The Landau-Problem on the $\phi$-Deformed Two-Torus

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The Landau-Problem on the $\theta$-Deformed Two-Torus

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Abstract

We study the Landau problem on the $\theta$-deformed two-torus and use well-known projective modules to obtain perturbed energy spectra. For a strong magnetic field $B$ the problem can be restricted to one particular Landau-level. First we represent generators of the algebra of the non-commutative torus $T^2_\theta$ as finite dimensional matrices. A second approach leads to a reducible representation with a $\theta$-dependent center. For a simple periodic potential, the rational part of the Hofstadter-butterfly spectrum is obtained.

Keywords: Landau problem, noncommutative torus

MSC2000: 81R60, 81S10

1 Introduction

The problem of a charged particle moving in two dimensions with a magnetic field $B$ applied perpendicular to the two-dimensional manifold is known as the (two-dimensional) Landau problem. For $B$ very large a perturbation
$V(x_i)$ of the free Hamiltonian $\mathcal{H}_0 = \frac{1}{2m}(\pi_1^2 + \pi_2^2)$ can be treated by projecting onto one Landau level. Due to this perturbation, the degeneracy of the levels will be lifted, but, since the energy gap between two separate Landau levels $\sim B/m$, is very large, the perturbation does not mix different Landau levels and the projection onto one level is justified.

The procedure of projecting onto one Landau level is known since a long time in solid state physics as the Peierls substitution [1].

We look how a $\theta$-deformation of the underlying manifold effects the Peierls substitution. We use the algebra over the $\theta$-deformed plane, $\mathcal{A}_\theta$, generated by two elements $\hat{x}_1$ and $\hat{x}_2$, which satisfy $[\hat{x}_1, \hat{x}_2] = 2\pi i \theta$. The phase-space of $\theta$-deformed quantum mechanics is generated by the coordinates $\hat{x}_i$ and momenta $\hat{p}_i$, $i = 1, 2$, subject to the relations

$$[\hat{x}_1, \hat{x}_2] = 2\pi i \theta, \quad [\hat{x}_k, \hat{p}_i] = i\delta_{k,i} \quad \text{and} \quad [\hat{p}_1, \hat{p}_2] = 0. \quad (1)$$

Like in the commutative case, a magnetic field is introduced by replacing the canonical momenta $\hat{p}_i$ by kinetic ones $\hat{p}_i$, which are covariant under local $U(1)$ gauge transformations $g = g(\hat{x}_i) = e^{i\phi(\hat{x}_i)}$, i.e. $\hat{p}_i \rightarrow \hat{\pi}_i = \hat{p}_i - A_i$, with $A_i = A_i(\hat{x}_1, \hat{x}_2)$ a $U(1)$ gauge potential transforming under $g$ according to $A_j \rightarrow A'_j = gA_jg^{-1} + ig(\partial_j g^{-1})$. Since $\hat{x}_1$ and $\hat{x}_2$ do not commute, also the coordinates have to be replaced by their covariant counterparts $\hat{x}_i \rightarrow \hat{\xi}_i = \hat{x}_i + 2\pi \theta \epsilon_{ij} A_j$. The commutation relations for the covariant phase-space coordinates read

$$[\hat{\xi}_1, \hat{\xi}_2] = 2\pi i \theta(1 + 2\pi \theta F), \quad [\hat{\xi}_k, \hat{\pi}_i] = i(1 + 2\pi \theta F)\delta_{k,i} \quad \text{and} \quad [\hat{\pi}_1, \hat{\pi}_2] = i F, \quad (2)$$

$$[\hat{\pi}_1, \hat{\pi}_2] = i F, \quad (3)$$

with $F = \partial_1 A_2 - i[A_1, A_2]$, the gauge field strength. For $1 + 2\pi \theta F \neq 0$, these relations are, up to some rescaling, equivalent to

$$[\hat{x}_1, \hat{x}_2] = 2\pi i \theta, \quad [\hat{x}_k, \hat{\pi}_i] = i\delta_{k,i} \quad \text{and} \quad [\pi_1, \pi_2] = iB, \quad (4)$$

with $B = F/(1 + 2\pi \theta F)$. For some detailed description see [2] for example.

In chapter 2 and 3 we use the well-known projective modules over $T^2_\theta$ and take a connection with constant curvature to obtain representations of the deformed tori algebra. Chapter 4 deals with a different reducible representation motivated by physics. We require that the magnetic translation operators commute with covariant coupled momenta and obtain a quantization condition. The flux turns out to be rational. For irrational deformation
parameter $\theta$, $B$ of equation (4) then is always irrational. Translation invariance is broken for nonzero magnetic flux. Using the results of chapters 2 and 4, we calculate the projection of the generators $U_j = e^{i\theta j}, j = 1, 2$ of $T^2_\theta$ onto one particular Landau level. The projected operators $U_j^{(\mu)}$ are represented as finite dimensional matrices.

In chapter 5 we calculate the energy corrections due to a small periodic potential $V(x_i) = v(\cos x_1 + \cos x_2)$. This yields the so-called Hofstadter butterfly [3] (or to be more precisely the rational part, i.e. the part arising from rational fluxes, of the Hofstadter butterfly).

2 Representation of $T^2_\theta$

The algebra generated by the elements $U_1$ and $U_2$ subject to the relation

$$U_1 U_2 = e^{-2\pi i \theta} U_2 U_1$$

(5)

is known as the algebra of functions over the non-commutative two-torus $T^2_\theta$ with deformation parameter $\theta$. Two unitary operators on a Hilbert space obeying equation (5) specify a representation of $T^2_\theta$, or, in other words, a module over $T^2_\theta$. Without loss of generality we assume the periods of the torus to be equal to $2\pi$. Further we restrict ourselves to irrational $\theta$ for the moment.

As shown by Connes and Rieffel [4, 5], for $\theta$ irrational, any $T^2_\theta$-module is either free (for constant magnetic field $B$, this corresponds to $B = 0$) or isomorphic to the module $E_{n,m}$ for some integers $n$ and $m$. The elements of the $T^2_\theta$-module $E_{n,m}$ will be Schwartz class functions $\phi_j(x) \in \mathcal{S}(\mathbb{R} \times \mathbb{Z}_m), x \in \mathbb{R}, j \in \mathbb{Z}_m$, with $\mathbb{Z}_m$ the cyclic group of order $m$. The action of the generators $U_i$ of $\phi_j(x)$ is given by

$$\begin{align*}
(U_1 \phi)_j(x) &= \phi_{j-1}(x - \frac{n}{m} - \theta), \\
(U_2 \phi)_j(x) &= e^{2\pi i (x - j n/m)} \phi_j(x),
\end{align*}$$

(6a)

(6b)

with some integer $n$.

The projective modules are classified by their $K$ theory group which is the rank two abelian group $\mathbb{Z}^2$. The set of classes of actual finite projective modules is the cone of positive elements, which for $\mathbb{Z}^2$ is

$$\{(x, y) \in \mathbb{Z}^2; x + \theta y > 0\}.$$
For $E_{n,m}$ the coordinates $x$ and $y$ are

\[ x = \sigma n, \quad y = \sigma m, \tag{8} \]

with $\sigma = \text{sgn}(n + m\theta)$ (c.f. [6]).

A magnetic field $B$ perpendicular to the torus is introduced via minimal coupling, i.e. the canonical momenta $p_j = -i\partial_j$ are replaced by kinetic ones $\pi_j = -i\nabla_j$, with $\nabla_j$ a connection on the torus with non-vanishing curvature such that

\[ [\pi_1, \pi_2] = iB. \tag{9} \]

On $E_{n,m}$ one can always construct connections with constant curvature:

\[ \nabla_1^{(0)} = \frac{im}{n + m\theta} x, \quad \nabla_2^{(0)} = \frac{1}{2\pi} \frac{\partial}{\partial x}, \tag{10} \]

satisfying $[\nabla_i^{(0)}, U_l] = i\delta_{ki}U_l$.

Next consider an algebra automorphism $T_\alpha$ of translations by $2\pi\alpha_1$ and $2\pi\alpha_2$ on $T_\theta^2$, defined by

\begin{align*}
U_1 &\rightarrow T_\alpha(U_1) = e^{2\pi i\alpha_1}U_1 \tag{11a} \\
U_2 &\rightarrow T_\alpha(U_2) = e^{2\pi i\alpha_2}U_2. \tag{11b}
\end{align*}

These automorphisms are inner iff $\alpha_i = k_i\theta + n_i$, $k_i, n_i \in \mathbb{Z}$. On $E_{n,m}$ this automorphism is represented by the adjointed action of the operator $T(\alpha)$, with

\[ (T(\alpha)\phi)_j = e^{-2\pi i\alpha_1} \frac{m}{n + m\theta} \phi_j(x - \alpha_2). \tag{12} \]

The action on the connections $\nabla_i^{(0)}$ is given by

\begin{align*}
\nabla_1^{(0)} &\rightarrow \nabla_1^{(\alpha)} = T_\alpha(\nabla_1^{(0)}) = \nabla_1^{(0)} + i\alpha_1 \frac{m}{n + m\theta}, \tag{13a} \\
\nabla_2^{(0)} &\rightarrow \nabla_2^{(\alpha)} = T_\alpha(\nabla_2^{(0)}) = \nabla_2^{(0)} - i\alpha_2 \frac{m}{n + m\theta}. \tag{13b}
\end{align*}

A review of the $T_\theta^2$-modules $E_{n,m}$ can be found in [7] for example. Using equation (9) one easily calculates a $\theta$-deformed quantization condition for the magnetic field strength $B$:

\[ 2\pi B = \frac{m}{n + m\theta}, \tag{14} \]

(see also [2] for example).
3 Representation of the Projected $U_i$

Using the representation of the kinetic momenta $\pi_j = -i \nabla_j^{(a)}$ on $T_\theta^2$, introduced in the previous section, we can calculate the eigenfunctions $\psi_{\mu,j}(x)$ of the free Landau Hamiltonian

$$\mathcal{H}_0 = \frac{1}{2}(\pi_1^2 + \pi_2^2),$$

$$\psi_{\mu,j}(x) = \langle x | \mu, j \rangle = \psi_\mu(x) \otimes \hat{e}_j = \mathcal{N}_\mu e^{-\frac{2\pi i}{n+m\theta j}(x - \alpha_2)} \otimes \hat{e}_j,$$

with $\phi_\mu(x) = e^{-x^2/2}H_{\mu}(x), \mathcal{N}_\mu = (2\mu-1)!^{-1/2} (2(n/m + \theta))^{-1/4}, H_{\mu}(x)$ the $\mu$th Hermite polynomial and $\hat{e}_j = \hat{e}_{j+m}$ the $(j \mod m)$th unit vector of $\mathbb{R}^m$. The eigenfunctions are orthonormal with respect to the scalar product

$$\langle \mu, j | \mu', j' \rangle = \left( \int_{-\infty}^{\infty} dx \psi_\mu^*(x) \psi_{\mu'}(x) \right) \otimes (\hat{e}_j \cdot \hat{e}_{j'}) = \delta_{\mu,\mu'}\delta_{j,j'}.$$  

Since $\psi_{\mu,j+m}(x) = \psi_\mu(x)$ each Landau level $\mu$ is spanned by $m$ orthonormal eigenfunctions $\psi_{\mu,j}(x), j = 0, \ldots, m-1$, i.e., each wave-function $\psi_\mu(x)$ of the $\mu$th Landau level can be represented as an $m$-dimensional vector $\tilde{c} = (c_0, c_1, \ldots, c_{m-1})$ by $\psi_\mu(x) = \sum_{j=0}^{m-1} x_j \langle x | \mu, j \rangle$. 

For fixed $\alpha_i$ and using the projector $P_\mu = \sum_{j=0}^{m-1} |\mu, j\rangle \langle \mu, j|$ onto the $\mu$th Landau level we get an $m$-dimensional representation $\rho^{(m,m)}$ of the projected generators of $T_\theta^2$, $U_1^{(\mu)} = P_\mu U_1 P_\mu$:

$$\rho^{(m,m)}(U_1^{(\mu)})_{j,j'} = c_\mu(\alpha_1) \delta_{j,(j'+1) \mod m}$$  

$$\rho^{(m,m)}(U_2^{(\mu)})_{j,j'} = c_\mu(\alpha_2) e^{-\frac{2\pi i}{\theta j} \hat{e}_j} \delta_{j,j'},$$

with $c_\mu(\alpha) = e^{-1/2\theta |\alpha| + 2\pi i \alpha / 2B} L_{\mu}(1/(2B))$ and $L_{\mu}$ the $\mu$th Laguerre polynomial. The commutation relation of the projected generators $U_1^{(\mu)}$ then reads

$$U_1^{(\mu)} U_2^{(\mu)} = e^{\frac{2\pi i}{\theta} \hat{e}_j} U_2^{(\mu)} U_1^{(\mu)} = e^\frac{4\pi i}{\hbar \theta} U_2^{(\mu)} U_1^{(\mu)}.$$

The representation of the (unprojected) $U_i$ is infinite dimensional since $\theta$ is irrational. There are infinitely many Landau levels, labeled by $\mu$, each
of which is \( m \)-fold degenerated. By projecting onto a finite dimensional subspace of the representation space, i.e. onto one Landau level, the \( U_i^{(\nu)} \) become finite dimensional matrices with some modified commutation relation (19).

The representation \( \rho^{(m,n)} \) is irreducible iff \( m \) and \( n \) are relatively prime. For \( \gcd(m, n) = d \), \( m' = m/d \) and \( n' = n/d \), \( \rho^{(m,n)} \) decomposes into \( d \) \( m' \)-dimensional representations \( \rho_0^{(m',n';j)} \), with

\[
\begin{align*}
\rho_0^{(m',n';j)}(U^{(\nu)}_1) &= e^{2\pi i \frac{j}{m'}} \rho^{(m',n')}((U^{(\nu)}_1)^j) \\
\rho_0^{(m',n';j)}(U^{(\nu)}_2) &= \rho^{(m',n')}((U^{(\nu)}_2)^j)
\end{align*}
\]

and \( j = 0, \ldots, d - 1 \).

4 Different Representation of \( U_i \)

Next we consider a different representation of \( T^2_\theta \) on the space of smooth functions over \( \mathbb{R}^2 \) motivated by physics, with

\[
\begin{align*}
(U_1 \psi)(x, y) &= e^{ix} \psi(x, y - \pi \theta), \quad \text{and} \quad (21a) \\
(U_2 \psi)(x, y) &= e^{iy} \psi(x + \pi \theta, y). \quad (21b)
\end{align*}
\]

We require special boundary conditions on \( \psi \):

\[
\begin{align*}
\psi(x + 2\pi n, y) &= e^{2\pi i \delta_1} \psi(x, y), \quad (22a) \\
\psi(x, y + n\Lambda) &= e^{2\pi i \delta_2 - imx} \psi(x, y), \quad (22b)
\end{align*}
\]

with \( \Lambda = 2\pi(1 + \frac{m\theta}{2\pi}) \). Equations (22) result from the study of magnetic translation operators (see below).

The kinetic momenta \( \pi_i \) will be represented (up to some gauge transformation) as

\[
\begin{align*}
(\pi_1 \psi)(x, y) &= (-i(1 - \pi \theta B)\partial_x + By)\psi(x, y) \quad \text{and} \quad (23a) \\
(\pi_2 \psi)(x, y) &= -i\partial_y \psi(x, y). \quad (23b)
\end{align*}
\]

In the representation (21), the commutant of \( T^2_\theta \) is generated by four
elements $Z_i, i = 1, \ldots, 4$ represented as

\[
(Z_1 \psi)(x, y) = e^{\frac{i}{n} \psi(x, y + \frac{\pi \theta}{n})}, \quad (24a)
\]

\[
(Z_2 \psi)(x, y) = e^{-\frac{iy + \pi \theta}{n}} \psi(x, y + \frac{\pi \theta}{n + m \theta}, y), \quad (24b)
\]

\[
(Z_3 \psi)(x, y) = \psi(x + 2\pi, y), \quad \text{and (24c)}
\]

\[
(Z_4 \psi)(x, y) = e^{i\frac{2\pi}{n} \psi(x, y + 2\pi(1 + \frac{m \theta}{2n}))}, \quad (24d)
\]

which fulfill the commutation relations $Z_k Z_i = e^{2\pi i \delta_{ki}} Z_i Z_k$, with

\[
\Theta = \frac{1}{n} \begin{pmatrix}
0 & -\frac{\delta}{n+m\delta} & -1 & 0 \\
\frac{\delta}{n+m\delta} & 0 & 0 & 1 \\
1 & 0 & 0 & -m \\
0 & -1 & m & 0
\end{pmatrix}. \quad (25)
\]

The entries of the matrix $\Theta$ have to be taken mod $\mathbb{Z}$, since they only appear in the exponent.

The magnetic translation operators $T_i, i = 1, 2$ are chosen such that they leave the Hamiltonian (15) invariant, i.e. they must commute with $\pi_i$. In the present gauge (23) this gives two generators of magnetic translations by the periods of the torus, $T_1 = Z_3$ and $T_2 = Z_4$.

Using a potential generated by $U_i$, the Hamiltonian and thus the physical setup is invariant under a translation by $2\pi n_1$ and $2\pi n_2$, respectively. Therefore we have $T_i^{n_i} \sim I$. This physical requirement yields a quantization of the magnetic flux per unit cell:

\[
2\pi \frac{2\pi n B}{1 - 2\pi B \delta} = 2\pi m \quad \Rightarrow \quad 2\pi B = \frac{m}{n + m \delta}, \quad (26)
\]

with some integer $m$ and $n = \gcd(n_1, n_2)$. Using this quantization condition we see that $T_1^n$ and $T_2^n$ commute with all the other operators and thus lie in the center of $T_0^\delta$. From the form of $\Theta$ in equation (25) we see that for $\delta$ irrational, the center of $T_0^\delta$ is generated by these two operators. In the representation (24), $T_1^n = e^{2\pi i \delta n \theta}$ and $T_2^n = e^{2\pi i \delta n \theta}$ due to the boundary conditions (22). Thus the representation of the center of $T_0^\delta$ is trivial for $\delta$ irrational. For $\theta \in \mathbb{Q}$, the center of $T_0^\delta$ is generated by $T_1^n (= Z_3^n), T_2^n (= Z_4^n), Z_1^\theta$ and $Z_2^\theta$, where $q$ is some integer depending on $\theta, m$ and $n$. Thus the center no longer has a trivial representation.
The energy spectrum of the free Landau-Hamiltonian (15) with the kinetic momenta \( \pi_\ell \) of (23) is given by \( \varepsilon_\mu = B(\mu + \frac{1}{2}) \), where \( \mu \in \mathbb{N} \) labels the Landau levels. The corresponding eigenfunctions of the \( \mu \)th Landau level have to be a superposition of functions \( \psi_{\mu,k} \)

\[
\psi_{\mu,k}(x, y) = N_\mu e^{i k x} e^{-\frac{B}{2}(y + \frac{1}{\pi \theta})^2 H_\mu(\sqrt{B}(y + \frac{1}{B} - \pi \theta) k)}, \tag{27}
\]

with \( N_\mu = (2^{2\mu} \pi B^{1/2} / B)^{-1/4} \).

Using the first boundary condition (22a), we get \( k = (l + \delta_1)/n \), with \( l \in \mathbb{Z} \). Thus any wave-function \( \Psi(x, y) \) can be written as

\[
\Psi(x, y) = \sum_{l=-\infty}^{\infty} c_l e^{i \frac{1 + 4 \delta_1}{\pi n} \phi_l(y), \tag{28}
\]

with \( \phi_l(y) = \phi(\sqrt{B}(y + \frac{1}{B} - \pi \theta) \frac{k_m + r + \delta_1}{n}) = \phi_{l+nm}(y - n \Lambda) \). The second boundary condition (22b) gives \( c_{l+nm} = e^{2\pi i \delta_2} c_l \). Replacing \( c_l = e^{-\frac{2\pi i \delta_2}{m} \phi_l} \), the \( \mu \)th Landau level is spanned by \( mn \) eigenfunctions

\[
\psi_{\mu,r}(x, y) = \langle x, y | \mu, r \rangle = N_\mu \sum_{k=-\infty}^{\infty} e^{-2\pi i \frac{km + \frac{r + \delta_1}{n}}{mn}} e^{i (km + \frac{r + \delta_1}{n}) x} \phi_{\mu}(\sqrt{B}(y + \frac{1}{B} - \pi \theta) \frac{k_m + r + \delta_1}{n}), \tag{29}
\]

with \( r = 0, \ldots, mn - 1, N_\mu = (2^{2\mu} \pi B^{1/2} / B)^{-1/4}, \phi_\mu(y) = e^{-y^2 / 2} H_\mu(y) \) and \( H_\mu(y) \) the \( \mu \)th Hermite polynomial. These eigenfunctions are orthonormal with respect to the scalar product

\[
\langle \mu, r | \mu', r' \rangle = \int_{-\infty}^{\infty} \frac{dx}{2\pi n} \int_0^{n \Lambda} dy \psi_{\mu,r}^*(x, y) \psi_{\mu',r'}(x, y) = \delta_{\mu,\mu'} \delta_{r,r'}. \tag{30}
\]

Translations by \( \vec{a} = (a_x, a_y) \) on the torus are given by an operator \( T(\vec{a}) \) with \( T(\vec{a})U_j = e^{i \alpha_j} U_j T(\vec{a}) \) and commutating with the kinetic momenta. In the presence of a magnetic field \( B \), these two conditions on commutation relations with \( U_i \) and \( \pi_i \) given in (21) and (23), respectively, yield a representation of the magnetic translation operator of the form

\[
(T(\vec{a})\psi)(x, y) = e^{-i \frac{B}{2\pi B \theta} \frac{a_y}{y} \psi(x - a_x, y - \frac{1 - \pi \theta B}{1 - 2\pi \theta B} a_y), \tag{31}
\]

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For $B \neq 0$ translations by an arbitrary vector $\vec{a}$ do not leave the space of sections, satisfying (22) for fixed $\delta$, invariant (as this would be the case for $B = 0$), as can be seen from the commutation relation

$$T_1^{n_1} T_2^{n_2} T(\vec{a}) = e^{-im(\delta_1, -\delta_2)} T(\vec{a}) T_1^{n_1} T_2^{n_2},$$  \hspace{1cm} (32)$$
with $k_i \in \mathbb{Z}$. One rather has to demand $\vec{a} = \frac{2\pi i}{m} (n_1, n_2)$ with $n_i \in \mathbb{Z}$ (cf. [8, Sec. 6]) for the commutative ($\theta = 0$) case.

Analog to the previous section, a projection onto the $\mu$th Landau level, using the projection operator $P_\mu = \sum_{r=0}^{m-1} |\mu, r\rangle \langle \mu, r|$ yields a $mn$-dimensional representation $\tilde{\rho}^{(m,n)}$ of the projected generators of $T_\theta^2, U_1^{(\mu)} = P_\mu U_1 P_\mu$:

$$\tilde{\rho}^{(m,n)}(U_1^{(\mu)})_{j,j'} = c_\mu(\frac{\delta_2}{m})\delta_{j,(j+n)} \mod mn$$  \hspace{1cm} (33a)$$
$$\tilde{\rho}^{(m,n)}(U_2^{(\mu)})_{j,j'} = c_\mu(-\frac{\delta_1}{m})e^{-\frac{2\pi i}{m}j\cdot j'},$$  \hspace{1cm} (33b)$$
with $c_\mu(\alpha) = e^{-1/(4B) + \frac{2\pi \alpha}{L_\mu}(1/(2B))}$ and $L_\mu$ the $\mu$th Laguerre polynomial. Again we choose fixed phases $\delta_i$. The representations $\rho^{(m,n)}$ are reducible and reduce to $n$ $m$-dimensional representations $\rho^{(m,n)}_{1^{(m,n)}}$, with

$$\rho^{(m,n)}_{1^{(m,n)}}(U_1^{(\mu)}) = \rho^{(m,n)}(U_1^{(\mu)}),$$  \hspace{1cm} (34a)$$
$$\rho^{(m,n)}_{1^{(m,n)}}(U_2^{(\mu)}) = e^{2\pi i \frac{\lambda}{n'}}\rho^{(m,n)}(U_2^{(\mu)}),$$  \hspace{1cm} (34b)$$
$j = 0, \ldots n - 1$ and $\rho^{(m,n)}$ given in (18). To be consistent with the definition of $\rho^{(m,n)}$ we have to replace $\alpha_1 \rightarrow \delta_2/m$ and $\alpha_2 \rightarrow -\delta_1/m$. For the sake of simplicity we set $c_\mu(\alpha) = 1$, keeping in mind, that we have to reintroduce this factor at the end of the calculations.

For $m$ and $n$ relatively prime, the representations $\tilde{\rho}^{(m,n)}_{1^{(m,n)}}$ are irreducible and unitary equivalent to $\rho^{(m,n)}$. For $\gcd(m, n) = d, m' = m/d$ and $n' = n/d$, $\tilde{\rho}^{(m,n)}_{1^{(m,n)}}$ decomposes into $d m'$-dimensional representations $\rho^{(m',n';\lambda)}_{1^{(m',n';\lambda)}}$, with

$$\rho^{(m',n';\lambda)}_{0^{(m',n';\lambda)}}(U_1^{(\mu)}) = e^{2\pi i \frac{\lambda}{n'}}\rho^{(m',n';\lambda)}_{1^{(m',n';\lambda)}}(U_1^{(\mu)}) = e^{2\pi i \frac{\lambda}{n'}}\rho^{(m',n';\lambda)}_{0^{(m',n';\lambda)}}(U_1^{(\mu)}),$$  \hspace{1cm} (35a)$$
$$\rho^{(m',n';\lambda)}_{1^{(m',n';\lambda)}}(U_2^{(\mu)}) = \rho^{(m',n';\lambda)}_{1^{(m',n';\lambda)}}(U_2^{(\mu)}) = e^{2\pi i \frac{\lambda}{m'n'}}\rho^{(m,n;\lambda';\lambda)}_{0^{(m,n;\lambda';\lambda)}}(U_2^{(\mu)}),$$  \hspace{1cm} (35b)$$
and $j = 0, \ldots dn - 1$ and $j' = 0, d - 1$. It is easy to see that $\rho^{(m,n;\lambda';\lambda)}_{0^{(m,n;\lambda';\lambda)}} \cong \rho^{(m,n;\lambda+1,\lambda')}_{1^{(m,n;\lambda+1,\lambda')}}$. Thus any (reducible) representation $\rho^{(m',n;d,n'd)}$, with $m$ and $n$
relatively prime, decomposes into a direct sum of irreducible representations
\[\hat{\rho}^{(md, nd)} = \bigoplus_{\nu=0}^{n-1} \bigoplus_{j=0}^{d-1} e^{2\pi i \frac{\nu}{nd}} \bigoplus_{j'=0}^{d-1} \rho^{(m, n; \lambda', \nu)}\],
with \[\rho^{(m, n; \lambda)}\] given in (20).

5 Energy Corrections Due to a Periodic Potential

Using the results of the previous sections, we calculate the energy corrections
to the \(\mu\)th Landau level due to a small periodic perturbation \(V(x_1, x_2)\) of the
free Hamiltonian (15). Provided the perturbation is small compared to the
energy gap between two different Landau levels, i.e. \(V\) does not mix between
states of two different Landau levels, we can use degenerate perturbation
theory up to first order. The corrections to the \(\mu\)th Landau level are obtained
by the eigenvalues of \(V\) projected onto this level.

Assume a simple periodic potential
\[V(x_1, x_2) = 2v(\cos x_1 + \cos x_2) = v(U_1 + U_1^\dagger + U_2 + U_2^\dagger),\] (37)

Using the representation \(\rho^{(m, n)}\) of (18), the projected potential \(V^{(\mu)} = P_\mu V P_\mu\)
is represented as an \(m \times m\)-dimensional matrix \(V^{(\mu m, n)} := \rho^{(m, n)}(V^{(\mu)})\). For
\(\gcd(m, n) = d > 1\), this matrix decomposes into \(d \frac{m}{d}\)-dimensional matrices
\[V^{(\mu m', n'; j/d)} := \rho^{(m', n'; j/d)}(V^{(\mu)}),\] with \(m' = m/d\) and \(n' = n/d\). In the follow-
ing we always assume \(m\) and \(n\) relatively prime and write \(md\) and \(nd\)
if we want to express that they have a common divisor \(d\). The matrices
\(V^{(\mu m', n'; j/d)}\), \(j = 0, \ldots, d-1\), have the explicit form
\[\left(V^{(\mu m', n; \lambda)}\right)_{r, r'} = c_\mu e^{2\pi i \frac{\lambda + 1}{m+1}} \delta_{r, (r+1) \mod m} + e^{-2\pi i \frac{\lambda + 1}{m+1}} \delta_{r, (r-1) \mod m}
+ 2 \cos\left(2\pi \frac{n}{m}r - 2\pi \alpha_2\right) \delta_{r, r'},\] (38)

with \(r, r' = 0, \ldots, m-1\), \(c_\mu = e^{-1/(4B)}L_\mu(1/(2B))\) and \(L_\mu\) the \(\mu\)th Laguerre
polynomial. According to Wilkinson [9] the eigenvalues \(\varepsilon\) of \(V^{(\mu m', n'; j/d)}\) are
obtained by an equation of the form
\[P(\varepsilon) = \cos 2\pi \left(\frac{j}{d} + m\alpha_1\right) + \cos 2\pi m\alpha_2,\] (39)
with \( P(\varepsilon) \) some \( m \)th order polynomial, independent of \( \alpha_i \) and \( j \).

Using the representation \( \tilde{\rho}^{(m,n)} \) of (33), the corresponding \( d^2mn \)-dimensional matrix \( V(\mu;m,d,n) := \tilde{\rho}^{(m,d,n)}(V(\mu)) \) decomposes into \( d^2n \) \( m \)-dimensional matrices \( V_0(\mu;m,n;j/d,j'/d) \), \( j, j' = 0, \ldots d - 1 \), with

\[
(V_0(\mu;m,n;j/d,j'/d))_{r,r'} = e^{-\frac{2\pi i}{m}(j+\delta_j)}\delta_{r,(r'+1) \mod m} + e^{-\frac{2\pi i}{m}(j-j')}\delta_{r,(r'-1) \mod m} + 2\cos(2\pi \frac{n}{m} r - 2\pi \frac{j'-\delta_1}{md})\delta_{r,r'}. \tag{40}
\]

Using the result of Wilkinson [9] again, one gets an equation of the form

\[
P(\varepsilon) = \cos \frac{2\pi}{d} (j + \delta_j) + \cos \frac{2\pi}{d} (j' - \delta_1). \tag{41}
\]

From equation (41) it follows that we can get rid of rational phases \( \delta_j = \frac{p_j}{q_j} \) in the boundary conditions (22), choosing a \( q \)-times enlarged super-cell, i.e. replacing \( d \to q_d \) and choosing \( j = p_1 \) and \( j' = p_2 \), respectively, which is clear from a physical point of view.

There is another fact we want to point out. To get the energy corrections to the \( \mu \)th Landau level due to the perturbation (37) for a fixed magnetic field, i.e. a fixed ratio \( \frac{m}{n} \), one has to calculate all eigenvalues of the matrices \( V(\mu;m,d,n) \) and \( V(\mu;m,d,n) \), respectively, for \( d = 1, 2, \ldots \). Using representation (40) and taking \( j, j' = 0, \ldots d - 1 \), with \( d \to \infty \), the (rational part of) the Hofstadter butterfly spectrum is obtained. By the representation (38) we take only \( j = 0, \ldots d - 1, d \to \infty \) and obtain only half of the values of \( P(\varepsilon) \) of equation (41). This is shown in figure 1 for \( \alpha = 0 \) (a) and \( \alpha = \frac{1}{2m} \) (b). Since we imposed toroidal boundary conditions on the configuration space, the magnetic flux is some rational number. So only the rational part, i.e. the part belonging to rational fluxes of the butterfly arises.

In either representation, the spectrum for fixed \( \alpha_i \), \( \delta_i \) is a pure point spectrum, which splits into \( m \) separate parts, in each of which the eigenvalues \( \varepsilon \) lie dense (for \( d \to \infty \)). In figure 2 the spectrum of \( V(\mu;m,d,n) \) for \( d \to \infty \) is plotted in units of the band width, \( \varepsilon = \frac{1}{2\pi} L_\mu \left( \frac{1}{2B} \right) \). One sees that the \( \theta \)-deformation of the underlying manifold has no effect on the spectrum (besides a rescaling \( B \to \frac{B}{1 - 2\pi nB} \)).
References


\[(q) \quad \frac{\nu q}{\tau} = \bar{\nu} \text{ and } \quad (v) \quad \theta = \bar{\nu} \text{ and } \quad \infty \leftarrow p \cdot w \cdots I = u \cdot \bar{\nu} \cdots I = w\]

For \( \frac{\nu q}{\tau} I \quad \bar{\nu} \text{ in units of the band width } \epsilon \)
Figure 2: Spectrum of $\sqrt{V_{\mu m, d}}$ in units of the band width $v e^{-\frac{\pi}{2\eta}} f_{\nu}(\frac{1}{2\eta})$ for $m = 1, \ldots 15, n = 1, \ldots m$ and $d \to \infty$. 