Compactness of the Solution Operator to $\delta$
on the Fock–Space in Several Dimensions

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COMPACTNESS OF THE SOLUTION OPERATOR TO $\overline{T}$
ON THE FOCK-SPACE IN SEVERAL DIMENSIONS

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Abstract. We consider the solution operator to $\overline{T}$ restricted to $(0,1)$-forms with coefficients in the Fock-space and more generally, weighted spaces of entire functions. We will show that the solution operator restricted to $(0,1)$-forms with coefficients in weighted spaces of entire functions can be interpreted as a Hankel-operator. Furthermore we will be able to investigate continuity and compactness of the solution operator. Unlike for one complex dimension, compactness fails in the case of several complex variables.

1. Preliminaries

It is well known that in many cases compactness of the solution operator to $\overline{T}$ only depends on the behavior of the solution operator on the closed subspace of forms with holomorphic coefficients (see [11],[12],[18],[17] and [19]).

In [19], compactness of the solution operator in the case of bounded convex domains is investigated. It is shown in [19] that compactness of the solution operator for $\overline{T}$ on $(0,1)$-forms implies that the boundary of $\Omega$ - in this case $\Omega$ is a bounded convex domain - does not contain any analytic variety of dimension greater or equal to 1. The proof requires the existence of a compact solution operator to $\overline{T}$ on $(0,1)$-forms with holomorphic coefficients. So in the case of bounded convex domains compactness of the solution operator restricted to $(0,1)$-forms with holomorphic coefficients implies already compactness of the solution operator on general $(0,1)$-forms.

A similar situation appears in [17] where the Toeplitz $C^\ast$-algebra $\mathcal{T}(\Omega)$ is considered and the relation between the structure of $\mathcal{T}(\Omega)$ and the $\overline{T}$-Neumann problem is discussed.

First, let us consider the Fock space:

$$\mathcal{F} = \left\{ f : f \text{ is entire and } \int_{\mathbb{C}^n} |f(z)|^2 e^{-\|z\|^2} d\lambda(z) < \infty \right\}.$$

Let
\[ e_k^2 = \int_{\mathbb{C}^n} |z|^{2k} e^{-|z|^2} d\lambda(z) \]
\[ = \int_{\mathbb{C}^n} |z_1|^{2k_1} \cdots |z_n|^{2k_n} e^{-(|z_1|^2 + \cdots + |z_n|^2)} d\lambda(z_1, \ldots, z_n). \]

Here \( k = (k_1, \ldots, k_n) \) is a multi-index. A special orthonormal system of \( \mathcal{F} \) is given by (see [1])

\[ \left\{ a_m(z) = \frac{z^m}{\sqrt{m!}} = \prod_k \frac{z_k^{m_k}}{\sqrt{m_k!}}, m = (m_1, \ldots, m_n) \right\}. \]

The reproducing kernel of \( \mathcal{F} \) is given by

\[ K(z, w) = \sum_{k=1}^{\infty} \phi_k(z) \overline{\phi_k(w)}, \]

where \( \{\phi_k\}_{k=1}^{\infty} \) is a complete orthonormal system. It is known, [1], that

\[ K(z, \omega) = \overline{K(\omega, z)}. \]

Inserting our special orthonormal system infers, that

\[ K(z, w) = \sum_{m=1}^{\infty} \frac{z^m}{\sqrt{m!}} \frac{\overline{w}^m}{\sqrt{m!}} = e^{\overline{w}z}. \]

Let \( L^2_{(0,q)}(|z|^2) \) be the space of \((0,q)\)-forms with coefficients in \( L^2(|z|^2) \). That is,

\[ L^2_{(0,q)}(|z|^2) = \left\{ \sum_J f_J d\bar{z}_J; f_J \in L^2(|z|^2) \right\}. \]

Here, the prime denotes summation over strictly increasing \( q \)-tuples \( J \), and

\[ d\bar{z}_J = d\bar{z}_{J_1} \wedge \cdots \wedge d\bar{z}_{J_q}. \]

The norm is

\[ \left\| \sum_J f_J d\bar{z}_J \right\|^2 = \sum_J \left\| f_J \right\|^2 d\mu. \]

with

\[ \frac{d\mu}{d\lambda} = e^{-|z|^2}. \]

Recall that the \( \bar{\partial} \)-operator acts via

\[ \bar{\partial} \left( \sum_J f_J d\bar{z}_J \right) = \sum_{j=1}^n \sum_J \frac{\partial f_J}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_j. \]
The derivatives are taken in the distribution sense, and the domain of $\overline{\partial}$ consists of those $(0, q)$-forms where the right hand side is in $L^2_{(0,q+1)}(|z|^2)$. For a global survey of the $\overline{\partial}$-operator on the Bergman space see [2], [3].

We will show that the canonical solution operator to the $\overline{\partial}$-equation on the Fock-space can in a certain way be interpreted as the Hankeloperator with the symbol $\tau$,

$$H_{\tau}(g) = (I - P)(\overline{g}).$$

Similar investigations have been made for the solution operator restricted to $(0, 1)$-forms with Bergmanspace-coefficients - see [4],[5],[6],[7],[8],[9] and [10] - and in the one-dimensional case for the Fock-space [11].

2. The solution operator as Hankel-operator

First we have to overcome some notational difficulties:

Definition 1. Let

$$\sqrt{\mathcal{F}} = \{ f : f \in \mathcal{F} \text{ and } \exists g \in \mathcal{F} \text{ with } f^2 = g \}.$$ 

Remark 1:

Consider the Hankel-operator with symbol $f$. That is,

$$H_f : L^2(\Omega, \phi) \rightarrow B^2(\Omega, \phi) \uparrow$$

$$H_f(g) = (I - P)(fg).$$

Here $P$ denotes the Bergmanprojection of $L^2(\Omega, \phi)$ onto the closed subspace $B^2(\Omega, \phi)$ and $\phi$ is a suitable weight function. If $\Omega$ is bounded and $f$ is continuous, we have no further problems with the domain and continuity of the Hankel-operator.

In the case of the Fock-space we have $\Omega = \mathbb{C}^n$ and $\phi = |z|^2$. The meaning of the set $\sqrt{\mathcal{F}}$ lies in the following consideration:

$$M_{\tau}g := \overline{g}.$$ 

Then the image of $\sqrt{\mathcal{F}}$ under $M_{\tau}$ lies in $L^2(\Omega, |z|^2)$, since

$$\int_{\mathbb{C}^n} |f|^2|\overline{z}|^2 e^{-|z|^2} d\lambda(z) \leq \int_{\mathbb{C}^n} |f|^2 |e^{-|z|^2} d\lambda(z) \int_{\mathbb{C}^n} |\overline{z}|^4 |e^{-|z|^2} d\lambda(z)$$

$$= \|g\| \|z^2\| < \infty.$$ 

Here $f^2 = g$ and $g \in L^2(\mathbb{C}^n, |z|^2)$.

Remark 2:
Every monom of the form $z^k$ belongs to $\sqrt{\mathcal{F}}$, since $z^k$ and $z^{2k}$ belong to $\mathcal{F}$. Moreover, the set $\sqrt{\mathcal{F}}$ is clearly dense in $\mathcal{F}$.

**Remark 3:**

In general multiplication-operators with continuous symbols are not globally defined. Consider for example $M_z$. A counterexample is easily constructed via taylor-expansion ([1]): Let

$$f(z) = \sum_m a[m]z^m.$$  

Then the norm can be expressed in the following way:

$$(f, f) = \sum_m [m!]^2 |a[m]|^2$$

Now choose

$$f(z) = \sum_m \frac{1}{m!(m+1)!} z^m.$$  

Then $f \in \mathcal{F}$, because

$$(f, f) = \sum_m \frac{1}{m!(m+1)!} < \infty,$$

but

$$(zf, zf) = \sum_m \frac{1}{m+1} = \infty.$$  

**Theorem 1.**

$$(1 - P)(\pi f) := H\pi f|_{\sqrt{\mathcal{F}}} = S_1|_{\sqrt{\mathcal{F}}}$$  

Here $S_1$ denotes the canonical solution operator to $\overline{\partial}$ on (0,1)-forms with holomorphic coefficients. Moreover, $S_1$ can be viewed as $L^2$-limit of the images of suitable elements of $\sqrt{\mathcal{F}}$.

**Proof:**

$$v(z) := \sum_{i=1}^n \overline{z_i}g_i(z)$$

It follows that

$$\overline{\partial}v = \sum_{i=1}^n \overline{\frac{\partial v}{\partial \overline{z}_i}}d\overline{z}_i$$

$$= \sum_{i=1}^n g_i d\overline{z}_i = g.$$  

The canonical solution operator is the one with minimum norm. In other words: a solution operator that is orthogonal to the kernel of $\overline{\partial}$. The kernel is exactly the Fock-space, so the Bergman projection and the projection onto the kernel of $\overline{\partial}$ coincide. Moreover, the solution operator is well defined.
It follows from [13], that the solution operator is bounded. Of course - as shown above - it has to coincide with the Hankel-operator on the dense set \( \sqrt{F_{(0,1)}} \). As a consequence of this the last statement of the theorem is obvious. \( \square \)

**Remark 4:**

We can use the Theorem 1 to interpret the canonical solution operator on \((0,1)\)-forms as an integral-operator - at least on the dense set \( \sqrt{F} \):

\[
S_{1}(g)(z) = (I - P)(\overline{\omega \mu})
\]

\[
= \sum_{i=1}^{n} \overline{z}_{i} g_{i}(z) - \int_{\mathbb{C}^{n}} K(z, w) \sum_{i=1}^{n} \overline{w}_{i} g_{i}(w) \, d\mu(w)
\]

\[
= \int_{\mathbb{C}^{n}} \sum_{i=1}^{n} \overline{w}_{i} g_{i}(w) K(z, w) - \left( \sum_{i=1}^{n} \overline{w}_{i} g_{i}(w) \right) K(z, w) \, d\mu(w)
\]

\[
= \int_{\mathbb{C}^{n}} K(z, w) (g(w), z - w) \, d\mu(w).
\]

Here we have made the following convention: For

\[
g(w) = \sum_{i=1}^{n} g_{i}(w) d\overline{z}_{i},
\]

let

\[
(g(w), z - w) := \sum_{i=1}^{n} g_{i}(w)(z_{i} - w_{i}).
\]

**Remark 5:**

In [11], a slightly different approach to the problem is chosen. It is shown that in the one dimensional case the map

\[
f \to F : A^{2}(|z|^{2}) \to L^{2}(|z|^{2}),
\]

where

\[
f(z) = \sum_{k=0}^{\infty} a_{k} z^{k}
\]

and

\[
F(z) = \overline{z} \sum_{k=0}^{\infty} a_{k} z^{k} - \sum_{k=0}^{\infty} \frac{\epsilon_{k}}{e_{k-1}} a_{k} z^{k-1},
\]
is a continuous linear operator, representing the canonical solution operator to $\overline{\partial}$ restricted to $A^2(\mathbb{C}, |z|^2)$.

We claim that Theorem 1 yields the same result—even in the more-dimensional case:
We have

$$f(z) = \lim_{n \to \infty} \sum_{k=0}^{n} a_k z^k$$

It follows from continuity of $S_1$ and theorem [1], that

$$S_1(f)(z) = S_1 \left( \lim_{n \to \infty} \sum_{k=0}^{n} a_k z^k \right)$$

$$= \lim_{n \to \infty} S_1 \left( \sum_{k=0}^{n} a_k z^k \right) = \lim_{n \to \infty} H_\overline{\pi} \left( \sum_{k=0}^{n} a_k z^k \right)$$

$$= \lim_{n \to \infty} (I - P) \left( \sum_{k=0}^{n} a_k z^k \right)$$

The claim follows from the following proposition. Theorem 1 just uses the known continuity of $S_1$ and the definition of the set $\sqrt{\mathcal{F}}$. So it overcomes the problems arising by investigating the multiplication-operator with $\overline{z}$. As a consequence of this the proof is more straight forward—compare [11] Proposition 1 and Proposition 2.

We now want to calculate the Bergman-projections of certain functions explicitly. In the following calculation we will make use of the taylor-expansion of the reproducing-kernel. Furthermore, we will need the following notation: Let $k = (k_1, \ldots, k_n)$ be a multi-index. Then let

$$k + 1_j = (k_1, \ldots, k_j + 1, \ldots, k_n).$$

**Proposition 1.** Let $f$ be the $(0, 1)$-form $f = z^j d\overline{z}_1$ where $j$ is a multi-index. Then

$$P(f \overline{z})(w) = \frac{c_j^2}{c_{j-1_1}^2} w^{j-1_1}.$$ 

**Proof:**
\[ P(f \overline{z})(w) = \int_{\mathbb{C}^n} K(w, z)(f(z), z) \, d\mu(z) \]
\[ = \int_{\mathbb{C}^n} \sum_{\mathbb{C}^m} \frac{z^n}{c^2_m} \, w^m \, (z_1, z^2) \, d\mu(z) \]
\[ = \int_{\mathbb{C}^n} \sum_{\mathbb{C}^m} \frac{z^{n+1}}{c^2_m} \, z^j \, w^m \, d\mu(z) \]
\[ = \sum_{\mathbb{C}^m} \frac{1}{c^2_m} \, w^m \, \int_{\mathbb{C}^n} \frac{z^{n+1}}{z^j} \, d\mu(z) \]
\[ = \sum_{\mathbb{C}^m} \frac{1}{c^2_m} \, w^m \, (z^j, z^{m+1}) \, \mathcal{F} \]
\[ = \sum_{\mathbb{C}^m} \frac{c^2_{m+1}}{c^2_m} \, \delta_{m+1, j} \, w^m = \frac{c^2_j}{c^2_{j-1}} \, w^{j-1}. \]

\[ \square \]

3. Noncompactness of \( S_1 \) on \((0, 1)\)-forms with coefficients in \( \mathcal{F} \)

We now want to show that the canonical solution-operator is not compact. In the one dimensional case the result can be found in [11].

**Theorem 2.** Let \( \{u_m(z)\} = \left\{ \frac{1}{c^2_m} \, d\overline{z}_1, m = (m_1, \ldots, m_n), m_1, \ldots, m_n \in \mathbb{N} \right\} \). Then

\[ S_1^* S_1(u_m)(w) = \left( \frac{c^2_{m+1}}{c^2_m} - \frac{c^2_m}{c^2_{m-1}} \right) \, u_m(w) \]

if \( m_1 > 1 \).

**Proof:** By Remark 4 we know that

\[ S_1(g)(z) = (I - P)(\overline{g}) \]
\[ = \int_{\mathbb{C}^n} K(z, w)(g(w), z - w) \, d\mu(w), \]

and the equalities hold at least for all \( g' \in \sqrt{\mathcal{F}} \) and \( g = g'd\overline{z}_1 \). One easily infers that

\[ S_1^*(g)(z) = -(I - P)(zg)d\overline{z}_1 \]
\[ = \int_{\mathbb{C}^n} K(w, z)(g(w), \overline{z} - \overline{w}) \, d\mu(z)d\overline{z}_1, \]

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because
\[
(S_1(g), h) = \int_{\mathbb{C}^n} \overline{h}(z) \int_{\mathbb{C}^n} K(z, w) (g(w), z - w) \, d\mu(w) \, d\mu(z)
\]
\[
= \int_{\mathbb{C}^n} K(z, w) \left( \int_{\mathbb{C}^n} g_1(w) (z_1 - w_1) \overline{h}(z) \, d\mu(z) \right) \, d\mu(w)
\]
\[
\int_{\mathbb{C}^n} g_1(w) \left( \int_{\mathbb{C}^n} K(w, z) \overline{h}(z) (z_1 - w_1) \, d\mu(z) \right) \, d\mu(w)
\]
\[
= (g, S_1^*(h))
\]

We know from Proposition 1, that
\[
P(fz)(w) = \frac{c_m^2}{c_{m-1}^2} w^{m-1}.
\]
So it follows
\[
S_1(u_m)(z) = \overline{u_m}(z) - \frac{c_m}{c_{m-1}^2} \frac{z^{m-1}}{c_{m-1}^2}.
\]
Hence
\[
S_1^* S_1(u_m)(w) = \int_{\mathbb{C}^n} K(w, z) \left( \overline{u_m}(z) - \frac{c_m}{c_{m-1}^2} \frac{z^{m-1}}{c_{m-1}^2} \right) (z_1 - w_1) \, d\mu(z)
\]
\[
= \int_{\mathbb{C}^n} \left( \sum \frac{z^k}{\sqrt{|k|!} \sqrt{|k|!}} \right) \left( \overline{u_m}(z) - \frac{c_m}{c_{m-1}^2} \frac{z^{m-1}}{c_{m-1}^2} \right) (z_1 - w_1) \, d\mu(z)
\]
and
\[
\int_{\mathbb{C}^n} \left( \sum \frac{w^k}{\sqrt{|k|!} \sqrt{|k|!}} \right) \left( \overline{u_m}(z) - \frac{c_m}{c_{m-1}^2} \frac{z^{m-1}}{c_{m-1}^2} \right) (z_1) \, d\mu(z)
\]
\[
= \int_{\mathbb{C}^n} \frac{z^{m+1}}{c_m} \left( \sum \frac{w^k}{\sqrt{|k|!} \sqrt{|k|!}} \right) \, d\mu(z)
\]
\[
- \frac{c_m}{c_{m-1}^2} \int_{\mathbb{C}^n} \frac{z^m}{c_m} \left( \sum \frac{w^k}{\sqrt{|k|!} \sqrt{|k|!}} \right) \, d\mu(z)
\]
\[
= \frac{w^m}{c_m} \int_{\mathbb{C}^n} |z|^{2m+1} \, d\mu(z) - \frac{w^m}{c_{m-1}^2} \int_{\mathbb{C}^n} |z|^{2m} \, d\mu(z)
\]
\[
= \left( \frac{c_{m+1}^2}{c_m^2} - \frac{c_m^2}{c_{m-1}^2} \right) u_m(w).
\]
Of course,
\[
\begin{align*}
\int_{\mathbb{C}^n} \left( \sum_{k} \frac{z^{[k]} \bar{w}^{[k]}}{\sqrt{|k|!} \sqrt{|k|!}} \right) \left( \overline{z_1 u_m(z)} - \frac{c_m z^{m-1}}{e_{m-1}} \right) (w_1) \, d\mu(z) \\
= w_1 \int_{\mathbb{C}^n} \left( \sum_{k} \frac{z^{[k]} \bar{w}^{[k]}}{\sqrt{|k|!} \sqrt{|k|!}} \right) \left( \overline{z_1 u_m(z)} - \frac{c_m z^{m-1}}{e_{m-1}} \right) d\mu(z) \\
= w_1 P(I - P)(z_1 u_m(z)) = 0.
\end{align*}
\]

\[\square\]

Now we can see, that $S_1$ is not compact on $(0,1)$-forms with Fock-space coefficients.

**Corollary 1.** $S_1$ is not compact on $(0,1)$-forms with Fock-space coefficients.

**Proof:**

By Theorem 2
\[
S_1^* S_1 (u_m)(w) = \left( \frac{c_{m+1}^2}{e_m^2} - \frac{c_m^2}{e_{m-1}^2} \right) u_m(w)
\]
if $m_1 > 1$.

The definition of the norm - inner product - on $(0,1)$-forms
\[
\left\| \sum_{j} f_j dz_j \right\|^2 = \sum_{j} \int_{\mathbb{C}^n} |f_j|^2 \, d\mu,
\]
shows that the set
\[
\{ u_m(z) \} = \left\{ \frac{z^m}{c_m}dz, m = (m_1, \ldots, m_n), m_1, \ldots, m_n \in \mathbb{N} \right\}
\]
is orthonormal.

Now remember the definition of $e_k$.
\[
e_k^2 = \int_{\mathbb{C}^n} |z|^{2k} e^{-|z|^2} \, d\lambda(z)
\]
\[
:= \int_{\mathbb{C}^n} |z_1|^{2k_1} \cdots |z_n|^{2k_n} e^{-(|z_1|^2 + \cdots + |z_n|^2)} \, d\lambda(z_1, \ldots, z_n)
\]
It is easily seen - standard calculation needed - that
\[
e_k = k!.
\]

Now
\[ \frac{c^2_{m+1}}{c^2_m} - \frac{c^2_m}{c^2_{m-1}} = \frac{m+1}{m!} - \frac{m!}{m-1} = m_1 + 1 - m_1 = 1. \]

It is known that \( S_1 \) is compact if and only if \( S_1^* S_1 \) is compact ([14]). \( S_1^* S_1 \) cannot be compact, because it has the special eigenvectors to the eigenvalue 1, which makes compactness impossible. See [16] or [15] for an excellent survey. \( \square \)

4. **A second proof for the non-compactness of \( S_1 \) on \( \mathcal{F}_{(0,1)} \)**

We want to give a second proof for the results of the last section. The idea of the following proof dates back to [18]. [18], investigates the compactness of the solution operator \( S \) and the Neumann-operator \( \mathcal{N} \) on the bidisc. That is,

\[ \mathbb{D} \times \mathbb{D} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}. \]

It turns out that both are not compact. In [18] a more general case is studied as well, namely the one of Reinhardt domains with flat boundary neighborhood. At the end of the section we will make some concluding remarks, which are quite instructive from a meta-mathematical point of view.

**Theorem 3.** \( S_1 \) is not compact on \((0,1)\)-forms with Fock-space coefficients, if the dimension of the coefficients is bigger than \(2\) - e.g. the functions depend on \( \geq 2 \) variables (second proof).

**Proof:**

Let

\[ u_{[m]}(z) = \frac{z^m}{\sqrt{|m|}} = \prod_k \frac{z_k^{m_k}}{\sqrt{m_k!}}. \]

Recall that this functions constitute a complete orthonormal-system. Now define

\[ \alpha_m(z, z') := u_{[m]}(z) d\overline{z}'. \]

Then

\[ \| \alpha_m \|_{\mathcal{F}_{(0,1)}} = \| u_{[m]} \| = \epsilon_0. \]

Clearly it follows with

\[ v_m(z, z') := u_{[m]}(z) \overline{z}' \]

that

\[ \overline{\partial} v_m = \alpha_m, \]

because
\[
\overrightarrow{v}_m = \sum_{i=1}^{n} \frac{\partial u_{[m]}}{\partial \overline{z}_i} d\overline{z} + u_{[m]}(z) d\overline{z} \\
= 0 + \alpha_m = \alpha_m
\]

Now we claim that we have even found the canonical solution operator. Indeed: Let \( h \in \mathcal{F} \). Then

\[
\int_{\mathbb{C}^{n+1}} u_m(z) \overline{h(z, z')} d\mu(z, z') \\
= c \int_{\mathbb{C}^{n}} u_m(z) \int_{\mathbb{C}} \overline{h(z, z')} e^{-|z'|^2 + |z|^2} d\overline{z'} \wedge dz' \wedge d\overline{z} \wedge dz = 0
\]

The last statement is for readers, that are not that familiar with the theory of Hilbert spaces of holomorphic functions, not obvious. So we take a closer look at the argument.

Let us consider

\[
\int_{\mathbb{C}} \overline{h(z, z')} e^{-|z'|^2} d\overline{z'} \wedge dz'.
\]

Using polar coordinates \( z' = re^{i\phi} \) one can see

\[
\int_{\mathbb{C}} \overline{h(z, z')} e^{-|z'|^2} d\overline{z'} \wedge dz' \\
= \lim_{n \to \infty} \int_{K_n(0)} \overline{h(z, z')} e^{-|z'|^2} d\overline{z'} \wedge dz' \\
= \lim_{n \to \infty} \int_{0}^{n} r e(r) \int_{\partial K_n(0)} \overline{h(z, z')} d\lambda(\phi) d\lambda(r) \\
= \lim_{n \to \infty} c \int_{0}^{n} 0 \times e(r) d\lambda(r) \\
= 0
\]

Since

\[\|v_m\|_{\mathcal{F}} = d \text{ and } v_m \perp v_{m'} \text{ for } m \neq m',\]

it follows that \( S_1 \) is not compact. Here \( d \) is a constant. \qed
Remark 6:

It is quite interesting that

\[ v_m(z, z') := u_{[m]}(z) \overline{z'} \]

has no holomorphic component. In fact in the last section it turned out, that

\[ P(f \overline{z})(w) = \frac{c^2}{c^2_{j-1}} w^j, \]

with \( f = z^j d \overline{z_j} \). The difference is, that \( u_{[m]}(z) \) is not a function in \( z' \) while \( f \) is one of \( z_1 \).

In [18], the compactness of the Neumann-operator is investigated as well. A main ingredient is the connection

\[ S = \overline{\partial}^* N \]

and the fact that \( N \) is the globally defined inverse of the densely defined operator

\[ \Box = \overline{\partial} \partial^* + \partial \overline{\partial} \]

and the form of \( \overline{\partial}^* = \frac{\partial}{\partial z} \). It seems to be more difficult to investigate the situation in the case of weighted spaces of entire functions. It is known ([13]), that in this case

\[ \overline{\partial}^* = -e^{\phi} \frac{\partial}{\partial z} (e^{-\phi} \alpha) \]

\( \forall \alpha \in Dom(\overline{\partial}^*) \) with

\[ Dom(\overline{\partial}^*) := \left\{ \alpha \in L^2(\mathbb{C}^n, \phi) : \frac{\partial \alpha}{\partial z} \in L^2(\mathbb{C}^n, \phi) \right\} \]

5. Other weighted spaces of entire functions

In this chapter we want to investigate the solution operator in a more general context. We do not restrict our attention to the case of \((0, 1)\)-forms with Fock-space coefficients. We will introduce other reproducing Hilbertspaces - namely the holomorphic functions that are in \( L^2(\mathbb{C}^n, |z|^m) \) for \( m \in \mathbb{N} \). The one dimensional case of this investigations can be found in [1].

The main difficulty - in comparison to the special case of the Fock-space \((m = 2)\) - will be that an analogy to Theorem 1 is not that easy to find. Concretely, the continuity of the solution operator is not so clear - it may be possible to adopt the Hörmander-theory, but it is not as obvious as in the case of \( m = 2 \), where the fact
\[ \frac{\partial^2 (|z|^2)}{\partial z \partial \overline{z}} = 1 \]

is a main ingredient.

It will turn out that the solution operator \( S \) is not compact for the case \( m \geq 2 \).

First we have to set some notations:

**Definition 2.**

\[ \mathcal{F}^m = \left\{ f : f \text{ is entire and } \int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^m} \, d\lambda(z) < \infty \right\} \]

\[ c_{k,m}^2 = \int_{\mathbb{C}^n} |z|^{2k} e^{-|z|^m} \, d\lambda(z) \]

\[ := \int_{\mathbb{C}^n} |z_1|^{2k_1} \cdots |z_n|^{2k_n} e^{-|z_1|^m + \cdots + |z_n|^m} \, d\lambda(z_1, \ldots, z_n) \]

*Here* \( k = (k_1, \ldots, k_n) \) is a multi-index.

As mentioned above one cannot expect a result like Theorem 1, but actually such a strong result is not needed for compactness investigations. We can therefore restrict our attention to the solution operator restricted to the elements of a complete orthonormal-system. A similarly result is valid - be aware that the \( c_{k,m} \) are not the same, they depend on the weight as indicated by the second index.

It is easy to see that these spaces \( (\mathcal{F}^m) \) are closed subspaces of the corresponding \( L^2 \)-spaces and that there exists a reproducing kernel, that has the form

\[ K(w, z) = \sum \frac{z^j \overline{w}^j}{c_{k,m}^2}. \]

**Proposition 2.** Let \( f \) be the \((0,1)\)-form \( f = z^j d\overline{z} \), where \( j \) is a multiindex. Then

\[ P(f \overline{z})(w) = \frac{c_{j,m}^2}{c_{j-1,m}^2} w^j. \]

**Proof:**

The proof is essentially the same as the one of Proposition 1. It is easily realized that it makes no difference - apart from notational problems - to consider forms of the form \( z^j d\overline{z} \) instead of \( z^j d\overline{z}_1 \). \( \Box \)
We can proof the non-compactness of the solution operator for the case \( m > 2 \). The proof is similar to the one of the case \( m = 2 \). We will only give a sketch of the proof and especially of the part, which is different.

**Theorem 4.** \( S_1^* S_1 \) is bounded on monoms of the form \( z_j \bar{d}z_j \) and therefore it can be viewed as a globally defined operator \( A \). \( S_1 \) fails to be compact on \( (0,1) \)-forms with \( F^m \)-coefficients for \( n > 1 \).

**Proof:**

As in the proof of Theorem 2 one can easily see that

\[
S^* S(u_n)(w) = \left( \frac{c_{n+1,j,m}^2}{c_{n,m}^2} - \frac{c_{n,m}^2}{c_{n-1,j,m}^2} \right) u_n(w)
\]

if \( n_1 > 1 \), where

\[
\{u_{m,n}(z)\} = \left\{ \frac{z^n}{c_{n,m}} \, d\bar{z}_j \right\}.
\]

Just note that for \( g = f \, d\bar{z}_j \),

\[
S(g)(z) = (I - P)(\bar{z}g)
\]

\[
= \int_{\mathbb{C}^n} K(z, w)(g(w), z - w) \, d\mu(w),
\]

\[
S^*(g)(z) = -(I - P)(zg)
\]

\[
= \int_{\mathbb{C}^n} K(w, z)(g(w), \bar{z} - \bar{w}) \, d\mu(w),
\]

and proceed exactly as in the proof of Theorem 2.

It still remains to show that

\[
\left( \frac{c_{n+1,j,m}^2}{c_{n,m}^2} - \frac{c_{n,m}^2}{c_{n-1,j,m}^2} \right) \to 0
\]

as \( n_j \to \infty \).

It is obvious - see the second part of the proof - that we just have to consider the one-dimensional case:

\[
\left( \frac{c_{n+1,m}^2}{c_{n,m}^2} - \frac{c_{n,m}^2}{c_{n-1,m}^2} \right) \to 0
\]

An easy substitution in the definition of the \( c_{n,m} \) yields that
\[
\Gamma \left( \frac{2n+4}{m} \right) \Gamma \left( \frac{2n+2}{m} \right) = \frac{\Gamma \left( \frac{2n+2}{m} \right)}{\Gamma \left( \frac{2n}{m} \right)} \to 0
\]

remains to be shown.

It follows from Stirlings formula

\[
n! \approx \left( \frac{n}{e} \right)^n \sqrt{2 \pi n},
\]

that (1) is true for \( m > 2 \). So the boundedness condition is clear.

Is is easily seen, that there are infinitely many orthogonal eigenvectors to
the same eigenvalue, so compactness fails. This follows from the definition
of the \( e_{k,m} \):

\[
e_{k,m}^2 = \int_{\mathbb{C}^n} |z_1|^{2k_1} \ldots |z_n|^{2k_n} e^{-\sum_{j=1}^n |z_j| m^j} \, d\lambda(z_1, \ldots, z_n).
\]

Here \( k = (k_1, \ldots, k_n) \) is a multi-index.

Now fix \( k_1 \) and consider the \((0,1)\)-form \( z^k d\overline{z}_1 \). The \( e_{k,m} \) can be calculated
more directly

\[
e_{k,m}^2 = \prod_{i=1}^n \int_{\mathbb{C}} |z_i|^{2k_1} e^{-|z_i| m} \, d\lambda(z_i)
\]

by just using the following argument inductively.
\[ c_{k_1,m}^2 := \left( \int_{\mathbb{C}} |z_1|^{2k_1} e^{-|z_1|^m} d\lambda(z_1) \right). \]

It follows with induction that

\[
c_{k,m}^2 = \prod_{i=1}^{n} \int_{\mathbb{C}} |z_i|^{2k_i} e^{-|z_i|^m} d\lambda(z_i)
= \prod_{i=1}^{n} c_{k_i,m}^2.
\]

Now we can compute

\[
\begin{align*}
\left( \frac{c_{k_1+1,m}^2}{c_{k_1,m}^2} - \frac{c_{k_1,m}^2}{c_{k_1-1,m}^2} \right)
&= \left( \frac{c_{k_1+1,m}^2 \prod_{i=2}^{n} c_{k_i,m}^2}{\prod_{i=1}^{n} c_{k_i,m}^2} - \frac{\prod_{i=1}^{n} c_{k_i,m}^2}{c_{k_1-1,m}^2 \prod_{i=2}^{n} c_{k_i,m}^2} \right) \\
&= \left( \frac{c_{k_1+1,m}^2}{c_{k_1,m}^2} - \frac{c_{k_1,m}^2}{c_{k_1-1,m}^2} \right).
\end{align*}
\]

Observing that this differences only depend on \( k_1 \) we see that there are infinitely many orthogonal eigenvectors to the same eigenvalue - namely

\[ \{ z^{k} d\overline{z} : (k_2, \ldots, k_n) \in \mathbb{N}^{n-1} \} \]

for the eigenvalue

\[ \frac{c_{k_1+1,m}^2}{c_{k_1,m}^2} - \frac{c_{k_1,m}^2}{c_{k_1-1,m}^2}. \]

This finishes the proof. \( \square \)

Up to this point we have not been able to investigate boundedness of the solution operator itself. The following Theorem will give an answer to this question. The idea of the proof is from [11].

**Theorem 5.** \( S_1 \) is bounded on the monoms of the form \( z^{k} d\overline{z} \) and therefore \( S_1 \) can be viewed as a globally defined bounded operator \( A \).
Proof:

We just compute the norms explicitly - we abbreviate $c_{n,m} = c_n$:

$$
\| S(u_n) \|^2 = \frac{1}{e_{n,m}^2} \int_{C^n} \left| \bar{z}_1 z^n - \frac{c_{n,m}^2}{c_{n-1,m}^2} z^{n-1} \right| e^{-|z|^m} \, d\lambda(z)
$$

$$
= \frac{1}{e_{n,m}^2} \int_{C^n} |z|^{2n-2|z|} \left( |z_1|^4 - \frac{2c_{n}^2 |z_1|^2}{c_{n-1}^2} + \frac{c_{n}^4}{c_{n-1}^4} \right) e^{-|z|^m} \, d\lambda(z)
$$

$$
= \frac{1}{e_{n,m}^2} \int_{C^n} |z|^{2n+2|z|} e^{-|z|^m} \, d\lambda(z) - \frac{2}{e_{n-1,m}^2} \int_{C^n} |z|^{2n} e^{-|z|^m} \, d\lambda(z)
$$

$$
+ \frac{e_{n}^2}{e_{n-1}^4} \int_{C^n} |z|^{2n-2|z|} e^{-|z|^m} \, d\lambda(z)
$$

$$
= \frac{c_{n+1}^2}{c_n^2} - \frac{c_{n}^2}{c_{n-1}^2}
$$

Just as in the proof of Theorem 4 one can see that the sequence

$$
\frac{c_{n+1}^2}{c_n^2} - \frac{c_{n}^2}{c_{n-1}^2}
$$

is bounded, because it tends to 0.

\[ \square \]

Remark 7:

In [11], a similar argument is used to show that the canonical solution operator on $(0,1)$-forms with holomorphic coefficients fails to be Hilbert-Schmidt.

6. Acknowledgments

I would like to thank Prof. Haslinger for his guidance during my thesis and for introducing me to the $\overline{\partial}$-Neumann problem.

References

[3] L. Hörmander, $L^2$ estimates and existence theorems for the $\overline{\partial}$ operator, Acta Mathematica 113 (1965), 89-152.

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