Quantum Gravity, Shadow States, and Quantum Mechanics

Abhay Ashtekar
Stephen Fairhurst
Joshua L. Willis

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Abhay Ashtekar,1,2,* Stephen Fairhurst,1,2,† and Joshua L. Willis1,‡

1Center for Gravitational Physics and Geometry
Department of Physics, The Pennsylvania State University
University Park, PA 16802-6300, USA

2Erwin Schrödinger Institute
9 Boltzmanngasse
1090 Vienna, Austria

3Theoretical Physics Institute
University of Alberta
Edmonton, Alberta T6G 2J1, Canada

Abstract

A program was recently initiated to bridge the gap between the Planck scale physics described by loop quantum gravity and the familiar low energy world. We illustrate the conceptual problems and their solutions through a toy model: quantum mechanics of a point particle. The model can also serve as a simple introduction to many of the ideas and constructions underlying quantum geometry. Maxwell fields will be discussed in the second paper of this series which further develops the program.

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*Electronic address: ashtekar@gravity.psu.edu
†Electronic address: sfairhurst@phys.ualberta.ca
‡Electronic address: jwillis@gravity.psu.edu
I. INTRODUCTION

Perhaps the central conceptual lesson of general relativity is *gravity is geometry*. There is no longer a background metric, no inert stage on which dynamics unfolds; like all physical fields, geometry is dynamical. Therefore, one expects that a fully satisfactory quantum gravity theory would also be free of a background space-time geometry. However, of necessity, a background independent description must use physical concepts and mathematical tools that are quite different from those normally used in low energy quantum physics. A major challenge, then, is to show that this low energy description does arise from the pristine, Planckian world in an appropriate sense. This challenge is now being met step by step in the context of loop quantum gravity. Some of the key ideas were summarized in [1], which in turn was motivated by [2]. They will now be discussed in detail and significantly extended in a series of papers, of which this is the first (see also [3, 4]). Our goal here is to illustrate, through a simple example, both the tension between the two frameworks and the new physical notions and mathematical techniques that are being used to resolve it.

Let us begin by listing some of the main issues and questions.

Loop quantum gravity is based on *quantum geometry*, the essential discreteness of which permeates all constructions and results. The fundamental excitations are 1-dimensional and polymer-like. A convenient basis of states is provided by spin networks. Low energy physics, on the other hand, is based on quantum field theories which are rooted in a flat space continuum. The fundamental excitations of these fields are 3-dimensional, typically representing wavy undulations on the background Minkowskian geometry. The convenient Fock-basis is given by specifying the occupation number in one particle states labelled by momenta and helicities. At first sight, the two frameworks seem disparate. Nonetheless, one would hope that the polymer description admits semi-classical states which approximate classical space-times as well as fluctuations on them represented by gravitons and other fields. On the other hand, since this perturbative description breaks down rather quickly because of closed graviton loops, the low energy results are not likely to emerge as first terms in a systematic expansion of a finite, full theory. *What then is the precise sense in which semi-classical states are to arise from the full theory?*

From a mathematical physics perspective, the basic variables of quantum geometry are holonomies (or Wilson loops) of the gravitational connection $A$ along 1-dimensional curves and fluxes of the conjugate momenta (the triads) $E$ across 2-surfaces. In the final quantum theory, the connection $A$ fails to be a well-defined operator(-valued distribution); only the holonomies are well-defined. In perturbative quantum field theories, by contrast, the vector potential operators are distributions, whence, a priori, their holonomies fail to be well-defined operators. Similarly, fluxes of electric field operators across 2-surfaces fail to be well-defined even on the Fock space of photons. Heuristically, then, it would appear that, even at a kinematic level, loop quantum gravity describes a ‘phase’ of gauge theories which is distinct from the one used in electrodynamics. Since it is generally believed that distinct phases carry distinct physics, it is natural to ask: *Is the well-tested, macroscopic ‘Coulomb phase’ of low energy gravity compatible at all with the Planck scale discreteness of quantum geometry?* If so, in what sense? How does it emerge from loop quantum gravity? Given the apparent deep differences, the procedure of extracting the ‘Coulomb phase’ from the fundamental Planckian description should be rather subtle.

Finally, a further technical but important complication arises from the detailed treatment
of dynamics. Solutions to the quantum Einstein equations (i.e., quantum constraints) do not belong to the so-called kinematical Hilbert space $\mathcal{H}_{\text{Poly}}$. This is not surprising: a similar situation occurs already for simple, quantum mechanical constrained systems. The kinematical Hilbert space provides the mathematical framework to construct well-defined operators which can be regarded as the quantum analogs of the classical constraint functions. If zero lies in the continuous part of the spectrum of these operators, none of the solutions to the quantum constraints are normalizable with respect to the kinematic inner product. (This is the case even for the simple constraint $p_x = 0$ in $\mathbb{R}^3$, and for the constraint $g^{ab}p_ap_b - \mu^2 = 0$ satisfied by a free particle in Minkowski space-time.) The solutions are distributional; they belong to the dual of a sub-space of ‘nice’ quantum states (e.g., the Schwartz space). The situation is completely analogous in quantum gravity. The ‘nice’ quantum states are typically taken to be finite linear combinations of spin network states and their space is denoted by Cyl (the space of ‘cylindrical’ functions of connections). Solutions to the quantum Einstein equations belong to its dual, Cyl*. There is an inclusion relation (providing a ‘Gel’fand-type’ triplet) Cyl $\subset \mathcal{H}_{\text{Poly}} \subset$ Cyl*. While the kinematical spin network states belong to Cyl, the physical states belong to Cyl*. Therefore, semi-classical states, capturing the low energy physics, should also be in Cyl*. The problem is that, as of now, Cyl* does not have a physically justified inner product; a definite Hilbert space structure is not yet available. Can one nonetheless hope to extract low energy physics already at this stage? In particular, can one test a candidate state in Cyl* for semi-classicality without access to expectation values?

The primary purpose of this paper is to analyze these and related issues using the simple example of a non-relativistic particle. We will find that the issues raised above arise also in this example and can be resolved satisfactorily. (For an analysis with similar motivation, but which emphasizes the role of constraints and discrete time evolutions, see [5].)

For readers who are not familiar with quantum geometry, this example can also serve as an introduction to the mathematical techniques used in that framework. However, as is typically the case with toy models, one has to exercise some caution. First, motivations behind various construction often become obscure from the restrictive perspective of the toy model, whence the framework can seem cumbersome if one’s only goal is to describe a non-relativistic particle. Secondly, even within mathematical constructions, occasionally external elements have to be brought in to mimic the situation in quantum geometry. Finally, because the toy model fails to capture several essential features of general relativity, there are some key differences between the treatment of the Hamiltonian and other constraints in the full theory and that of the Hamiltonian operator in the toy model. With these caveats in mind, the toy model can be useful in understanding the essential differences between our background independent approach and the Fock-space approach used in Minkowskian, perturbative quantum field theory.

We will begin with the usual Weyl algebra generated by the exponentiated position and momentum operators. The standard Schrödinger representation of this algebra will play the role of the Fock representation of low energy quantum field theories and we will construct a new, unitarily inequivalent representation—called the polymer particle representation—in which states are mathematically analogous to the polymer-like excitations of quantum geometry. The mathematical structure of this representation mimics various features of quantum geometry quite well; in particular there are clear analogs of holonomies of connections and fluxes of electric fields, non-existence of connection operators, fundamental discreteness, spin
networks, and the spaces Cyl and Cyl\(^{\dagger}\). At the basic mathematical level, the two descriptions are quite distinct and, indeed, appear to be disparate. Yet, we will show that states in the standard Schrödinger Hilbert space define elements of the analog of Cyl\(^{*}\). As in quantum geometry, the polymer particle Cyl\(^{*}\) does not admit a natural inner product. Nonetheless, as indicated in [1], we can extract the relevant physics from elements of Cyl\(^{*}\) by examining their shadow, which belong to the polymer particle Hilbert space \(\mathcal{H}_{\text{poly}}\). This physics is indistinguishable from that contained in Schrödinger quantum mechanics in its domain of applicability.

These results will show that, in principle, one could adopt the viewpoint that the polymer particle representation is the ‘fundamental one’—it incorporates the underlying discreteness of spatial geometry—and the standard Schrödinger representation corresponds only to the ‘coarse-grained’ sector of the fundamental theory in the continuum approximation. Indeed, this viewpoint is viable from a purely mathematical physics perspective, i.e., if the only limitation of Schrödinger quantum mechanics were its failure to take into account the discrete nature of the Riemannian geometry. In the real world, however, the corrections to non-relativistic quantum mechanics due to special relativity and quantum field theoretic effects largely overwhelm the quantum geometry effects, hence the above viewpoint is not physically tenable. Nonetheless, the results for this toy model illustrate why an analogous viewpoint can be viable in the full theory: Although the standard, low energy quantum field theory seems disparate from quantum geometry, it can arise, in a systematic way, as a suitable semi-classical sector of loop quantum gravity.

The paper is organized as follows. Section II recalls a few essential notions from quantum geometry which motivate our construction of the polymer particle representation. This representation is constructed in detail in section III. In section IV we show that the standard coherent states of the Schrödinger theory can be regarded as elements of Cyl\(^{*}\), introduce the notion of ‘shadow states’ and use them to show that the elements of Cyl\(^{*}\) defined by the coherent states are, in a precise sense, semi-classical from the perspective of the ‘fundamental’ polymer particle representation. In section V we discuss dynamics in the polymer particle representation. To define the kinetic energy term in the Hamiltonian, one can mimic the procedure used to define the Hamiltonian constraint operator in quantum general relativity. However, in the toy model, this requires the introduction of a new structure by hand, namely a fundamental length scale, which can be regarded as descending from an underlying quantum geometry. The resulting dynamics is indistinguishable from the standard Schrödinger mechanics in the domain of applicability of the non-relativistic approximation. Deviations arise only at energies which are sufficiently high to probe the quantum geometry scale. In particular, shadows of the Schrödinger energy eigenstates are excellent approximations to the ‘more fundamental’ polymer eigenstates.

In the second paper in this series we will show that we can extend these ideas to quantum field theory, where the familiar low energy physics can be extracted from a more fundamental theory based on quantum geometry. The two appendices of the present paper contain some technical material which will be important to that analysis.

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\(^{1}\) Of course, since this is only a simple, ‘toy example’, it does not capture all the subtleties. In particular, we will see that a number of distinct notions in quantum geometry often coalesce to a single notion in the example.
II. QUANTUM GEOMETRY

This summary of quantum geometry will enable the reader to see the parallels between quantum geometry and the polymer particle representation constructed in section III. It will be used primarily to motivate our constructions in subsequent sections. Our discussion will be rather brief and, in particular, we will omit all proofs. (These can be found, e.g., in [6-17].) A detailed understanding of this material is not essential to the rest of the paper.

In diffeomorphism invariant theories of connections, the phase space consists of pairs of fields \((A, E)\) on a 3-manifold \(\Sigma\), where \(A_x^i\) are connection 1-forms which take values in the Lie-algebra of the structure group \(G\), and \(E^a_x\) are 'electric fields' which are vector densities with values in the dual of the Lie algebra. For the purpose of this paper, it suffices to restrict ourselves to two special cases: i) \(G = \text{SU}(2)\), used in quantum geometry, and, ii) \(G = \text{U}(1)\) used in quantum Maxwell theory. In either case, the 'elementary' classical observables are taken to be holonomies \(A_x^i\) along paths \(e\) defined by \(A\) and fluxes \(E_S\) of electric fields across 2-surfaces \(S\). From the perspective of the standard Hamiltonian formulation of field theories, these functions are 'singular': Since they are supported on 1-dimensional curves and 2-dimensional surfaces, respectively, we are in effect using distributional smearing functions. Nonetheless, the symplectic structure on the classical phase space endows them with a natural Lie bracket and the resulting Lie-algebra is taken as the point of departure in quantum theory.

The Hilbert space of states can be constructed in two ways. In the first, one uses the fact that, as usual, the configuration variables \(A_x^i\) give rise to an Abelian \(C^*\)-algebra \(\mathcal{A}\), called the holonomy algebra. One then introduces a natural (diffeomorphism invariant) positive linear functional on it and uses the Gel'fand-Naimark-Segal (GNS) construction to obtain a Hilbert space \(\mathcal{H}_{\text{Pol}}\) of states and a representation of \(\mathcal{A}\) on it. Finally, self-adjoint electric flux operators are introduced on \(\mathcal{H}_{\text{Pol}}\) using the heuristic idea that \(E\) should be represented by \(-i\hbar \delta / \delta A\).

The second approach is more explicit. One begins by specifying the space \(\text{Cyl}\) of 'nice' functions of connections. Fix a graph \(\gamma\) on the 3-manifold \(\Sigma\) with \(N\) edges. A connection \(A\) associates to each edge \(e\) a holonomy \(A_x^i \in G\). The space of \(N\)-tuples \((A_1, \ldots, A_N)\) defines a configuration of the gauge theory restricted to the graph \(\gamma\) and will be denoted by \(\mathcal{A}_\gamma\). Clearly, \(\mathcal{A}_\gamma\) is isomorphic with \(G^N\). Now, given a smooth, complex-valued function \(\psi\) on \(G^N\), we can define a function \(\Psi\) of connections in an obvious fashion:

\[
\Psi(A) = \psi(A_1, \ldots, A_N).
\]

The space of these functions is denoted \(\text{Cyl}_\gamma\). Elements of \(\text{Cyl}_\gamma\) have knowledge only of the connection \textit{restricted to} \(\gamma\). The space \(\text{Cyl}\) of all cylindrical functions is obtained by simply considering all possible graphs \(\gamma\):

\[
\text{Cyl} = \bigcup_{\gamma} \text{Cyl}_\gamma.
\]

\footnote{From the viewpoint of the algebraic approach, which has been so successful in quantum field theory in curved space-times, working with a specific Hilbert space representation may seem restrictive. However, the algebraic approach is not so well-suited for systems, like general relativity, with non-trivial constraints. More importantly, there is no loss of generality in working with the above representation because it is singled out essentially by the requirement of diffeomorphism covariance [18].}
Thus, each element of Cyl depends only on holonomies of the connection along edges of some finite graph \( \gamma \) but the graph can vary from one function to another. Had we restricted ourselves to a fixed graph \( \gamma \), the theory would have been equivalent to a lattice gauge theory on a (generically irregular) ‘lattice’ \( \gamma \). However, since we allow all possible graphs, we are dealing with a field theory, with an infinite number of degrees of freedom, of all connections on \( \Sigma \).

The next step is to introduce an inner product on Cyl. For this, we simply use the induced Haar measure \( \mu_H^{(N)} \) on \( \mathcal{A}_\gamma \approx G^N \): Given any two functions \( \Psi_1 \) and \( \Psi_2 \) on \( \mathcal{A}_\gamma \), we set

\[
(\Psi_1, \Psi_2) = \int_{\mathcal{A}_\gamma} \bar{\psi}_1 \psi_2 \, d\mu_H^{(N)}.
\]  

(II.1)

Using properties of the Haar measure one can verify that this definition is unambiguous, i.e., if \( \Psi_1 \) and \( \Psi_2 \) are cylindrical with respect to another graph \( \gamma' \), the right side of (II.1) is unchanged if we replace \( \gamma \) with \( \gamma' \). This prescription provides us with an Hermitian inner product on all of Cyl because, given any \( \Psi_1, \Psi_2 \in \text{Cyl} \), there exists a (sufficiently large) graph \( \gamma \) such that \( \Psi_1, \Psi_2 \in \text{Cyl}_\gamma \). The Cauchy completion of Cyl with respect to this inner product provides the required Hilbert space \( \mathcal{H}_{\text{Foly}} \) of all quantum states, obtained in the first method via the GNS construction.

Because we consider all possible graphs on \( \Sigma \) in its construction, \( \mathcal{H}_{\text{Foly}} \) is very large. However, it can be decomposed into convenient finite dimensional sub-spaces. Each of these subspaces is associated with a labelling of edges of a graph \( \gamma \) by non-trivial irreducible representations of \( G \). Thus, in the case when \( G = SU(2) \), let us label each edge \( e \) of \( \gamma \) with a non-zero half-integer (i.e., spin) \( j_e \). Then, there is a finite dimensional sub-space \( \mathcal{H}_{\gamma, j} \) such that

\[
\mathcal{H}_{\text{Foly}} = \bigoplus_{\gamma, j} \mathcal{H}_{\gamma, j}.
\]  

(II.2)

This is called the spin network decomposition of \( \mathcal{H}_{\text{Foly}} \). Although \( \mathcal{H}_{\text{Foly}} \) is very large, practical calculations are feasible because each of the sub-spaces \( \mathcal{H}_{\gamma, j} \) can be identified with the Hilbert space of a spin-system which is extremely well understood. In the case when \( G = U(1) \), we label each edge \( e \) with a non-zero integer \( n_e \). The Hilbert space \( \mathcal{H}_{\gamma, n} \) is now 1-dimensional, spanned by the function

\[
\Psi(A) = e^{in_1 b_1} \cdots e^{in_N b_N}
\]

where \( e^{i \theta_n} \) is the holonomy of the connection \( A \) along the edge \( e_m \). These functions are called flux network states and by replacing \( j \) by \( n \) in (II.2) one now obtains a decomposition of \( \mathcal{H}_{\text{Foly}} \) in terms of 1-dimensional orthonormal subspaces.

As in any Schrödinger description, quantum states in \( \mathcal{H}_{\text{Foly}} \) can be regarded as square integrable functions on the quantum configuration space. In systems with finite number of degrees of freedom, the quantum configuration space is normally the same as the classical one. However, for systems with an infinite number of degrees of freedom, there is typically a significant enlargement: while classical configurations are represented by smooth fields, quantum configurations are distributional. This occurs also in our case: \( \mathcal{H}_{\text{Foly}} = L^2(\mathcal{A}, d\mu_c), \) where \( \mathcal{A} \) is a suitable completion of the space \( \mathcal{A} \) of smooth connections and \( \mu_c \), a regular measure on it. An element \( \tilde{A} \) of \( \tilde{\mathcal{A}} \) is called a generalized connection. It associates with every oriented path \( \epsilon \) in \( \Sigma \) an element \( \tilde{A}(\epsilon) \) of \( G \), the holonomy along \( \epsilon \) subject only to two
conditions: i) \( \bar{A}(e_1 \circ e_2) = \bar{A}(e_1) \bar{A}(e_2) \); and, ii) \( \bar{A}(e^{-1}) = [\bar{A}(e)]^{-1} \). Note that the assignment \( e \rightarrow \bar{A}(e) \) can be arbitrarily discontinuous, whence the quantum configuration space \( \bar{A} \) is a genuine extension of the classical configuration space \( A \). Nonetheless, in a natural topology, \( A \) is dense in \( \bar{A} \), whence \( \bar{A} \) can be regarded as a suitable completion of \( A \). However, as is typically the case in field theories, the measure \( \mu_c \) is concentrated on genuinely generalized connections; all the smooth configurations in \( A \) are contained in a set of zero measure.

The measure \( \mu_c \) is completely defined by the family of measures \( \mu_H^{(N)} \) on \( A, \approx G^N \); because \( \mu_H^{(N)} \) are mutually consistent in a precise sense, they can be ‘glued together’ to obtain \( \mu_c \). Indeed, every measure on \( \bar{A} \) arises as a consistent family of measures on \( A \). More generally, structures in the full quantum theory are constructed as consistent families of structures on \( A \), or \( Cyl \). In particular, many of the physically interesting operators on \( H_{Poly} \)—such as the holonomies \( \hat{A} \), the fluxes \( \hat{E} \) of \( E \) across \( S \), area operators \( \hat{A}_S \) associated with 2-surfaces \( S \), and volume operators \( \hat{V}_R \) associated with spatial regions \( R \)—arise as consistent families of operators on \( Cyl \). Therefore, their properties can be explored in terms of their actions on finite dimensional spaces \( H_{\gamma, x} \) (or \( H_{\gamma, n} \)).

While the above structures suffice to discuss quantum kinematics, as pointed out in the Introduction, an additional notion is needed in the discussion of quantum dynamics: solutions to the quantum Einstein’s equations do not belong \( \mathcal{H} \) because they fail to be normalizable. Their natural home is \( Cyl^* \), the algebraic dual of \( Cyl \). We have a natural inclusion:

\[ Cyl \subset H_{Poly} \subset Cyl^* \]

To discuss physical states and explore the physically relevant semi-classical sector, then, we are led to focus on \( Cyl^* \).

We will see in section III that the essential features of these constructions and results are mirrored in a transparent way in the ‘polymer particle representation’ of a non-relativistic point particle.

III. \textbf{SCHRÖDINGER AND POLYMER PARTICLE FRAMEWORKS}

The physical system we wish to consider is a particle moving on the real line \( \mathbb{R} \). (It is straightforward to extend our discussion to \( \mathbb{R}^n \).) A natural point of departure for quantum theory is provided by the Weyl-Heisenberg algebra. To each complex number \( \zeta \) associate an operator \( W(\zeta) \) (which will turn out to be a product of the exponentiated position and momentum operators) and consider the free vector space \( W \) generated by them. Introduce a product on \( W \) via:

\[ W(\zeta_1)W(\zeta_2) = e^{\frac{i}{2} \ln \zeta \zeta} W(\zeta_1 + \zeta_2), \]

and an involution \( * \) via

\[ [W(\zeta)]^* = W(-\zeta). \]

This is the Weyl-Heisenberg \( * \)-algebra of non-relativistic quantum mechanics. Here, as in the mathematical literature, we have chosen \( \zeta \) to be dimensionless.

In the physics literature, one implicitly introduces a length scale \( d \) and ‘splits’ operators \( W(\zeta) \) by setting

\[ W(\zeta) = e^{\frac{i}{2} \lambda \mu} U(\lambda) V(\mu) \]
where $\zeta = \lambda d + i(\mu/d)$. Thus, $U(\lambda) = W(\lambda d)$ and $V(\mu) = W(i\mu/d)$. The operators $U(\lambda)$ and $V(\mu)$ satisfy $[U(\lambda)]^* = U(-\lambda)$, $[V(\mu)]^* = V(-\mu)$ and the product rule:

$$U(\lambda_1)U(\lambda_2) = U(\lambda_1 + \lambda_2), \quad V(\mu_1)V(\mu_2) = V(\mu_1 + \mu_2),$$

$$U(\lambda)V(\mu) = e^{-i\lambda\mu} V(\mu)U(\lambda).$$ (III.3)

Therefore, in any representation of $W$, the Hilbert space carries 1-parameter unitary groups $U(\lambda), V(\mu)$. To fix notation and make a detailed comparison, we will first recall the standard Schrödinger representation of this algebra and then introduce the polymer particle representation in some detail.

**A. The Schrödinger representation**

The celebrated Stone-von Neumann theorem ensures us that every irreducible representation of $W$ which is weakly continuous in the parameter $\zeta$ is unitarily equivalent to the standard Schrödinger representation, where the Hilbert space is the space $L^2(\mathbb{R}, d\bar{x})$ of square integrable functions on $\mathbb{R}$ (where $\bar{x}$ is dimensionless). $W(\zeta)$ are represented via:

$$\hat{W}(\zeta)\psi(\bar{x}) = e^{\frac{i\zeta}{\hbar}} e^{i\alpha\bar{x}} \psi(\bar{x} + \beta),$$ (III.4)

where $\zeta = \alpha + i\beta$. This is an irreducible representation of $W$. Furthermore, the $\hat{W}(\zeta)$ are all unitary (i.e., satisfy $[\hat{W}(\zeta)]^* = [\hat{W}(\zeta)]^{-1}$) and weakly continuous in $\zeta$ (i.e., all matrix elements of $\hat{W}(\zeta)$ are continuous in $\zeta$).

In physics terms, the Hilbert space $\mathcal{H}_{Sch}$ is the space of square integrable functions of $x = \frac{\bar{x}}{d}$ and the action of these operators is given by

$$\hat{U}(\lambda) \psi(x) = e^{i\lambda x} \psi(x) \quad \text{and} \quad \hat{V}(\mu) \psi(x) = \psi(x + \mu)$$ (III.5)

for all $\psi \in \mathcal{H}_{Sch}$. Now, the 1-parameter unitary groups $\hat{U}(\lambda)$ and $\hat{V}(\mu)$ are weakly continuous in the parameters $\lambda, \mu$. This ensures that there exist self-adjoint operators $\hat{x}$ and $\hat{p}$ on $\mathcal{H}_{Sch}$ such that

$$\hat{U}(\lambda) := e^{i\lambda \hat{x}} \quad \text{and} \quad \hat{V}(\mu) = e^{i\frac{\mu}{\hbar} \hat{p}}.$$ (III.6)

We conclude with two remarks:

i) The Schrödinger representation can be obtained using the Gel’fand-Naimark-Segal (GNS) construction with the positive linear (or, ‘expectation-value’) functional $F_{Sch}$ on $W$:

$$F_{Sch}(W(\zeta)) = e^{-\frac{1}{\hbar}\frac{\zeta^2}{d^2}}.$$ (III.7)

The expectation values of $\hat{U}$ and $\hat{V}$ are given by:

$$F_{Sch}(U(\lambda)) = e^{-\frac{1}{\hbar}\frac{\lambda^2}{d^2}} \quad \text{and} \quad F_{Sch}(V(\mu)) = e^{-\frac{1}{\hbar}\frac{\mu^2}{d^2}}.$$ (III.8)

The corresponding GNS ‘vacuum’ (i.e., cyclic) state $\psi_{Sch}$ is simply

$$\psi_{Sch}(x) = \left(\pi d^2\right)^{-\frac{1}{4}} e^{-\frac{x^2}{2d^2}}.$$
ii) For definiteness, we have presented the Schrödinger representation using position wave functions \( \psi(x) \). In terms of momentum wave functions \( \psi(k) \), which will be more useful in the next subsection, we have:

\[
\hat{U}(\lambda)\psi(k) = \psi(k - \lambda), \quad \text{and} \quad \hat{V}(\mu)\psi(k) = e^{i\mu k} \psi(k) \tag{III.9}
\]

and the GNS cyclic state is given by:

\[
\psi_{\text{Sch}}(k) = \left( \frac{\pi}{d^2} \right)^{-\frac{1}{4}} e^{-\frac{k^2 d^2}{2}}.
\]

B. The polymer particle representation

We are now ready to introduce the polymer particle representation of the Weyl-Heisenberg algebra which is unitarily inequivalent to the Schrödinger. This construction must, of course, violate one or more assumptions of the Stone-von Neumann uniqueness theorem. It turns out that only one assumption is violated: in the new representation, the operator \( V(\mu) \) will not be weakly continuous in \( \mu \), whence there will be no self-adjoint operator \( \hat{p} \) such that \( V(\mu) = \exp(i\mu \hat{p}) \). While the unavailability of the standard momentum operator seems alarming at first, this is just what one would expect physically in the absence of a spatial continuum. More precisely, if the spatial Riemannian geometry is to be discrete (as, for example, in loop quantum gravity), one would not expect the operator \( p = -ih d/\partial x \) to exist at a fundamental level. The key question is whether one can nonetheless do quantum mechanics and reproduce the well-tested results. This is a difficult question with many technical subtleties. But, as we will see in sections IV and V, the answer is in the affirmative: by adopting the viewpoint that the natural arena for quantum theory is the analog of \( \text{Cyl}^* \), one can recover results of Schrödinger quantum mechanics in the domain of its validity.

To bring out the similarity and differences with quantum geometry, we will construct the Hilbert space of states, \( \mathcal{H}_{\text{Poly}} \), in steps, using the same terminology. A graph \( \gamma \) will consist of a countable set \( \{x_i\} \) of points on the real line \( \mathbb{R} \) with the following two properties: i) The \( x_i \) do not contain sequences with accumulation points in \( \mathbb{R} \); and, ii) there exist constants \( \ell, \rho \) such that the number \( n(I) \) of points in any interval \( I \) of length \( \ell(I) \) is bounded by \( n(I) \leq \rho \ell(I) \). The two technical conditions will ensure convergence of certain series; see section IV.\(^3\)

Denote by \( \text{Cyl}_\gamma \) the space of complex valued functions \( f(k) \) of the type:

\[
f(k) = \sum_j f_j e^{-ix_j k} \tag{III.10}
\]

on \( \mathbb{R} \), where \( x_j \) are real and \( f_j \) are complex numbers with a suitable fall-off. To simplify the later specification of domains of operators, we will choose the fall-off to be such that \( \sum_j |x_j|^{2n} |f_j|^2 < \infty \) for all \( n \). \( \text{Cyl}_\gamma \) is a vector space (which is infinite dimensional if the number of points in \( \gamma \) is infinite). We will say that functions \( f(k) \) in \( \text{Cyl}_\gamma \) are cylindrical.

\(^3\) In the earlier version of this paper, we only had condition i). We thank Jacob Yngvasson for pointing out that it does not suffice and Chris Fewster and Jerzy Lewandowski for the precise formulation of ii).
with respect to $\gamma$. Thus, each cylindrical state is a discrete sum of plane waves; it fails to belong to the Schrödinger Hilbert space. The real number $k$ is the analog of connections in quantum geometry and the plane wave $\exp(-ikx_j)$ can be thought of as the ‘holonomy of the connection $k$ along the edge $x_j$’.

Next, let us consider all possible graphs, where the points (and even their number) can vary from one graph to another, and denote by Cyl the infinite dimensional vector space of functions on $\mathbb{R}$ which are cylindrical with respect to some graph. Thus, any element $f(k)$ of Cyl can be expanded as in (III.10), where the uncountable basis $\exp(-ix_jk)$ is now labeled by arbitrary real numbers $x_j$. Let us introduce a natural, Hermitian inner product on Cyl by demanding that $\exp(-ix_jk)$ are orthonormal:

$$<e^{-ix_jk}|e^{-ix_jk}> = \delta_{x_i,x_j}. \quad (\text{III.11})$$

(Note that the right side is the Kronecker $\delta$ and not the Dirac distribution.) Denote by $\mathcal{H}_{\text{Pol}}$ the Cauchy completion of Cyl. This is the Hilbert space underlying our representation.

To summarize, $\mathcal{H}_{\text{Pol}}$ is the Hilbert space spanned by countable linear combinations $\sum_{j=1}^{\infty} f_j \exp(-ix_jk)$ of plane waves in the momentum space, subject to the condition $\sum_{j=1}^{\infty} |f_j|^2 < \infty$, where $\{x_j\}$ is an arbitrary countable set of real numbers, which can vary from one state to another. Even more succinctly, $\mathcal{H}_{\text{Pol}} = L^2(\mathbb{R}, d\mu_d)$, where $\mathbb{R}$ is the real line equipped with discrete topology and $\mu_d$ is the natural discrete measure on it.

The Weyl-Heisenberg algebra $\mathbf{W}$ is represented on $\mathcal{H}_{\text{Pol}}$ in the same manner as in the Schrödinger representation:

$$\hat{W}(\zeta) f(k) = [e^{\frac{i\zeta}{\hbar}} U(\lambda) V(\mu)] f(k) \quad (\text{III.12})$$

where, as before, $\zeta = \lambda d + i(\mu/d)$ and the action of $\hat{U}$ and $\hat{V}$ is given by (see (III.9))

$$\hat{U}(\lambda) f(k) = f(k - \lambda) \quad \text{and} \quad \hat{V}(\mu) f(k) = e^{i\mu k} f(k). \quad (\text{III.13})$$

It is straightforward to check that these operators provide a faithful, irreducible representation of $\mathbf{W}$ on $\mathcal{H}_{\text{Pol}}$. Each $\hat{U}(\lambda)$ and $\hat{V}(\mu)$ is unitary.

The structure of this representation becomes more transparent in terms of eigenkets of $\hat{U}(\lambda)$. Let us associate with the basis elements $\exp(-ix_jk)$ a ket $|x_j\rangle$ and, using the textbook heuristic notation, express $\exp(-ix_jk)$ as a generalized scalar product:

$$(k, x_j) = e^{-ix_jk}$$

Then, $\{|x_j\rangle\}$ is an orthonormal basis and the action of the basic operators $\hat{U}$ and $\hat{V}$ is given by:

$$\hat{U}(\lambda) |x_j\rangle = e^{i\lambda x_j} |x_j\rangle \quad \text{and} \quad \hat{V}(\mu) |x_j\rangle = |x_j - \mu\rangle. \quad (\text{III.14})$$

It is straightforward to verify that $\hat{U}(\lambda)$ is weakly continuous in $\lambda$ whence there exists a self-adjoint operator $\hat{x}$ on $\mathcal{H}_{\text{Pol}}$ with $\hat{U}(\lambda) = \exp(i\lambda \hat{x})$. Its action can now be expressed as:

$$\hat{x} |x_j\rangle = x_j |x_j\rangle \quad (\text{III.15})$$

just as one would expect. However, there is an important difference from the Schrödinger representation: The eigenkets of $\hat{x}$ are now normalizable, and hence elements of the Hilbert space itself. In this sense the eigenvalues are ‘discrete’.
By contrast, although the family $\hat{V}(\mu)$ provides a 1-parameter unitary group on $\mathcal{H}_{\text{Poly}}$, it fails to be weakly continuous in the parameter $\mu$. This follows from the fact that, no matter how small $\mu$ is, $\langle x_j | \hat{x} \rangle$ and $\langle x_j | \hat{V}(\mu) | x_j \rangle$ are orthogonal to one another, whence

$$\lim_{\mu \to 0} \langle x_j | \hat{V}(\mu) | x_j \rangle = 0,$$

while $\hat{V}(\mu = 0) = 1$ and $\langle x_j | x_j \rangle = 1$. Thus, there is no self-adjoint operator $\hat{p}$ on $\mathcal{H}_{\text{Poly}}$ satisfying the second of eqs. (III.6).

Finally, this representation can be obtained via Gel’fand-Naimark-Segal construction, using the following positive linear (or expectation value) functional on the Weyl-Heisenberg algebra $\mathcal{W}$:

$$F_{\text{Poly}}(W(\zeta)) = \begin{cases} 1 & \text{if } \text{Im } \zeta = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (\text{III.16})$$

In terms of $U(\lambda)$ and $V(\mu)$, we have:

$$F_{\text{Poly}}(U(\lambda)) = 1 \quad \forall \lambda,$$

$$F_{\text{Poly}}(V(\mu)) = \begin{cases} 1 & \text{if } \mu = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (\text{III.17})$$

The corresponding cyclic state is simply $|\psi_\epsilon \rangle = |x_\epsilon = 0\rangle$. Note that, in contrast to the Schrödinger positive linear functional $F_{\text{Sch}}$, no scale had to be introduced in the definition of $F_{\text{Poly}}$. This is the analog of the fact that the corresponding positive linear functional in quantum geometry is diffeomorphism invariant.

We will conclude with a few remarks.

i) The step by step procedure used above brings out the fact that the polymer particle description captures many of the mathematical features of quantum geometry, but now for a very simple physical system. Our notation is geared to reflect the analogies. Thus, sets $\gamma = \{x_i\}$ are the analogs of graphs of quantum geometry; individual points $x_j$, the analogs of edges; the continuous, momentum variable $k$, the analog of connections; $\exp(-ixjk)$ the analog of the holonomy along an edge; $\text{Cyl}_x$, the analog of the space of cylindrical functions associated with a graph and $\text{Cyl}$ the space of all cylindrical functions of quantum geometry; and the $|x_j\rangle$ the analogs of spin network states. Indeed, we again have the Hilbert space decomposition analogous to (II.2):

$$\mathcal{H}_{\text{Poly}} = \bigoplus_x \mathcal{H}_x$$

where $\mathcal{H}_x$ are the 1-dimensional subspaces spanned by our basis vectors $|x\rangle$. (The decomposition is thus analogous to that in the $U(1)$ case).

ii) What is the situation with operators? The basic operators of quantum geometry—holonomies and fluxes of the electric field—are respectively analogous to the operators $\hat{V}(\mu)$ and $\hat{x}$ on $\mathcal{H}_{\text{Poly}}$. The commutator between $\hat{x}$ and $\hat{V}(\mu)$,

$$[\hat{x}, \hat{V}(\mu)] = -\mu \hat{V}(\mu), \quad (\text{III.18})$$

11
is completely parallel to the commutator between electric fields and holonomies in quantum geometry. Just as \( \hat{\mathcal{V}}(\mu) \) are unitary but \( \hat{p} \) does not exist, holonomies are unitarily represented but the connection operator does not exist. Like \( \hat{x} \), the electric flux operators are unbounded self-adjoint operators with discrete eigenvalues. (However, in the case of electric fluxes, the set of eigenvalues is a discrete subset of the real line, equipped with its standard topology.) It is this discreteness that leads to the loss of continuum in the quantum Riemannian geometry which in turn ‘justifies’ the absence of the standard momentum operator \(-i\hbar d/dx\) in the polymer particle example.

iii) Recall that in quantum geometry, elements of \( \mathcal{H}_{\text{Poly}} \) can be represented as functions on a compact space \( \hat{\mathcal{A}} \), the quantum configuration space obtained by a suitable completion of the classical configuration space \( \mathcal{A} \). What is the situation with respect to \( \mathcal{H}_{\text{Poly}} \)? Now, the classical configuration space is just the real line \( \mathbb{R} \) (of momenta \( k \)). The quantum configuration space turns out to be the Bohr compactification \( \hat{\mathbb{R}}_{\text{Bohr}} \) of \( \mathbb{R} \) (discovered and analyzed by the mathematician Harald Bohr, Niels’ brother). All quantum states are represented by square integrable functions on \( \hat{\mathbb{R}}_{\text{Bohr}} \) with respect to a natural measure \( \mu_c \); \( \mathcal{H}_{\text{Poly}} = L^2(\hat{\mathbb{R}}_{\text{Bohr}}, d\mu_c) \).

Finally, as in quantum geometry, \( \hat{\mathbb{R}}_{\text{Bohr}} \) is also the Gel’fand spectrum of the Abelian \( C^* \)-algebra of ‘holonomy’ operators \( V(\mu) \). (For details on the Bohr compactification, see [19].)

IV. RELATION BETWEEN SCHRÖDINGER AND POLYMER DESCRIPTIONS: KINEMATICS

Elements of the polymer Hilbert space \( \mathcal{H}_{\text{Poly}} \) consist of discrete sums \( f(k) = \sum_j f_j \exp(-ixjk) \) of plane waves. Therefore, it follows that the intersection of \( \mathcal{H}_{\text{Poly}} \) with \( \mathcal{H}_{\text{Sch}} \) consists just of the zero element. While each provides an irreducible, unitary representation of the Weyl-Heisenberg algebra, the two Hilbert spaces are ‘orthogonal’. Therefore, one might first think that the standard physics contained in the Schrödinger representation cannot be recovered from the polymer framework. We will now show that this is not the case.

As explained in the introduction, the key idea is to focus on \( \text{Cyl}^* \), the algebraic dual\(^4\) of \( \text{Cyl} \). Since \( \text{Cyl} \subset \mathcal{H}_{\text{Poly}} \), it follows that we have:

\[ \text{Cyl} \subset \mathcal{H}_{\text{Poly}} \subset \text{Cyl}^*. \]

We will denote the elements of \( \text{Cyl}^* \) by upper case letters, e.g., \( (\Psi) \), and their action on elements \( |f\rangle \) of \( \text{Cyl} \) simply with a juxtaposition, e.g. \( (\Psi) \) maps \( |f\rangle \) to the complex number \( (\Psi)|f\rangle \).

The Weyl-Heisenberg algebra has a well-defined action on \( \text{Cyl} \), and hence by duality, on

\(^4\) As in quantum geometry, we are taking the algebraic dual just for simplicity. When the framework is further developed, one would introduce an appropriate topology on \( \text{Cyl} \) (which is finer than that of \( \mathcal{H}_{\text{Poly}} \)) and define \( \text{Cyl}^* \) as the space of linear functions on \( \text{Cyl} \) which are continuous in this topology. The algebraic dual is ‘too large’ but this fact is not relevant here: since our main goal is to represent all semi-classical Schrödinger states by elements of \( \text{Cyl}^* \) we can just ignore the fact that the algebraic dual also contains other ‘unwanted’ states.
However, this representation is far from being irreducible. In particular, \( \mathcal{H}_{\text{Poly}} \) is contained in \( \text{Cyl}^* \) and provides us with an irreducible representation. More importantly for what follows, the Schwartz space \( \mathcal{S} \), a dense subspace of \( \mathcal{H}_{\text{Sch}} \) consisting of smooth functions on \( \mathbb{R} \) which, together with all their derivatives fall off faster than any inverse polynomial in \( x \), is also embedded in \( \text{Cyl}^* \). (This follows from the two technical conditions in the definition of a graph and, of course, the definition of Cyl.) Since all coherent states belong to \( \mathcal{S} \) and they form an over-complete basis in \( \mathcal{H}_{\text{Sch}} \), Schrödinger quantum mechanics is somehow encoded in \( \text{Cyl}^* \). Our task is to analyze this encoding.

We will often use the fact that \( \mathcal{S} \) is stable under Fourier transform; i.e., \( \psi(x) \in \mathcal{S} \) if and only if its Fourier transform \( \tilde{\psi}(k) \in \mathcal{S} \). The embedding of \( \mathcal{S} \) in \( \text{Cyl}^* \) is given just by the Schrödinger scalar product: each element \( \psi \in \mathcal{S} \) defines an element \( (\Psi| \text{of Cyl}^* \) via

\[
(\Psi| \sum_j f_j e^{-ix_jk}) = \frac{1}{\sqrt{2\pi}} \sum_j f_j \int dk \tilde{\psi}(k)e^{-ix_jk} = \sum_j f_j \tilde{\psi}(x_j)
\]  

(IV.2)

where \( \tilde{\psi}(k) \) is the Fourier transform of \( \psi(x) \). Thus, although elements of Cyl fail to be normalizable in the Schrödinger Hilbert space, their Schrödinger inner product with elements of \( \mathcal{S} \) is well-defined and naturally leads to a linear map from Cyl to \( \mathbb{C} \).

Can we exploit the fact that \( \mathcal{S} \) is embedded in \( \text{Cyl}^* \) to extract the physics of Schrödinger quantum mechanics from \( \text{Cyl}^* \)? At first sight, there appears to be a key problem: \( \text{Cyl}^* \) is not equipped with a scalar product. We could restrict ourselves just to \( \mathcal{S} \subset \text{Cyl}^* \) and introduce on it the Schrödinger scalar product by hand. But this would just be an unnecessarily complicated way of arriving at the Schrödinger representation. More importantly, in non-perturbative quantum gravity, we do not have the analog of the Schrödinger Hilbert space and, furthermore, indications are that its perturbative substitute, the graviton Fock space, is ‘too small’. Therefore, for our polymer particle toy model to be an effective tool, we should not restrict ourselves to a ‘small’ subspace of it such as \( \mathcal{S} \). Rather, we should work with the full \( \text{Cyl}^* \) and use only that structure which is naturally available on it. Thus, our challenge is to show that standard quantum physics can be extracted from \( \text{Cyl}^* \) directly, without making an appeal to the Schrödinger Hilbert space. Known facts about the Schrödinger representation can be used only to motivate various constructions, but not in the constructions themselves.

In quantum gravity, a principal open problem is that of existence of semi-classical states. Therefore, in the rest of this section we will focus on the problem of isolating elements of \( \text{Cyl}^* \) which correspond to the standard coherent states of Schrödinger quantum mechanics and extracting their physics using only those structures which are naturally available in the polymer framework. Hamiltonians and their various properties will be discussed in the next section.

A. Isolating semi-classical states

Fix a classical state, i.e., a point \( (x_c, p_c) \) in the classical phase space. In Schrödinger quantum mechanics, the corresponding semi-classical states are generally represented by
coherent states peaked at this point. In these states, the product of uncertainties in the basic observables \( \hat{x} \) and \( \hat{p} \) is minimized, \( (\Delta \hat{x})(\Delta \hat{p}) = \hbar/2 \), and furthermore, in suitable units, these uncertainties are distributed ‘equally’ among the two observables. To obtain a specific coherent state, one has to specify these units, or, in physical terms, ‘tolerance’—the uncertainties in \( x \) and \( p \) we can tolerate. Let us therefore introduce a length scale \( d \) and ask that the uncertainty \( \Delta x \) in \( \hat{x} \) be \( d/\sqrt{2} \) and that in \( \hat{p} \) be \( \hbar/(\sqrt{2}d) \). (In the case of an harmonic oscillator, \( d \) is generally taken to be \( \sqrt{\hbar/m\omega} \). However, in this section on kinematics, it is not necessary to restrict ourselves to a specific system.) Set

\[
\zeta_o = \frac{1}{\sqrt{2d}} \left( x_o + \frac{i d^2}{\hbar} p_o \right) = \frac{1}{\sqrt{2d}} \left( x_o + \frac{ik_o d^2}{\hbar} \right)
\]

where, from now on, we will use \( k_o := p_o/\hbar \). Then, the standard coherent state \( \psi_{\zeta_o} \) is generally obtained by solving the eigenvalue equation

\[
\hat{a} \psi_{\zeta_o}(x) \equiv \frac{1}{\sqrt{2d}} \left( \hat{x} + \frac{i d^2}{\hbar} \hat{p} \right) \psi_{\zeta_o}(x) = \zeta_o \psi_{\zeta_o}(x),
\]

where \( \hat{a} \) is the annihilation operator and \( c \) is a normalization constant. Since \( \psi_{\zeta_o} \in \mathcal{S} \), it canonically defines an element \( \Psi_{\zeta_o} \) of \( \text{Cyl}^r \). Our first task is to isolate this \( \Psi_{\zeta_o} \) using just the polymer framework. The second task, that of analyzing its properties and specifying the sense in which it is a semi-classical state also from the polymer perspective, will be taken up in the next subsection.

Now, in the polymer framework, the operator \( \hat{p} \) fails to be well-defined. Therefore, we cannot introduce the creation and annihilation operators used in the above construction. However, recall that the operators \( \hat{U}(\lambda), \hat{V}(\mu) \) and \( \hat{x} \) are well-defined on \( \text{Cyl} \) and hence also on \( \text{Cyl}^r \). We can therefore reformulate (IV.3) by an equivalent eigenvalue equation in terms of these operators. Since the equation is now to be imposed on \( \text{Cyl}^r \), we have to replace the annihilation operator \( \hat{a} \) by its adjoint, \( \hat{a}^\dagger \), the creation operator. Now, using the Baker-Hausdorff-Campbell identity in \( \mathcal{H}_{\text{Sch}} \), we have:

\[
e^{\sqrt{2} a^\dagger a} = e^{\sqrt{2} \hat{a}^\dagger} V(-\alpha d) e^{-\sqrt{2} \hat{a}}.
\]

where the factor of \( \sqrt{2} \) is introduced just for technical simplification and \( \alpha \) is an arbitrary real number. Note that the operators on the right side are all well-defined on \( \text{Cyl}^r \).

Collecting these ideas motivated by results in the Schrödinger representation, we are now led to seek the analog of coherent states in \( \text{Cyl}^r \) by solving the eigenvalue equation:

\[
(\Psi_{\zeta_o}) \left[ e^{\sqrt{2} \hat{a}^\dagger} V(-\alpha d) e^{-\sqrt{2} \hat{a}} \right] = e^{\sqrt{2} \zeta_o} (\Psi_{\zeta_o}),
\]

for all real numbers \( \alpha \). Note that, to capture the full content of the original eigenvalue equation (IV.3), it is essential to allow arbitrary \( \alpha \) in the exponentiated version (IV.5).

To obtain the solution, it is convenient to use a basis in \( \text{Cyl}^r \). Recall first that any element \( f \) of \( \text{Cyl} \) can be expanded out as a discrete sum, \( f = \sum_j f_j |x_j\rangle \), where the \( f_j \) are complex
coefficients and the $x_j$ real numbers. Therefore, the action of any element $(\Psi |$ of Cyl$^*$ is completely determined by the action $(\Psi | x) = \overline{\Psi}(x)$ of $(\Psi |$ on all basis vectors $| x \rangle$. That is, $(\Psi |$ can be expanded as a continuous sum

$$(\Psi | = \sum_x \overline{\Psi}(x) | x \rangle \quad \text{(IV.6)}$$

where the dual basis $(x |$ in Cyl$^*$, labeled by real numbers $x$, is defined in an obvious fashion:

$$(x | x_j \rangle = \delta_{x,x_j}.$$ 

Note that, although there is a continuous sum in (IV.6), when operating on any element of Cyl only a countable number of terms are non-zero.

Using (IV.6) in (IV.5), it is straightforward to show that the coefficients $\Psi_{\zeta \alpha}(x)$ must satisfy:

$$\overline{\Psi}_{\zeta \alpha}(x + \alpha d) = \exp \left[ \sqrt{2} \overline{\zeta} \zeta - \frac{\alpha x}{d} + \frac{\alpha^2}{2} \right] \overline{\Psi}(x) \quad \text{(IV.7)}$$

for all real numbers $\alpha$. It is easy to verify that this equation admits a solution which is unique up to a normalization factor. The general solution is given by:

$$(\Psi_{\zeta \alpha} | = \tilde{c} \sum_x \left[ e^{-\frac{\alpha x \zeta}{2d}} e^{-i\alpha(x-x_0)} \right] (x | \text{.} \quad \text{(IV.8)}$$

As one might have hoped, the coefficients in this expansion are the same as the expression (IV.4) of the coherent state wave function in the Schrödinger representation. Note that, to obtain a unique solution (up to a multiplicative constant), it is essential to impose (IV.7) for all real numbers $\alpha$.

To summarize, by using the standard procedure in the Schrödinger representation as motivation, we wrote down an eigenvalue equation directly in Cyl$^*$ to single out a candidate semi-classical state $(\Psi_{\zeta \alpha} |$ peaked at a generic point $(x_0, p_0)$ of the classical phase space. Since this is a linear equation, one cannot hope to restrict the overall normalization of the solution. Up to this trivial ambiguity, however, the solution is unique. We will refer to it as a polymer coherent state. As one might have hoped, this polymer coherent state is just the element $(\Psi_{\zeta \alpha} |$ of Cyl$^*$ defined by the standard coherent state $\psi_{\zeta \alpha} \in \mathcal{S}$ in $\mathcal{H}_{\text{Sch}}$. Note that this is not an assumption but the result of a self-contained calculation that was carried out entirely in Cyl$^*$. However, at this stage, it is not a priori obvious that $(\Psi_{\zeta \alpha} |$ is a semi-classical state from the polymer perspective, especially because we no longer have access to the Schrödinger scalar product. This issue will be discussed in the next subsection.

**B. Shadow States**

For simplicity, in this subsection we will restrict ourselves to the candidate semi-classical state $(\Psi_o |$ corresponding to $\zeta = 0$. (The general case is completely analogous and discussed in subsection IV.C.) Our task is to show that this state is sharply peaked at $x=0$ and $p=0$ using only the polymer framework. However, right at the outset we encounter two difficulties. Firstly, the operator $\hat{p}$ is not defined in the polymer framework. We will therefore have to
define a ‘fundamental operator’ on $\mathcal{H}_{\text{Pol}^6}$, which is approximated by $\hat{p}$ of the Schrödinger representation. The second difficulty is that, since there is no inner product on Cyl*, the required expectation values cannot be defined on it. To overcome this obstacle, we will use graphs as ‘probes’ to extract physical information from elements $(\Psi)$ of Cyl*. More precisely, we will ‘project’ each $(\Psi)$ to an element $|\Psi_{\text{shad}}^{\gamma}\rangle$ in Cyl$_\gamma$ and analyze properties of $(\Psi)$ in terms of its shadows $|\Psi^{\text{shad}}\rangle$. Each shadow captures only a part of the information contained in our state, but the collection of shadows can be used to determine the properties of the full state in Cyl*.

Let us begin by defining the required projection $\hat{P}_\gamma$ from Cyl* to Cyl$_\gamma$:

$$(\Psi|\hat{P}_\gamma := \sum_{x_j \in \gamma} \Psi(x_j) |x_j\rangle) \equiv |\Psi^{\text{shad}}_{\gamma}\rangle.$$ (IV.9)

The ket $|\Psi^{\text{shad}}_{\gamma}\rangle$ defines the shadow cast by the element $(\Psi)$ of Cyl* on the graph $\gamma$ in the sense that

$$(\Psi|f_\gamma) = \langle \Psi^{\text{shad}}_{\gamma}|f_\gamma\rangle$$

where the left side is the result of the action of an element of Cyl* on an arbitrary element $f_\gamma$ of Cyl$_\gamma$, and the right side is the scalar product on Cyl$_\gamma$. Our task is to analyze properties of the shadows

$$|\Psi^{\text{shad}}_{\text{shad}}\rangle := (\Psi\langle\hat{P}_\gamma).$$

of our candidate semi-classical state. The essential idea is to say that $|\Psi\rangle$ is semi-classical if physical observables of interest have expected mean values with small uncertainties in its shadows $|\Psi^{\text{shad}}_{\gamma}\rangle$ on sufficiently refined graphs $\gamma$.

To make this notion precise, we need to select: i) A suitable family of graphs; ii) a class of observables of interest; and, iii) acceptable ‘tolerances’ for mean-values and uncertainties of these observables. We will restrict ourselves to shadows on regular lattices 5 with sufficiently small lattice spacing (as discussed below). For definiteness, as in Schrödinger quantum mechanics, the class $\mathcal{C}$ of observables of interest will consist just of position and momentum operators. Tolerances $\tau$ will be determined by the physical parameters of the system under consideration (i.e., the length scale $d$ of subsection IV.A).

We will say that a state $(\Psi) \in \text{Cyl}^*$ is semi-classical with respect to $\mathcal{C}$ and peaked at a point $(x,p)$ of the classical phase space, if within specified tolerances $\tau$, the ‘expectation values’ of any operator $\hat{A} \in \mathcal{C}$ equals the classical value $A(x,p)$ and the fluctuations are small; i.e., if

$$\frac{|\langle\Psi|\hat{A}|\Psi^{\text{shad}}_{\gamma}\rangle|}{|\Psi^{\text{shad}}_{\gamma}\rangle|^2} = A(x,p)(1 + \tau_A^{(1)}) \quad \text{and} \quad \frac{|\langle\Psi|\hat{A}^2|\Psi^{\text{shad}}_{\gamma}\rangle|^2}{|\Psi^{\text{shad}}_{\gamma}\rangle^2} - \left(\frac{|\langle\Psi|\hat{A}|\Psi^{\text{shad}}_{\gamma}\rangle|}{|\Psi^{\text{shad}}_{\gamma}\rangle|^2}\right)^2 \leq \tau_A^{(2)}$$ (IV.10)

for all sufficiently refined graphs $\gamma$. Here $|f\rangle$ is the norm of the state $|f\rangle$ in $\mathcal{H}_{\text{Pol}^6}$, and $\tau_A^{(i)}$ are the tolerances assigned to the observable $\hat{A}$. The meaning of the equation is clearer if the

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5 Quantum geometry considerations imply that, to probe semi-classicality, we should only use those graphs in which the number of points in any macroscopic interval is proportional to the length of the interval. Regular lattices offer the simplest way to achieve this. A priori one may be concerned that this is ‘too small a class’. But the results of this section show that it suffices.
operators are thought as acting on the candidate semi-classical state \(|\Psi\rangle\) in Cyl* by duality. Thus, in the first equality, the ‘expectation value’ of \(\hat{A}\) in the candidate semi-classical state \(|\psi\rangle\) is evaluated by the action of \(|\Psi\rangle\) (in Cyl*) on the shadow \(|\Psi_{\gamma}^{\text{shad}}\rangle\) of \(|\Psi\rangle\) on the graph \(\gamma\).

If the action of the operator \(\hat{A}\) leaves Cyl* invariant, as one might hope, this ‘expectation value’ reduces to the more familiar expression \(<\Psi_{\gamma}^{\text{shad}}|\hat{A}|\Psi_{\gamma}^{\text{shad}}\rangle\). However, for more general operators, the two expressions do not agree and \(<\Psi|\hat{A}|\Psi_{\gamma}^{\text{shad}}\rangle\) turns out to be the better measure of the expectation value.

Let us then work with infinite regular lattices with spacing \(\ell\), where \(\ell\) is chosen to be sufficiently small (see below). The shadow of our candidate semi-classical state \(|\Psi_{\gamma}\rangle\) on the regular graph is given by:

\[
|\Psi_{\gamma,\ell}^{\text{shad}}\rangle = c \sum_{n \in \mathbb{Z}} e^{-\frac{n^2 \ell^2}{2d^2}} |n\ell\rangle ,
\]  

(IV.11)

where \(c\) is an arbitrary constant. We can now compute the expectation values and fluctuations of various operators in detail and examine if the state can be regarded as semi-classical. On general grounds, one would hope to obtain good agreement with the standard coherent state of Schrödinger quantum mechanics provided the lattice spacing \(\ell\) is much smaller than the length scale \(d\) that defines our tolerance. We will show that, although there are subtleties, this expectation is borne out. However, let us first pause to examine whether this requirement is physically reasonable. As an example, consider the vibrational oscillations of a carbon monoxide molecule. These are well described by a harmonic oscillator with parameters

\[m \approx 10^{-26} \text{ kg} \quad \text{and} \quad \omega \approx 10^{15} \text{ Hz} \]

The textbook treatment of the harmonic oscillator implies that we cannot require the tolerance \(d\) for \(\dot{x}\) to be smaller than

\[d_{\text{min}} = \sqrt{\frac{\hbar}{m\omega}} \approx 10^{-12} \text{ m} ; \]

if we did, the resulting state would spread out quickly under quantum evolution. On the other hand, since no evidence of spatial discreteness has been observed at particle accelerators, the quantum geometry viewpoint requires us to choose \(\ell < 10^{-19} \text{ m}\), and we may even wish to move \(\ell\) all the way down to the Planck scale \((\ell_p = 1.6 \times 10^{-35} \text{ m})\). Thus, our assumption that \(\ell \ll d\) is well justified. Working in this regime, we will now show that the quantities computed using (IV.11) agree to leading order with the standard Schrödinger coherent state. The corrections are of order \(\ell^2/d^2 < 10^{-14}\) and, furthermore, appear in the regime in which Schrödinger quantum mechanics is inapplicable due to, e.g., relativistic effects.

Let us begin with the norm of the shadow of the polymer coherent state:

\[
\langle\Psi_{\gamma,\ell}^{\text{shad}}|\Psi_{\gamma,\ell}^{\text{shad}}\rangle = |c|^2 \sum_{n=-\infty}^{\infty} e^{-\frac{n^2 \ell^2}{d^2}} .
\]  

(IV.12)

Here, we have used the fact that \(\langle x_i| x_j \rangle = \delta_{x_i,x_j}\) to simplify the double sum to a single one. Now, since \(\ell \ll d\), the exponential on the right hand side of (IV.12) decays very slowly, whence we can not estimate the norm by keeping just a few terms in the sum. Fortunately,
however, we can use the Poisson re-summation formula:

$$
\sum_n g(x + n) = \sum_{n=-\infty}^{\infty} e^{2\pi i x n} \int_{-\infty}^{\infty} g(y) e^{-2\pi i y n} dy , \quad (\text{IV.13})
$$

for all functions $g(y)$ which are suitably well behaved for the sums to converge. We will take

$$
g(y) = e^{-\frac{y^2}{d^2}} \quad \text{and} \quad x = 0 .
$$

Then it is straightforward to calculate

$$
\langle \Psi^{\text{shad}}_{\ell, \ell} | \Psi^{\text{shad}}_{\ell, \ell} \rangle = |c|^2 \frac{\sqrt{\pi} d}{\ell} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi^2 n^2 d^2}{\ell^2}} \approx |c|^2 \frac{\sqrt{\pi} d}{\ell} \left( 1 + 2e^{-\frac{\pi^2 d^2}{\ell^2}} \right) , \quad (\text{IV.14})
$$

where we have used $(d/\ell) \gg 1$ to truncate the series after the second term.

Next we turn to the expectation value of and fluctuations in $\dot{x}$. For semi-classicality, the expectation value should be close to zero and the fluctuations of the order $d/\sqrt{2}$. For expectation values, we obtain

$$
\langle \Psi_{\ell, \ell} | \dot{x} | \Psi^{\text{shad}}_{\ell, \ell} \rangle = |c|^2 \sum_n (n\ell) e^{-\frac{n^2 \ell^2}{d^2}} = 0 , \quad (\text{IV.15})
$$

due to antisymmetry in $n$. This result agrees exactly with that obtained from the Schrödinger coherent state. Let us turn to the calculation of fluctuations. We have

$$
\langle \Psi_{\ell, \ell} | \dot{x}^2 | \Psi^{\text{shad}}_{\ell, \ell} \rangle = |c|^2 \sum_n (n\ell)^2 e^{-\frac{n^2 \ell^2}{d^2}}
$$

$$
= |c|^2 \frac{\sqrt{\pi} d^3}{2\ell} \sum_n e^{-\frac{\pi^2 n^2 d^2}{\ell^2}} \left( 1 - 2e^{-\frac{\pi^2 d^2}{\ell^2}} \right) , \quad (\text{IV.16})
$$

where we have once again made use of the Poisson re-summation formula. By combining the results of (IV.16) and (IV.14), we can obtain the fluctuations in $\dot{x}$,

$$
(\Delta x)^2 := \frac{\langle \Psi_{\ell, \ell} | \dot{x}^2 | \Psi^{\text{shad}}_{\ell, \ell} \rangle}{\langle \Psi^{\text{shad}}_{\ell, \ell} \rangle^2} - \frac{\langle (\Psi_{\ell, \ell} | \dot{x}^2 | \Psi^{\text{shad}}_{\ell, \ell} \rangle)^2}{\langle \Psi^{\text{shad}}_{\ell, \ell} \rangle^2} \approx \frac{d^2}{2} \left( 1 - \frac{4\pi^2 d^2}{\ell^2} e^{-\frac{\pi^2 d^2}{\ell^2}} \right) , \quad (\text{IV.17})
$$

where we have made use of the fact that the expectation value of $\dot{x}$ is zero. Hence, we see that the fluctuations in $\dot{x}$ satisfy our ‘tolerance’ requirement. Indeed, to leading order, they agree with the those in the standard coherent states of the Schrödinger framework and the sub-leading terms are extremely small, going to zero as $\ell/d$ tends to zero. Interestingly, these corrections actually decrease the uncertainty in $x$ for the discrete case.

Thus, we see that our candidate semi-classical state $|\Psi_{\ell, \ell}\rangle$ is indeed sharply peaked at $x = 0$. What about the momentum? As mentioned above, there is no natural analog of the Schrödinger momentum operator $\hat{p}$ on $\mathcal{H}_{\text{Poly}}$. Thus, the viewpoint is that the standard $\hat{p}$ operator is a ‘low energy’ construct. There are several operators in the ‘fundamental description’ whose action on ‘low lying states’ is approximated by $\hat{p}$. Here, we will choose
one and test for semi-classicality of \( \langle \Psi_c \rangle \). As one might hope, the difference between candidate choices is manifest only at such high energies that the Schrödinger quantum mechanics is inapplicable there.

To define an analog of the Schrödinger momentum operator, we will use a standard strategy from lattice gauge theories. We first note that, classically, if \( k \mu \) is small then we can expand \( \exp(-i k \mu) \) as

\[
\exp(-i k \mu) = 1 - i k \mu - \frac{k^2 \mu^2}{2} + \cdots
\]

whence

\[
\frac{\exp(-i k \mu) - \exp(i k \mu)}{-2i \mu} = k + O(k^2 \mu) .
\]

In quantum theory, then, it seems natural to define the analog of the momentum operator in a similar way. Choose a sufficiently small value \( \mu_c \) of \( \mu \) (with \( \ell \leq \mu_c \ll d \)) and define the momentum operator on \( \mathcal{H}_{\text{Poly}} \) as \( \hat{p} = \hbar \hat{K}_{\mu_c} \), with

\[
\hat{K}_{\mu_c} := \frac{i}{2 \mu_c} (\hat{V}(\mu_c) - \hat{V}(-\mu_c)) .
\]

(The simpler definition \( \hat{K}_{\mu_c} = (i/2 \mu_c)(\hat{V}(\mu_c) - 1) \) is not viable because this operator fails to be self-adjoint.) With this definition in hand, let us examine the expectation value and fluctuations in \( \hat{K}_{\mu_c} \). \( \langle \Psi \rangle \) will be semi-classical also for momentum if the expectation value of \( \hat{K}_{\mu_c} \) is close to zero and the fluctuation is of the order \( 1/\sqrt{2d} \).

Now, a direct calculation in the polymer Hilbert space yields

\[
\langle \hat{V}(\mu) \rangle := \frac{\langle \Psi_c | \hat{V}(\mu) | \Psi_{\text{shad}} \rangle_{\mu,c}}{\| \Psi_{\text{shad}} \|^2} = e^{-k^2 \mu^2/4d^2} \left( 1 + 2 e^{-\pi^2 d^2 / \ell^2} \left[ \cos \left( \frac{\pi \mu}{\ell} \right) - 1 \right] \right) ,
\]

for any value of \( \mu \). Using this result, it is straightforward to show that

\[
\langle \hat{K}_{\mu_c} \rangle = 0
\]

because of the antisymmetry between \( \hat{V}(\mu_c) \) and \( \hat{V}(-\mu_c) \) in our definition (IV.20). Next, let us analyze the fluctuations

\[
\langle \hat{K}_{\mu_c}^2 \rangle = \frac{1}{4 \mu_c^2} \left( 2 - \langle \hat{V}(2\mu_c) \rangle - \langle \hat{V}(-2\mu_c) \rangle \right) .
\]

Substituting \( \mu = \pm 2\mu_c \) in the above expression (IV.21), we obtain

\[
\langle \hat{K}_{\mu_c}^2 \rangle \approx \frac{1}{2\mu_c^2} \left[ 1 - e^{-\mu_c^2 / \sigma^2} \right] \approx \frac{1}{2d^2} \left[ 1 - \left( \frac{\mu_c^2}{2d^2} \right) \right] ,
\]

where we have used the fact \( \mu_c, \ll d \) to expand in powers of \( \mu_c/d \) in the last step. Recalling that the expectation value of \( \hat{K}_{\mu_c}^2 \) is zero, we obtain the fluctuations in \( \hat{K} \) as

\[
(\Delta \hat{K}_{\mu_c})^2 \approx \frac{1}{2d^2} \left[ 1 - \left( \frac{\mu_c^2}{2d^2} \right) \right] .
\]
Since the approximate momentum operator is given by $\hbar \hat{K}_{\mu e}$, we conclude that the state is sharply peaked at $p = 0$ and the fluctuations are within the specified tolerance.

Finally, collecting the results for $\Delta x$ and $\Delta k$, we obtain the uncertainty relations for the shadow of the polymer semi-classical state:

$$(\Delta x)^2(\Delta K_{\mu e})^2 = \frac{1}{4} \left[ 1 - \left( \frac{\mu_e}{2d^2} \right)^4 + O \left( \frac{\mu_e^4}{d^4} \right) \right]. \quad (IV.26)$$

Note that the corrections to the standard uncertainty relations at order $(\mu_e/d)^2$ decrease the uncertainty. This can occur because the commutator between the position and the approximate momentum operator is not simply a multiple of identity. Such modifications of the uncertainty relations have also been obtained in string theory. Our discussion shows that the effect is rather generic.

To summarize, in subsection IV A, we found candidate semi-classical states $|\Psi_{\zeta}\rangle$ in $Cyl^+$ working entirely in the polymer particle framework. In this sub-section, we showed that the polymer coherent state $|\Psi_\zeta\rangle$ is semi-classical in the polymer sense: its shadows on sufficiently refined lattices are sharply peaked at the point $(x=0, p=0)$ of the classical phase space. Furthermore, the fluctuations in $x$ and $p$ are essentially the same as those in the Schrödinger coherent state $\psi_\zeta$ of (IV.4). There are deviations, but in the regime of applicability of Schrödinger quantum mechanics, they are too small to violate experimental bounds.

C. General coherent states

Let us now analyze the properties of general polymer coherent states $|\Psi_\zeta\rangle$ with $\zeta = \frac{1}{\sqrt{2d}} (x + id^2k)$. Calculations of expectation values and fluctuations proceed in a very similar manner to those described above for $|\Psi_\zeta\rangle$. (The only difference arises from the fact that we may not have a point in our graph at the position $x$. However, this affects the sub-leading terms.) Therefore, we will simply state the final results:

1. The norm of the state is given by

$$\langle \Psi_{\zeta,\ell}^{\text{had}} | \Psi_{\zeta,\ell}^{\text{had}} \rangle = |c|^2 \sqrt{\frac{\pi d}{\ell}} \left( 1 + O \left( \frac{\pi^2 d^2}{\ell^2} \right) \right). \quad (IV.27)$$

2. The expectation value of the $\hat{x}$ operator is

$$\langle \hat{x} \rangle := \frac{\langle \Psi_{\zeta,\ell}^{\text{had}} | \hat{x} | \Psi_{\zeta,\ell}^{\text{had}} \rangle}{\langle \Psi_{\zeta,\ell}^{\text{had}} | \Psi_{\zeta,\ell}^{\text{had}} \rangle} = x \left[ 1 + O \left( \frac{\pi^2 d^2}{\ell^2} \right) \right]. \quad (IV.28)$$

Thus, the expectation value of position is $x$ within the tolerance $\tau_x^{(1)} = e^{-\frac{\pi^2 d^2}{\ell^2}}$.

3. The fluctuation in $x$ is

$$(\Delta x)^2 = \frac{d^2}{2} \left[ 1 + O \left( \frac{\pi^2 d^2}{\ell^2} \right) \right]. \quad (IV.29)$$

So, the leading term, $d/\sqrt{2}$, in the fluctuation in $x$ is the same as in the Schrödinger coherent states. Also, the sub-leading terms are independent of $\zeta$, i.e., are the same for all polymer coherent states.
4. One can evaluate the $\hat{K}_{\mu_e}$ operator on an arbitrary coherent state. The result is,

$$\langle \hat{K}_{\mu_e} \rangle = k \left( 1 + \mathcal{O}(k^2 \mu_e^2) + \mathcal{O}\left(\frac{\ell^2}{d^2}\right) \right). \quad (IV.30)$$

Thus, we now encounter a new situation. The tolerance $\tau_{\hat{K}_{\mu_e}}^{(1)}$ is acceptably small only if $k \mu_e \ll 1$. In this case, we obtain an uncertainty relation similar to the one in (IV.26). However, for $k \mu_e \sim 1$ our states do not satisfy the semi-classicality requirement. But note that the non-relativistic approximation — and hence the motivation for including $\hat{K}_{\mu_e}$ in the list $\mathcal{C}$ of observables— breaks down long before one reaches such high momenta. (In the case of the CO molecule, for example, this would correspond to the energy level $n \geq 10^{14}$.)

To summarize, we have introduced polymer coherent states $| \Psi_\zeta \rangle$ and investigated their properties using their shadows $| \Psi_\zeta^{(l)} \rangle$. Given a tolerance $d$ for $\hat{x}$, an uniform graph can serve as a suitable ‘probe’ provided the lattice spacing $\ell$ is chosen so that $\ell / d \ll 1$. As far as semi-classical states are concerned, systems which can be treated adequately within non-relativistic quantum mechanics can also be well-described by the polymer particle framework, without any reference to the Schrödinger Hilbert space.

Remark: Recall that the normalized Schrödinger coherent states $| \psi_\zeta \rangle$ form an orthonormal basis in $\mathcal{H}_{\text{Sd}}$, providing a convenient resolution of the identity:

$$\int dk \int dx \langle \psi_\zeta | \psi_\zeta \rangle = I. \quad (IV.31)$$

Does a similar result hold for the shadow coherent states $| \Psi^{\text{shad}}_{\zeta,\ell} \rangle$ in the Hilbert space $\mathcal{H}_{\text{Poly}}^{\ell}$ restricted to the graph? A priori, it would appear that there is a potential problem. Since

$$| \Psi^{\text{shad}}_{\zeta,\ell} \rangle = e^{i k \cdot (x - d)} e^{i \zeta (x - d)} | n \ell \rangle$$

where $\zeta = \frac{1}{\sqrt{2d}}(x + id^2 k)$, it follows that the projection operators

$$P_{\zeta} := \frac{| \Psi^{\text{shad}}_{\zeta,\ell} \rangle \langle \Psi^{\text{shad}}_{\zeta,\ell} |}{\| | \Psi^{\text{shad}}_{\zeta,\ell} \rangle \|^2}$$

defined by the shadow coherent states are periodic: $P_{\zeta} = P_{k'}$, where $k' = k + (2\pi N)/\ell$. Thus, while the label $k$ took values on the entire real line in (IV.31), with shadow coherent states in $\mathcal{H}_{\text{Poly}}^{\ell}$, it can only take values in $[-\pi / \ell, \pi / \ell]$. Therefore, one might be concerned that, because of the ‘effective momentum cut-off’ we may not have ‘sufficient’ coherent states for the standard over-completeness to hold. However, it turns out that this concern is misplaced. $\mathcal{H}_{\text{Poly}}^{\ell}$ is sufficiently small because of the restriction to a fixed lattice for an exact over-completeness of the desired type to hold [21, 22]:

$$\int_{-\pi / \ell}^{\pi / \ell} \frac{dk}{2\pi} \int_{-\infty}^{\infty} dx \ P_{\zeta} = \sum_n | n \ell \rangle \langle n \ell | = I_{\ell}, \quad (IV.32)$$

where $I_{\ell}$ is the identity operator on $\mathcal{H}_{\text{Poly}}^{\ell}$.
V. RELATION BETWEEN SCHRÖDINGER AND POLYMER DESCRIPTIONS: HAMILTONIANS

A. The conceptual setting

Since a secondary goal of this paper is to illustrate strategies used in loop quantum gravity, let us begin by recalling the situation with the Hamiltonian constraint of quantum general relativity [23]. The main term in the classical constraint is of the form $\text{Tr} \, E^a E^b F_{ab}$, where, as explained in section II, the triad fields $E$ are the analogs of $x$ in the polymer particle example and $F_{ab}$ is the curvature of the gravitational connection $A$, the analog of $k$. While $E$'s and holonomies of $A$ are well-defined operators on the quantum geometry Hilbert space, connections are not. Therefore, $F_{ab}$ has to be expressed in terms of holonomies. Given a spin network state, at each vertex, one introduces new edges, creating ‘small’ loops and expresses $F_{ab}$ in terms of holonomies along these small loops (taking care of appropriate ‘area factors’). The resulting operator initially depends on the choice these new edges. However, while acting on diffeomorphism invariant states (in Cyl’), the dependence on the details of these edges drops out. Thus, on states of physical interest, the final Hamiltonian constraint does not make explicit reference to details such as the lengths and ‘positions’ of the new edges.

Let us now turn to the polymer particle. Now, the classical Hamiltonian is of the form

$$H = \frac{\hbar^2}{2m} \dot{x}^2 + V(x), \quad (V.1)$$

where $V(x)$ is a potential. Since the operator $\dot{x}$ is well-defined in the polymer framework, the main technical problem is that of defining the operator analog of $\dot{x}^2$. Thus the situation is analogous to that with the Hamiltonian constraint, described above. Again, we will need to introduce some extra structure (which is invisible classically), this time to define the analog of $\dot{x}^2$ in terms of ‘holonomies’ $V(\mu)$ of the ‘connection’ $k$ on the full Hilbert space $\mathcal{H}_{\text{Poly}}$. From a mathematical viewpoint, the obvious choice is an ‘elementary length’ $\mu_c$. Physically, this is motivated by the expectation that such a scale will be provided by a deeper theory (such as quantum geometry) through an underlying discreteness. From now on, we will adopt the viewpoint that this discreteness is fundamental, whence observationally only those $V(\mu)$ are relevant for which $\mu = N\mu_c$, for an integer $N$.

Given $\mu_c$, we will set

$$\hat{H} = \frac{\hbar^2}{2m} \hat{\mu}_c^2 + V(\dot{x}) \quad \text{where} \quad \hat{\mu}_c^2 = \frac{1}{\mu_c^2} \left( 2 - V(\mu_c) - V(-\mu_c) \right) \quad (V.2)$$

Unfortunately, in this example, we do not have a useful analog of the diffeomorphism invariance of loop quantum gravity which can help remove the dependence on this extra structure. Therefore, the final Hamiltonian operator on $\mathcal{H}_{\text{Poly}}$ will continue to depended on $\mu_c$; the reference to the additional structure does not go away. This is simply a consequence of the fact that a toy model can not mimic all aspects of the richer, more complicated theory, whence, to carry out constructions which are analogous to those in the full theory, certain structures have to be introduced ‘externally’. However, we will see that, if one chooses the discreteness scale $\mu_c < 10^{-13}m$ as in Section IV B, in the domain of validity of non-relativistic quantum
mechanics, predictions derived from (V.2) are indistinguishable from those of Schrödinger quantum mechanics and therefore in agreement with experiments. In contrast to results of section IV, this holds for fully quantum mechanical results, not just the semi-classical ones.

At first sight, this may seem obvious. However, the detailed analysis will reveal that certain subtleties arise and have to be handled carefully. These issues provide concrete hints for the precise procedure required to compare the polymer and continuum theories in the much more complicated context of quantum geometry. Thus, while the toy model is constructed to mimic the situation in the full theory, its concrete results, in turn, provide guidance for the full theory.

Since the key difficulties in the polymer description involve the kinetic term, to illustrate the similarity and differences between the polymer and Schrödinger dynamics it suffices to work with a fixed potential. To facilitate the detailed comparison, in this paper we will focus on the harmonic oscillator potential. (Results on general potentials will be discussed elsewhere [24].)

Remarks: i) In the semi-classical considerations of the last section, we had to find a ‘fundamental’ operator on $\mathcal{H}_{\text{Foly}}$ which is the analog of the Schrödinger momentum operator. Technically, the situation with the kinetic term in the Hamiltonian, discussed above, is completely analogous. However, there is a conceptual difference: whereas the operator $\hat{K}_{\mu_c}$ was used only for semi-classical purposes, $\hat{H}$ is to govern ‘fundamental dynamics’ on $\mathcal{H}_{\text{Foly}}$. Therefore, it has to be constructed and analyzed more carefully. In particular, $\hat{K}_{\mu_c}^2 \neq \hat{K}_{\mu_c}^2$; we will see that the latter choice gives an unwanted degeneracy in the eigenvalues of $\hat{H}$.

ii) Since the final Hamiltonian now depends on $\mu_c$, in the polymer description, $\mu_c$ now has a fundamental significance. This strengthens the viewpoint that the algebra of physical observables is generated only by $\hat{V}(N\mu_c)$ and $\hat{x}$.

B. Eigenvalues and eigenstates of $\hat{H}$ in $\mathcal{H}_{\text{Foly}}$

Recall that a general element $|\Psi\rangle$ of $\mathcal{H}_{\text{Foly}}$ can be expanded out as $|\Psi\rangle = \sum_x |\Psi(x)|x\rangle$ (where $\Psi(x)$ is non-zero only at a countable set of points). Therefore, the eigenvalue equation $\hat{H} |\Psi\rangle = E |\Psi\rangle$ becomes a difference equation on the coefficients $\Psi(x)$:

$$\psi(x + \mu_c) + \psi(x - \mu_c) = \left[ 2 - \frac{2E \mu_c^2}{\hbar \omega \frac{d^2}{dx}} + \frac{x^2 \mu_c^2}{d^2} \right] \psi(x). \quad (V.3)$$

The form of this equation implies that a basis of solutions is given by states of the type

$$|\Psi_{x_o}\rangle = \sum_{m=-\infty}^{\infty} \psi_{x_o}^{(m)} |x_o + m\mu_c\rangle \in C_{y, x_o},$$

where $\alpha^{x_o}$ is the regular lattice consisting of points $x_o + m\mu_c$ with $x_o \in [0, \mu_c)$. For these states, the difference equation reduces to a recursion relation

$$\psi_{x_o}^{(m+1)} + \psi_{x_o}^{(m-1)} = \left[ 2 - \frac{2E \mu_c^2}{\hbar \omega \frac{d^2}{dx}} + \frac{(x_o + m\mu_c)^2 \mu_c^2}{d^2} \right] \psi_{x_o}^{(m)}. \quad (V.4)$$
The full polymer Hilbert space $\mathcal{H}_{\text{Poly}}$ can be decomposed as a direct sum of separable Hilbert spaces $\mathcal{H}_{\text{Poly}}^{x_0}$, 

$$\mathcal{H}_{\text{Poly}} = \bigoplus_{x_0 \in \mathbb{R}} \mathcal{H}_{\text{Poly}}^{x_0},$$

and the above energy eigenstates belong to the sub-space $\mathcal{H}_{\text{Poly}}^{x_0}$ of $\mathcal{H}_{\text{Poly}}$. Note that since the observable algebra is now generated by $\hat{x}$ and $\hat{V}(N \mu_c)$, observables can not mix states belonging to distinct $\mathcal{H}_{\text{Poly}}^{x_0}$; each of these Hilbert spaces is superselected. Hence, we can focus on one at a time and find all eigenvalues and eigenstates of the Hamiltonian in it.

1. The case $x_0 = 0$

Let us consider the $x_0 = 0$ case first. If $E$ is to be an eigenvalue of $\hat{H}$, the coefficients $\Psi_0^{(m)}$ must fall off sufficiently fast for $\ket{\Psi_0}$ to be normalizable. It turns out that the simplest way to get a control on this fall-off is to make a ‘Fourier transform’ and go to the momentum representation. Recall from section III B that for each real number $k$, there is an element $(k)$ of $\mathbb{C}^*$ defined by: $(k \ x) = e^{-ikx}$. Given any energy eigenstate $\ket{\Psi_0} \in \mathcal{H}_{\text{Poly}}^0$, we can evaluate its ‘Fourier transform’

$$\psi(k) := (k | \Psi_0) = \sum_{m=-\infty}^{\infty} \Psi_0^{(m)} e^{-ikm\mu_0}$$

(V.5)

where $k \in (-\frac{\mu_0}{\mu_c}, \frac{\mu_0}{\mu_c})$; by construction $\psi(k)$ is periodic. The eigenvalue equation (V.4) now becomes

$$\frac{d^2 \Psi_0(k)}{dk^2} + 2\frac{d}{k^2} \left( \frac{E}{\hbar \omega} + \frac{d^2}{\mu_0^2} \left[ \cos(k\mu_0) - 1 \right] \right) \Psi_0(k) = 0.$$  

(V.6)

Thus, the difference equation (V.3) in the position space becomes a differential equation in the momentum space. Setting

$$\phi := \frac{k\mu_0 + \pi}{2}, \quad h := \frac{4d^2}{\mu_0^2}, \quad \text{and} \quad b := h \cdot \frac{2E}{\hbar \omega},$$

(V.7)

the equation simplifies to:

$$\frac{d^2 \Psi_0(\phi)}{d\phi^2} + (b - h^2 \cos^2(\phi)) \Psi_0(\phi) = 0.$$  

(V.8)

This is precisely the well-studied Mathieu’s equation. From basic theory of differential equations we conclude that (V.8) does admit solutions. However, since the Fourier transforms (V.5) of states in the position representation are necessarily periodic, the question is whether the solutions $\Psi_0(\phi)$ are periodic (with period $\pi$). If they are, we may take the inverse Fourier transform and recover a state $\ket{\Psi_0} \in \mathcal{H}_{\text{Poly}}^0$; by Parseval’s theorem this state must be normalizable. Thus, the question of whether $\Psi_0^{(m)}$ have appropriate fall-off reduces to whether solutions $\Psi_0(\phi)$ to Mathieu’s equation are periodic.

We can now appeal to the general theory of ordinary differential equations with periodic coefficients—specifically, Floquet’s theorem—to conclude that: i) there is a discrete
infinite of periodic solutions with the required period $\pi$; and (ii) each of the corresponding energy eigenvalues is non-degenerate. (See [26] for the general theory; [27] for application to Mathieu’s equation.) Let us denote the allowed eigenvalues by $E_{0,n}$ and the corresponding eigenstates in $\mathcal{H}_{\text{Poly}}^0$ by $|\Psi_{0,n}\rangle = \sum m \mu_m |m\mu_m\rangle$. The question now is how these eigenvalues and eigenstates are related to those of the Schrödinger theory.

By examining our definition of parameters in (V.7), we see that the ratio of $\mu_c/d$ in which we are interested corresponds to very large values of $h$. We can then employ an asymptotic formula [27] for the $b$ coefficients:

$$b \sim (2n+1)h - \frac{2n^2 + 2n + 1}{4} + O(h^{-1}).$$

By substituting this expansion back into our definition (V.7) of the $b$ coefficients we obtain the following expansion for the energy eigenvalues $E_{0,n}$:

$$E_n \sim (2n+1)\frac{\hbar\omega}{2} - \frac{2n^2 + 2n + 1}{16} \left(\frac{\mu_c}{d}\right)^2 \frac{\hbar\omega}{2} + O\left(\frac{\mu_c^4}{d^4}\right).$$

Thus, in the limit $\mu_c/d \to 0$, the $E_n$ reduce to the Schrödinger eigenvalues, but for the ‘physical’ nonzero value of $\mu_c/d$, there is a correction introduced by the ‘fundamental’ discreteness. We see from this equation that the first correction to the eigenvalue is negative and of order $\mu_c^2/d^2$. Using the very conservative value $10^{-13}$ m of $\mu_c$ for a carbon monoxide molecule we conclude that these corrections are significant only when $n \approx 10^7$, i.e., when the vibrational energy of the oscillator is $\approx 10$ MeV, or, in classical terms, the average vibrational velocity is $10^{14}$ ms$^{-1}$. Thus, while the corrections are conceptually important, in the domain of validity of non-relativistic quantum mechanics they are too small to have been observed.

Next, let us compare the eigenstate $|\Psi_{0,n}\rangle$ with the shadow $|\Psi_{0,n}^{\text{shad}}\rangle$ of the $n$th Schrödinger eigenstate on the graph $\alpha^0$. Unfortunately, we cannot carry out this task analytically because closed form expressions for the Mathieu functions are not available. Therefore, let us calculate the norm of

$$|\Delta \Psi_{0,n} := |\Psi_{0,n}^{\text{shad}}\rangle - |\Psi_{0,n}\rangle$$

numerically and study its behavior as a function of $n$ and $\mu_c/d$. It turns out [24] that the log-log plot of the norm of $|\Delta \Psi_{0,n}\rangle$ against $\mu_c$ is linear for $10^{-8} < (\ell/d) < 1$ and $n \leq 10$, a range of parameters for which the computation can be readily performed. By a least squares analysis we can then verify that:

$$\langle \Delta \Psi_{0,n} | \Delta \Psi_{0,n}\rangle^{\frac{1}{2}} \sim (n+1)^{1.35} \left(\frac{\mu_c}{d}\right)^{1.10},$$

These numerical results together with the analytic knowledge that the difference equation (V.3) with which we began is itself a standard discretization of Hermite’s equation strongly suggest that for $n \ll 10^7$, the exact eigenstates are experimentally indistinguishable from the shadows of the Schrödinger eigenstates on the graph $\alpha^0$. However, since this evidence is not as mathematically clear-cut as other results of this paper, we will examine this issue from a different angle in sub-section V.C.

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2. General $x_o$

Let us now consider the energy eigenstates in the Hilbert space $\mathcal{H}^{x_o}$ for a general value of $x_o \in (0, \mu_0)$. Now, the momentum wave function is given by

$$\psi_{x_o}(k) := \braket{k | \Psi_{x_o}} \sum_{m=-\infty}^{\infty} \psi_{x_o}^{(m)} e^{-ikx_o} e^{ikm\mu_o}$$  \quad (V.13)$$

Thus the momentum space wave function $\psi_{x_o}(k)$ is no longer periodic in $k$ but satisfies instead:

$$\psi_{x_o} \left( \frac{\pi}{\mu_o} \right) = e^{2\pi i x_o} \psi_{x_o} \left( -\frac{\pi}{\mu_o} \right).$$  \quad (V.14)$$

The differential equation that an energy eigenstate must satisfy continues to be (V.6) and by simple redefinitions of parameters we are again led to the standard Mathieu equation. Thus, the only difference between the $x_o = 0$ and $x_o \neq 0$ cases lies in the boundary conditions that the solutions are to satisfy. Again, thanks to the very exhaustive literature available on Mathieu’s equation [26, 27]\(^6\), we conclude that: i) there is a discrete infinity of solutions satisfying (V.14); ii) each of the corresponding energy eigenvalues is non-degenerate; and iii) the eigenvalues are very close to those in the Schrödinger theory with corrections which become $O(1)$ only when energy levels corresponding to $n \approx 10^7$.

To summarize, the full polymer Hilbert space $\mathcal{H}_{\text{Poly}}$ can be decomposed in to orthogonal, separable subspaces $\mathcal{H}_{\text{Poly}}^{x_o}$, each left invariant by the algebra of observables. The energy eigenvalue equation can therefore be solved on these subspaces independently. In all cases, there is a discrete infinity of eigenvalues; they are very close to the eigenvalues of the Schrödinger theory in its domain of validity; and each eigenvalue is non-degenerate. There is numerical evidence that the eigenstates in $\mathcal{H}_{\text{Poly}}^{x_o}$ are very close to the shadows of the Schrödinger eigenstates (which naturally belong to $Cyl^+$) on graphs $\alpha^{x_o}$.

**Remarks:** 1. Recall that for the kinetic energy term $\tilde{H}_{\text{kin}}$ in the Hamiltonian, we used the operator $\tilde{K}_{\mu_e}$ of (V.2) rather than the square $\tilde{K}_{\mu_e}^2$ of the operator $\hat{K}_{\mu_e}$ of (IV.20). Both alternatives appear to be viable from the classical standpoint. However, had we chosen $\tilde{K}_{\mu_e}^2$ in place of $\tilde{K}_{\mu_e}^2$, we would have found a 2-fold degeneracy in the eigenvectors irrespective of how small $n$ is because, in effect, the coefficients $\psi_{x_o}^{(m)}$ for even and odd $m$ would have decoupled in the eigenvalue equation. Hence, from a quantum mechanical perspective, only the choice $\tilde{K}_{\mu_e}^2$ is experimentally viable. This situation is familiar from lattice gauge theories but brings out the fact that the polymer framework has to be set up rather delicately; small $\mu_e/d$ does not automatically ensure that the polymer results would be close to the continuum ones.

2. While all ‘low lying’ eigenvalues are very close to $\hbar \omega (n + \frac{1}{2})$, eigenvalues in different sectors $\mathcal{H}_{\text{Poly}}^{x_o}$ differ from one another slightly. Suppose for a moment that the only limitation of Schrödinger quantum mechanics comes from the fact that it ignores the inherent discreteness implied by quantum geometry. Since the polymer particle model accounts for this discreteness, it would then be the ‘fundamental’ theory underlying Schrödinger quantum mechanics. Then, we would be led to conclude that the detailed energy levels of physical

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\(^6\) See also [28], especially chapter IV and the graphs and accompanying discussions on pages 40 and 97–98.
harmonic oscillators would be sensitive to the physical, underlying quantum geometry; i.e.,
depend on the graph which best captures the fundamental discreteness along the line of
motion of the oscillator.

3. Our construction was motivated by the way the Hamiltonian constraint is treated in
full general relativity. Note however, a qualitative difference. Solutions to the Hamiltonian
constraints fail to belong to the polymer Hilbert space of quantum geometry because zero
fails to be a discrete eigenvalue of those operators. In the case of a harmonic oscillator,
by contrast, the full spectrum is discrete. Therefore, now the eigenvectors belong to the
polymer Hilbert space and Cyl* is relevant only in making contact with the Schrödinger
quantum mechanics. Had we considered a free particle instead, as in the Schrödinger theory,
the spectrum of the polymer Hamiltonian operator

$$\hat{H}_{\text{free}} = (\hbar^2 / 2m) \hat{K}_{\mu_c}^2$$

would have been continuous. The eigenvectors would no longer be normalizable in \( \mathcal{H}_{\text{Poly}} \)
but belong to Cyl*. For energies \( E \ll \hbar^2 / (2m \mu_c^2) \), they are practically indistinguishable
from plane waves in the sense that their shadows on sufficiently refined regular graphs (with
separation \( \sim \mu_c \)) are very close to those of plane waves. However, as one would expect, for
higher energies, the ‘fundamental description’ introduces major corrections.

C. Shadows of Schrödinger eigenstates

In this sub-section we will further explore the relation between the polymer and Schrödiger
energy eigenstates. For definiteness let us restrict ourselves to the \( x_c = 0 \) case, i.e., to the
Hilbert space \( \mathcal{H}_{\text{Poly}}^0 \) associated with the graph \( \alpha^0 \).

The shadows on \( \alpha^0 \) of the Schrödinger eigenstates \( |\Psi_n\rangle \) are given by

$$|\Psi_{0,n}^{\text{shad}}\rangle = c_n \sum_m H_n \left( \frac{m \mu_c}{d} \right) e^{-\frac{m^2 \mu_c^2}{2d^2}} |m \mu_c\rangle.$$  \hspace{1cm} (V.15)

where \( c_n \) is the normalization constant. The main result of this sub-section is that these
shadows satisfy the eigenvalue equation of the polymer Hamiltonian \( (V.2) \) to an excellent
degree of approximation if \( n \ll 10^7 \).

The action of the Hamiltonian \( (V.2) \) on an arbitrary state \( |\Psi\rangle = \sum_m \psi(m) |m \mu_c\rangle \) can be
easily calculated. The result is:

$$\hat{H} |\Psi\rangle = \frac{\hbar \omega d^2}{2 \mu_c^2} \sum_m \left[ \left( 2 + \frac{m^2 \mu_c^4}{d^4} \right) \psi(m) - \left( \psi(m + 1) + \psi(m - 1) \right) \right] |m \mu_c\rangle.$$  \hspace{1cm} (V.16)

Let us begin with the shadow ground state \( |\Psi_{0,0}^{\text{shad}}\rangle \). We have:

$$\hat{H} |\Psi_{0,0}^{\text{shad}}\rangle = \frac{\hbar \omega d^2}{2 \mu_c^2} c_0 \sum_m \left[ \left( 2 + \frac{m^2 \mu_c^4}{d^4} \right) e^{-\frac{m^2 \mu_c^2}{2d^2}} - e^{-\frac{m^2 \mu_c^2}{2d^2}} \left( e^{-\frac{m \mu_c^2}{d^2}} + e^{-\frac{m \mu_c^2}{d^2}} \right) \right] |m \mu_c\rangle.$$  \hspace{1cm} (V.17)
To make the structure of the right side more transparent, let us expand the last three exponentials and keep the lowest few terms:

\[
\hat{H} \left| \Psi_{0,0}^{(\text{shad})} \right> = \frac{\hbar \omega}{2 \mu_c^2} c_0 \sum_m e^{-\frac{m^2 \mu_c^2}{2 d^2}} \times \left[ \left( 2 + \frac{m^2 \mu_c^4}{d^4} \right) - \left( 2 - \frac{\mu_c^2}{4d^2} + \frac{\mu_c^4}{4d^4} - \frac{\mu_c^6}{24d^6} - \frac{m^2 \mu_c^4}{2d^4} - \cdots \right) \right] | m \mu_c >
\]

\[
= \frac{\hbar \omega}{2} \left| \Psi_{0,0}^{(\text{shad})} \right> + \frac{\hbar \omega}{2} \sum_m e^{-\frac{m^2 \mu_c^2}{2 d^2}} \left[ -\frac{\mu_c^2}{d^2} + \frac{\mu_c^4}{6d^4} + \frac{2m^2 \mu_c^4}{d^4} - \cdots \right] | m \mu_c >.
\]

Thus, we have

\[
\hat{H} \left| \Psi_{0,0}^{(\text{shad})} \right> = \frac{\hbar \omega}{2} \left[ \left| \Psi_{0,0}^{(\text{shad})} \right> + \left| \delta \Psi_{0,0} \right> \right]
\]

where \( | \delta \Psi_0 \rangle \) is \( c_0/4 \) times the last sum in (V.18). Since this ‘remainder’ proportional to \( \mu_c^2/d^2 \), in the series in square brackets only terms with large \( m \) make significant contributions and these terms are severely damped by the exponential multiplicative factor. Hence it is plausible that \( \langle \delta \Psi_0 | \delta \Psi_0 \rangle \ll 1 \), i.e., that the shadow state is very nearly an eigenstate of \( \hat{H} \). We will first establish that the situation is similar for all excited states and then show that the expectation on smallness of the remainder term is correct for all eigenstates.

Let us then act on the shadow (V.15) of the \( n \)-th excited state with the Hamiltonian. We obtain:

\[
\hat{H} \left| \Psi_{0,n}^{(\text{shad})} \right> = \frac{\hbar \omega}{2 \mu_c^2} c_n \sum_m e^{-\frac{m^2 \mu_c^2}{2 d^2}} \left\{ \left( 2 + \frac{4m^2 \mu_c^4}{d^4} \right) H_n \left( \frac{m \mu_c}{d} \right) - e^{-\frac{m^2 \mu_c^2}{2 d^2}} \times \left[ e^{-\frac{m \mu_c^2}{d^2}} H_n \left( \frac{m \mu_c}{d} + \frac{\mu_c}{d} \right) + e^{\frac{m \mu_c^2}{d^2}} H_n \left( \frac{m \mu_c}{d} - \frac{\mu_c}{d} \right) \right] \right\}
\]

This expression can be simplified using the basic recurrences satisfied by the Hermite polynomials and by expanding the exponentials using Taylor’s theorem. As with the ground state, we can then conclude

\[
\hat{H} \left| \Psi_{0,n}^{(\text{shad})} \right> = \frac{2n + 1}{2} \hbar \omega \left| \Psi_{0,n}^{(\text{shad})} \right> + \frac{\hbar \omega}{2} \left| \delta \Psi_n \right>
\]

where \( \left| \delta \Psi_n \right> \) can be evaluated explicitly. To bound its norm we use the fact that the sums appearing in the norm are Riemann sums for integrals that may be evaluated analytically. In this way it is possible to prove that:

\[
\langle \delta \Psi_{n,\mu_c} | \delta \Psi_{n,\mu_c} \rangle^{1/2} = \frac{\sqrt{35}}{48} \left( 2n^4 + 4n^3 + 10n^2 + 8n + 3 \right)^{1/2} \left( \frac{\mu_c}{d} \right)^2 + O \left( n^3 \left( \frac{\mu_c}{d} \right)^4 \right).
\]

We see immediately that \( \langle \delta \Psi_n | \delta \Psi_n \rangle^{1/2} \) approaches zero if we let \( \mu_c/d \) approach zero. For finite \( \mu_c/d \), its value depends on \( n \) and, as one would expect on physical grounds, is of order unity when \( n \sim d/\mu_c \). In the case of the molecular vibrations of carbon monoxide considered above, this corresponds to \( n \sim 10^7 \). It is obvious that (among other things) the
approximation that $V(x)$ can be described by the simple harmonic oscillator potential will break down long before this energy level $n$ is reached.

For the bound (V.22) to be useful, we must know when the $O(n^3 (\mu_e/d)^4)$ term is negligible. This can easily be investigated numerically, and it is found that the asymptotic behavior (V.22) is attained almost as soon as $\mu_e/d < 1$. To give some examples, one finds that for the ground state, even when $\mu_e/d$ is as large as 0.1, equation (V.22) is accurate to less than a percent and the magnitude of the norm of $|\delta \Psi_n|$ is about $2.2 \times 10^{-3}$. For $n = 9$ and $\mu_e/d = 10^{-3}$, equation (V.22) is accurate to one part in $3 \times 10^{-5}$ when $\mu_e/d = 10^{-3}$, and the magnitude of the norm of $|\delta \Psi_n|$ is $2.13 \times 10^{-13}$. Thus, not only does the norm of $|\delta \Psi_n|$ approach zero as $\mu_e/d$ approaches zero, but it also quickly approaches the asymptotic behavior of equation (V.22).

To summarize, we have shown that the shadows of the Schrödinger energy eigenstates on the graph $\alpha^0$ are eigenstates of the polymer Hamiltonian $\hat{H}$ to a high degree of approximation at ‘low’ energies. Quantum geometry effects manifest themselves only at energy levels as high as $n \sim 10^7$, i.e., long beyond the validity of non-relativistic approximation. This result complements our findings in Section V B where we compared the exact eigenstates of the polymer Hamiltonian with the shadows of the Schrödinger Hamiltonian.

VI. DISCUSSION

We began, in section I, by raising three conceptual issues of a rather general nature that arise in relating background independent approaches to quantum gravity with low energy physics: i) What is the precise sense in which semi-classical states arise in the full theory? ii) Is the fundamental Planck scale theory, with an in-built fundamental discreteness, capable of describing also the low energy physics rooted in the continuum, or, does it only describe an entirely distinct phase? iii) Can one hope to probe semi-classical issues without a canonical inner product on the space of physical states $Cyl^+$? To probe these issues in a technically simpler context, we introduced the ‘polymer framework’ in a toy model—a non-relativistic particle—where the same questions arise naturally. In the context of the model, we found encouraging answers to all three questions: although at first the polymer description seems far removed from the standard Schrödinger one, the second can be recovered from the former in detail.

Specifically, we could: a) give a criterion to select the coherent states entirely within the polymer framework and, using their shadows, demonstrate in detail that they are sharply peaked about the corresponding classical states; and, b) introduce the Hamiltonian operators in the polymer framework and show that their eigenvalues and eigenfunctions are indistinguishable from those of the continuum, Schrödinger theory within its domain of validity. Logically, one can forego the continuum theory, work entirely with the polymer description, and compare the theoretical predictions with experimental results. However, since we already know that Schrödinger quantum mechanics reproduces the experimental results within its domain of validity, it is simpler in practice to verify agreement with the Schrödinger results.

As one might physically expect, since the polymer framework ‘knows’ about the underlying discreteness, it predicts corrections to the Schrödinger framework which become significant when the energies involved are sufficiently high to probe that discreteness. Thus, we have a concrete mathematical model, inspired by loop quantum gravity, which realizes the idea that
a fundamental theory can be radically different from the continuum theory both conceptually and technically and yet reproduce the familiar low energy results.

The broad strategy we followed, including the use of shadow states, was already outlined in the general program [1]. Notions needed in this analysis are all available in field theories as well as full quantum gravity. The details of the polymer particle toy model provide concrete hints for these more complicated theories. For example, it may seem ‘obvious’ in that calculations on sufficiently refined graphs should reproduce the continuum answers. Our analysis showed that this is not necessarily the case. Naively, one would have used the operator \((\hbar^2/2m)\hat{K}^2\) as the kinetic part of the quantum Hamiltonian. However, this choice would have given a two-fold degeneracy for all eigenstates of the polymer Hamiltonian including the ‘low energy’ ones, while in the Schrödinger theory, all eigenstates are non-degenerate. This is a concrete illustration of how the requirement that the theory should reproduce predictions of well-established theories in the low energy regime can be used to discriminate between choices available in the construction of the ‘fundamental’ framework. A second example arises from a cursory examination of the form of the polymer particle Hamiltonian. While the potential continues to be unbounded as in the Schrödinger theory, the kinetic part of the Hamiltonian is now bounded. Therefore, at first, it seems that the kinetic term will not be able to ‘catch up’ for large \(x\) to produce normalizable solutions to the eigenvalue equation \((\hbar^2/2m)\hat{K}^2 + x^2)|\Psi\rangle = E|\Psi\rangle\). Furthermore, this expectation can be ‘confirmed’ by numerical solutions to the difference eigenvalue equation (V.4). However the careful examination of section V.B, involving the Mathieu equation in the momentum representation, showed that the expectation is incorrect and the divergence of \(|\Psi\rangle\) one encounters in computer calculations is just a numerical artifact. Finally, even at the kinematical level, there is a subtlety: a priori, it is not at all obvious that any calculation to select semi-classical states \(|\Psi\rangle\) in Cyl, carried out entirely within the polymer framework, will reproduce the standard coherent states. One could indeed be working in an inequivalent ‘phase’ of the theory and thus find that there are no semi-classical states at all or discover states which are semi-classical in a certain well-defined sense but distinct from the standard coherent states (as in, e.g., [3]). The fact that the Schrödinger semi-classical states can be recovered in the polymer framework is thus non-trivial and suggests how standard low energy physics could emerge from the polymer framework. Thus, our analysis provides useful guidelines for more realistic theories, pointing out potential pitfalls where care is needed and suggesting technical strategies.

Finally, there are also some conceptual lessons. First, we saw concretely that recovery of semi-classical physics entails two things: isolation of suitable states and a suitable coarse graining. In the toy model, the coarse graining scale was set by our tolerance \(d\) and continuum physics emerges only when we coarse grain on this scale. A second lesson is that the availability of a scalar product on the space of physical states is not essential at least for semi-classical considerations: The framework of shadow states—with its Wilsonian overtones—provides an effective strategy to recover low energy physics. In the next paper in this series we will see that these ideas can be naturally elevated to the quantum Maxwell theory.
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APPENDIX A: THE DISPLACEMENT OPERATOR \( \hat{V}(\mu) \) AND HOLOMIES

Recall that the displacement operators \( \hat{V}(\mu) \) are the analogs of holonomy operators in Maxwell theory and quantum geometry. In this Appendix, we collect some properties of displacement operators which will be useful in the discussion of holonomies in the subsequent papers.

We begin by recalling the commutator between \( \hat{x} \) and \( \hat{V}(\mu) \):

\[
[\hat{x}, \hat{V}(\mu)] = -\mu \hat{V}(\mu).
\]  

(A.1)

This equation gives rise to interesting uncertainty relations, even though \( \hat{V}(\mu) \) are unitary rather than self-adjoint [20]. To obtain these, let us decompose \( \hat{V} \) into the sum of two Hermitian operators,

\[
\hat{V}(\mu) = \hat{C}(\mu) + i\hat{\mathcal{S}}(\mu).
\]  

(A.2)

It is straightforward to obtain the commutation relations between \( \hat{C}, \hat{\mathcal{S}} \) and \( \hat{x} \) from (A.1) as

\[
[\hat{x}, \hat{C}(\mu)] = -i\mu \hat{\mathcal{S}}(\mu) \quad \text{and} \quad [\hat{x}, \hat{\mathcal{S}}(\mu)] = i\mu \hat{C}(\mu).
\]

Therefore, we can obtain uncertainty relations between \( \hat{C}, \hat{\mathcal{S}} \) and \( \hat{x} \):

\[
(\Delta x)^2 (\Delta C(\mu))^2 \geq \frac{\mu^2}{4} \langle \hat{\mathcal{S}}(\mu) \rangle^2 \quad \text{and} \quad (\Delta x)^2 (\Delta S(\mu))^2 \geq \frac{\mu^2}{4} \langle \hat{C}(\mu) \rangle^2
\]  

(A.3)

Now, it is natural to define the uncertainty in \( \hat{V} \) as

\[
(\Delta V)^2 := \langle V^4 V \rangle - \langle V \rangle^2 = 1 - (\langle C \rangle^2 + \langle S \rangle^2),
\]

(A.4)

where the second expression follows from the unitarity of \( \hat{V} \) and the definitions of \( \hat{C} \) and \( \hat{\mathcal{S}} \) (A.2). Finally, combining (A.4) and (A.3) we obtain the desired uncertainty relation

\[
(\Delta x)^2 \frac{(\Delta V(\mu))^2}{1 - (\Delta V(\mu))^2} \geq \frac{\mu^2}{4}.
\]  

(A.5)
It is natural to ask how close the semi-classical states of section IV.A come to saturating this bound. Let us begin by considering the state $|\Psi_c\rangle$ peaked at $(x=0, k=0)$. The ‘expectation value’ of $\hat{V}(\mu)$ in $|\Psi_c\rangle$ and its shadow $|\Psi_{c}^{\text{bad}}\rangle$ on a regular lattice with spacing $\ell$ is given in (IV.21) as

$$\langle \hat{V}(\mu) \rangle \approx e^{-\frac{\mu^2}{4d^2}} \left( 1 + e^{-\frac{\pi^2 d^2}{\ell^2}} \left[ 2 \cos \left( \frac{\pi \mu}{\ell} \right) - 2 \right] \right).$$

Then, it is straightforward to evaluate the fluctuations of $\hat{V}(\mu)$ as

$$(\Delta \hat{V}(\mu))^2 := 1 - |\langle \hat{V}(\mu) \rangle|^2 \approx 1 - e^{-\frac{\mu^2}{2d^2}},$$

where we have neglected corrections of order $\exp(-\frac{\pi^2 d^2}{\ell^2})$. Combining (A.6) with the fluctuations in $x$ (IV.17), we obtain:

$$(\Delta x)^2 \cdot \frac{(\Delta V(\mu))^2}{1 - (\Delta V(\mu))^2} \approx \left( \frac{d^2}{2} \right) \cdot \left( 1 - e^{-\frac{\mu^2}{2d^2}} \right).$$

(A.7)

Thus, for a general $\mu$, we are not close to saturation. However, if $\mu \ll d$, we can expand in powers of $\mu/d$ to obtain:

$$(\Delta x)^2 \frac{(\Delta V(\mu))^2}{1 - (\Delta V(\mu))^2} = \frac{\mu^2}{4} \left( 1 + O \left( \frac{\mu^2}{d^2} \right) \right).$$

(A.8)

Thus, in this case, the uncertainty relation (A.5) is indeed saturated, modulo terms of the order $(\mu/d)^2$. If $\mu \sim \ell$, a similar result can be obtained for general coherent states peaked at any value of momentum $k$, even when $k$ approaches $\pi/\ell$. Note that this in marked contrast to the uncertainty relation between $\hat{x}$ and $\hat{K}_{\mu}$, which is similarly saturated only if $k \mu < 1$.

Finally, a natural question is whether the ‘expectation value’ of $\hat{V}(\mu)$ can be used to determine the momentum of the system when it is in a semi-classical state. In a semi-classical state labelled by $\zeta = \frac{1}{\sqrt{2d}}(x + id^2 k)$, the ‘expectation value’ of $\hat{V}(\mu)$ is given by

$$\langle \hat{V}(\mu) \rangle = e^{-\frac{\mu^2}{4d^2}} e^{-ik\mu} \left[ 1 + O \left( e^{-\frac{\pi^2 d^2}{\ell^2}} \right) \right].$$

(A.9)

An obvious strategy is to just define the ‘expected momentum’ $\tilde{k}$ in the quantum state $|\Psi_\zeta\rangle$ to be:

$$\langle \hat{V}(\mu) \rangle = |\langle \hat{V}(\mu) \rangle| e^{-i\mu k},$$

(A.10)

i.e., to associate the momentum $k$ with the phase of the $\hat{V}$ operator. Clearly, modulo corrections $O(\exp -\pi^2 d^2/\ell^2)$, $\tilde{k}$ equals $k$. Moreover, this result holds even if $k \sim \pi/\ell$. The $|\langle \hat{V}(\mu) \rangle|$ factor in our expression (A.10) may seem surprising. However, it does not arise because of the polymer nature of the Hilbert space we are considering; it is necessary also in the Schrödinger representation. Note also that our expression (A.6) for the variation of $V$, implies that $|\langle \hat{V}(\mu) \rangle|$ must be less than one if $\mu \neq 0$. Otherwise the fluctuation in $V$ will vanish, i.e., we will have a state of definite momentum and the uncertainty relation (A.5) would imply that the state must have infinite spread in $x$.

Techniques introduced in this appendix will be useful when it comes to examining expectation values and fluctuations of holonomies in Maxwell and gravitational semi-classical states.
In the main body of the paper we introduced ‘fundamental operators’ such as $\hat{K}_{\mu}$ and $\hat{H}$ on the entire polymer Hilbert space $\mathcal{H}_{F_{\text{poly}}}$ and analyzed their properties. In field theories, by contrast, one often ties operators to the energy scale under consideration and constructs from them ‘an effective field theory’ à la Wilson. Such constructions are likely to play an important role in relating quantum field theories on quantum geometries with low energy physics. Therefore, in this appendix, we will extend some of the considerations of sections IV and V by allowing operators which are tied to the lattice spacing under consideration. For example, by setting $\hat{K}_\ell = (i/2\ell)(\hat{V}(\ell) - \hat{V}(-\ell))$, we obtain a family of ‘momentum’ operators $\hat{K}_\ell$, one for each regular lattice. The dependence of such operators on $\ell$ in the limit $\ell \to 0$ will enable us to relate our constructions to the Wilsonian renormalization group flow. We will now examine properties of such families of operators and introduce the notion of ‘approximate consistency in the low energy regime’, which will be useful in the analysis of field theories in later papers.

As mentioned in section II, operators on the full Hilbert space $\mathcal{H}_{F_{\text{poly}}}$ in quantum geometry often arise from consistent families of operators on the Hilbert spaces $\{\mathcal{H}_\gamma\}$ associated to graphs $\gamma$ [9]. However, since we will be interested in ‘low energy’ states that lie in $\text{Cyl}^*$ but not in $\mathcal{H}_{F_{\text{poly}}}$, we will use as our starting point the consistency of families of operators on $\text{Cyl}^*$. This concept is defined naturally using the duality between Cyl and $\text{Cyl}^*$. Specifically, if we are given a family of operators $\{\hat{O}_\gamma\}$ defined on each Hilbert space $\mathcal{H}_\gamma$, then this family is said to be consistent on $\text{Cyl}^*$ if, given any state $|\Psi\rangle \in \text{Cyl}^*$, any two graphs $\gamma$ and $\gamma'$ such that $\gamma \subseteq \gamma'$, and any state $|\phi_\gamma\rangle \in \text{Cyl}_\gamma$, the following holds:

$$
(\Psi | \hat{O}_\gamma | \phi_\gamma) = (\Psi | \hat{O}_{\gamma'} \Pi_{\gamma'/\gamma} | \phi_\gamma) \quad \text{.} \tag{B.1}
$$

Here $\Pi_{\gamma'/\gamma}$ denotes the pull-back from $\text{Cyl}_\gamma$ to the larger Hilbert space $\text{Cyl}_{\gamma'}$. This condition serves to ensure that the matrix elements of the operator are independent of the graph $\gamma$ used to calculate them, i.e., that there is a single operator $\hat{O}$ on $\text{Cyl}^*$ such that $(\Psi | \hat{O} | \phi_\gamma) = (\Psi | \hat{O}_\gamma | \phi_\gamma)$ for all graphs $\gamma$. In the polymer particle example, several important operators are consistent on $\text{Cyl}^*$, including the position operator $\hat{x}$ and the displacement operator $\hat{V}(\mu)$.

However, the new families of operators such as $\hat{K}_\ell$, defined above, do not form a consistent family. Neither would be the family of Hamiltonian operators $\hat{H}_\ell$ on $\mathcal{H}_\ell$, if their definitions were similarly tied to the lattice spacing. (The Hamiltonians defined in lattice gauge theory are typically of this type.) To examine such families of operators, we must weaken our definition of consistency on $\text{Cyl}^*$.

We do so in two directions. First, since the momentum operators are intimately connected to differentiation, we cannot expect a weakened form of (B.1) to hold for arbitrary states in $\text{Cyl}^*$, but only for ‘low energy ones’, i.e., states that are elements of $\mathcal{S}$. Second, we do not require expectation values in (B.1) to be exactly equal, but instead only that the norm of their difference should be small. Finally, as in the main text, we will only consider regular lattices. We then say that a family of operators $\{\hat{O}_\gamma\}$ defined on regular lattices $\gamma$ is approximately consistent on low energy states if, given a constant $\alpha_0$ (with same dimensions

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We are grateful to Jurek Lewandowski for pointing out the utility of this definition. 

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as \( \hat{O} \), an \( \epsilon > 0 \), and any two states \( \psi(x), \phi(x) \in \mathcal{S} \), there exists a regular lattice \( \gamma \) such that for any regular lattice \( \gamma' \) that is a refinement of \( \gamma \), the following holds:

\[
\left| \langle \psi | \hat{O}_{\gamma} \Phi_{\gamma}^{\text{shad}} \rangle - \langle \psi | \hat{O}_{\gamma'} \Phi_{\gamma'}^{\text{shad}} \rangle \right| < \alpha \epsilon.
\]

(B.2)

Note that, in this definition, it is essential that we divide by the appropriate norms since states in \( \text{Cyl}^* \) are not normalized.

It is obvious from this definition that any consistent family of operators is automatically approximately consistent on low energy states. Moreover, it is not hard to show that the family of momentum operators \( \hat{K}_\ell \) and Hamiltonians \( \hat{H}_\ell \) satisfies this definition as well. The proof follows from the fact that the two matrix elements in (B.2) (divided by the appropriate norms) are Riemann sums for the same integrals. Hence, since they converge in the \( \ell \to 0 \) limit to the same thing, they form a Cauchy net and equation (B.2) follows (see [24] for details).

Thus, we have generalized the usual notion of consistent families of operators to important families of operators that do not form a consistent family, thus allowing us to use techniques in analyzing such operators that are similar to those that have played such an important role in quantum geometry. This generalization will be useful in subsequent papers on the relation between ‘polymer field theories’ on quantum geometry and the familiar low energy field theories in the continuum.

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