A Solution Formula to ĵ
Using the Bergman–Projection

Georg Schneider

Vienna, Preprint ESI 1210 (2002)  
September 27, 2002

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at
A SOLUTION FORMULA TO \( \overline{\partial} \) USING THE BERGMAN-PROJECTION

GEORG SCHNEIDER

ABSTRACT. We consider the solution operator to the \( \overline{\partial} \)-operator restricted to forms with Fock-space and Bergman-space coefficients. We will develop a general solution formula using arguments of the theory of reproducing Hilbert-spaces. Some existing solution formulas will turn out to be special cases of our result.

1. Preliminaries

In many cases non-compactness of the solution operator already happens when the solution operator is restricted to the corresponding subspace of holomorphic functions. (See [11], [12], [16], [15] and [17].)

It is pointed out in [17] that compactness of the solution operator for \( \overline{\partial} \) on \((0,1)\)-forms implies that the boundary of \( \Omega \) - in this case \( \Omega \) is a bounded convex domain - does not contain any analytic variety of dimension greater or equal to 1. The proof uses that there is a compact solution operator to \( \overline{\partial} \) on \((0,1)\)-forms with holomorphic coefficients. In this case compactness of the solution operator restricted to \((0,1)\)-forms with holomorphic coefficients implies already compactness of the solution operator on general \((0,1)\)-forms.

A similar situation appears in [15] where the Teoplitz \( C^* \)-algebra \( T(\Omega) \) is considered and the relation between the structure of \( T(\Omega) \) and the \( \overline{\partial} \)-Neumann problem is discussed.

Our work is motivated by the fact that in some cases the solution operator can be interpreted as the Hankel operator. See for example [4], [5], [6], [7], [8], [9] and [10].

Our solution operator may be quite helpful for compactness investigations. The question of compactness of the solution operator is of interest for various reasons; see [2] for an excellent survey.

Recall that we understand under the Fock-space \( \mathcal{F} \) the space of holomorphic functions that are square integrable with respect to the weight function \( e^{-|z|^2} \). That is

\[
\text{2000 Mathematics Subject Classification. Primary 32W05; Secondary 32A15.}
\]

\text{Key words and phrases. \( \overline{\partial} \)-Neumann problem, Fock-space, Hankel-operator.}
$\mathcal{F} = \left\{ f : f \text{ is entire and } \int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2} d\lambda(z) < \infty \right\}.$

Let us consider the following notations:

$$
e_k^2 = \int_{\mathbb{C}^n} |z|^2 e^{-|z|^2} d\lambda(z)
:= \int_{\mathbb{C}^n} |z_1|^{2k_1} \cdots |z_n|^{2k_n} e^{-\left(|z_1|^2 + \cdots + |z_n|^2\right)} d\lambda(z_1, \ldots, z_n),$$

Here $k = (k_1, \ldots, k_n)$ is a multiindex.

It is well known [1], that

$$\left\{ u[m](z) = \frac{z^m}{\sqrt{m!}} = \prod_k \frac{z_k^{m_k}}{\sqrt{m_k!}}, m = (m_1, \ldots, m_n) \right\}$$

constitutes a complete orthonormal system for $\mathcal{F}$. The reproducing kernel is given by

$$K(z, w) = \sum_{k=1}^{\infty} \phi_k(z) \overline{\phi_k(w)},$$

where $\{\phi_k\}_{k=1}^{\infty}$ is a complete orthonormal system. It is known [1], that

$$K(z, \omega) = \overline{K(\omega, z)},$$

$$f(z) = \int_{\mathbb{C}^n} K(z, \omega) f(\omega) d\lambda(\omega).$$

Inserting our special orthonormal system one can see, that

$$K(z, w) = \sum_{m} \frac{z^m}{\sqrt{m!}} \overline{w^m} = e^{z \overline{w}}.$$ 

Let $L^2_{(0,q)}(|z|^2)$ be the space of $(0,q)$-forms with coefficients in $L^2(|z|^2)$. That is

$$L^2_{(0,q)}(|z|^2) = \left\{ \sum_{j} f_j d\overline{z}_j; f_j \in L^2(|z|^2) \right\}$$

Here, the prime denotes summation over strictly increasing $q$-tuples $J$, and

$$d\overline{z}_J = d\overline{z}_{i_1} \wedge \cdots \wedge d\overline{z}_{i_q}.$$ 

The norm is

$$\left\| \sum_{j} f_j d\overline{z}_J \right\|^2 = \sum_{j} \int_{\mathbb{C}^n} |f_j|^2 d\mu,$$

with

$$\frac{d\mu}{d\lambda} = e^{-|z|^2}.$$ 

Recall that the $\overline{\partial}$-operator acts via
\[
\mathcal{D} \left( \sum_{j} f_j \overline{d\xi_j} \right) = \sum_{j=1}^{n} \sum_{j} \frac{\partial f_j}{\partial \overline{\xi_j}} \overline{d\xi_j} \wedge d\xi_j.
\]

The derivatives are taken in the distribution sense, and the domain of \( \mathcal{D} \) consists of those \((0, q)\)-forms where the right hand side is in \( L^2_{(0, q+1)}(\Omega) \).

For a global survey of the \( \overline{\partial} \)-operator on the Bergman space see [2], [3].

2. The main result

Several solution formulas to \( \overline{\partial} \) are known at the moment. See for instance [14] for a solution formula restricted to \((0, q)\)-forms with Bergman-space coefficients of a bounded pseudoconvex domain. Furthermore solution formulas can be found as well in [12]. There the special case of \((n, n - 1)\)-forms is considered and \( n \) is the number of different variables on which the coefficients depend and only the restriction to Bergman-space coefficients is considered. The aim of this paper is to develop a solution formula to \( \overline{\partial} \) restricted to \((p, q)\)-forms with Fock-space coefficients. In the discussion after the proof of the main result we will realize the solution formula given in [12] as a special case of our result. It will have the property

\[
(Sv)_j \perp \mathcal{F}
\]

and

\[
\overline{\partial} Sv = v \quad \forall v \in \mathcal{F}_{(p, q)}.
\]

Here \( S \) denotes our solution formula and the second condition just means, that \( S \) is a solution formula to \( \overline{\partial} \) restricted to \((p, q)\)-forms with Fock-space coefficients. Condition (1) is quite similar to the condition that the operator is the canonical solution operator - but it is not quite the same. In many cases this difference is not important for compactness investigations. So the new formula could be quite useful in constructing counterexamples, because we will use the Bergman-projection instead of the projection onto the kernel of \( \overline{\partial} \). This projection can be interpreted as an integral operator and it is much easier to understand.

At the end of the section we will ensure ourselves that the formula of [14] is also valid for \((p, q)\)-forms with Fock-space coefficients and we will see that both formulas coincide in the one-dimensional case. We will also recognize the formula from [12] as a special case of our new one. These considerations are quite instructive, because the proof of our theorem will turn out to be quite "algebraic". A part of the proof dates back to [14]. The difference is just, that we do not make use of the projection formula

\[
1 - P' = S\overline{\partial}.
\]

Instead we will use results from the theory of reproducing Hilbert-spaces.

Let

\[
I - P' = S\overline{\partial}.
\]
\[ H^2_0(C^n, |z|^2) := \text{ker} \overline{\partial} = \{ f \in L^2_0(C^n, |z|^2) : \overline{\partial} f = 0 \}. \]

Since \( \overline{\partial} \) is a densely defined, closed operator,

\[ H^2_0(C^n) = \text{ker} \overline{\partial} \]

is a closed subspace of \( L^2(C^n, |z|^2) \). So the orthogonal projection onto \( H^2_0(C^n, |z|^2) \) exists:

\[ P' : L^2_0(C^n, |z|^2) \rightarrow H^2_0(C^n, |z|^2). \]

Remember that

\[ \mathcal{F}_q = \{ f \in L^2_0(C^n, |z|^2) : f_I \text{ is entire } \forall I = (i_1, \ldots, i_q) \}. \]

The following example shows that \( \mathcal{F}_q \neq H^2_0(C^n, |z|^2) \). Let

\[ f(z_1, z_2) = f_1(z_1, z_2) d\bar{z}_1 + f_2(z_1, z_2) d\bar{z}_2 \]

\[ := (z_1 + \bar{z}_2) d\bar{z}_1 + (z_2 + \bar{z}_1) d\bar{z}_2. \]

Now it follows easily from the definition of \( \overline{\partial} \), that \( \overline{\partial} f = 0 \). So \( f \in H^2_0(C^n, |z|^2) \). Of course \( f \) has not holomorphic coefficients. Now we will formulate the main result of this section:

**Theorem 1.** Let \( f \in \mathcal{F}_{p,q+1} \). Then we have

\[ [S_{q+1}](f) = \frac{1}{q+1} (-1)^p \sum_{|I|=p} \left( \int_{C^n} K(z, w) \right. \]

\[ \times \left( \sum_{|K|=q} \left( g_{I,(i_0)K} d\bar{z}_I \wedge d\bar{z}_K \right) , z - w \right) d\mu(w) \]

is a solution operator to \( \overline{\partial} \) and

\[ [S_{q+1}](f) = \frac{1}{q+1} (-1)^p \sum_{|I|=p} \left( \sum_{i_0=1}^n \left( H^2_0 \left( \sum_{|K|=q} \left( g_{I,(i_0)K} d\bar{z}_I \wedge d\bar{z}_K \right) \right) \right) \right) \]

\[ =: \frac{1}{q+1} (-1)^p \sum_{|I|=p} \left( \sum_{i_0=1}^n \left( (I - P) \left( \sum_{|K|=q} \left( g_{I,(i_0)K} d\bar{z}_I \wedge d\bar{z}_K \right) \right) \right) \right). \]

Here the inner product is defined by

\[ \left( \sum_{|K|=q} \left( g_{I,(i_0)K} d\bar{z}_K \right) , z - w \right) := \sum_{i_0=1}^n \left( \sum_{|K|=q} \left( g_{I,(i_0)K} d\bar{z}_K \right) \right) (\bar{z}_{i_0} - w_{i_0}) \]

and
\[ g_{1,M} := f_1 J e^M \]

with

\[ e^M = sgn(\pi), \]

where \( \pi \) denotes the permutation which maps \( J \) to \( M \).

If there is no such \( \pi \) then set \( e^M = 0 \). Furthermore, let \( g_{1,(i_0)K} = g_{1,J} \) if \( J = (i_0, K) \).

For more details of the notation see the following proof.

**Proof:**

The proof uses some notational tools from [14] (see remarks before). We shall try to work out the idea of the proof: Let

\[ f = \sum_{|J|=q} f_J d\bar{z}_J = \sum_{|J|=q} f_J d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q} \]

be a \((0,q)\)-form with holomorphic coefficients. Let us consider the special case

\[ f_I = f_{(1,\ldots,q)} := f_I \]

and

\[ f_J = 0 \text{ if } J \neq I. \]

Then of course

\[ \overline{\partial} f := \sum_{|J|=q} f_J d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q \]

is a solution to \( \overline{\partial} \), because

(3) \[ \overline{\partial}(\overline{\partial} f) = f_J d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q. \]

Now let us consider a general \((p,q)\)-form:

\[ f = \sum_{|I|=p,|J|=q} f_{I,J} d\bar{z}_J. \]

Let

\[ J^{+1} := \{(i_0,\ldots,i_q) \in \{1,\ldots,n\}^{q+1} : i_0 < \cdots < i_q \} \]

and

\[ M^{+1} := \{(i_0,\ldots,i_q) \in \{1,\ldots,n\}^{q+1} : i_k \neq i_j \text{ for } k \neq j \}. \]

Now we can introduce an equivalence-relation: For \( J, M \in M^{+1} \) define

\[ J \sim M \iff \{j_1,\ldots,j_q\} = \{m_1,\ldots,m_q\} \]
Of course, this is an equivalence-relation on $\mathcal{M}^{q+1}$. For $J \in \mathcal{M}^{q+1}$ let

$$\mathcal{M}_J^{q+1} := \{ M \in \mathcal{M}^{q+1} : J \sim M \}$$

and of course we have

$$\mathcal{M}^{q+1} = \bigcup_{J \in \mathcal{J}^{q+1}} \mathcal{M}_J^{q+1}.$$ 

For a general $(p, q + 1)$-form we infer

$$f = \sum_{|\ell| = p, |\mathcal{J}| = q+1} f_{\ell, \mathcal{J}} d z_{\ell} \wedge d \bar{z}_{\mathcal{J}} = \sum_{|\ell| = p, \mathcal{J} \in \mathcal{J}^{q+1}} f_{\ell, \mathcal{J}} d z_{\ell} \wedge d \bar{z}_{\mathcal{J}}$$

and

$$\sum_{|\ell| = p, \mathcal{J} \in \mathcal{M}^{q+1}} f_{\ell, \mathcal{J}} d z_{\ell} \wedge d \bar{z}_{\mathcal{J}} = \sum_{|\ell| = p, \mathcal{J} \in \mathcal{J}^{q+1}} \sum_{K \in \mathcal{M}_J^{q+1}} f_{\ell, K} d z_{\ell} \wedge d \bar{z}_{K}.$$ 

We have several further consequences of our definitions:

$$\mathcal{M}_J^{q+1} \cap \mathcal{J}^{q+1} = \{ J \}$$

Furthermore, the map

$$M \mapsto J$$

is well defined. Here $M \in \mathcal{M}^{q+1}$ and $J \in \mathcal{M}_M^{q+1} \cap \mathcal{J}^{q+1}$. It is clear, that

$$|\mathcal{M}_J^{q+1}| = (q + 1)!$$

and

$$|\mathcal{M}_J^{q+1}| = (q + 1)! |\mathcal{J}^{q+1}|.$$ 

Now we have overcome the notational problems, so that we can apply the idea of the proof to general $(p, q + 1)$ forms with holomorphic coefficients.

$$(q + 1)! f = (q + 1)! \sum_{|\ell| = p, |\mathcal{J}| = q+1} f_{\ell, \mathcal{J}} d z_{\ell} \wedge d \bar{z}_{\mathcal{J}}$$

$$= (q + 1)! \sum_{|\ell| = p, \mathcal{J} \in \mathcal{J}^{q+1}} f_{\ell, \mathcal{J}} d z_{\ell} \wedge d \bar{z}_{\mathcal{J}}$$

$$= \sum_{|\ell| = p, \mathcal{J} \in \mathcal{J}^{q+1}} \left[ f_{\ell, \mathcal{J}} d z_{\ell} \wedge d \bar{z}_{\mathcal{J}} + \ldots + f_{\ell, \mathcal{J}} d z_{\ell} \wedge d \bar{z}_{\mathcal{J}} \right] \frac{1}{(q+1)!}.$$ 

Now we use $\mathcal{M}_J^{q+1} = \{ M_1, \ldots, \mathcal{M}(q+1) \}$. Furthermore, let $\epsilon_M = s g n(\pi)$ where $\pi$ denotes the permutation which maps $J$ to $M$ if $J \sim M$ and $\epsilon_M = 0$ if $J \neq M$.

So we can see
\[
\sum_{|I|=p, J \in J^{s+1}} \left[ f_{I,J} d^s z_I \wedge d\bar{z}_J + \ldots + f_{I,J} d^{(s+1)} z_I \wedge d\bar{z}_J \right]^{(s+1)!} \\
= \sum_{|I|=p, J \in J^{s+1}} \left[ f_{I,J} c^M_{J} d^s z_I \wedge d\bar{z}_{M_1} + \ldots + f_{I,J} c^M_{J(q+1)} d^{(q+1)} z_I \wedge d\bar{z}_{M_{(q+1)}} \right]^{(q+1)!} \\
= \sum_{|I|=p, J \in J^{s+1}} \sum_{M \in M^{q+1}_J} f_{I,J} c^M_J d^s z_I \wedge d\bar{z}_M.
\]

Let \( g_{I,M} := f_{I,J} c^M_J \) with \( J \in M^{q+1}_J \cap J^{q+1} \). So we infer

\[
\sum_{|I|=p, J \in J^{q+1}} \sum_{M \in M^{q+1}_J} f_{I,J} c^M_J d^s z_I \wedge d\bar{z}_M \\
= \sum_{|I|=p} \sum_{M \in M^{q+1}} g_{I,M} d^s z_I \wedge d\bar{z}_M \\
= \sum_{|I|=p} \left( \sum_{i_0, i_1, \ldots, i_q=1}^n g_{I,i_0,i_1,\ldots,i_q} d^s z_I \wedge d\bar{z}_{i_0,i_1,\ldots,i_q} \right) \\
= \sum_{|I|=p} \left( \sum_{i_0=1}^n \sum_{i_1, \ldots, i_q=1}^n g_{I,(i_0)i_1,\ldots,i_q} d^s z_I \wedge d\bar{z}_{i_0,i_1,\ldots,i_q} \right) \\
= \sum_{|I|=p} \sum_{L \in L^q} \left( \sum_{i_0=1}^n \sum_{K \in J^q} \sum_{L \in L^q} g_{I,(i_0)L} d^s z_I \wedge d\bar{z}_{i_0} \wedge d\bar{z}_L \right) \\
= (-1)^p \sum_{|I|=p} \left( \sum_{i_0=1}^n \sum_{K \in J^q} \sum_{L \in L^q} g_{I,(i_0)L} c^L_K d^s z_I \wedge d\bar{z}_{i_0} \wedge d\bar{z}_L \right) \\
= (-1)^p \sum_{|I|=p} \left( \sum_{i_0=1}^n \sum_{K \in J^q} \left( g_{I,(i_0)L} c^L_K d^s z_I \wedge d\bar{z}_{i_0} \wedge d\bar{z}_L \right) \\
+ \ldots + g_{I,(i_0)L} c^{L_{q'}}_K d^s z_I \wedge d\bar{z}_{i_0} \wedge d\bar{z}_{L_{q'}} \right) \\
\]
\[\begin{align*}
&= (-1)^p \sum_{\mid l \mid = p} \left( \sum_{i_0 = 1}^{n} \sum_{K \in J^i} \left( g_{l,(i_0)K} d\bar{z}_{i_0} \wedge d z_I \wedge d \bar{z}_K + \ldots 
+ g_{l,(i_0)K} d\bar{z}_{i_0} \wedge d z_I \wedge d \bar{z}_K \right) \right) \\
&= (-1)^p \sum_{\mid l \mid = p} \left( \sum_{i_0 = 1}^{n} \sum_{\mid K \mid = q} 'q! g_{l,(i_0)K} d\bar{z}_{i_0} \wedge d z_I \wedge d \bar{z}_K \right) .
\end{align*}\]

Here we used \( \mathcal{M}_L^q = \{ L_1, \ldots, L_q \} \). Now we can easily calculate \( S_{q+1} \):

\[\begin{align*}
(q + 1)! [S_{q+1}] (f) &= (q + 1)! [S_{q+1}] \left( \sum_{\mid l \mid = p} \sum_{\mid J \mid = q} 'f_{l, J} d z_I \wedge d \bar{z}_J \right) \\
&= [S_{q+1}] \left( (-1)^p \sum_{\mid l \mid = p} \sum_{i_0 = 1}^{n} \sum_{\mid K \mid = q} 'q! g_{l,(i_0)K} d\bar{z}_{i_0} \wedge d z_I \wedge d \bar{z}_K \right) \\
&= (-1)^p \sum_{\mid l \mid = p} \sum_{i_0 = 1}^{n} [S_{q+1}] \left( \sum_{\mid K \mid = q} 'q! g_{l,(i_0)K} d\bar{z}_{i_0} \wedge d z_I \wedge d \bar{z}_K \right) \\
&= (-1)^p \sum_{\mid l \mid = p} \sum_{i_0 = 1}^{n} H_{\bar{\omega}} \left( \sum_{\mid K \mid = q} 'q! g_{l,(i_0)K} d\bar{z}_{i_0} \wedge d z_I \wedge d \bar{z}_K \right) \\
\end{align*}\]

Here we used the fact that

\[\begin{align*}
H_{\bar{\omega}} \left( \sum_{\mid K \mid = q} 'g_{l,(i_0)K} d\bar{z}_{i_0} \wedge d z_I \wedge d \bar{z}_K \right) \\
= \sum_{\mid K \mid = q} 'H_{\bar{\omega}} \left( g_{l,(i_0)K} d\bar{z}_{i_0} \wedge d z_I \wedge d \bar{z}_K \right) .
\end{align*}\]

So finally

\[\begin{align*}
[S_{q+1}] (f) (z) &= \frac{1}{q + 1} (-1)^p \sum_{\mid l \mid = p} \sum_{i_0 = 1}^{n} H_{\bar{\omega}} \left( \sum_{\mid K \mid = q} 'g_{l,(i_0)K} d\bar{z}_{i_0} \wedge d z_I \wedge d \bar{z}_K \right) \\
&= \frac{1}{q + 1} (-1)^p \sum_{\mid l \mid = p} \sum_{i_0 = 1}^{n} \int_{\mathcal{B}} K(z, w) \left( \sum_{\mid K \mid = q} 'g_{l,(i_0)K} d\bar{z}_{i_0} \wedge d \bar{z}_K \right) (z_{i_0} - w_{i_0}) \, d\mu(w)
\]
\[
= \frac{1}{q+1} (-1)^p \sum_{|I| = p} \int_{\mathbb{C}^n} K(z, w) \left( \sum_{|K| = q} g_{I, \{\omega\} K} dz_1 \wedge d\overline{z}_K \right) (z_0 - w_0) d\mu(w)
\]

\[
= \frac{1}{q+1} (-1)^p \sum_{|I| = p} \int_{\mathbb{C}^n} K(z, w) \left( \sum_{|K| = q} g_{I, \{\omega\} K} dz_1 \wedge d\overline{z}_K \right) , z - w d\mu(w)
\]

The last calculation finishes the proof. \(\square\)

3. Special cases and further possible investigations

Now we want to compare Theorem 1 to several existing solution formulas. First we want to ensure ourselves, that a similar formula to the one of [14] also holds for the case of \((0, q)\)-forms with Fock-space coefficients. The main ingredient will be the existence of the so called projection formula. The proof turns out to be quite the same as the one of Theorem 1. So we will omit technical difficulties.

Furthermore, the solution formula is also valid in the case of \((p, q)\)-forms with Bergman-space \(B^2(\Omega)\) coefficients. Here

\[
B^2(\Omega) = \left\{ f : f \text{ is holomorphic and } \int_{\Omega} |f(z)|^2 d\lambda(z) < \infty \right\}
\]

and \(\Omega\) is a bounded domain.

The next proposition will yield the projection formula. As mentioned in [14] we just have to convince ourselves that the solution operator can be globally defined on \(\overline{\partial}\)-closed forms which in fact follows from the Hörmander-theory. Nevertheless, we will carry out the proof for the convenience of the reader. First we will formulate the proposition needed from [13].

**Proposition 1.** Let \(\Omega\) be a pseudoconvex open set in \(\mathbb{C}^n\). Then the equation

\[
\overline{\partial} u = f
\]

has (in the sense of distribution theory) a solution \(u \in L^2_{(0,q)}(\Omega, loc)\) for every \(f \in L^2_{(0,q+1)}(\Omega, loc)\) with \(\overline{\partial} f = 0\).

Even more is true [13]:

9
Proposition 2. Let $\Omega$ be a pseudoconvex open set in $\mathbb{C}^n$ and let $\phi$ be a real valued function in $C^2(\Omega)$ such that

$$c \sum_{j=1}^n |w_j|^2 \leq \sum_{j,k=1}^n \frac{\partial^2 \phi}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k,$$

$\forall z \in \Omega$ and $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$.

Here $c$ is a positive continuous function in $\Omega$. If

$$g \in L^2_{(0,1)}(\Omega) \text{ and } \overline{\partial} g = 0,$$

it follows that one can find $u \in L^2_{(0,1)}$ with $\overline{\partial} u = g$ and

$$\int |u|^2 e^{-\phi} \ d\lambda \leq 2 \int |g|^2 \frac{e^{-\phi}}{c} \ d\lambda$$

Proposition 3. Let $P'$ be the projection onto the kernel of $\overline{\partial}$. Then

$$P' = I - S\overline{\partial}.$$

Proof:

Let $g \in L^2_{(0,1)} := L^2_{(0,1)}(\Omega)$ be a closed form. It follows from the proposition above that

$$\exists f \in L^2_{(0,1)} \text{ with } \overline{\partial} f = g.$$ 

Now consider the map

$$S : g \mapsto (I - P')(f) \in H^2_{q}.$$

The rest of the proof is just functional-analysis:

We have

$$P' f = f - S\overline{\partial} f \forall f \in L^2_{q}$$

because

$$\overline{\partial}(f - S\overline{\partial} f) = 0 \Rightarrow f - S\overline{\partial} f \in H^2_{q} \Rightarrow P' f = f - S\overline{\partial} f$$
The same combinatoric argument can be applied to derive in connection with proposition 3 the following theorem:

**Theorem 2.** Let \( f \in \mathcal{F}_{(0,q+1)} \). Then the canonical solution operator \( S_{q+1} \) is given by

\[
[S_{q+1}](f) = \frac{1}{q+1} \sum_{i_0=1}^{n} \left( H_{\overline{\Omega}} \left( \sum_{|K|=q} g_{(i_0)} K \, d\overline{z}_K \right) \right)
\]

\[
= \frac{1}{q+1} \sum_{i_0=1}^{n} \left( (I - P') \overline{\Omega} \left( \sum_{|K|=q} g_{(i_0)} K \, d\overline{z}_K \right) \right).
\]

Here \( P' \) denotes the projection onto the kernel of \( \overline{\mathcal{F}} \), and

\[ g_M := f_J e^M_j \text{ if } J \sim M \]

and

\[ g_M := 0 \text{ if } J \not\sim M. \]

Analyzing the proof of Theorem 1 one can see that only the holomorphy has been used. So we can formulate the same result for the solution operator restricted to forms with Bergman-space coefficients.

**Theorem 3.** Let \( f \in B^2_{(p,q)} \). Then a solution operator \( S_{q+1} \) is given by

\[
[S_{q+1}](f) = \frac{1}{q+1} (-1)^p \sum_{|I|=p} \left( \int_{\Omega} K_{\Omega}(z, w) \right)
\]

\[
\left( \sum_{|K|=q} g_{I,(i_0)} K_{I} \, d\overline{z}_I \wedge d\overline{z}_K, z - w \right) \, d\mu(w)
\]

and

\[
[S_{q+1}](f) = \frac{1}{q+1} (-1)^p \sum_{|I|=p} \sum_{i_0=1}^{n} \left( H_{\overline{\Omega}} \left( \sum_{|K|=q} g_{I,(i_0)} K_{I} \, d\overline{z}_I \wedge d\overline{z}_K \right) \right)
\]

\[
= \frac{1}{q+1} (-1)^p \sum_{|I|=p} \sum_{i_0=1}^{n} \left( (I - P) \overline{\Omega} \left( \sum_{|K|=q} g_{I,(i_0)} K_{I} \, d\overline{z}_I \wedge d\overline{z}_K \right) \right).
\]

Here \( K_{\Omega}(z, w) \) is the reproducing kernel of \( \Omega \).
Several solution formulas are known until now [12]. We can identify them as special cases of Theorem 3:

The canonical solution operator

\[ S_1 : B^2_{(0,1)}(\Omega) \to L^2(\Omega) \]

has the form

\[ S_1(g)(z) = \int_\Omega \mathcal{K}_\Omega(z, w) (g(w), z - w) \, d\lambda(w). \]

Here

\[ (g(w), z - w) = \sum_{j=1}^n g_j(w)(\overline{z}_j - \overline{w}_j) \]

and

\[ z = (z_1, \ldots, z_n), \quad w = (w_1, \ldots, w_n). \]

Remark:

It is clear that our solution operator on \((0,1)\)-forms is the canonical one.

There is a second solution formula in [12] which is a special case of Theorem 3:

Let \( u \) be the \((n, n-1)\)-form

\[ u = \sum_{j=1}^n u_j dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge [d\overline{z}_j] \wedge \cdots \wedge d\overline{z}_n \]

where

\[ u_j(z) = \frac{(-1)^n + j - 1}{n} \int_\Omega (\overline{z}_j - \overline{w}_j) \mathcal{K}_\Omega(z, w) \tilde{\omega}(w) \, d\lambda(w). \]

Then

\[ \overline{\partial} u = \omega \]

where

\[ \omega = \tilde{\omega} \, dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n. \]

4. Acknowledgments

I would like to thank Prof. Haslinger for his guidance during my thesis and for introducing me to the \( \overline{\partial} \)-Neumann problem.
REFERENCES


INSTITUT FÜR MATHEMATIK, UNIVERSITÄT WIEN, STRUDLHOFGASSE 4, A-1090 WIEN, AUSTRIA; GEORG.SCHNEIDER@UNIVIE.AC.AT