid theory. If the measure $\mu_0$ is finite then both factors has type $\text{II}_1$. In this case we have bicyclic vector for both factors, namely - a characteristic function of $X_0$, thus the coupling constant equal to 1. If the measure $\mu_0$ is not finite (sigma-finite) then type of factor is $\text{II}_\infty$. All these facts can be extended from the case of countable group to the case of locally compact group but we do not settle on. As was mentioned above the representation of the crossed-product algebra generated by joint actions of $G \times G^\circ$ on $(X, \mu)$ is irreducible (so called Koopmans representation).

The ordinary constructions of the regular representation (neither in terms of crossed-product nor in terms of the theory of groupoid) can not cover the general scheme of our first paragraph, the point is that our assumptions about the actions of the group does not demand that the first action in a sense uniquely defined the second action.
3. HEISENBERG CASE

3.1. Z-action on $\mathbb{R}$ and Heisenberg group.

In the first example of our scheme which is different from von Neumann case
we consider the trivial action of $\mathbb{Z}$ on $\mathbb{R}$ by the shifts. We will see that it is closed
to the classical context. Nevertheless only recently L.Faddeev [2] pointed out on
its role in the theory of quantum fields theory and quantum groups as an example
of the situation when we need to divide classical Heisenberg-Weyl representations
of canonical commutative relations (Heisenberg group) onto two parts. So he had
considered the factors of type $II_1$ which are naturally appeared in this representation.
In the papers of M.Rieffel and A.Connes ([3, 4]) about rotation algebra there
were closely related considerations. But I did not find their the analysis of factors.

The example can be considered also as the factor-representation of quantum
torus, (or of rotation algebra) which is not spatially isomorphic to well-known
factor-representation of it and also as factor-representation of discrete Heisenberg
group. But the main context which we use below is the terms of classical
Heisenberg group.

Let $\lambda_1, \lambda_2$ are two real nonzero numbers; consider the actions of the groups $G = \mathbb{Z}\lambda_1$ and $H = \mathbb{Z}\lambda_2$ on the real line $X = \mathbb{R}$ by the shifts $x \rightarrow x + \lambda_1, x + \lambda_2$. Let $\mu$ is
Lebesgue measure on $X = \mathbb{R}$, so we already have all needed data for general scheme
of the paragraph 1: two commuting dissipative actions with invariant measure.
We can assume that $\lambda_1, \lambda_2 > 0$. As the fundamental domains we can choose the
semiopen intervals $[0, \lambda_1)$ and $[0, \lambda_2)$, so the space of the orbits (quotient spaces)
of the groups are the circles $\mathbb{R}/\mathbb{Z}\lambda_1$ and $\mathbb{R}/\mathbb{Z}\lambda_2$. Algebras $C_G$ (corresp. $C_H$) are the
algebra of the periodic functions with the periods correspondingly $\lambda_1$ and $\lambda_2$.
The action of the first group $G$ on the second circle and the second group $H$ on the
first circle are isomorphic of course to the rotation on some angle, more exactly the
actions of the first and second groups are isomorphic to the rotation correspondingly
on the angles $\{\lambda_2/\lambda_1\}$ and $\{\lambda_1/\lambda_2\}$ (enter of the ratio) of the unit circle. Thus we
have the representations in $L^2(\mathbb{R}, \mu)$ of both algebras -shortly the first algebra
generated as $W^*$-algebra by two pairs of operators

$$(V_1 f)(x) = \exp\{i2\pi \lambda_1^{-1} x\} f(x) \quad \text{and} \quad (U_1 f)(x) = f(x + \lambda_2)$$

and

$$(V_2 f)(x) = \exp\{i2\pi \lambda_2^{-1} x\} f(x) \quad \text{and} \quad (U_2 f)(x) = f(x + \lambda_1)$$

Remark that without diminishing the generality we can put $\lambda_1 = 1$ by choosing
suitable normalization of $\mathbb{R}$ and denoting by $x' = \lambda_2$; so we obtain the following
two pairs of the operators

$$(V'_1 f)(x) = \exp\{i2\pi x\} f(x) \quad \text{and} \quad (U'_1 f)(x) = f(x + x'),$$

and

$$(V'_2 f)(x) = \exp\{i2\pi x^{-1}\} f(x) \quad \text{and} \quad (U'_2 f)(x) = f(x + 1).$$

But it is more convenient for us not to specialized $\lambda_1, \lambda_2$.

If one wants to reduce these systems to the rotations of the unit circle then
obtains after normalization two rotations on the angles $\{\lambda\}$ and on $\{\lambda^{-1}\}$ (here
\{\} means \textit{enrich}, In [2] was used $-\lambda^{-1} \text{ instead of } \lambda^{-1}$, but there no difference in between because this is simply the changing of sign of the generator of the group $\Gamma$.

Suppose now that $\lambda \equiv \frac{\lambda}{\lambda_0}$ is irrational.

The main fact is these to dynamical systems can be formulated as follow:

\textbf{Theorem 2.} Consider an irrational positive $\lambda \in \mathbb{R}$ and two pairs of the operators above $(U_1, V_1)$ and $(U_2, V_2)$ then:

1. Each pair of operators generated in $L^2(\mathbb{R}, m)$ $W^*$-algebras which is von Neumann factor $A_\lambda$ and $A_{\lambda^{-1}}$ of type $\Pi_1$ and which are mutual commutants in $\mathcal{B}(L^2(\mathbb{R}))$.

2. The coupling constant of the first factors is equal to $\lambda$ and to $\lambda^{-1}$ for the second factor. (in initial terms $\frac{\lambda}{\lambda_0}$ and $\frac{\lambda}{\lambda_0}$ correspondingly.)

3. Four operators $V_1, U_1, V_2, U_2$ generated the irreducible infinite dimensional representation of the Heisenberg group (see below).

The claims 1. and 2. follow from the theorem 1; we explain only item 3 and remarkable connection with Heisenberg group.

Let $\mathcal{H}$ is the quotient of the three-dimensional Heisenberg group $\mathbb{H}^\mathbb{R}$ of the nilpotent matrices over $\mathbb{R}$ but over the subgroup $\mathbb{Z}$ of the center, so topologically this group is the product of the real plane and unit the circle. We will denote the elements of the group $\mathcal{H}$ as a triples:

$$\mathcal{H} = \{(a, b, a); \quad a, b \in \mathbb{R}, \quad a \in S^1 = \mathbb{R}/\mathbb{Z}\},$$

with multiplication (we use multiplicative notation for the third coordinate):

$$(a, b, a) \cdot (a', b', a') = (a + a', b + b', a \cdot a' \cdot \exp\{2\pi i ab'\}).$$

Choose real nonzero numbers $\lambda_1 \lambda_2$ and define the class of the discrete subgroups of the group $\mathcal{H} = \Gamma(\lambda_1, \lambda_2)$:

$$\Gamma(\lambda_1, \lambda_2) = \{(m\lambda_1, n\lambda_2^{-1}, \exp\{2\pi ir\frac{\lambda_1}{\lambda_2}\}), \quad m, n, r \in \mathbb{Z}\}$$

We need to consider the pairs of such groups $\Gamma(\lambda_1, \lambda_2)$ and $\Gamma(\lambda_2, \lambda_1)$, so we have

$$\Gamma(\lambda_2, \lambda_1) = \{(m'\lambda_2, n'\lambda_1^{-1}, \exp\{2\pi ir'\frac{\lambda_2}{\lambda_1}\}), \quad m', n', r' \in \mathbb{Z}\}$$

Simple calculation shows that the groups $\Gamma(\lambda_1, \lambda_2)$ and $\Gamma(\lambda_2, \lambda_1)$ commute and moreover, each subgroup is a centralizer of the second subgroup inside of the group $\mathcal{H}$. In the case when the number $\lambda \equiv \frac{\lambda}{\lambda_0}$ is irrational one, two groups together generate topologically the whole group $\mathcal{H}$.

Remark that for all pairs $(\lambda_1, \lambda_2)$ with irrational ratio $\lambda$ the group $\Gamma(\lambda_1, \lambda_2)$ is isomorphic to the \textit{discrete Heisenberg group} $\mathbb{H}^\mathbb{Z} = \{(m, n, p), m, n, p \in \mathbb{Z}\}$ which is discrete subgroup of $\mathbb{H}^\mathbb{R}$.

Now let us consider the canonical irreducible representations of the group $\mathcal{H}$ in the Hilbert space $L^2(\mathbb{R}, \mu)$ (here $\mu$ is Lebesgue measure). They are parameterized by Planck constant which is for this group is a nonzero integer: $n \in \mathbb{Z} \setminus \{0\}$.
nonidentical characters of the center and the corresponding unitary operators have a form:

\[(U_{a,b,a}f)(x) = a^n \exp(2\pi i a x) f(x+b), \quad x \in \mathbb{R}, \quad f \in L^2(\mathbb{R}), \quad a, b \in \mathbb{R}, \quad a \in \mathbb{R}/\mathbb{Z}.
\]

Denote this representation as \( \rho_n \).

**Lemma 3** Suppose that the number \( \frac{a}{\lambda} \) is irrational.

The restrictions of the irreducible representation \( \rho_n \) of the group \( \mathcal{H} \) onto the subgroups \( \Gamma(\lambda_1, \lambda_2) \) and \( \Gamma(\lambda_2, \lambda_1) \) are the factor-representations of those groups both of which have type \( \Pi_1 \), and the corresponding factors are mutual commutants in \( B(L^2) \). The coupling constant is equal to \( \frac{a}{\lambda_1} \) for the first factor and \( \frac{a}{\lambda_2} \) for the second one.

For \( n = 1 \) the representation \( (\rho_1) \) evidently coincides with representations defined in the formulation of the theorem 2.

All the claims can be checked directly. This is the example which have been considered in [2]. So we have decomposed irreducible representation of Hiesenberg group onto two factors of type \( \Pi_1 \) each of which naturally appeared as a restriction of the representation onto the commuting lattices of Heisenberg groups.

For various pairs \( \lambda_1, \lambda_2 \) we obtain a different decompositions of that irreducible representation, the ratios \( \lambda \) and \( \lambda^{-1} \) is the coupling constant - a complete spatial invariant of this decomposition. Remark that coupling uniquely defined by subgroups. The representations \( \rho_n \) for other \( n \) has the same structure. Of course we can consider these representations as representation of whole Heisenberg group \( \mathcal{H} \). The case of arbitrary (not integer) Planck constant also does not give anything new except another normalization of the subgroups \( \Gamma \). We do not stop on this.

### 3.2. An analytical corollary.

The theorem about coupling constant gives one pure analytic fact perhaps unknown from Fourier analysis.

**Theorem 4.** (Cyclic vector for coordinate and impulse.) Consider to unitary operators \( V \) and \( U \) in \( L^2(\mathbb{R}, \mu) \) (\( \mu \) is normalized on the interval \([0, 1]\) Lebesgue measure).

\[(Vf)(x) = \exp(i2\pi x) f(x) \quad \text{and} \quad (Uf)(x) = f(x + \gamma).
\]

Suppose that \( \gamma \) is irrational. Then the following two conditions are coincided:

1. \( \gamma < 1 \)
2. There exists a function \( f \in L^2(\mathbb{R}, \mu) \) of the norm 1, such that the orbit of \( f \) under the group generated by \( V \) and \( U \) is a total set in \( L^2(\mathbb{R}, \mu) \). (=linear hull of orbit is everywhere dense). In another word: \( f \) is a cyclic element for the algebra generated by those operators.

**Proof.**

\[\square\]

1. \( \Rightarrow 2 \). Let put \( f = \chi_{[0, \gamma)} \) -characteristic function of the interval. It is clear that linear hull of the \( V \)-orbit of this function gives a dense subset in norm-topology in the subspace \( L^2([0, \gamma)) \subset L^2(\mathbb{R}, \mu) \). Then operator \( U \) shifts it along \( \mathbb{R} \). This part of is course trivial.

2. \( \Rightarrow 1 \). Here we use the calculation of the coupling constant which was done above. The commutant of the \( W^* \)-algebra generated by the operators \( V \) and \( U \) is
the $W^*$-algebra generated by the operators $V'$ and $U'$:
\[
(V'f)(x) = \exp\{2\pi i \gamma^{-1}\} f(x), \quad (Uf)(x) = f(x + 1).
\]

Consequently the coupling constant of the first factor is equal to $\gamma$ and by the theorem which were mentioned above -see [1] - existence of the cyclic vector is equivalent to the inequality $\gamma < 1$. ■

Of course if $\gamma > 1$ then cyclic vector exist for the commutant and for factor itself the cyclic vector for commutant is the separating vector (=vector which do not annulate any nonzero operator form the factor). So using this equivalence it is enough to prove that in the case when $\gamma < 1$ each nonzero vector annulates at least one element of he factor.

The quoted theorem based on the theory of von Neumann algebras and specially on the theory of multiplicity of factor-representation.

But the question how to prove this nontrivial fact directly in terms of classical analysis as I know, is open and could be not so difficult. My conversations with the specialists in Fourier analysis gave me expression (may be wrong) that the question was not even posed and studied seriously.

We can notice that the coupling constant in our examples has very special values: if we fix a $\lambda < 1$ as a size of shift on the real line above then this constraction gave only value $\lambda$ or $\lambda^{-1}$ as a coupling constant. It is not clear if there are a natural construction for splitting together shifts with an arbitrary (even belonging to $SL(2,\mathbb{Z})$-orbit of $\lambda$) values of coupling constant. Of course all factors are hyperfinite and starting from the pair of factor and its commutant with given coupling constant we can choose the structure of crossed-product with needed dynamical systems in both factors, but such a construction looks rather artificial.

3.3. Connection with factor-representations of quantum torus and discrete Heisenberg group

Consider rotation $C^*$-algebra (=quantum 2-torus) $A_\theta$, where $\theta$ is irrational positive number less that one and algebra topologically generated by two unitary elements $u, v$ with relation:
\[
UV = \exp\{2\pi i \theta\} VU.
\]

It is well-known that this algebra has only one nontrivial trace which generated the regular representation of it as crossed product -exactly as in paragraph 2 above). It is known K-functor $K_0(A_\theta) = \mathbb{Z}^2$, and positive elements in it - $\{(m, n) : m \cdot \theta + n > 0\}$ and even projectors which realized spokespersons classes in $K_0$. But if we want to classify factor-representation not up to isomorphism but up to spatial isomorphism then we have continuum of the factor representations which parameterized by positive real number which is coupling constant. So the question arises - how naturally realize such factor-representations. Here I want to mention only that the construction above gives such realization for special coupling constant $-\theta$. As we have mentioned the groups $\Gamma(\lambda_1, \lambda_2)$ for irrational ratio $\lambda$ are isomorphic to the discrete Heisenberg group $H^2$, so consequently we constructed factor-representations of it. I do not know if the question about description of list of all factor-representations up to spatial isomorphism had been discussed, although this group has nonsmooth dual (=antiliminaire=has no good description of the irreducible unitary representations, nevertheless the list of characters and more generally of factor-representations is definitely visible.
4. REMARKS ON NONCOMMUTATIVE GROUPS (TWO-SIDED GENERAL SCHEME)

Consider the pairs of dynamical systems in the sense of paragraph 1 for more general class groups. It is easy to generalize construction from the case of $\mathbb{R}$ to $\mathbb{R}^d$: two groups $\mathbb{Z}^d$ act by the shifts on the vectors $(\lambda_1, \lambda_2, \ldots, \lambda_d)$ and $(\lambda'_1, \lambda'_2, \ldots, \lambda'_d)$ correspondingly. It will be the same effect and instead of three dimensional Heisenberg group above we must consider $2d+1$-dimensional Heisenberg group. The same scheme can be done for any locally-compact abelian not compact i.e. additive group of $p$-adic numbers an so on.

In fact in those the example the factors turn out to be factors generated by the restriction of irreducible representations of some group (Heisenberg group) onto two commuting subgroups. But this is impossible for general noncommutative groups because in general there are no commuting subgroup like $\Gamma(\lambda_1, \lambda_2)$ which topologically generated whole group. For example in the semisimple groups there are no commuting lattices $\Gamma_1$, $\Gamma_2$ with finite volume of the fundamental domain.

But instead of the factorization of the irreducible representation of the group itself as it was before with Heisenberg group we can use our scheme in another way: to factorize the representation of the crossed-product of the group on the space of function on the group of homogeneous space. Here I will define such a variant of the general scheme.

Choose in the locally compact non compact unimodular group $G$ two lattices $\Gamma_1$ and $\Gamma_2$ which intersect by 1 and consider left of actions of $\Gamma_1$ and right actions of $\Gamma_2$ on the group $G$ with Haar measure $m$. Because left and right actions commute we are in the situation of our scheme. In order to have ergodicity we need the following condition: the product two lattices $\Gamma_1$ and $\Gamma_2$ topologically generate whole group $G$, which means the group generated by $\Gamma_1$ and $\Gamma_2$ is everywhere dense in the group $G$.

In Hilbert space $L^2(G, m)$ we have two factors $B_\gamma$ which is generated by multipliers on the bounded functions on $G/\Gamma_1$ and by operators of the right shift on $\gamma \in \Gamma_1$, and factor $B_\gamma$ generated by multipliers of the functions on $\Gamma_1 \setminus G$ and by operators on the left shift on $\gamma \in \Gamma_2$. These four family of operators defined an irreducible representation on the crossed-product algebra $(\Gamma_1 \times \Gamma_2) \ltimes C_b(G)$, where $C_b(G)$ is the space of all bounded functions on $G$.

Here are some examples of this situation and it is very interesting to study this. Look again on the Heisenberg group and its subgroup $H^2$ - discrete Heisenberg group and $\Gamma(\lambda_1, \lambda_2)$. Then in the space $L^2(H^2)$ we can define two factors. This is different with example in the beginning of this section - here subgroups $\Gamma_{1,2}$ act on the space (nilpotent manifold).

Consider a semisimple group, say $SL(2, \mathbb{R})$ and its lattices $\Gamma_1 = SL(2, \mathbb{Z})$ and

$$\Gamma_2 = \{ \gamma = \left( \begin{array}{cc} m & n \lambda \\ p \lambda^{-1} & q \end{array} \right) \}$$

where $\lambda$ is fixed irrational number and the matrix

$$\left( \begin{array}{cc} m & n \\ p & q \end{array} \right)$$

is an arbitrary element of the lattice $\Gamma_1 = SL(2, \mathbb{Z})$. Then we will obtain actions which in the framework of our scheme gives non-hyperfinite factors because those
lattices are not amenable groups. In this example coupling constant is equal to 1 but it easy to construct the lattices with given positive number as a ratio of the volumes of the fundamental domains or a coupling constant for corresponding factors.

5. REALIZATION OF THE FACTOR-REPRESENTATIONS OF INFINITE SYMMETRIC GROUP.

Now I want to give an example of the factor-representations of infinite symmetric group with nontrivial coupling constant. But before let us discuss this problem from more general point of view.

The interest to factor-representations of type II₁ based on the fact that the traces in factor gives a finite characters of the group. And for some groups the set of traces is sufficiently large for he goals of harmonic analysis. The theory of traces (=finite characters) is a natural generalization of the theory characters of the finite and compact groups.

Nevertheless as we had mentioned the spatial trace exists only when coupling constant of factor is equal to one. So the additional question arises: to find a natural realization of factor-representation with given coupling constant. Of course such factor-representations are quasiequivalent = algebraically equivalent to the factor-representations with the trace, but it that equivalence is not so evident a priori. So citation is the following - each finite factor-representation is algebraically equivalent to the factor-representations with spatial trace and consequently it could be formally realized as subrepresentation of the direct sum of the several copies of that trace representation. But such realization looks very formally. Below we will give an explicit example of the natural realization for infinite symmetric group which looks like a generalization of our method for construction of the trace representation for that group ([5]).

To construct not tautological (not GNS) realization of any factor-representations of type II₁ for the given group with given trace is not obvious task, and an additional problem - particularly with given coupling constant. Sometimes it is possible to use the crossed-product (or groupoid, or orbit) construction for example for representations of solvable group. But nontrivial cases of the use of such constructions appeared in the asymptotic theory of representations of classical groups. In [5] we suggested the model of factor representations of infinite symmetric infinite unitary groups etc. where the structure of the crossed-product appeared in a very special role.

I will repeat here that construction in a new reduction and then change it in order to construct the needed example with coupling constant. The main idea has wide applications for example for the theory of representation of factor-representations of the group of matrices over finite fields.

Let $X_0$ be the set of the \textit{eventually symmetric sequences} of symbols from countable or finite alphabet $A$:

\[ X_0 = \{ x = \{ x_i \}_{i \in \mathbb{Z}_{\leq 0}} : \ x_i \in A \quad i \in \mathbb{Z}_{\leq 0} \} \]

and for any $x$

\[ x_{-i} = x_i \text{ for sufficiently large } i = i(x) \geq 0 \]

The subset of the true symmetric functions $X_0^\circ = \{ \{ x_i \} : x_i = x_{-i}, i = 1 \ldots \}$ is equipped with Bernoulli (product) measure $\mu_0$, which is defined by probability
measure \( \nu \) on \( A \). Two copies of the group \( S_\infty \) acts separately on positive and negative parts of the sequences and these actions evidently commute. So we have the action of direct product \( S_\infty \times S_\infty \). Let us define \( \sigma - \text{finite} \) a measure on \( X_0 \) which is extend measure \( \mu_0 \) onto \( X_0 \) as invariant under the action of \( S_\infty \times S_\infty \). Now let us consider a representation of the group \( S_\infty \times S_\infty \) in the Hilbert space \( L^2 (X_0, \mu) \) by the operators of substitution. We are just in the situation of our general scheme of the paragraph 1.

**Theorem** [5] If the values on the one-point sets of the measure \( \nu \) on the countable (or finite) set \( A \) are all nonequal then the representations of each of two symmetric groups are factor representations of type and are commutant one of another. The representation of the direct sum \( S_\infty \times S_\infty \) is irreducible.

**Remark.** The theorem was formulated and proved in the terminology of groupoid theory (or orbit theory) in [5]. The main point and nontrivial fact contained in this theorem is the following: we obtain the factors as a \( W^* \)-closure of the operators of the group only! In another word: the \( W^* \)-closure of representation of crossed-product of each of two symmetric groups with the algebras of corresponding functions (as in the scheme of the paragraph 1,2) is the same as \( W^* \)-closure of the group algebras; in some mysterious way the needed structure of crossed-product appeared in \( W^* \)-closure of the representation of group algebra automatically. The explanation of that effect as well as the proof of the theorem is the following: the action of the groups is not free and characteristic function \( \chi \) of the set \( X_{\#} \) is bicyclic element; linear combinations of the shifts approximately distinguished the points of the \( X_0 \). This example give of course the von Neumann (orbit) construction and coupling constant is equal to one, see paragraph 2. The simplest concrete example is the following: \( A = \{0,1\} \), \( \nu = \{\alpha, 1 - \alpha\} \), \( 0 < \alpha < 1/2 \).

In the case of multiple values (i.e., above if \( \alpha = 1/2 \)) the permutations of the symbols with equal values of the measure \( \nu \) give additional symmetry; the element \( \chi \) is not bicyclic, in this case commutant to both factors is more wider and representation of \( S_\infty \times S_\infty \) is reducible. It is easy to describe decompositions in that case also.

The model above is a new (with respect to ([5])) model of the representation in the space of the functions on the set of eventually symmetric sequences; for our main goals here (to construct a representation with nontrivial coupling constant) this model is more suitable for generalizations than groupoid one.

Now let us consider the realization of factor-representation with nonunit coupling constant. Let \( r \) is positive integer and \( X_r \) be a set of all two-sided infinite sequences \( x = \{x_i\}_{i \in \mathbb{Z}} \) with the following symmetry: for each sequence \( x \) there exist \( k = k(x) \geq 0 \) such that

\[
x_{i-k} = x_{i+k+r}, \quad i = 1, \ldots
\]

("shifted eventual symmetry") If \( r = 0 \) then \( X_r \) is the set of \textit{eventually symmetric sequences} \( X_0 \) defined above. Define again two actions of infinite symmetric group - the first acts by permutations of negative (left) part of sequences \( x \) and the second - on the positive (right) part as before. Evidently the actions are commute. Now

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2. This fact is nothing to do with well-known fact that each AF-algebras has a structure of crossed product; i.e. here we have completely different structure of crossed-product which appeared for more deep and individual reason.
define a sigma-finite measure $\mu'$ on $X_r$ as following: first of all the measure $\mu''_t$ is defined on the set of sequences $X_{k,r} \equiv \{x : x_{-i} = x_{i+r}, i = 1, \ldots \}$ (symmetric shifted sequences) as a Bernoulli measure as above with measure $\nu$ as a multiplier. Then the extension $\mu'$ of the measure $\mu''_t$ on whole space is defined by invariance under the actions of the group.

The fundamental domain for the left action could be put as the set $X_{k,r},$ and for the right action the subset $X_{k',r} \subset X_k,r$, with the additional condition on the values of the coordinates $x_i$ for $i = 1, \ldots, r,$ namely this finite subsequence must be monotonic; all zeros stay in the beginning and ones in the end of sequence (like $(0, \ldots, 0, 1 \ldots 1)$). The measure of the fundamental domain for the left action by definition is 1, it is easy to check that the measure of the fundamental domain for the right action is equal to $\frac{2^r}{\mathcal{A}}.$ So, for $r > 1$ we have different volumes for the fundamental domains. The definition of the representation of the group $S_\infty \times S_\infty$ in the space $L^2(X_r, \mu)$ is the same as above and again if the values of measure $\nu$ on the alphabet $A$ is different then we have the same conclusions as in the theorem:

**Theorem**

Under the denotations above the representation of the double of the group $S_\infty \times S_\infty$ is irreducible and its restrictions on the subgroups $S_\infty \times 1$ and $1 \times S_\infty$ are factor representations of infinite symmetric group of type $\Pi_1$ which are mutual commutants with coupling constants $\frac{r+1}{\mathcal{A}}.$

As before we obtain factor as $W^\ast$-closure of the representation of the groups itself which is the same as $W^\ast$-closure of crossed product. Remark that we obtain here only the special values of the coupling constants - $\frac{r+1}{\mathcal{A}}$ and inverse. It is not clear if other constants can be obtained in the natural way. The example of the constructed representation is important also as example of explicit realization of "tame"representations of the group $S_\infty$ in the sense of G. Olshansky, presumably, each tame representations is a factor representation of type $\Pi_1$ with some coupling constant.

KEY WORDS: Coupling constant, dynamical systems, factors, Heisenberg group, infinite symmetric group

Список литературы