Transversally Harmonic Maps
Between Manifolds with Riemannian Foliations

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Abstract

We consider leaf preserving maps between manifolds equipped with Riemannian foliations. We construct a transversal tension field for such maps and define transversally harmonic maps. Then we give some examples of such maps using the suspension construction.

1 Introduction

Let \((M_1, \mathcal{F}_1), (M_2, \mathcal{F}_2)\) be two manifolds with Riemannian foliations and \(f : M_1 \to M_2\) a smooth map preserving the leaves. A Riemannian foliation is defined locally by Riemannian submersions. Let \(U_1\) be an open subset in

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$M_1$ and $U_1 \to \mathcal{U}_1$ a Riemannian submersion defining locally the foliation $\mathcal{F}_1$. Let $U_2$ be an open subset of $M_2$ and $U_2 \to \mathcal{U}_2$ a Riemannian submersion defining locally the foliation $\mathcal{F}_2$. These open sets may be taken in such a way that $f(U_1) \subset U_2$. Then there exists the map $\overline{f}: \mathcal{U}_1 \to \mathcal{U}_2$ such that the following diagram

$$
\begin{array}{ccc}
U_1 & \xrightarrow{f} & U_2 \\
\downarrow & & \downarrow \\
\mathcal{U}_1 & \xrightarrow{\overline{f}} & \mathcal{U}_2 
\end{array}
$$

commutes. It is natural from the point of view of foliations to define transversally harmonic maps as such that each induced map $\overline{f}$ is harmonic. Such a definition does not depend on the choice of local Riemannian submersion defining the Riemannian foliation. This follows from the fact that the holonomy pseudogroup associated with the Riemannian foliations consists of local isometries, cf. section 2.2. In the present paper we approach this problem from another point of view. We construct a global transversal tension field of a map $f$ and then define transversally harmonic maps as those maps for which this tension field vanishes. Many questions about harmonic maps posed by Eells and Sampson in their celebrated paper, cf. [7], may be reformulated for the transversally harmonic maps; for instance, the question of a representation of a homotopy class of a continuous foliated map by a transversally harmonic map and many other question which were and have been under research during last years.

We give many examples of transversally harmonic maps in the last section of our paper. In particular we apply the suspension method to obtain transversally harmonic maps. As partial results we get also examples of harmonic maps and harmonic morphisms. That happens when we apply the suspension procedure to the maps which are harmonic morphisms and harmonic immersions.

The subject of harmonic maps between foliated manifolds was considered by a few authors, cf. [9], [8] and the references there, though usually in a different context: for instance, when the foliation in one of the manifolds consists of points.

The inspiration of our paper is the R.T. Smith thesis, cf. [20, 4] where, among other interesting ideas, there is proved the so called reduction theorem for harmonic maps. This theorem relates the harmonicity of an equivariant map, with respect the an action of Lie groups, to the harmonicity of the
induced map between the spaces of orbits.

We can look at transversally harmonic maps in another light. The space of leaves of a Riemannian foliation can be very complicated topological space. Any foliated map $f$ induces a continuous map $\overline{f}$ between the leaf spaces. Therefore it is impossible to develop any geometry on such a space in a direct way. One of the approaches is to define geometrical objects on the leaf space as foliated objects on the foliated manifold. So we can look at foliated manifolds as desingularisation of leaf spaces.

2 Preliminaries

In the section we gather some useful facts about harmonic maps, foliations and Riemannian foliations in particular. In the last part of the section we study the foliated geometry of the natural bundle of a Riemannian foliation in the spirit of P. Molino, which was developped by the second author in [22], cf. also [23].

2.1 Harmonic maps - basic definitions

If $f : M_1 \rightarrow M_2$ is a differentiable map between Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$ then the differential $df$ is a section of the bundle $T^*M_1 \otimes f^{-1}TM_2 \rightarrow M_1$ where $f^{-1}TM_2$ denotes the pull-back bundle via the map $f$. The bundle $T^*M_1 \otimes f^{-1}TM_2 \rightarrow M_1$ carries the connection induced by the Levi-Civita connections on $M_1$ and $M_2$. Then the map $f$ is said to be harmonic if and only if $\text{trace} \nabla df = 0$; the trace is calculated with respect to the Riemannian metric on $M_1$. Therefore harmonic maps may be defined as critical points of the energy functional. In fact, let $\epsilon(f) = \frac{1}{2} \| df \|^2$ be the smooth function on $M_1$ which to any point $x$ of $M_1$ associates the Hilbert-Schmidt norm of the differential at that point. The function $\epsilon(f)$ is called the energy density and it is a Lagrangian which arises naturally in the Riemannian geometry, cf. [15]. If $M_1$ is compact then the energy of $f$ is defined as

$$E(f) := \int_{M_1} \epsilon(f) d\nu_{g_1}$$

where $d\nu_{g_1}$ is the measure determined by the Riemannian metric $g_1$ on $M_1$. Then $f$ is harmonic if and only if it is a critical point of the energy functional. If $M_1$ is not compact then a harmonic map is a critical point of the energy

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with respect to the compactly supported variations. The map \( \text{trace} \nabla df \) is called the tension field of \( f \) and usually is denoted by \( \tau(f) \). The equation \( \text{trace} \nabla df = 0 \) is the Euler-Lagrange equation of the variational problem defined by the energy functional.

All the above definitions apply to the case when the manifolds considered are pseudo-Riemannian. The main difference is that the Euler-Lagrange equations are not any more of the elliptic but of the hyperbolic type. For details about harmonic maps, cf. [7, 4, 5, 6].

### 2.2 Foliations—basic definitions

Let \( \mathcal{F} \) be a foliation on a Riemannian n-manifold \((M, g)\). Then \( \mathcal{F} \) is defined by a cocycle \( U = \{U_i, f_i, g_{ij}\}_{i \in I} \) modeled on a q-manifold \( N_0 \) where

- \( \{U_i\}_{i \in I} \) is an open covering of \( M \);
- \( f_i: U_i \to N_0 \) are submersions with connected fibers;
- \( g_{ij}: N_0 \to N_0 \) are local diffeomorphisms of \( N_0 \) such that \( f_i = g_{ij}f_j \) on \( U_i \cap U_j \).

The connected components of the trace of any leaf of \( \mathcal{F} \) on \( U_i \) consist of fibers of \( f_i \). The open subsets \( N_i = f_i(U_i) \subset N_0 \) form a q-manifold \( N = \bigsqcup N_i \) which can be considered to be a transverse manifold for the foliation \( \mathcal{F} \). The pseudogroup \( \mathcal{H}_N \) of local diffeomorphisms of \( N \) generated by \( g_{ij} \) is called the holonomy pseudogroup of the foliated manifold defined by the cocycle \( U \).

If we have an atlas of \( M \) consisting of charts adapted to the foliation \( \mathcal{F} \), i.e. \((U, \varphi), \varphi: U \to \mathbb{R}^p \times \mathbb{R}^q \) such that \( \varphi = (\varphi^1, \varphi^2) = (x_1, \ldots, x_p, y_1, \ldots, y_q) \) and the vector fields \( \partial/\partial x_1, \ldots, \partial/\partial x_p \) span the bundle \( T\mathcal{F} \) tangent to the leaves of the foliation \( \mathcal{F} \), then the role of a submersion defining \( \mathcal{F} \) on \( U \) is played by \( \varphi^2: U \to \mathbb{R}^q \).

A vector field \( X \) on \( M \) is called an infinitesimal automorphism (i.a. for short) of \( \mathcal{F} \) if and only if \([X, Y] \in T\mathcal{F} \) for any vector field \( Y \in T\mathcal{F} \).

Let \( T\mathcal{F}^\perp \) be a subbundle supplementary to the tangent bundle \( T\mathcal{F} \). If there is a Riemannian metric on \( M \), then \( T\mathcal{F}^\perp \) can be the orthogonal complement of \( T\mathcal{F} \) with respect to this Riemannian metric.

Let a section \( X \) of \( T\mathcal{F}^\perp \) on \( U_i \) be an i.a. of \( \mathcal{F} \), then it defines a unique vector field \( \bar{X} \) on \( N_i \) such that \( df_i(X) = \bar{X} \). And vice versa any vector field \( \bar{X} \) on \( N_i \) defines a section \( X \) of \( T\mathcal{F}^\perp \) on \( U_i \) such that \( df_i(X) = \bar{X} \), which is
an i.a. of $\mathcal{F}$. For any open subset $U$ of $M$ the set of such i.a. on $U$ is denoted by $\Gamma_i(U)$.

It is a consequence of the fact that the bundle map

$$df_i|T\mathcal{F}^\perp: T\mathcal{F}^\perp \to TN_i$$

is a linear isomorphism on each fiber. So i.a. of $\mathcal{F}$ which are section of $T\mathcal{F}^\perp$ are basic (foliated) sections of this bundle.

Likewise the dual bundle $(T\mathcal{F}^\perp)^*$ to $T\mathcal{F}^\perp$ is fibre-wise isomorphic to $TN^*$ via $(df_i)^*$. Therefore both vector bundles $T\mathcal{F}^\perp$ and $(T\mathcal{F}^\perp)^*$ are foliated by the foliations $\mathcal{F}^\perp$ and $\mathcal{F}^{\perp*}$, respectively, whose leaves are covering spaces of leaves of $\mathcal{F}$ and the submersions $df_i|T\mathcal{F}^\perp$ and $((df_i)^*)^{-1}|(T\mathcal{F}^\perp)^*$ are constant on the leaves of the corresponding foliations, cf. [22]. This implies that any tensor product of these bundles is also foliated by a foliation whose leaves are covering spaces of leaves of $\mathcal{F}$. The details can be found in the above mentioned paper of the second author. Therefore foliated sections of $(T\mathcal{F}^\perp, \mathcal{F}^\perp)$ are just basic orthogonal vector fields and foliated sections of $(T\mathcal{F}^{\perp*}, \mathcal{F}^{\perp*})$ are basic 1-forms.

We recall that a smooth function $f: M \to \mathbb{R}$ is $\mathcal{F}$-basic if and only if it is constant on leaves, which is equivalent to the condition $df(X) = 0$ for any vector $X \in T\mathcal{F}$ (tangent to leaves of the foliation $\mathcal{F}$).

### 2.3 Riemannian foliations

Suppose that $(M, \mathcal{F}, g)$ is a Riemannian foliation. It means that locally there exist a family of Riemannian submersions $\varphi : (U, g) \to (\overline{U}_\varphi, \overline{g}_\varphi)$ such that $U$ is open in $M$, $\overline{g}_\varphi$ is a Riemannian metric on a manifold $\overline{U}_\varphi$ and the intersections of the leaves of $\mathcal{F}$ with $U$ are fibres of the submersion $\varphi$. In the language of cocycles it means that there exists a cocycle $\mathcal{U}$ such that the transverse manifold $N$ is a Riemannian manifold, the transformations $g_{ij}$ are local isometries and the submersions $f_i$ are Riemannian.

Then the sheaf $\Gamma_i(\mathcal{F}^\perp)$ of basic sections of the vector bundle $T\mathcal{F}^\perp \to M$ may be described in a simpler way. If $V$ is an open subset of $M$ then $X \in \Gamma_i(V)$ if and only if for each $(U, \varphi)$ local Riemannian submersion defining $\mathcal{F}$ we have that the restriction of $X$ to $U$ is projectable via the map $\varphi$ on a vector field $\overline{X}$ on $\overline{U}$.

**Definition 2.1 (cf. [16]).** A transversal partial connection on $(M, \mathcal{F}, g)$ is
a sheaf operator $D$ such that for each $U$ open subset of $M$

$$D : \Gamma_b(U, T\mathcal{F}^\perp) \times \Gamma_b(U, T\mathcal{F}^\perp) \rightarrow \Gamma_b(U, T\mathcal{F}^\perp)$$

and for each $X, Y, Z \in \Gamma_b(U, T\mathcal{F}^\perp)$ $D_Z$ is $\mathbb{R}$-linear and each $f, h \in C_0^\infty(U)$ we have that

- $D_{fX+hy}Z = fD_XY + hD_XZ$
- $D_X$ is $\mathbb{R}$ linear
- $D_X fY = X(f)Z + fD_XY$ (transversal Leibniz rule)

Let $\nabla$ be the Levi-Civita connection of $g$ then for each $U$ an open subset of $M$ and $X, Y \in \Gamma_b(U, \mathcal{F}^\perp)$ there is defined

$$D_X Y = (\nabla_X Y)\perp.$$

It is easy to verify that $D$ is a partial transversal connection on $(M, g, \mathcal{F})$. Let $(U, \varphi)$ be a Riemannian submersion defining the foliation $\mathcal{F}$ on an open set $U$. Suppose that $X, Y \in \Gamma_b(U, T\mathcal{F}^\perp)$ and $\overline{X}, \overline{Y}$ be the push forward vector fields via the map $\varphi$. Then there is a well known property of Riemannian foliations stating that

$$d\varphi(D_X Y) = (\nabla^\varphi_X Y),$$

(2.1)

cf. [21], where $\nabla^\varphi$ is the Levi-Civita connection of the metric $\varphi$. It is worthwhile to observe that $(\nabla_X Y)\perp$ is a local basic section of $T\mathcal{F}^\perp$.

The partial basic connection has also the following local description. If $X_1, \ldots, X_q$ is a local basis of basic sections of $T\mathcal{F}^\perp \rightarrow M$, then any local basic section $X$ of this bundle may be written as $X = \sum_{a=1}^q a_a X_a$, where $a_1, \ldots, a_q$ are smooth functions on $U$. It is easy to observe that if $X$ is a foliated section, then automatically these functions are basic; in fact

$$a_a = g(X, X_a) = g(\overline{X}, \overline{X}_a) \circ \varphi.$$

Suppose that

$$D_{X_i} X_j = \sum_{a=1}^q \Gamma^k_{ij} X_k$$
then it follows that the functions $\Gamma^k_{ij}$ have to be basic and are projectable onto the Christoffel symbols of the connection $\nabla^{\mathcal{F}}$ of the induced metric $\mathcal{g}$ on $\varphi(U)$ i.e., $\Gamma^k_{ij} = \overline{\Gamma^k_{ij}} \circ \varphi$ where

$$\nabla^{\mathcal{F}}_{X_i} X_j = \sum_{k=1}^{q} \Gamma^k_{ij} X_k.$$ 

Let

$$(T\mathcal{F}^\perp)^* \rightarrow M$$ (2.2)

be the dual bundle to the normal bundle to the foliation. Then we consider the sheaf of the basic sections of the bundle (2.2). Explicitly, the sheaf of the basic sections of (2.2) consists of $X^i$ where $X$ is a basic vector field which is orthogonal to the foliation. If $U$ is open in $M$ then the sections of $\Gamma(U, (T\mathcal{F}^\perp)^*)$ are just basic forms with respect to the foliation $\mathcal{F}$.

The connection $D$ extends in a natural way over the local basic sections of the bundle $(T\mathcal{F}^\perp)^* \rightarrow M$ by requiring the contraction condition: if $\omega \in \Gamma(U, (T\mathcal{F}^\perp)^*)$ and $X, Y \in \Gamma(U, T\mathcal{F}^\perp)$ then $D$ is the unique connection on $\Gamma(U, (T\mathcal{F}^\perp)^*)$ such that $(D_X \omega)Y = X(\omega(Y)) - \omega(D_X Y)$. The partial transversal connection $D$ on $(T\mathcal{F}^\perp)^*$ may be described locally in the following way: if $X, \ldots, X_q$ is a basis of orthonormal basic local sections of the bundle $T\mathcal{F}^\perp$ and

$$D_{X_\alpha} X_\beta = \sum_{\gamma=1}^{q} \Gamma^\gamma_{\alpha\beta} X_\gamma \quad \text{and} \quad D X_\beta = \sum_{\alpha, \gamma=1}^{q} \Gamma^\gamma_{\alpha\beta} X^i_\alpha \otimes X_\gamma$$ (2.3)

for some basic functions $\Gamma^\gamma_{\alpha\beta}$ then

$$D_{X_\alpha} X^i_\beta = -\sum_{\gamma=1}^{q} \Gamma^\gamma_{\alpha\beta} X^i_\gamma \quad \text{and} \quad D X^i_\beta = -\sum_{\alpha, \gamma=1}^{q} \Gamma^\gamma_{\alpha\beta} X^i_\alpha \otimes X^j_\gamma.$$ (2.4)

Moreover $\Gamma^\gamma_{\alpha\beta} = \overline{\Gamma^\gamma_{\alpha\beta}} \circ \varphi$ where $\overline{\Gamma^\gamma_{\alpha\beta}}$ are the Christoffel symbols of the Levi-Civita connection of the metric $\mathcal{g}$ on $\overline{U}$. From the formulas (2.3) it is easy to get the following corollary.

**Lemma 2.1.** If $s$ is a basic local section of $(T\mathcal{F}^\perp)^* \rightarrow M$ then

$$Ds = \sum_{\alpha, \beta=1}^{q} \xi_{\alpha\beta} X^i_\alpha \otimes X^j_\beta$$

where $\xi_{\alpha\beta}$ are local basic functions on $M$. 

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2.4 Basic connection on the pull-back bundle

Let \( (M_i, F_i, g_i) \) be Riemannian manifolds with Riemannian foliations for \( i = 1, 2 \). Let \( \nabla^i \) be the Levi-Civita connections of the respective metrics and \( D^i \) the induced transversal partial connections on the orthogonal complement bundles \( T F_i^\perp \to M_i \) where \( i = 1, 2 \). Suppose that \( f : M_1 \to M_2 \) is a smooth leaf preserving map; this means that \( df(T F_1) \subset T F_2 \). Then there are given natural bundle maps

\[
\pi_i : T F_i^\perp \to TM_i, \quad \Pi_i : TM_i \to T F_i^\perp
\]

for \( i = 1, 2 \), where \( \pi_i \) is the inclusion of \( T F_i^\perp \) in \( TM_i \) and \( \Pi_i \) is the orthogonal projection of \( TM_i \) onto \( T F_i^\perp \). The map \( f \) defines the pull-back bundle \( f^{-1} T F_2^\perp \to M_1 \). Any local basic section of the bundle \( T F_2^\perp \) pulls back, via the map \( f \), to a section of the bundle \( f^{-1} T F_2^\perp \). The set of such pulled sections is a presheaf.

**Definition 2.2.** The sheaf of basic sections of the bundle \( f^{-1} T F_2^\perp \) is the sheaf of sections of this bundle which is generated, over the sheaf of rings of the basic functions over \( M_1 \), by the presheaf of the pulled back basic sections of \( T F_2^\perp \).

Let \( U_i \subset M_i \) be open subsets and let \( \varphi_i : (U_i, g_i) \to (\overline{U}_i, \overline{g}_i) \) be a Riemannian submersions on \( M_i \) which define locally the Riemannian foliation \( F_i \) for \( i = 1, 2 \). Suppose also that \( f(U_1) \subset U_2 \). Let \( X_1, \ldots, X_{r_1} \) and \( Y_1, \ldots, Y_{r_2} \) be the basic sections of \( T F_1^\perp \) over \( U_1 \) and \( T F_2^\perp \) over \( U_2 \), respectively. Then \( X_1, \ldots, X_{r_1} \) are projectable via \( \varphi_1 \) on the frame \( \overline{X}_1, \ldots, \overline{X}_{r_1} \) and \( Y_1, \ldots, Y_{r_2} \) are projectable via the map \( \varphi_2 \) on the frame \( \overline{Y}_1, \ldots, \overline{Y}_{r_2} \). Then we have the following well-known lemma.

**Lemma 2.2.** There exists the unique map \( \overline{f} : \overline{U}_1 \to \overline{U}_2 \) such that the following diagram

\[
\begin{array}{ccc}
U_1 & \xrightarrow{f} & U_2 \\
\varphi_1 \downarrow & & \downarrow \varphi_2 \\
\overline{U}_1 & \xrightarrow{\overline{f}} & \overline{U}_2 
\end{array}
\]  

(2.5)

commutes. \( \square \)
The sheaf of basic sections of the bundle $f^{-1}T\mathcal{F}_2^\perp$ may be described locally in the following way: $\Gamma^1(U, f^{-1}T\mathcal{F}_2^\perp)$ consists of all

$$s = \sum_{\alpha=1}^{g} a_{\alpha} Y_\alpha \circ f$$

where $a_1, \ldots, a_g$ are basic functions on $U_1$.

**Lemma 2.3.** Let $X$ be a local basic section of $T\mathcal{F}_1^\perp \to M_1$ then $\Pi_2 df(X)$ is a basic section of the bundle $f^{-1}T\mathcal{F}_2^\perp$.

**Proof.** There exist smooth functions $a_1, \ldots, a_g$ on $U_1$ such that

$$\Pi_2 df(X) = \sum_{\alpha=1}^{g_2} a_\alpha (Y_\alpha \circ f).$$

We observe that for each $\alpha = 1, \ldots, g_2$ we have that

$$a_\alpha = g_2(\Pi_2 df(X), Y_\alpha \circ f) = \frac{\partial}{\partial x_\alpha} (d_\varphi_2 df(X), d_\varphi_2 (Y_\alpha \circ f)) = \frac{\partial}{\partial x_\alpha} (\overline{df(X)}, \overline{Y_\alpha \circ f}) \circ \varphi_1.$$

Then it follows that $a_\alpha$ are basic functions because they are constant on the leaves of $\mathcal{F}_1$ for $\alpha = 1, \ldots, g_2$. Hence $df(X)$ is a basic section of the bundle $f^{-1}T\mathcal{F}_2^\perp$. \hfill \Box

The basic sections of $f^{-1}T\mathcal{F}_2^\perp$ may be also described in the following way.

**Lemma 2.4.** A local section $s$ of $f^{-1}T\mathcal{F}_2^\perp \to M_1$ is basic if and only if it is locally projectable on a section $\pi$ such that the following diagram

$$\begin{array}{ccc}
\mathcal{F}_1^\perp & \xrightarrow{df_2} & \mathcal{T}^1 U_1 \\
| s & \downarrow & | \pi \\
U_1 & \xrightarrow{\varphi_1} & U_2
\end{array}$$

commutes.

The partial connections $D^2$ on $T\mathcal{F}_2^\perp$ determines in a unique way a partial pull-back connection $D^2$ on $f^{-1}(T\mathcal{F}_2^\perp) \to M_1$, we use here the same symbol for the connection on the transversal bundle and the pull-back bundle. The connection $D^2$ on the pull-back bundle is defined in a unique way by the following natural condition: if $Y$ is a local basic section of $T\mathcal{F}_2^\perp$ and $X$ is a local basic section of $T\mathcal{F}_1^\perp$ then $D^2_X(Y \circ f) = (D^2_{df(X)^\perp} Y) \circ f$. 

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Lemma 2.5. If $s$ is a section belonging to $\Gamma_s(U_1, f^{-1}(T\mathcal{F}_2^\perp))$ which projects via the bundle map, cf. diagram (2.6), onto the section $\overline{s}$ then

$$d\varphi_2(D_X^2 s) = \nabla^2_{\overline{X}\overline{s}}$$

(2.7)

where $X$ is a basic section of $T\mathcal{F}_1^\perp \to U_1$, $\overline{X}$ is the corresponding section of $TU_1 \to U_1$ and $\nabla^2$ is the pull-back, via the map $\overline{f}$, of the Levi-Civita connection of $(\overline{\mathcal{T}}_2, \overline{\mathcal{Y}}_2)$.

We may write down the explicit formulas for the covariant derivative of the pulled-back sections using the frames of basic vector fields. In fact if $\Gamma_{ij}^k$, $\Gamma_{\alpha\beta}^{\gamma}$ denote the Christoffel symbols of the partial foliated connections on $M_1$ and $M_2$, respectively, i.e.

$$\nabla^1_{X_i} X_j = \sum_{k=1}^{q_1} \Gamma^1_{ij} X_k \quad \text{and} \quad \nabla^2_{Y_\beta} Y_\alpha = \sum_{\gamma=1}^{q_2} \Gamma^{2\gamma}_{\alpha\beta} Y_\gamma$$

and

$$D^2_{X_\alpha} (Y_\beta \circ f) = \sum_{l, \gamma=1}^{q_2} A_{\alpha l} \Gamma^{2\gamma}_{l\beta} (Y_\gamma \circ f)$$

(2.8)

where $A_{\alpha l}$ are smooth functions on $U_1$ such that $\Pi_\alpha df(X_\alpha) = \sum_{l=1}^{q_2} A_{\alpha l}(Y_l \circ f)$, then from (2.8) it follows that

$$D^2(Y_\beta \circ f) = \sum_{\alpha=1}^{q_1} \sum_{l,\gamma=1}^{q_2} A_{\alpha l} \Gamma^{2\gamma}_{l\beta} X^l_\alpha \otimes (Y_\gamma \circ f).$$

(2.9)

Therefore for any basic section we have the following.

Lemma 2.6. If $s$ is a basic section of of the bundle $f^{-1}T\mathcal{F}_2^\perp \to M_1$ then

$$D^2 s = \sum_{\alpha=1}^{q_1} \sum_{l=1}^{q_2} \zeta_{\alpha l} X^l_\alpha \otimes (Y_l \circ f)$$

where $\zeta_{\alpha l}$ are local basic functions on $M_1$

Next we consider the bundle

$$(T\mathcal{F}_1^\perp)^* \otimes_{M_1} f^{-1}T\mathcal{F}_2^\perp \to M_1.$$  

(2.10)
The sheaf of basic sections of bundle (2.10) is defined as the unique sheaf of sections generated, over the sheaf of basic functions over $M_1$, by the tensor products $X \otimes s$ where $s$ is a basic section of $f^{-1}T\mathcal{F}_2^\perp$ and $X$ is a basic section of $(T\mathcal{F}_1^\perp)^*$.

**Lemma 2.7.** A local section $s$ of $(T\mathcal{M}_1)^* \otimes f^{-1}T\mathcal{F}_2^\perp \rightarrow M_1$ is basic if and only if it is locally projectable on a section $\tilde{s}$ of the bundle $(T\mathcal{U}_1)^* \otimes \tilde{f}^{-1}T\mathcal{U}_2 \rightarrow U_1$, via the following bundle map:

\[
\begin{array}{c c c}
(T\mathcal{F}_1^\perp)^* \otimes f^{-1}T\mathcal{F}_2^\perp & \xrightarrow{\mathrm{d}\phi_1 \otimes \mathrm{d}\phi_2} & (T\mathcal{U}_1)^* \otimes \tilde{f}^{-1}T\mathcal{U}_2 \\
\uparrow s & & \uparrow \tilde{s} \\
U_1 & \xrightarrow{\phi_1} & U_1.
\end{array}
\quad (2.11)
\]

Since there are partial basic connections $D^2$ and $D^1$ in $f^{-1}(T\mathcal{F}_2^\perp) \rightarrow M_1$ and $(T\mathcal{F}_1^\perp) \rightarrow M_1$, respectively, then there is the naturally induced connection $D$ on the bundle (2.10). In fact, for each local basic section $\omega \otimes s$ of the bundle (2.10) we put $\mathcal{D}_X \omega \otimes s = \mathcal{D}_X \omega \otimes s + \omega \otimes \mathcal{D}_X s$ which is a basic section of (2.10). The definition of $D$ extends to all sections of (2.10) using the Leibniz rule over the sheaf of basic smooth functions over $M_1$.

Next we consider the bundle

\[
(T\mathcal{F}_1^\perp)^* \otimes (T\mathcal{F}_1^\perp)^* \otimes f^{-1}T\mathcal{F}_2^\perp \rightarrow M_1.
\quad (2.12)
\]

The basic sections of the bundle (2.12) are those which are generated by sections of the form $s_1 \otimes s_2 \otimes (Y \circ f)$ where $s_1, s_2$ are local basic sections of the bundle $(T\mathcal{F}_2^\perp)^* \rightarrow M_2$ and $Y$ is a local basic section of $T\mathcal{F}_2^\perp \rightarrow M_2$. Then we have the following lemma.

**Lemma 2.8.** If $s$ is a basic section of the bundle $(T\mathcal{F}_1^\perp)^* \otimes f^{-1}T\mathcal{F}_2^\perp \rightarrow M_1$ then the map

\[
(X, Y) \rightarrow (\mathcal{D}_X s)Y
\]

is a basic section of the bundle (2.12).

*Proof.* Let $s = \mu s_1 \otimes (Y \circ f)$ be a local basic section of the bundle (2.10) i.e. $\mu$ is a local basic function on $M_1$, $s_1$ is a local basic section of $(T\mathcal{F}_1^\perp)^* \rightarrow M_1$ and $Y$ is a local basic section of $(T\mathcal{F}_2^\perp)^* \rightarrow M_2$ then

\[
\mathcal{D}s = d\mu \otimes s_1 \otimes (Y \circ f) + \mu Ds_1 \otimes (Y \circ f)
\]

\[+ \mu s_1 \otimes \mathcal{D}(Y \circ f).\]
Then we have that $d\mu$, $Ds_1$ and $D(Y \circ f)$ are local basic section of the respective bundles over $M_1$. $\square$

**Corollary 2.1.** The covariant derivative $D(\Pi_2 df U_1)$ is a global basic section of the bundle (2.12).

*Proof.* The corollary follows from the fact that $\Pi_2 df U_1$ is a global basic section of the bundle (2.10) and from the construction of the connection $D$ in that bundle. $\square$

### 3 Transversally harmonic maps

It is the main section of the paper. The first part contains a general characterisation of transversally harmonic maps using the map induced on the level of transverse manifolds. In the second part we prove that ‘being transversally harmonic’ is a transverse property, i.e. two foliated maps which induce isometrically equivalent maps on the level of transverse manifolds are transversally harmonic at the same time. In the last part we discuss the suspension construction as a means to construct transversally harmonic maps.

#### 3.1 General results

We define the *transversal second fundamental form of the map $f$* as $D(\Pi_2 df U_1)$ which is a section of the bundle (2.12). There is a strict relationship between the transversal second fundamental form of $f$ and the second fundamental form of the induced map $\overline{f}$. Using the notations of Lemma 2.2 we obtain the following property.

**Lemma 3.1.** Let $Z_1, Z_2$ be two local basic vector fields on $U_1$ which projects, via the map $\varphi_1$ on vector fields $\overline{Z}_1, \overline{Z}_2$. Then

$$\Pi_2(D(\Pi_2 df U_1)(Z_1, Z_2)) = (\nabla df)(\overline{Z}_1, \overline{Z}_2)$$

(3.1)

where $f$ is the induced map between $U_1$ and $U_2$.

*Proof.* The assertion follows immediately from equations (2.1) and (2.7). $\square$

**Lemma 3.2.** The transversal second fundamental form is symmetric.
Proof. In fact, for each \( X, Y \in T_x\mathcal{F}_1^\perp \) we have that

\[
d\varphi_2 D(\Pi_2 df\mathcal{U}_1)(X,Y) = (\nabla df)(X,Y)
= (\nabla df)(Y,X)
= d\varphi_2 D(\Pi_2 df\mathcal{U}_1)(Y,X).
\]

Since \( d_{f(x)}\varphi_2 \) is an isomorphism when restricted to \( T_{f(x)}\mathcal{F}_2^\perp \), then it follows that \( D(\Pi_2 df\mathcal{U}_1) \) is symmetric. \( \Box \)

**Definition 3.1.** A map \( f \) is said to be **transversally harmonic** if and only if the trace of the transversal second fundamental form vanishes. This trace is called the **transversal tension field** of \( f \) and is denoted by \( \tau_b(f) \).

If \( X_{1x}, \ldots, X_{q_1x} \) is an orthonormal basis of the space \( T_x\mathcal{F}_1^\perp \) then

\[
\tau_b(f)_x = \text{trace}_{T\mathcal{F}_2^\perp} D(\Pi_2 df\mathcal{U}_1) = \sum_{a=1}^{q_1} D(\Pi_2 df\mathcal{U}_1)(X_a, X_a)
\]

and \( \tau_b(f) \) is a section of the bundle \( f^{-1}T\mathcal{F}_2^\perp \rightarrow \mathcal{M}_1 \).

There is a close relationship between the transversal tension field of \( f \) and the tension fields of the induced maps \( \mathcal{F} \) which can be obtained using the local submersions defining the foliations \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \). In fact, from Lemma 3.1 it follows that

\[
d\varphi_2 \tau_b(f)_x = \tau(\mathcal{F})_x.
\]  (3.2)

and equation (3.2) is valid for each of the foliations defining local submersions \( \varphi_i : U_i \rightarrow \mathcal{U}_i \) such that \( f(U_1) \subset U_2 \). Then we have the following theorem.

**Theorem 3.1.** The map \( f \) is transversally harmonic if and only if \( \mathcal{F} \) is harmonic.

*Proof.* The theorem follows immediately from equation (3.2) and from the property that \( d_{f(x)}\varphi_2 \) is an isomorphism when restricted to \( T_{f(x)}\mathcal{F}_2^\perp \). \( \Box \)

There is no if and only if relationship between the harmonicity and the transversal harmonicity of \( f \). Clearly transversal harmonicity does not imply harmonicity; we have the following example.
Example 3.1. Let \((B_1, g_1), (B_2, g_2), (F_1, h_1), (F_2, h_2)\) be Riemannian manifolds. We consider foliations on \(B_1 \times F_1\) and \(B_2 \times F_2\), given by the projections on the first component \(\pi_1 : B_1 \times F_1 \to B_1\), \(\pi_2 : B_2 \times F_2 \to B_2\), respectively. The projections \(\pi_1\) and \(\pi_2\) are clearly Riemannian fibrations and the foliations are also Riemannian. Let \(f : B_1 \times F_1 \to B_2 \times F_2\) be a smooth map preserving the leaves of the foliations. Then \(f\) has to be of the form \(f(x, y) = (f_1(x), f_2(x, y))\) for all \(x \in B_1\) and \(y \in F_1\) where \(f_1 : B_1 \to B_2\), \(f_2 : B_1 \times F_1 \to F_2\) are smooth. Then for the product Riemannian metrics on \(B_1 \times F_1\) and \(B_2 \times F_2\) easy calculations show that the tension field of \(f\) is equal to

\[
\tau(f) = (\tau(f_1), \tau(f_2|_{B_1}) + \tau(f_2|_{F_1}))
\]

(3.3)

where \(\tau(f_1)\) is the tension field at \(x\) of \(f_1 : B_1 \to B_2\), \(\tau(f_2|_{B_1})\) is the tension field at \(x\) of the map \(x \to f_2(x, y)\), while \(y\) is fixed and \(\tau(f_2|_{F_1})\) is the tension field at \(y\) of the map \(y \to f_2(x, y)\) whilst \(x\) is fixed, cf. [20, 4]. It follows that the harmonicity of \(f = (f_1, f_2)\) is equivalent to the condition that \(f_1\) is harmonic and \(\tau(f_2|_{B_1}) + \tau(f_2|_{F_1}) = 0\). Equation (3.3) is satisfied when the vertical and horizontal contributions to the tension field annihilate each other. On the other hand we have an easy observation that if \(f_1\) is harmonic and \(f_2|_{B_1}, f_2|_{F_1}\) are harmonic for each \(x \in B_1, y \in B_2\) then \(f\) is harmonic. Hence it follows that there are a lot of maps \(f\) which are transversally harmonic but not harmonic.

Moreover, the harmonicity of a map does not imply the transversal harmonicity. We shall construct an example of a harmonic map which is not transversally harmonic using a warp product of two manifolds, cf. Example 3.2. This is a natural extension of Example 3.1. First we recall some of the definitions and properties of a warped product, cf. [18]. Let \((B, g), (F, h)\) be two Riemannian manifolds and \(\alpha : B \to \mathbb{R}\) a smooth map. Then on the product manifold \(B \times F\) we define a Riemannian metric tensor by \(\tilde{g} = g \oplus e^{2\alpha}h\). Vector fields on \(B\) may be naturally considered as vector fields on \(B \times F\); in the same way vector fields on \(F\) may be naturally considered as vector fields on \(B \times F\). Let \(\nabla^g\) be the Levi–Civita connection on \(B\) and \(\nabla^h\) be the Levi–Civita connection on \(F\). Let \(X, Y\) be vector fields on \(B\) and \(V, W\) vector fields on \(F\). Then we have the following formulas for the Levi–Civita
connection $\nabla_{\tilde{\mathbf{g}}}$ of the metric $\tilde{\mathbf{g}}$:

$$
\nabla_{\tilde{\mathbf{g}}} Y = \nabla_{\hat{\mathbf{g}}} Y
$$

$$
\nabla_{\tilde{\mathbf{g}}} V = \nabla_{\hat{\mathbf{g}}} X = X(\alpha) V
$$

$$
\nabla_{\tilde{\mathbf{g}}} W = -h(V, W) \text{grad}_{\hat{\mathbf{g}}} \alpha + \nabla_{\hat{\mathbf{g}}} W.
$$

(3.4)

The following fact is well-known.

**Lemma 3.3.** The vertical foliation given by the trivial fibration $B \times F \to B$ is a Riemannian foliation of the Riemannian manifold $(B \times F, \tilde{g})$.

Let $(B_k, g_k)$, $(F_k, h_k)$ be four Riemannian manifolds where $k = 1, 2$. Suppose that there are given $\alpha_k : B_k \to \mathbb{R}$ two smooth warping functions, and two smooth maps $f_1 : B_1 \to B_2$ and $f_2 : B_1 \times F_1 \to F_2$. Then the map $f = (f_1, f_2) : B_1 \times F_1 \to B_2 \times F_2$ sends the vertical foliation of $B_1 \times F_1$ into the vertical foliation of $B_2 \times F_2$. It means that the following diagram

$$
\begin{array}{ccc}
B_1 \times F_1 & \xrightarrow{f} & B_2 \times F_2 \\
\downarrow & & \downarrow \\
B_1 & \xrightarrow{h} & B_2
\end{array}
$$

(3.5)

commutes where the vertical arrows in (3.5) are the natural projections on the first components of the cartesian product. We consider the Riemannian metric tensors $g_k + e^{2\alpha_k} h_k$ on the product manifolds $B_k \times F_k$, for $k = 1, 2$. The tension field of $f$ may be calculated using equations (3.4). Easy but rather tedious calculations give the following formula:

$$
\tau(f) = \tau(f_1) + \tau(f_2|_{B_1}) - \|df_2|_{B_1}\|^2 (\text{grad}_{g_2} \alpha_2) \circ f_1
$$

$$
+ e^{-2\alpha_1} \tau(f_2|_{F_1}) - e^{-2\alpha_1} \|df_2|_{F_1}\|^2 (\text{grad}_{g_2} \alpha_2) \circ f_1
$$

$$
+ \text{dim} F_1 e^{-2\alpha_1} df_2 (\text{grad}_{g_1} \alpha_1).
$$

(3.6)

In formula (3.6) $\tau(f_1)$ denotes the tension field of $f_1$ with respect to the metrics $g_1$ and $g_2$, $\tau(f_2|_{B_1})$ denotes the tension field of $f_2|_{B_1}$ with respect to the metrics $g_1$ and $h_2$, $\tau(f_2|_{F_1})$ denotes the tension field of $f_2|_{F_1}$ with respect to the metrics $h_1$ and $h_2$. Moreover $\|df_2|_{B_1}\|$ is the Hilbert–Schmidt norm induced by the metrics $g_1$ and $h_2$, $\|df_2|_{F_1}\|$ is the Hilbert–Schmidt norm induced by the metrics $h_1$ and $h_2$.

From the construction of the transversal tension field of a map between foliated manifolds follows the following corollary.
Corollary 3.1. The map $f$ is transversally harmonic if and only if $\tau(f_1) = 0$.

Moreover we have the following special case of formula (3.6):

**Corollary 3.2.** If the warping map $\alpha_1$ vanishes then

$$\tau(f) = \tau(f_1) + \tau(f_2|_{B_1}) - \|df_2\|^2(\text{grad}_{g_2^*}\alpha_2) \circ f_1$$

(3.7)

where the Hilbert–Schmidt norm $\|df_2\|$ is taken with respect to the $g_1 \oplus h_1$ metric on $B_1 \times F_1$ and $h_2$ metric on $F_2$.

**Example 3.2.** Suppose that $B_1 = B_2 = F_1 = F_2 = \mathbb{R}$, $\alpha_1 \equiv 0$, $\alpha_2(x) = x$, $f_1(x) = x^2$, $f_2(x, y) = 2y$ and $f(x, y) = (f_1(x), f_2(x, y))$. The foliations on $B_1 \times F_1$ and $B_2 \times F_2$ are given by the projections on $B_1$ and $B_2$, respectively. The map $f$ is a smooth leaf preserving map from $B_1 \times F_1$ to $B_2 \times F_2$. Since $\alpha_1$ vanishes then the metric on $B_1 \times F_1$ is just product one. The warp product metric on $B_2 \times F_2$ is given, in the standard coordinates $(x, y) \in \mathbb{R}^2$, by the formula $dx^2 + e^{2x}dy^2$. Then we apply formula (3.7) and get that $\tau(f) = 0$, $\tau(f_1) = 2 \neq 0$. Hence $f$ is a map which is harmonic but not transversally harmonic.

On the other hand we have the following theorem.

**Theorem 3.2.** Let $f : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ be a leaf preserving harmonic map between two manifolds with Riemannian foliations. Moreover, we suppose that the foliation $\mathcal{F}_1$ has all its leaves minimal, the foliation $\mathcal{F}_2$ is totally geodesic and $f$ is horizontal, i.e. for all $df(T\mathcal{F}_1) \subset T\mathcal{F}_2$, then $f$ is transversally harmonic.

**Proof.** Since the properties of harmonicity and transversal harmonicity are local then we consider open subsets $U_i \subset M_i$, Riemannian fibrations $\pi_i : U_i \to \overline{U}_i$, $i = 1, 2$, such that the foliations on $U_i$ are fibres of the submersions $\pi_i$ and $f(U_1) \subset U_2$. Then we have the following commutative diagram

$$
\begin{array}{ccc}
U_1 & \xrightarrow{f} & U_1 \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
\overline{U}_1 & \xrightarrow{\overline{f}} & \overline{U}_1
\end{array}
$$

(3.8)
where \( \overline{f} \) is a naturally induced smooth map. We have the following formula for the tension field of the composition of maps, cf. [20, 4],

\[
\tau(\pi_1 \circ \overline{f}) = d\overline{f}(\tau(\pi_1)) + \text{trace} \nabla d\overline{f}(d\pi_1, d\pi_1)
 \]

\[
= \text{trace}_{T_{\overline{U}_1}} \nabla d\overline{f}(d\pi_1, d\pi_2) = \tau(\overline{f}),
\]

the last equality follows from the fact that the trace over the tangential part to the leaves vanishes since the leaves are minimal in \( M_1 \). On the other hand we have that

\[
\tau(\pi_2 \circ f) = d\pi_2(\tau(f)) + \text{trace} \nabla d\pi_2(df, df)
 \]

\[
= \text{trace}_{T_{\overline{U}_1}} \nabla d\pi_2(df, df) + \text{trace}_{T_{\overline{U}_1}} \nabla d\pi_2(df, df) = 0
\]

where the last equality holds because \( \tau(f) = 0 \) and \( \text{trace}_{T_{\overline{U}_1}} \nabla d\pi_2(df, df) = 0 \) because \( \pi_2 \) is a Riemannian fibration and the Levi-Civita connection on \( \overline{U}_2 \) is projected from \( U_2 \). Then from the commutativity of the diagram (3.8) we have the result.

\[ \square \]

### 3.2 Transversally harmonic maps and holonomy pseudogroups

Let \( f: (M_1, \mathcal{F}_1) \rightarrow (M_2, \mathcal{F}_2) \) be a smooth foliated map. Suppose that \( \mathcal{U} = \{U_i, \varphi_i, g_{ij}\} \) is a cocycle defining the foliation \( \mathcal{F}_1 \) and let denote by \( \mathcal{V} = \{V_a, \psi_a, h_{a\beta}\} \) a cocycle defining the foliation \( \mathcal{F}_2 \) such that for any \( i \in I \) there exists \( \alpha(i) \in A \) such that \( f(U_i) \subset V_{\alpha(i)} \). Let \( \tilde{U}_i = \varphi_i(U_i) \) and \( \tilde{V}_a = \psi_a(V_a) \). Then the manifold \( N_1 = \bigsqcup \tilde{U}_i \) is a transverse manifold of the foliation \( \mathcal{F}_1 \) and \( N_2 = \bigsqcup \tilde{V}_a \) is a transverse manifold of the foliation \( \mathcal{F}_2 \), cf. [22, 23]. The transformations \( g_{ij} \) generate a pseudogroup \( \mathcal{H}_1 \) which is called the holonomy pseudogroup of \( \mathcal{F}_1 \) associated to the cocycle \( \mathcal{U} \) and the transformations \( h_{a\beta} \) generate a pseudogroup \( \mathcal{H}_2 \) which is called the holonomy pseudogroup of \( \mathcal{F}_2 \) associated to the cocycle \( \mathcal{V} \).

On the level of transverse manifolds the map \( f \) induces a smooth map \( \overline{f} \) as for any \( i \in I \) the following diagram is commutative:

\[
\begin{array}{ccc}
U_i & \xrightarrow{f(U_i)} & V_{\alpha(i)} \\
\varphi_i \downarrow & & \downarrow \psi_{\alpha(i)} \\
\overline{U}_i & \xrightarrow{\overline{f}_{\alpha(i)}} & \overline{V}_{\alpha(i)}
\end{array}
\]

(3.9)
The map $\tilde{f}: N_1 \to N_2$ is defined as follows:

$$\tilde{f}|_{\mathcal{U}_i} = \tilde{f}_{\alpha(i)}.$$ 

The map $\tilde{f}$ has the following interesting property:

Take two open sets $U_i$ and $U_j$ such that $U_i \cap U_j \neq \emptyset$ then $\tilde{f}(U_i \cap U_j) \subset V_{\alpha(i)} \cap V_{\alpha(j)}$.

The intersection $U_i \cap U_j$ covers the open subset $U_{ji}$ in $U_i$ and the open subset $U_{ij}$ in $U_j$. Likewise $V_{\alpha(i)} \cap V_{\alpha(j)}$ covers $V_{\alpha(i)\alpha(j)}$ in $V_{\alpha(i)}$ and $V_{\alpha(i)\alpha(j)}$ in $V_{\alpha(j)}$. Moreover, the map $g_{ji}: U_{ji} \to U_{ij}$ is a diffeomorphism and likewise $h_{\alpha(j)\alpha(i)}: V_{\alpha(j)\alpha(i)} \to V_{\alpha(i)\alpha(j)}$ is a diffeomorphism. Then

$$h_{\alpha(j)\alpha(i)}f_{\alpha(i)}|_{U_{ji}} = f_{\alpha(j)\alpha(i)}g_{ji}|_{U_{ji}}.$$

To describe better the properties of the induced map $\tilde{f}$ we introduce the notion of a morphism between pseudogroups, cf. [12].

A family $\mathcal{K}$ of smooth local maps from $N_1$ into $N_2$ is called a morphism of $(N_1, \mathcal{H}_1)$ into $(N_2, \mathcal{H}_2)$ if

1. any $k \in \mathcal{K}$ is a smooth map $k: W \to N_2$ where $W$ is an open subset of $N_1$;

2. the domains of $k \in \mathcal{K}$ form an open covering of $N_1$;

3. for any $k \in \mathcal{K}$, $k: W \to N_2$ and any open subset $W' \subset W$ the restriction $k|_{W'} \in \mathcal{K}$;

4. for any family of maps $k_i \in \mathcal{K}$ such that the map $k = \cup k_i$ is well-defined, $k \in \mathcal{K}$;

5. for any $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$, the map $h_2^{-1} \circ k \circ h_1 \in \mathcal{K}$.

A morphism $\Phi$ is an equivalence of pseudogroups $\mathcal{H}_1$ and $\mathcal{H}_2$ if $\Phi^{-1}$ is also a morphism of $\mathcal{H}_2$ into $\mathcal{H}_1$.

Let $\tilde{N}_1$ be a smooth manifold with a pseudogroup $\tilde{\mathcal{H}}_1$ which is equivalent to $\mathcal{H}_1$ and let $\tilde{N}_2$ be a smooth manifold with a pseudogroup $\tilde{\mathcal{H}}_2$ which is equivalent to $\mathcal{H}_2$, cf. [12]. Let $\Phi$ be the equivalence between $\mathcal{H}_1$ and $\tilde{\mathcal{H}}_1$ and let $\Psi$ be the equivalence between $\mathcal{H}_2$ and $\tilde{\mathcal{H}}_2$. Then for any $\varphi, \varphi' \in \Phi$ and $g \in \mathcal{H}_1$, $g' \in \tilde{\mathcal{H}}_1$

$$\varphi' \circ g \circ \varphi^{-1} \in \tilde{\mathcal{H}}_1$$
and 
\[ \varphi^{-1} \circ g' \circ \varphi \in \mathcal{H}_1. \]
Likewise for any \( \psi, \psi' \in \Psi \) and \( g \in \mathcal{H}_2, g' \in \mathcal{H}_2 \)
\[ \psi' \circ g \circ \psi^{-1} \in \mathcal{H}_2 \]
and
\[ \psi'^{-1} \circ g' \circ \psi \in \mathcal{H}_2. \]
With these definitions in mind the following proposition is obvious.

**Proposition 3.1.** Let \( \mathcal{K} \) be a morphism of \((N_1, \mathcal{H}_1)\) into \((N_2, \mathcal{H}_2)\). Then the maps \( \psi \circ k \circ \varphi^{-1}, \varphi \in \Phi, \psi \in \Psi \) define a morphism of \((\tilde{N}_1, \tilde{\mathcal{H}}_1)\) into \((\tilde{N}_2, \tilde{\mathcal{H}}_2)\), which we denote \( \Psi \circ \mathcal{K} \circ \Phi^{-1} \).

The next lemma clarifies the relation between the maps induced on the level of transverse manifolds for various choices of cocycles defining the foliations.

**Lemma 3.4.** Let \( f : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2) \) be a foliated map. Let \( \mathcal{U} \) and \( \mathcal{U}' \) be two cocycles defining the foliation \( \mathcal{F}_1 \) and let \( \mathcal{V} \) and \( \mathcal{V}' \) be two cocycles defining the foliation \( \mathcal{F}_2 \). Let \((N_1, \mathcal{H}_1), (\tilde{N}_1, \tilde{\mathcal{H}}_1), (N_2, \mathcal{H}_2), (\tilde{N}_2, \tilde{\mathcal{H}}_2)\) be the corresponding transverse manifolds and holonomy pseudogroups. Let \( \Phi : (N_1, \mathcal{H}_1) \to (\tilde{N}_1, \tilde{\mathcal{H}}_1) \) be an equivalence of the pseudogroups and
\[ \Psi : (N_2, \mathcal{H}_2) \to (\tilde{N}_2, \tilde{\mathcal{H}}_2) \]
be the other equivalence of the pseudogroups. Let \( \mathcal{K}(f) : (N_1, \mathcal{H}_1) \to (N_2, \mathcal{H}_2) \)
be the morphism induced by the foliated map \( f \) and let \( \tilde{\mathcal{K}}(f) : (\tilde{N}_1, \tilde{\mathcal{H}}_1) \to (\tilde{N}_2, \tilde{\mathcal{H}}_2) \)
be the second morphism induced by the foliated map \( f \). Then
\[ \tilde{\mathcal{K}}(f) = \Psi \circ \mathcal{K}(f) \circ \Phi^{-1}. \]

Now let us return to the situation of our particular interest - that of pseudogroups of local isometries.

Let \( \mathcal{H}_1 \) be a pseudogroup of local isometries of a Riemannian manifold \((N_1, g_1)\). Let \( \mathcal{H}_1 \) be a pseudogroup of local transformations of the manifold \( \tilde{N}_1 \) which is equivalent to the pseudogroup \( \mathcal{H}_1 \). Then there is a Riemannian metric \( \tilde{g}_1 \) on \( \tilde{N}_1 \) for which \( \mathcal{H}_1 \) is a pseudogroup of local isometries and the equivalence between \( \mathcal{H}_1 \) and \( \mathcal{H}_1 \) consists of local isometries of \((N_1, g_1)\) into \((\tilde{N}_1, \tilde{g}_1)\), cf. [23].

Combining Lemma 3.4 with the above remark we get the following theorem.
**Theorem 3.3.** Let \((M_1, \mathcal{F}_1), (M_2, \mathcal{F}_2), (\tilde{M}_1, \tilde{\mathcal{F}}_1), (\tilde{M}_2, \tilde{\mathcal{F}}_2)\) be four foliated manifolds by Riemannian foliations and \(f: (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)\) and

\[
\tilde{f}: (\tilde{M}_1, \tilde{\mathcal{F}}_1) \to (\tilde{M}_2, \tilde{\mathcal{F}}_2)
\]

be two foliated maps. Let us also assume that the holonomy pseudogroups of \((M_1, \mathcal{F}_1)\) and \((\tilde{M}_1, \tilde{\mathcal{F}}_1)\) are equivalent as well as those of \((M_2, \mathcal{F}_2)\) and \((\tilde{M}_2, \tilde{\mathcal{F}}_2)\). Then the map \(f\) is transversally harmonic if and only if the map \(\tilde{f}\) is.

**Proof.** The map \(f\) is harmonic if and only if the induced morphism \(K(f)\) consists of harmonic maps (between the transverse manifolds for some (and then for any) choice of the cocycles defining the foliations. As the corresponding pseudogroups are equivalent the second induced map consists of harmonic maps for the transported Riemannian metric. Therefore the map \(\tilde{f}\) is transversally harmonic for any bundle-like metric inducing the given Riemannian metrics on the transverse manifolds.

To close this section we present a result on foliation preserving maps of codimension one Riemannian foliations. Such a foliation is defined by a closed 1-form without zeros and therefore admits a global i.a. \(X\) without zeros. The transverse Riemannian metric \(g\) is given by \(g(X, X) = 1\), so the flow \(\varphi_t\) of \(X\) is transversally isometric. Let \(f: M \to M\) be a transversally harmonic smooth map which preserves the foliation \(\mathcal{F}\). There exists \(s \in \mathbb{R}\) such that \(\varphi_s f(L_0) = L_0\). Therefore the map \(H: [0, 1] \times M \to M\), \(H(t, x) = \varphi_t f(x)\) is a homotopy between \(f\) and \(\varphi_s f\) which consists of transversally harmonic maps. The foliation \(\mathcal{F}\) is developable onto \(\mathbb{R}\) with the standard metric. \(\varphi_s f\) induces an harmonic map \(\tilde{f}: \mathbb{R} \to \mathbb{R}\), therefore \(\tilde{f}\) has to be of the form \(x \mapsto ax + b\). Since it commutes with the holonomy group \(\Gamma\) of isometries (i.e. with some non-trivial translations), it itself must be a translation. If \(\tilde{f}\) maps one orbit of \(\Gamma\) into itself, then it maps any orbit of \(\Gamma\) into itself. Thus we have proved the following result.

**Theorem 3.4.** Let \(M\) be a compact manifold foliated by a codimension one Riemannian foliation. Any transversally harmonic map of \((M, \mathcal{F})\) into itself is homotopic via transversally harmonic maps to a map which maps each leaf into itself.
3.3 The suspension construction

We recall the construction of a suspension which can be found in [17] and use it for constructing foliation preserving maps between foliated manifolds.

Let $(F, g)$ be Riemannian manifold and $\text{Isom}(F, g)$ be the group of its isometries. Let us choose any smooth manifold $S$ and let $\pi_1(S) = G$ be its fundamental group and $h$ be a representation of the group $G$ into $\text{Isom}(F, g)$. Let us take the Cartesian product $\hat{S} \times F$, where $\hat{S}$ is the universal covering of $S$. The group $G$ acts on this product via the deck transformations on $\hat{S}$ and the representation $h$ on $F$:

$$(s, v), \gamma = (s, \gamma, h(\gamma)(v))$$

The product $\hat{S} \times F$ is equipped with the Riemannian metric $g_e \times g$, where $g_e$ is any Riemannian metric lifted from $S$ to $\hat{S}$. The action of $G$ on $\hat{S} \times F$ is isometric for this Riemannian metric.

The action of the group $G$ is totally discontinuous and the quotient manifold we denote by $\hat{M}(S, F; h)$. It is a fibre bundle over $S$ with the standard fibre $F$ which admits a foliation $\mathcal{F}_M$ transverse to the fibres. Its leaves are covering spaces of $S$. In the induced Riemannian metric the foliation by the fibres is totally geodesic, cf. [13], and the foliation $\mathcal{F}$ is Riemannian.

Let $(F_1, g_1)$, $(F_2, g_2)$, $(F_1, h_1)$, $(F_2, h_2)$ be four Riemannian manifolds. Let denote by $G_i$ the fundamental group of the manifold $S_i$, and by $h_i$ a representation of the group $G_i$ into the group $\text{Isom}(F_i, g_i)$ of isometries of the Riemannian manifold $(F_i, g_i), i=1,2$. Let $f: S_1 \to S_2$ be a smooth map. Let $\tilde{f}: \hat{S}_1 \to \hat{S}_2$ be its lift to the universal coverings of these manifolds. Then $\tilde{f}$ is $(G_1, G_2)$-equivariant. Denote by $\pi_1(f): G_1 \to G_2$ the map induced by $f$ on the fundamental groups of the manifolds. Let $h_i: G_i \to \text{Isom}(F_i, g_i)$ be the representations of the groups $G_i$, $i=1,2$, respectively. Let us choose a map $\phi: F_1 \to F_2$ which is $(G_1, h_1; G_2, h_2)$-equivariant, i.e. $\phi(v, h_1(\gamma)) = \phi(v), h_2(\pi_1(f)(\gamma))$ for any $v \in F_1$ and any $\gamma \in G_1$. The map $\tilde{\psi}: \hat{S}_1 \times F_1 \to \hat{S}_2 \times F_2$ defined as $\tilde{\psi} = (f, \phi)$ is $(G_1, G_2)$-equivariant. Therefore it induces a map $\psi: M_1(S_1, F_1; h_1) \to M_2(S_2, F_2; h_2)$.

The following lemmas are fundamental for constructing transversally harmonic maps.

**Lemma 3.5.** The map $\phi: F_1 \to F_2$ is harmonic if and only if $\psi$ is transversally harmonic.
Proof. Our lemma follows from the fact that the following diagram

\[
\begin{array}{ccc}
\tilde{S}_1 \times F_1 & \xrightarrow{\tilde{f} \times \phi} & \tilde{S}_2 \times F_2 \\
\downarrow & & \downarrow \\
M_1 & \xrightarrow{\psi} & M_2
\end{array}
\]

(3.10)

commutes, in which the vertical maps are local isometries. In fact, the map \(\tilde{f} \times \phi\) projects on \(\psi\) and the foliations on \(\tilde{S}_k \times F_k\) projects on the foliations on \(M_k, k = 1, 2\). Then the map \(\tilde{f} \times \psi\) is transversally harmonic if and only if \(\psi\) is transversally harmonic. But the transversal harmonicity of \(\tilde{f} \times \phi\) is equivalent to the harmonicity of \(\phi\).

\[\square\]

Proposition 3.2. The maps \(f : S_1 \rightarrow S_2, \phi : F_1 \rightarrow F_2\) are harmonic if and only if the induced suspended map \(\psi : M_1(S_1, F_1, h_1) \rightarrow M_2(S_2, F_2, h_2)\) is harmonic.

Proof. Once again the lemma follows from the commutativity of the diagram (3.10) and the fact that the projections are local isometries. \[\square\]

The diagram (3.10) permits us to prove a similar result for harmonic morphisms. Harmonic morphisms are generalisation of harmonic maps and are characterised by the property that they carry germs of harmonic functions to germs of harmonic functions. A map is a harmonic morphism if and only if it is harmonic and horizontally conformal, cf. [10]. Basic properties and examples of harmonic morphism are can be found in [4, 5, 6]. The vast and updated bibliography of the theory of the harmonic morphisms is on the web page, cf. [11].

Proposition 3.3. Suppose that at least one of the maps \(f\) or \(\phi\) is a local diffeomorphism. Then the maps \(f : S_1 \rightarrow S_2, \phi : F_1 \rightarrow F_2\) are harmonic morphisms if and only if the induced suspended map \(\psi : M_1(S_1, F_1, h_1) \rightarrow M_2(S_2, F_2, h_2)\) is a harmonic morphism.

Proof. From the previous observation it follows that \(\psi\) is harmonic if and only if both \(f\) and \(\phi\) are so. If one of the maps \(f\) or \(\phi\) is an immersion, apart from a discrete set of points, then the map \(\tilde{f} \times \phi\) in diagram (3.10) is horizontally conformal if both \(f\) and \(\phi\) are horizontally conformal. Hence our proposition follows. \[\square\]
Let $F : [0, 1] \times S_1 \to S_2$ be a continuous map such that for each $t \in [0, 1]$ $F_t : S_1 \to S_2$ is smooth. Moreover, we suppose that there exists a continuous map $\Phi : [0, 1] \times F_t \to F_2$ such that for each $t \in [0, 1]$ the map $\Phi_t : F_t \to F_2$ is smooth. Let $h_i : \pi_1(S_i) = G_i \to \text{Isom}(F_i)$, $i = 1, 2$ be representations of the fundamental groups such that $\Phi_t$ is $(G_1, h_1; G_2, h_2)$-equivariant for each $t \in [0, 1]$. We observe that the map $(F_t)_* : G_1 \to G_2$ does not depend on the parameter $t$. Let $t_0$ be such that $\Phi_{t_0}$ is harmonic. Then we have the following

**Proposition 3.4.** The continuous map

$$\Psi : [0, 1] \times M_i(S_1, F_t, h_1) \to M_2(S_2, F_t, h_2)$$

induced by $\Phi$ is transversally harmonic.

**Remark 3.1.** If the action $h_1 : \pi_1(S_1) \to \text{Isom}(F_1)$ is trivial, which happens for example when $S_1$ is simply connected, then $M_i(S_1, F_t; h_1)$ is isometric to $S_1 \times F_1$ and the map $\tilde{f} \times \phi$ is pushed down to the map $f \times \phi$. Then it follows that $\psi : M_1 \to M_2$ is a composition of $f \times \phi$ with the projection $\tilde{S}_2 \times F_2$ onto $M_2$. The latter map is a local isometry. Hence, to get interesting examples of transversally harmonic maps it is necessary to look for manifolds $F_i$ with non-trivial isometric action of $\pi_1(S_i)$ on them.

## 4 Examples

In this section we present examples of transversally harmonic maps between foliated manifolds obtained from Hopf fibrations (considered as foliated manifolds) using the suspension construction. Finally we present a construction which permits to find transversally harmonic maps between nilmanifolds.

**Example 4.1.** Let $S_1 = S_2 = \mathbb{S}^1$ be the unit circle and $f : S_1 \to S_2$ be a smooth map. Then $\pi(S_1) = \mathbb{Z}$ and there exits $n \in \mathbb{Z}$ such that $\pi_1(f)(m) = mn$ for each $m \in \mathbb{Z}$. Actually $f$ is smoothly homotopic to the map $f_z : \mathbb{S}^1 \to \mathbb{S}^1$ such that $f_z(z) = z^n$ where $z \in \mathbb{S}^1 \subset \mathbb{C}$. Then we consider the map $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(y) = \alpha y + \beta$. It is clear that $\phi$ is harmonic. Then we define two homomorphisms $h_i : \mathbb{Z} \to \text{Isom}(\mathbb{R})$, $(i = 1, 2)$ such that $h_1(m)x = x + \gamma mn$ and $h_2(m)x = x + \alpha \gamma m$. Then $\tilde{S}_1 = \mathbb{R}$. We have that

$$M_1 = \frac{\mathbb{R} \times \mathbb{R}}{\mu_1} \quad M_2 = \frac{\mathbb{R} \times \mathbb{R}}{\mu_2}.$$
Then we have the following map \( \psi : M_1 \to M_2 \) such that \( \psi([x, y]) = [\tilde{f}(x), \alpha y + \beta] \) is a transversally harmonic map. If we consider a particular case of the map \( f = f_c \) then the suspension map is given by \( \psi([x, y]) = [2\pi n \alpha y, \alpha y + \beta] \).

**Example 4.2.** Let \( S_1 = S_2 = \mathbb{S}^1 \) and \( F_1 = F_2 = \mathbb{S}^1 \) with their standard Riemannian structures. Suppose that there is given a smooth map \( f : S_1 \to S_2 \) such that \( f_*(m) = nm \) where \( n \in \mathbb{Z}^* \). Suppose also that there is given a map \( \phi : F_1 \to F_2 \) such that \( \phi(z) = wz^k \) where \( w \in \mathbb{S}^1 \subset \mathbb{C} \); this map is harmonic for any \( k \in \mathbb{Z} \). Let us consider the representations \( \mu_j : \mathbb{Z} \to \text{Isom}(F_j) \) \((j = 1, 2)\) given by \( \mu_1(m)z = q^{nm}z \) and \( \mu_2(m)z = q^{km}z \) where \( q \in \mathbb{S}^1 \subset \mathbb{C} \). Then it follows that the map \( \phi \) is equivariant with respect to the actions \( \mu_1 \) and \( \mu_2 \), and we have the transversally harmonic map \( \psi : M_1 \to M_2 \) such that \( \psi([x, z]) = [\tilde{f}(x), wz^k] \). We observe that in this case

\[
M_1 = \frac{\mathbb{R} \times \mathbb{S}^1}{\mu_1}, \quad M_2 = \frac{\mathbb{R} \times \mathbb{S}^1}{\mu_2}.
\]

In a particular case when \( f(z) = f_c(z) = z^n \) we get that \( \psi \) is a local diffeomorphism which is harmonic. Hence \( \psi \) is a harmonic map which is locally a diffeomorphism. In particular \( \psi \) is a harmonic morphism. Moreover the map \( \psi \) acts as follows: \( \psi([x, z]) = [2\pi n, wz^k] \).

**Example 4.3.** Let \( S_1 = S_2 = \mathbb{S}^1 \), \( F_1 = \mathbb{S}^3 \), \( F_2 = \mathbb{S}^2 \) with their standard Riemannian structures. Suppose that there is given a smooth map \( f : S_1 \to S_2 \) such that \( f_*(m) = nm \) where \( n \in \mathbb{Z} \). Let \( \phi : \mathbb{S}^3 \to \mathbb{S}^2 \) be the Hopf fibration. Then we consider the representations \( \mu_j : \mathbb{Z} \to \text{Isom}(F_j) \) \((j = 1, 2)\) given by:

\[
\mu_1(m)(z_1, z_2) = (q_1^{nm}z_1, q_2^{nm}z_2) \quad \text{for} \quad (z_1, z_2) \in \mathbb{S}^3 \subset \mathbb{C}^2 \quad \text{where} \quad q_1, q_2 \in \mathbb{S}^1 \subset \mathbb{C};
\]

\[
\mu_2(m)(a, z) = (a, q_1^{nm}z_2 \overline{q_2}) \quad \text{for} \quad (a, z) \in \mathbb{S}^2 \subset \mathbb{R} \times \mathbb{C}.
\]

It is easy to observe that \( \mu_1 \) and \( \mu_2 \) act by isometries. Then the map \( \phi \) is equivariant with respect to the actions \( \mu_1 \) and \( \mu_2 \). Hence we get a transversally harmonic map \( \psi : M_1 \to M_2 \). The map \( \psi \) is given explicitly by: \( \psi([x, (z_1, z_2)]) = [\tilde{f}(x), (|z_1|^2 - |z_2|^2, 2z_1\overline{z}_2)] \).

We observe that in this case

\[
M_1 = \frac{\mathbb{R} \times \mathbb{S}^3}{\mu_1}, \quad M_2 = \frac{\mathbb{R} \times \mathbb{S}^2}{\mu_2}.
\]

In the particular case when \( f(z) = f_c(z) = z^n \) we get that \( f_c \) is a local diffeomorphism which is harmonic. Since the Hopf fibration \( \phi \) is a harmonic morphism then \( \psi \) is a harmonic morphism too. In this case the map \( \psi \) is given by: \( \psi([x, (z_1, z_2)]) = [2\pi n, (|z_1|^2 - |z_2|^2, 2z_1\overline{z}_2)] \).

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The above example may be extended to other Hopf fibrations. We present that in the following two examples.

**Example 4.4.** Let $S_1 = S_2 = \mathbb{S}^1$, $F_1 = \mathbb{S}^7$, $F_2 = \mathbb{S}^4$ with their standard Riemannian structures; $f : S_1 \to S_2$ such that $f_*(m) = nm$ where $n \in \mathbb{Z}$. Let $\phi : \mathbb{S}^7 \to \mathbb{S}^4$ be the Hopf fibration. We consider the representations $\mu_j : \mathbb{Z} \to \text{Isom}(F_j)$ ($j = 1, 2$) given by: $\mu_1(m)(z_1, z_2) = (q_1^{nm}z_1, q_2^{nm}z_2)$ for $(z_1, z_2) \in \mathbb{S}^7 \subset \mathbb{H}^2$ where $q_1, q_2 \in \mathbb{S}^3 \subset \mathbb{H}$; $\mu_2(m)(a, z) = (a, q_1^m \overline{zq_2^m})$ for $(a, z) \in \mathbb{S}^4 \subset \mathbb{R} \times \mathbb{H}$. It is easy to observe that $\mu_1$ and $\mu_2$ act by isometries. Then the map $\phi$ is equivariant with respect to the actions $\mu_1$ and $\mu_2$. Hence we get a transversally harmonic map $\psi : M_1 \to M_2$. In this case we have that

$$M_1 = \frac{\mathbb{R} \times \mathbb{S}^7}{\mu_1}, \quad M_2 = \frac{\mathbb{R} \times \mathbb{S}^4}{\mu_2}.$$ 

In the particular case when $f(z) = f_*(z) = z^n$ the map $\phi$ is a harmonic morphism too because $f_*$ is a local harmonic diffeomorphism and $\phi$ is a harmonic morphism.

In the similar way we construct transversally harmonic maps from the Hopf fibration $\mathbb{S}^{15} \to \mathbb{S}^8$.

**Example 4.5.** Let $S_1 = S_2 = \mathbb{S}^1$, $F_1 = \mathbb{S}^{15}$, $F_2 = \mathbb{S}^8$ with their standard Riemannian structures. Suppose that there is given a smooth map $f : S_1 \to S_2$ such that $f_*(m) = nm$ where $n \in \mathbb{Z}$. Let $\phi : \mathbb{S}^{15} \to \mathbb{S}^8$ be the Hopf fibration. We consider the representations $\mu_j : \mathbb{Z} \to \text{Isom}(F_j)$ ($j = 1, 2$) given by: $\mu_1(m)(z_1, z_2) = (q_1^{nm}z_1, q_2^{nm}z_2)$ for $(z_1, z_2) \in \mathbb{S}^{15} \subset \mathbb{O}^2$ where $\mathbb{O}$ denote the Cayley numbers and $q_1, q_2$ are Cayley numbers of length one; $\mu_2(m)(a, z) = (a, q_1^m \overline{zq_2^m})$ for $(a, z) \in \mathbb{S}^8 \subset \mathbb{R} \times \mathbb{O}$. Hence we get an equivariant map $\phi : M_1 \to M_2$ which is transversally harmonic. In the case when $f = f_*$ the map $\phi$ is a harmonic morphism. We observe also that

$$M_1 = \frac{\mathbb{R} \times \mathbb{S}^{15}}{\mu_1}, \quad M_2 = \frac{\mathbb{R} \times \mathbb{S}^8}{\mu_2}.$$ 

**Example 4.6.** Let $S_1 = \mathbb{S}^1$, $S_2 = \mathbb{S}^1 \times \mathbb{S}^1$, with their standard product Riemannian structures. We consider a map $f : S_1 \to S_2$ be such that the induced homotopy morphism $f_* : \mathbb{Z} \to \mathbb{Z}^2$ acts as follows: $f_*(m) = (n_1m, n_2m)$. Then we put $F_1 = \mathbb{S}^3$ and $F_2 = \mathbb{S}^2$. We define $\mu_j : \pi_1(S_j) \to \text{Isom}(F_j)$ ($j = 1, 2$) in the following way: $\mu_1(m)(z_1, z_2) = (q_1^{nm}z_1, q_2^{nm}z_2)$ where $(z_1, z_2) \in \mathbb{S}^3 \subset \mathbb{C}^2$. 

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and $q_1, q_2$ are unit complex numbers; $\mu_2(m_1, m_2)(a, z) = (a, q_1^{m_1} \bar{q}_2^{m_2})$ where $(a, z) \in S^2 \subset \mathbb{R} \times \mathbb{C}$. It is easy to observe that $\mu_1, \mu_2$ act by isometries. The Hopf fibration $\phi : S^3 \to S^2$ is an equivariant map with respect to the actions $\mu_1$ and $\mu_2$. Hence, via the suspension construction, we get a transversally harmonic map

$$\psi : \frac{\mathbb{R} \times S^3}{\mu_1} \to \frac{\mathbb{R}^2 \times S^2}{\mu_2}.$$ 

If the map $f$ is a harmonic immersion then the map $\psi$ is a harmonic morphism.

The above example may be generalized for the foliations with the higher dimension of the leaves applying the Hopf fibrations defined by the quaternions $\mathbb{H}$ and octonions $\mathbb{O}$.

**Example 4.7.** We consider $S_1, S_2$ and $f : S_1 \to S_2$ as in Example 4.6. Then we consider the Hopf fibration $\phi : F_1 = S^7 \to F_2 = S^4$. We define the actions $\mu_j : \pi_1(S_j) \to \text{lsom}(F_j)$ by the same formulas as in Example 4.6 though considering $q_i, q_i$ the unit quaternions. In this way we get transversally harmonic map

$$\psi : \frac{\mathbb{R} \times S^7}{\mu_1} \to \frac{\mathbb{R}^2 \times S^4}{\mu_2}.$$ 

Since the Hopf fibrations are harmonic morphisms then $\psi$ is a harmonic morphism if $f$ is a harmonic immersion; this happens, for instance, when $f(z) = (z^{m_1}, z^{m_2})$.

**Example 4.8.** If in Example 4.6 we substitute quaternions with the octonians and use similar formulas to define the actions the fundamental groups on $F_1$ and $F_2$ then we get a transversally harmonic map

$$\psi : \frac{\mathbb{R} \times S^{15}}{\mu_1} \to \frac{\mathbb{R}^2 \times S^8}{\mu_2}.$$ 

**Example 4.9.** Let $S_1 = SO(n_1), S_2 = SO(n_2)$ with $n_1 \leq n_2$ and let $f : S_1 \to S_2$ be a natural immersion i.e.

$$f(A) = \begin{pmatrix} A & 0 \\ 0 & I_{n_2-n_1} \end{pmatrix}$$

where $I_{n_2-n_1}$ denotes the identity matrix with $n_2 - n_1$ columns and rows. The map is a harmonic immersion with respect to the bi-invariant metrics
on $S_1$ and $S_2$. Moreover $f$ induces the identity on the fundamental groups $f_* = \text{Id} : \mathbb{Z}_2 \to \mathbb{Z}_2$. Then $f$ lifts to the harmonic immersion $\tilde{f} : \text{Spin}(n_1) \to \text{Spin}(n_2)$. We put $F_1 = S^{k_1}$, $F_2 = S^{k_2}$ where $k_1 \leq k_2$ and $\phi : F_1 \to F_2$ the canonical totally geodesic immersion. Hence $\phi$ is also a harmonic map. Then we define $f(x)(a) = (-1)^a x$ for each $a \in \mathbb{Z}_2$ and each $x \in F_i$ $(i = 1, 2)$. Then the map $\tilde{f} \times \phi$ is equivariant with respect to the action of $\mathbb{Z}_2$ on $\text{Spin}(n_1) \times F_1$ and $\text{Spin}(n_2) \times F_2$. Then, by passing to the quotient, we have constructed a transversally harmonic map

$$\psi : \text{Spin}(n_1) \times_{\mathbb{Z}_2} S^{k_1} \to \text{Spin}(n_2) \times_{\mathbb{Z}_2} S^{k_2}.$$ 

We observe that the map we have constructed is a harmonic morphism because it is harmonic immersive.

**Remark 4.1.** The definition of the transversally harmonic map may be easily extend to the case of the Riemannian foliations on the pseudo-Riemannian manifolds. We use the Hopf fibrations specifically to produce examples of a transversally harmonic maps and harmonic morphisms in the Riemannian case. There exist Hopf fibrations for the pseudo-spheres in the indefinite case and they are proved to be also harmonic maps, cf. [14, 3]. Hence the indefinite Hopf fibrations may be used to construct transversally harmonic maps and harmonic morphisms in the indefinite case.

We complete our paper with an observation considering another possibility of constructing transversally harmonic maps using subgroups of nilpotent groups.

Let $\Gamma_i$, $i=1,2$, be finitely generated torsion-free subgroups of simply connected nilpotent Lie groups $N_i$. Then there exist simply connected nilpotent Lie groups $U_i$ admitting $\Gamma_i$ as cocompact subgroups and submersions $h_i : U_i \to N_i$ which induce the natural inclusions of $\Gamma_i$ into $U_i$ and $N_i$, respectively. Let us denote by $M_i(U_i, \Gamma_i)$ the compact nilmanifold $U_i/\Gamma_i$. The submersion $h_i$ induces a foliation $\mathcal{F}_i$ on the nilmanifold $M_i(U_i, \Gamma_i)$. The geometrical properties of these foliations were studied in [1, 2].

Any homomorphism $\tilde{f}$ of $N_1$ into $N_2$ which maps $\Gamma_1$ into $\Gamma_2$, according to Theorem 2.11 of [19] can be lifted to a homomorphism $\tilde{f}$ of $U_1$ into $U_2$ such that $\tilde{f}h_1 = h_2f$. This property ensures that $\tilde{f}$ induces a foliated map $f : (M_i(U_i, \Gamma_i), \mathcal{F}_i) \to (M_i(U_i, \Gamma_i), \mathcal{F}_i)$. The holonomy pseudogroup of $(M_i(U_i, \Gamma_i), \mathcal{F}_i)$, $i=1,2$, is equivalent to the pseudogroup of action of $\Gamma_i$ on $N_i$ and the induced morphism is the one defined by the map $\tilde{f}$. Therefore if $\tilde{f}$ is harmonic the lifted map $f$ is transversally harmonic.

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References


