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Abstract

The classic Poincaré inequality bounds the $L^2$-norm of a function, $f$, orthogonal to a given function $g$ in a domain $\Omega$, in terms of some $L^p$-norm of its gradient in $\Omega$. Suppose we now remove a set $\Gamma$ from $\Omega$ and concentrate our attention on $\Lambda = \Omega \setminus \Gamma$. This new domain might not even be connected and hence no Poincaré inequality can generally hold for it. This is so even if the volume of $\Gamma$ is arbitrarily small. A Poincaré inequality does hold, however, if one makes the additional assumption that $f$ has a finite $L^p$-norm on the whole of $\Omega$, not just on $\Lambda$. The important point is that the Poincaré inequality thus obtained bounds the $L^2$-norm of $f$ in terms of the $L^p$-norm of the gradient on $\Lambda$ (not $\Omega$) plus an additional term that goes to zero as the volume of $\Gamma$ goes to zero. This error term depends on $\Gamma$ only through its volume.

Another direction in which we generalize the Poincaré inequality is to the operator $\nabla + iA(x)$ in place of the usual $\nabla$. (Here, $A$ is a given vector field.) Along the way we present some conjectures and open problems.

1 Introduction

The simplest Poincaré inequality refers to a bounded, connected domain $\Omega \subset \mathbb{R}^n$, and a function $f \in L^2(\Omega)$ whose distributional gradient is also in $L^2(\Omega)$ (namely, $f \in W^{1,2}(\Omega)$). While it is false that there is a finite constant $S$, depending only on $\Omega$, such that

$$\int_{\Omega} |f|^2 \leq S \int_{\Omega} |\nabla f|^2,$$

for all $f$, such an inequality does hold if we impose the additional condition that $\int_{\Omega} f = 0$. The constant $S$ depends on $\Omega$, but it is independent of $f$. In fact, $1/S$ is the second eigenvalue of the Laplacian in $\Omega$ with Neumann boundary conditions. This is merely a consequence of Bessel’s inequality.

A simple generalization of (1) is to replace the condition $\int_{\Omega} f = 0$ by the condition $\int_{\Omega} fg = 0$, where $g$ is any $L^2(\Omega)$ function that is not orthogonal to the lowest Neumann

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eigenfunction of the Laplacian, i.e., $\int_{\Omega} g = 1$. Then (1) becomes

$$\int_{\Omega} |f - \int_{\Omega} fg| \leq S' \int_{\Omega} |\nabla f|^2,$$

(2)

where $S' \geq S$ depends on $g$ as well. However, $S' = S$ when $g = 1/|\Omega|$, where $| \cdot |$ generally denotes volume (Lebesgue measure) of a set. We will meet a similar, but different condition on $g$ in Corollary 3.

This inequality can be generalized. See, e.g., [1, Thms. 8.11 and 8.12].

**Theorem 1 (Poincaré inequalities for $W^{1,p}(\Omega)$).** Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, open set with the cone property. Let $1 \leq q \leq \infty$ and let $\max \{1, qn/(n + q)\} \leq p \leq \infty$ if $q < \infty$ and $n < p \leq \infty$ if $q = \infty$. Let $g$ be a function in $L^p(\Omega)$, $p' = p/(p - 1)$, such that $\int_{\Omega} g = 1$. Then there is a constant $S_{p,q} > 0$, which depends on $\Omega$, $g$, $p$, $q$, such that for any $f \in W^{1,p}(\Omega)$

$$\left\| f - \int_{\Omega} fg \right\|_{L^q(\Omega)} \leq S_{p,q} \| \nabla f \|_{L^{p'}(\Omega)}.$$

(3)

**Remarks.** $W^{1,p}(\Omega)$ consists of those complex valued functions in $L^p(\Omega)$ whose distributional derivatives are also functions in $L^p(\Omega)$. The limiting case $q = np/(n - p)$ requires Sobolev’s inequality in the proof, and hence the inequality is known as the Poincaré-Sobolev inequality in this case. One implication of (3) is that $f \in L^p(\Omega)$ if $\nabla f \in L^p(\Omega)$. The domain $\Omega$ is required to have the “cone property” (see, e.g., [2]), i.e., each point of $\Omega$ is the vertex of a spherical cone with fixed height and angle, and which is situated in $\Omega$. Note that the constants $S_{p,q}$ depend not only on the volume of $\Omega$, but also on its shape. The Poincaré inequality is often presented as (3) with $q = p$ and with $g = 1/|\Omega|$. In this case $\int_{\Omega} fg$ is usually written as $\overline{f}$ or $\langle f \rangle$. A generalization to $W^{m,p}(\Omega)$ for $m > 1$ is possible (see, e.g., [1]).

Now we turn to the problem that concerns us in this paper. It was motivated by a treatment of the quantum mechanical many-body problem, specifically, the first proof of Bose-Einstein condensation in a physically realistic situation [3]. Further developments require a version of the Poincaré inequality in which $\nabla$ is replaced by a connection on a $U(1)$ bundle, namely $\nabla \rightarrow \nabla + iA$, where $A$ is a one-form, or vector field. The generalization to this situation leads to a proof of superfluidity for the same physical system [6]. These extensions of the classic Poincaré inequality are interesting in their own right and raise further questions in analysis. Our main results are Theorems 3 and 4.

The problem concerns the following obstruction to the use of the Poincaré inequality: Let us remove a small set $\Gamma$ from $\Omega$ and concentrate our attention on $\Lambda = \Omega \setminus \Gamma$. This new domain might not even be connected and hence no Poincaré inequality can generally hold for $\Lambda$, no matter how small $|\Gamma|$ might be.

A trivial example is to let $\Omega$ be a unit square in $\mathbb{R}^2$ and to let $\Gamma$ be a thin annulus in $\Omega$ of outer radius $1/2$ and inner radius $1/2 - \varepsilon$. Take $g(x) = 1$. We can take $f = 1$ inside the disc of radius $1/2 - \varepsilon$ and $f = 0$ elsewhere. Thus, regardless of how small $\varepsilon$ may be, the right side of (3), with $\Omega$ replaced by $\Lambda$, will be zero while the left side is positive. Another, perhaps more interesting example is one in which $\Lambda$ is connected but fails to satisfy Theorem 1 because the cone property is absent. This can be accomplished with a small $\Gamma$ that is topologically a ball, but which has a sufficiently rough surface (see, e.g., [1, Sect. 8.7]).
The smallness of $|\Gamma|$ cannot restore the Poincaré inequality in $\Lambda$. A generalized Poincaré inequality does hold, however, if one makes the additional assumption that $f$ has a finite $L^p$ gradient norm on the whole of $\Omega$, not just on $\Lambda$. The important point is that the Poincaré inequality thus obtained bounds the $L^p$-norm of $f$ in terms of the $L^p$ gradient norm on $\Lambda$ (not $\Omega$) plus an additional error term. Furthermore — and this is also important — the effective Poincaré inequality that holds for $\Lambda$ approaches that given in (3) as the volume of $\Gamma$ tends to zero — in a manner that depends only on $|\Gamma|$ and not on its shape.

The inequalities in this paper can obviously be extended in various ways, e.g., to smooth compact manifolds [4], weighted Sobolev spaces [5] or $W^{m,p}(\Omega)$ for $m > 1$, but we resist the temptation to do so here. In fact, in a physics application Theorem 3 is needed for a cube in $\mathbb{R}^3$ with a pair of opposite faces identified [6].

## 2 The Generalized Poincaré Inequalities

We begin by stating the least general formuates of our main Theorems 3 and 4, but one with the advantage that we can obtain it simply from Theorem 1.

**Theorem 2 (First generalization of Poincaré inequalities).** Let $\Omega$, $g$, $p$, $q$ be as in Theorem 1, and let \( \tilde{q}_n = \max\{1, qn/(n + q)\} \). Let $\Lambda \subset \Omega$ be a measurable subset of $\Omega$ and let $\Gamma = \Omega \setminus \Lambda$.

There are constants $S^{p,q}$ (generally different from $S_{p,q}$), depending only on $\Omega$, $g$, $p$ and $q$, but not on $\Lambda$, such that for all $f \in W^{1,p}(\Omega)$

$$
\left\| f - \int_{\Omega} f g \right\|_{L^q(\Omega)} \leq S^{p,q} \left[ \left\| \nabla f \right\|_{L^p(\Lambda)} + |\Omega|^{1/p-1/\tilde{q}_n} \left\| \nabla f \right\|_{L^q(\Gamma)} \right] \tag{4}
$$

if $1 \leq q < \infty$ and $\tilde{q}_n \leq p \leq \infty$. One can take $S^{p,q} = S_{\tilde{q}_n,q}|\Omega|^{1/\tilde{q}_n-1/p}$.

For $q = \infty$, there exists constants $\hat{S}^{p,r}$ such that

$$
\left\| f - \int_{\Omega} f g \right\|_{L^\infty(\Omega)} \leq \hat{S}^{p,r} \left[ \left\| \nabla f \right\|_{L^p(\Lambda)} + |\Omega|^{1/p-1/r} \left\| \nabla f \right\|_{L^r(\Gamma)} \right] \tag{5}
$$

for all $n < r \leq p \leq \infty$. One can take $\hat{S}^{p,r} = S_{p,\infty}|\Omega|^{1/r-1/p}$.

The reader might justly wonder how the volume of $\Gamma$ plays a role in the error term, as claimed in the Introduction. The following corollary displays this dependence; its proof applies mutatis mutandis to Theorems 3 and 4 as well.

**Corollary 1 (Explicit volume dependence).** Under the assumptions of Theorem 2

$$
\left\| f - \int_{\Omega} f g \right\|_{L^q(\Omega)} \leq S^{p,q} \left[ \left\| \nabla f \right\|_{L^p(\Lambda)} + \left( \frac{|\Gamma|}{|\Omega|} \right)^{1/\tilde{q}_n-1/p} \left\| \nabla f \right\|_{L^q(\Gamma)} \right] \tag{6}
$$

if $1 \leq q < \infty$ and $\tilde{q}_n \leq p \leq \infty$.

For $q = \infty$,

$$
\left\| f - \int_{\Omega} f g \right\|_{L^\infty(\Omega)} \leq \hat{S}^{p,r} \left[ \left\| \nabla f \right\|_{L^p(\Lambda)} + \left( \frac{|\Gamma|}{|\Omega|} \right)^{1/r-1/p} \left\| \nabla f \right\|_{L^r(\Gamma)} \right] \tag{7}
$$

for all $n < r \leq p \leq \infty$. 

Proof of Corollary 1. Apply Hölder’s inequality to the rightmost norms in (4) and (5). □

Remarks.

1. As a special case, we can assume that $g(x) = 0$ for $x \in \Gamma$ in Corollary 1, and use the simple fact that $\| \cdot \|_{L^q(\Omega)} \leq \| \cdot \|_{L^q(\mathcal{A})}$ to obtain
\[
\left\| f - \int_{\mathcal{A}} fg \right\|_{L^q(\Omega)} \leq S^{p,q} \left[ \left\| \nabla f \right\|_{L^p(\mathcal{A})} + \left( \frac{|\Gamma|}{|\Omega|} \right)^{1/q - 1/p} \left\| \nabla f \right\|_{L^p(\Gamma)} \right]
\] (8)
when $q < \infty$, and similarly for (5). The virtue of (8) is that it is an inequality that depends only on $\mathcal{A}$, except for an error term. We emphasize again that the constants $S^{p,q}$ do not depend on $\mathcal{A}$, but only on $\Omega$ and $g$.

2. A weaker inequality is obtained by substituting $\| \nabla f \|_{L^p(\Omega)}$ for $\| \nabla f \|_{L^p(\Omega)}$ on the right sides of (6) and (7). In this way the dependence on $\Gamma$ is solely through its volume (for any given value of $\| \nabla f \|_{L^p(\Omega)}$).

3. Theorem 2 is a corollary of Theorem 1. This is in contrast to our general results, Theorems 3 and 4, which do not appear to follow directly from Theorem 1. We leave it as an open problem to deduce Theorems 3 and 4 as corollaries of Theorem 1, if possible.

4. For $1 \leq q_n/(q + n) < n$ the exponents of $|\Gamma|$ appearing in Corollary 1 are optimal. This can be seen as follows. If $f$ is supported in a small ball of volume $|\Gamma|$, the corresponding minimal ‘energy’ $\| \nabla f \|_{L^p} / \| f \|_{L^q}$ is of the order $|\Gamma|^{1/p - 1/q - 1/n}$. Inequality (6) cannot hold for a larger exponent, since an $f$ supported on two non-intersecting small balls of volume $|\Gamma|$, with $\int f g = 0$, would violate (6) for small enough $|\Gamma|$.

If $q = \infty$ or $q < n/(n - 1)$ the optimal dependence on the volume of $\Gamma$ remains an open problem.

Proof of Theorem 2. By Theorem 1 with $1 \leq q < \infty$ we have that
\[
\left\| f - \int_{\mathcal{A}} fg \right\|_{L^q(\Omega)} \leq S^{q_n,q} \left\| \nabla f \right\|_{L^{q_n}(\Omega)} \cdot
\] (9)
We estimate the right side by the triangle inequality
\[
\left\| \nabla f \right\|_{L^{q_n}(\Omega)} \leq \left\| \nabla f \right\|_{L^{q_n}(\mathcal{A})} + \left\| \nabla f \right\|_{L^{q_n}(\Gamma)} \cdot
\] (10)
Hölder’s inequality implies that, for any $p \geq q_n$,
\[
\left\| \nabla f \right\|_{L^{q_n}(\mathcal{A})} \leq \| f \|_{L^p(\mathcal{A})} |\mathcal{A}|^{1/q_n - 1/p}.
\] (11)
Using $|\mathcal{A}| \leq |\Omega|$ this proves (4), with $S^{p,q} = S^{q_n,q} |\Omega|^{1/q_n - 1/p}$. The same proof works for $q = \infty$, with $\tilde{S}^{p,q} = S_{p,\infty} |\Omega|^{1/p - 1/p}$. □
Our second theorem generalizes the Poincaré inequality in another direction, namely the inclusion of a vector field (or connection) \( A \). For simplicity we assume that \( A \) is a bounded, measurable function \( A : \Omega \to \mathbb{R}^n \), but a weaker condition will certainly suffice (see, e.g., [1, Sect. 7.20]).

We start by formulating this generalization in the \( L^2 \) to \( L^2 \) setting, as in (2). Let us replace \( \nabla \) by \( \nabla + iA(x) \) on the right side of (1) and (2) and let \( E_A \) denote the lowest Neumann eigenvalue of \( H = -((\nabla + iA(x))^2) \) on \( \Omega \), i.e.,

\[
E_A = \inf \left\{ \| (\nabla + iA) f \|_{L^2(\Omega)}^2 : f \in W^{1,2}(\Omega), \| f \|_{L^2(\Omega)} = 1 \right\}.
\]  

(12)

There is a finite number (which could be greater than one) of \( L^2(\Omega) \) normalized eigenfunctions corresponding to this \( E_A \), as shown in the appendix.

In the following, we will restrict ourselves to the case \( n \geq 2 \) because \( n = 1 \) presents nothing new: \( A \) can be eliminated by a unitary transformation, namely \( f(x) \to f(x) \exp(-i \int_\Omega A(y) dy) \).

When \( A = 0 \) the multiplicity of \( E_A \) is 1, and the eigenfunction is \( \phi_0 = \) constant. In this case the replacement of \( f \) by \( f - \int f g \) as in (2) has the same qualitative effect as restricting the inequality to functions \( f \) whose \( L^2(\Omega) \) distance to \( \phi_0 \) is bounded below by a fixed multiple of the \( L^2(\Omega) \) norm of \( f \). The fixed multiple depends on \( g \), of course, but so does the constant \( S' \) appearing in (2).

For \( A \neq 0 \) the multiplicity can be greater than 1, and we adopt the second viewpoint in this case. First, we define the ground state manifold \( M_A \) to be the linear subspace spanned by the eigenvectors corresponding to \( E_A \). We obtain an inequality for functions whose distance \( d_A \) to \( M_A \) exceeds a certain value \( \delta > 0 \), i.e.,

\[
d_A(f) := \inf_{\phi \in M_A} \| f - \phi \|_{L^2(\Omega)} \geq \delta \| f \|_{L^2(\Omega)}.
\]  

(13)

Because of the discrete spectrum of \( H \), it is true for such functions that

\[
\int_\Omega |f(x)|^2 \, dx \leq S_\delta \left[ \int_\Omega |(\nabla + iA(x)) f|^2 \, dx - E_A \int_\Omega |f(x)|^2 \, dx \right],
\]  

(14)

with \( S_\delta \) depending on \( \delta, A \) and \( \Omega \). Explicitly, for \( \delta = 1 \) we have \( 1/S_1 = E_A^{(2)} - E_A \), where \( E_A^{(2)} \) is the smallest eigenvalue above \( E_A \).

If \( M_A \) is one-dimensional, spanned by the normalized eigenfunction \( \phi_A \), we can go back to our original formulation and take \( g \) to be an \( L^2(\Omega) \) function satisfying \( \int_\Omega g \phi_A = 1 \), (see the discussion after (1)). An easy generalization of (2) is then

\[
\int_\Omega \left| f(x) - \phi_A(x) \left[ \int_\Omega f g \right] \right|^2 \, dx \leq S_\delta \left[ \int_\Omega |(\nabla + iA(x)) f|^2 \, dx - E_A \int_\Omega |f(x)|^2 \, dx \right].
\]  

(15)

If \( g = \overline{\phi_A} \) then the optimal constant is \( 1/S_\delta = E_A^{(2)} - E_A \).

Our goal is to generalize (14) to the case of a punctured \( \Omega \). This will be done in Theorem 3. Before doing this, we need the following Lemma. It establishes the Poincaré inequalities for functions that vanish on a set of positive measure.

**Lemma 1 (Poincaré inequalities for functions with small support).** Let \( \Omega, p \) and \( q \) be as in Theorem 1, and let \( 0 < \delta < 1 \). Then there is a finite number \( \overline{S}_{p,q} > 0 \), which depends on \( \Omega, \delta, p, q \), such that for any \( f \in W^{1,p}(\Omega) \) with \( \{ x : f(x) \neq 0 \} \leq \overline{S}_{p,q}(1 - \delta) \)

\[
\| f \|_{L^q(\Omega)} \leq \overline{S}_{p,q} \| \nabla f \|_{L^p(\Omega)}.
\]  

(16)
Proof. Since $\Omega$ is bounded it suffices to prove this Lemma for the largest possible $q$, given $p$. In particular, it is sufficient to consider the case $q \geq 1$. From Theorem 1 we know that
\[
\left\| f - \frac{1}{|\Omega|} \int_{\Omega} f \right\|_{L^q(\Omega)} \leq S_{p,q} \left\| \nabla f \right\|_{L^p(\Omega)}
\]
for the $p$'s and $q$'s in question. By the triangle inequality
\[
\left\| f \right\|_{L^q(\Omega)} \leq \left\| f - \frac{1}{|\Omega|} \int_{\Omega} f \right\|_{L^q(\Omega)} + |\Omega|^{1/q-1} \left\| \int_{\Omega} f \right\|. \tag{18}
\]
By Hölder’s inequality and the assumption on the support of $f$
\[
\left\| \int_{\Omega} f \right\| \leq \left\| f \right\|_{L^q(\Omega)} \left\{ x : f(x) \neq 0 \right\}^{1-1/q} \leq \left\| f \right\|_{L^q(\Omega)} \left( (1 - \delta) |\Omega| \right)^{1-1/q}.
\]
Inserting (18) and (19) in (17) we arrive at
\[
\left\| f \right\|_{L^q(\Omega)} \leq S_{p,q} \left( 1 - (1 - \delta)^{1-1/q} \right)^{-1} \left\| \nabla f \right\|_{L^p(\Omega)}. \tag{20}
\]
\]

\textbf{Theorem 3 (Second $L^2$ generalization of Poincaré inequality).} For $n \geq 2$ let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, open set that has the cone property, and let $E_A$ and $M_A$ be as explained above. Let $\Lambda \subset \Omega$ be a measurable subset of $\Omega$, $\Gamma = \Omega \setminus \Lambda$, and let $0 < \delta \leq 1$. For any $\varepsilon > 0$ there exists a positive constant $C$, depending on $\Omega$, $A$, $\delta$ and $\varepsilon$ (but not on $\Lambda$ and $\Gamma$) such that, for every $f \in W^{1,2}(\Omega)$ satisfying $d_A(f) \geq \delta \left\| f \right\|_{L^2(\Omega)}$
\[
\left\| (\nabla + iA)f \right\|^2_{L^2(\Omega)} + C \left( (\nabla + iA)f \right)^2_{L^2(\Gamma)} \geq \left( \frac{1}{S_\varepsilon + \varepsilon} + E_A \right) \left\| f \right\|^2_{L^2(\Omega)}, \tag{21}
\]
where $r = 2n/(n+2)$ and $S_\varepsilon$ is the optimal constant in (14).

The following is similar to Corollary 1, and follows immediately from Theorem 3 by using Hölder’s inequality.

\textbf{Corollary 2 (Explicit volume dependence).} Under the same assumptions as in Theorem 3,
\[
\left\| (\nabla + iA)f \right\|^2_{L^2(\Lambda)} + C |\Lambda|^{2/n} \left( (\nabla + iA)f \right)^2_{L^2(\Gamma)} \geq \left( \frac{1}{S_\varepsilon + \varepsilon} + E_A \right) \left\| f \right\|^2_{L^2(\Omega)}, \tag{22}
\]

\textbf{Proof of Theorem 3.} Assume that the assertion of the theorem is false. Then there exists a sequence of triples $(C_j, f_j, \Gamma_j)$, with $\int_{\Omega} |f_j|^2 = 1$, $d_A(f_j) \geq \delta$ and $\lim_{j \to \infty} C_j = \infty$, such that
\[
\lim_{j \to \infty} \left( \left\| (\nabla + iA)f_j \right\|^2_{L^2(\Lambda_j)} + C_j \left( (\nabla + iA)f_j \right)^2_{L^2(\Gamma_j)} \right) < 1/S_\varepsilon + E_A, \tag{23}
\]
where we denoted $\Lambda_j = \Omega \setminus \Gamma_j$. This implies that $\left\| (\nabla + iA)f_j \right\|_{L^2(\Gamma_j)} \to 0$ as $j \to \infty$.

We claim that it is no restriction to assume that $\lim_{j \to \infty} |\Gamma_j| = 0$. If this is not the case, define $\gamma_j \subset \Gamma_j$ by
\[
\gamma_j = \left\{ x \in \Gamma_j : \left\| (\nabla + iA(x))f_j(x) \right\| \geq \left\| (\nabla + iA)f_j \right\|^{1/2}_{L^2(\Gamma_j)} \right\}. \tag{24}
\]
Note that $|\gamma_j| \leq \|(\nabla + iA)f_j\|_{L^2(\Gamma_j)}^{1/2} \rightarrow 0$ as $j \rightarrow \infty$. Moreover,
\[
\|(\nabla + iA)f_j\|_{L^2(\Gamma_j \setminus \gamma_j)}^2 \leq |\Omega|\|(\nabla + iA)f_j\|_{L^2(\Gamma_j)} ,
\] (25)
which also goes to zero as $j \rightarrow \infty$. Therefore (23) holds with $\Lambda_j$ and $\Gamma_j$ replaced by $\Lambda_j \cup (\Gamma_j \setminus \gamma_j)$ and $\gamma_j$, respectively.

It is therefore enough to consider the case $\lim_{j \rightarrow \infty} |\Gamma_j| = 0$. By passing to a subsequence we can assume that $\sum_j |\Gamma_j|$ is finite.

For some fixed $N$ let $\Sigma_N = \Omega \setminus \bigcup_{j \geq N} \Gamma_j$. Note that $f_j$ is bounded in $L^2(\Omega)$ and $(\nabla + iA)f_j$ is bounded in $L^2(\Sigma_N)$ and, therefore, we can choose a subsequence such that $f_j \rightharpoonup f$ weakly in $L^2(\Omega)$ and also $(\nabla + iA)f_j \rightharpoonup (\nabla + iA)f$ weakly in $L^2(\Sigma_N)$. By weak lower semicontinuity of norms and the fact that $\Sigma_N \subset \Lambda_j$ for $j \geq N$,
\[
\liminf_{j \rightarrow \infty} \int_{\Lambda_j} |(\nabla + iA)f_j|^2 \geq \liminf_{j \rightarrow \infty} \int_{\Sigma_N} |(\nabla + iA)f_j|^2 \geq \int_{\Omega} |(\nabla + iA)f|^2 .
\] (26)
This holds for all $N$ and, since $\Sigma_N \subset \Sigma_{N+1}$ and $|\bigcup_N \Sigma_N| = |\Omega|$,
\[
\liminf_{j \rightarrow \infty} \int_{\Lambda_j} |(\nabla + iA)f_j|^2 \geq \int_{\Omega} |(\nabla + iA)f|^2 .
\] (27)

Suppose that we knew that $f_j \rightharpoonup f$ strongly in $L^2(\Omega)$. Then clearly $d_A(f) \geq \delta \|f\|_{L^2(\Omega)} = \delta$, so the right side of (27) would be $\geq 1/S_\delta + E_A$ by (14), and thereby contradict (23) and establish (21).

In the following, we will show that $f_j \rightharpoonup f$ strongly in $L^2(\Omega)$. For $M > 0$, define
\[
f_j^M(x) = \min \{ M, |f_j(x)| \}
\] (28)
and
\[
h_j^M(x) = |f_j(x)| - f_j^M(x).
\] (29)
Note that both $f_j^M$ and $h_j^M$ are in $W^{1,2}(\Omega)$. Moreover,
\[
\left| \left\{ x : h_j^M(x) \neq 0 \right\} \right| = \left| \left\{ x \in \Omega : |f_j(x)| > M \right\} \right| \leq \frac{\|f_j\|_{L^2(\Omega)}^2}{M^2} = \frac{1}{M^2} .
\] (30)
By choosing $M$ larger that $2|\Omega|^{-1/2}$ we can use Lemma 1 to conclude that
\[
\|h_j^M\|_{L^2(\Omega)} \leq S \|
abla h_j^M\|_{L^\infty(\Omega)}
\] (31)
for some constant $S$ independent of $M$ and $j$. Note that the intersection of the two sets $\alpha_j := \{ x : \nabla f_j^M(x) \neq 0 \}$ and $\beta_j := \{ x : \nabla h_j^M(x) \neq 0 \}$ has measure zero. Therefore
\[
\|\nabla h_j^M\|_{L^\infty(\Omega)} = \|\nabla f_j\|_{L^\infty(\beta_j)} \leq \| (\nabla + iA)f_j\|_{L^2(\alpha_j)} |\beta_j|^{1/n} + \| (\nabla + iA)f_j\|_{L^2(\Gamma_j)} ,
\] (32)
where we used again Hölder’s inequality and also the diamagnetic inequality $|\nabla |f(x)| \leq |(\nabla + iA)(x)f(x)|$ (see [1, Thm. 7.21]). By (30), $|\beta_j| \leq 1/M^2$. This fact, together with (31), (32), and (23) implies that
\[
\limsup_{j \rightarrow \infty} \|h_j^M\|_{L^2(\Omega)} \leq S(E_A + 1/S_\delta)^{1/2} M^{-2/n} .
\] (33)
From (23) we see that $(\nabla + iA)f_j$ is a bounded sequence in $L^r(\Omega)$ and, since $A$ is bounded by assumption, the same is true for $\nabla f_j$. Hence we can apply the Rellich-Kondrashov theorem (see, e.g., [7, Thm. 6.2]) to conclude that, modulo choosing a subsequence, $f_j \to f$ strongly in $L^{2-\nu}(\Omega)$ for any $0 < \nu \leq 1$, and therefore

$$\int_\Omega |f|^{2-\nu} = \lim_{j \to \infty} \int_\Omega |f_j|^{2-\nu}. \quad (34)$$

By definition of $f_j^M$,

$$\int_\Omega |f_j|^{2-\nu} \geq \int_\Omega |f_j^M|^{2-\nu} \geq \frac{1}{M^\nu} \int_\Omega |f_j^M|^2. \quad (35)$$

Using (33) we therefore obtain

$$\int_\Omega |f|^{2-\nu} \geq \frac{1}{M^\nu} \left(1 - S(E_A + 1/S_\delta)^{1/2} M^{-2/n}\right)^2,$$  
and hence

$$\int_\Omega |f|^2 = \lim_{\nu \to 0} \int_\Omega |f|^{2-\nu} \geq \left(1 - S(E_A + 1/S_\delta)^{1/2} M^{-2/n}\right)^2. \quad (37)$$

Since $M$ can be chosen arbitrarily large, this shows that $\|f\|_{L^2(\Omega)} = 1$, and finishes the proof.

Remarks.

1. By the same argument as in Remark 4 after Theorem 2 the exponent $2/n$ in (22) is sharp, i.e., it cannot be increased. This shows also that (3) cannot hold for $r < 2n/(n+2)$.

2. As we stated before, $A$ can be eliminated when $n = 1$. Theorem 3 is nevertheless true in this case, but with $r = 1$ instead of $2n/(n+2) = 2/3$. It represents an improvement over Theorem 2 because of the optimal constant in (21). Also, (22) is true in this case, with an exponent 1. It is natural to conjecture that (22) holds up to the critical exponent $2 = 2/n$ for $n = 1$. Unfortunately, the method of proof presented here does not allow for this generalization.

3. The constant appearing on the right side of (21) is optimal, up to an $\varepsilon$. This is in contrast to Theorem 2, in which the constant $S^{p,q}$ is left unspecified. The simple proof that we gave of Theorem 2 does not allow us to relate $S^{p,q}$ to the optimal constant for the usual Poincaré inequality for $A = \Omega$ (although we can relate it to $S_{\delta,q}$). Thus, even in the $A = 0\}$ case, the more complicated proof of Theorem 3 has the advantage of yielding information about the sharp constant.

It is also clear that (21) cannot hold for $\varepsilon = 0$. The constant $C'$ has to go to infinity as $\varepsilon \to 0$. Otherwise, the inequality would be violated by the $f$ that yields equality in (14).

We can now use the result of Theorem 3 to get an inequality of the form (15) that holds for all $f \in W^{1,2}(\Omega)$. The proof is obtained by replacing $f$ in (22) by $f - \phi_A \int_\Omega f g$ and using the Cauchy-Schwarz inequality.
Corollary 3 (Analogue of Corollary 1). Let $\Omega$, $\Gamma$, $\Lambda$ and $E_A$ be as in Theorem 3, and suppose that $\mathcal{M}_A$ is one-dimensional, spanned by the normalized eigenfunction $\phi_A$ corresponding to $E_A$. Let $g$ be an $L^2(\Omega)$ function satisfying $\int_{\Omega} g \phi_A = 1$. For any $\varepsilon > 0$ there exists a positive constant $C$ depending on $\Omega$, $A$, $g$ and $\varepsilon$ (but not on $\Lambda$ and $\Gamma$) such that, for every $f \in W^{1,2}(\Omega)$,

$$
\|(\nabla + iA)f\|_{L^2(\Omega)}^2 + C|\Gamma|^{2/n} \|(\nabla + iA)f\|_{L^2(\Gamma)}^2 + C \left| \int_\Omega fg \right| \|(\nabla + iA)f\|_{L^2(\Gamma)} \|(\nabla + iA)\phi_A\|_{L^2(\Gamma)}^2 \geq E_A \|f\|_{L^2(\Omega)}^2 + \frac{1}{S^\gamma + \varepsilon} \|f - \phi_A \int_{\Omega} fg\|_{L^2(\Omega)}^2, \tag{38}
$$

with $S^\gamma$ the optimal constant in (15).

Remarks.

1. For regular enough boundary of $\Omega$ it follows from elliptic regularity that $(\nabla + iA)\phi_A$ is in fact a bounded function [8]. This allows us to replace $\|(\nabla + iA)\phi_A\|_{L^2(\Gamma)}$ by const. $|\Gamma|^{1/2}$. In any case, $\|(\nabla + iA)\phi_A\|_{L^2(\Gamma)}$ goes to zero as $|\Gamma| \to 0$.

2. As in Remark 1 after Theorem 2 one can consider the special case where $g$ vanishes on $\Gamma$ to obtain an inequality that depends on $\Gamma$ only via its volume. It has to be noted, however, that $E_A$ is defined on the whole of $\Omega$ and not just on $\Lambda$.

3. Strictly speaking, Corollary 3 is not really a corollary of Theorem 3 because of the optimal constant $S^\gamma$ appearing in (38). From (22) we can only infer (38) with $S^\gamma$ replaced by $S_\delta$, for some $\delta$ depending on $g$. However, by imitating the proof of Theorem 2 one can show that (38) holds.

Analogous to Theorem 2, Theorem 3 has a natural extension to general $p$'s and $q$'s, such that the Rellich-Kondrashov theorem holds. We also restrict ourselves to the case $p > 1$, i.e., we consider the cases $1 \leq q \leq \infty$ and $\max\{1, qn/(n + q)\} < p < \infty$. This excludes the critical case $p = \max\{1, qn/(n + q)\}$, and the possibility of this extension is an open problem.

In this general case the energy $E_A^{p,q}$ is defined by

$$
E_A^{p,q} = \inf \left\{ \|(\nabla + iA)f\|_{L^p(\Omega)} : f \in W^{1,p}(\Omega), \|f\|_{L^q(\Omega)} = 1 \right\}, \tag{39}
$$

in analogy with (12). The ground state manifold $\mathcal{M}_A^{p,q}$ is given by the set of minimizers of (39), which can be shown to be non-empty by applying the Rellich-Kondrashov theorem. (Here $p > 1$ is important.) Note that, in general, this will not be a linear space.

The relevant distance to $\mathcal{M}_A^{p,q}$ is now the $L^q(\Omega)$ distance

$$
d_A^{p,q}(f) := \inf_{\phi \in \mathcal{M}_A^{p,q}} \|f - \phi\|_{L^q(\Omega)}. \tag{40}
$$

For functions with $d_A^{p,q}(f) \geq \delta \|f\|_{L^q(\Omega)}$, the inequality

$$
\|f\|_{L^q(\Omega)}^p \leq S_\delta^{p,q} \left[ \|(\nabla + iA)f\|_{L^p(\Omega)}^p - E_A^{p,q} \|f\|_{L^p(\Omega)}^p \right], \tag{41}
$$
holds for some positive $S^p_\delta$ depending on $\delta$. Inequality (41) can be proved in the same way as the original Theorem 1, see, e.g., [1, Thm. 8.11].

The generalization to punctured domains is the following theorem, whose proof imitates that of Theorem 3.

**Theorem 4** (Third $L^p-L^q$ generalization of Poincaré inequalities). Let $n \geq 2$, $1 < q < \infty$ and $\max\{1, qn/(n+q)\} < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected open set with the cone property, and let $E^p_\Omega$ and $\mathcal{M}^p_\Omega$ be as explained above. Let $A \subset \Omega$ be a measurable subset of $\Omega$, $\Gamma = \Omega \setminus A$, and let $0 < \delta \leq 1$. For any $\varepsilon > 0$ there exists a positive constant $C$, depending on $\Omega$, $A$, $p$, $q$, $\delta$ and $\varepsilon$ (but not on $\Lambda$ and $\Gamma$) such that, for every $f \in W^{1,p}(\Omega)$ satisfying $d_\delta^A(f) \geq \delta \|f\|_{L^p(\Omega)}$

$$\|\nabla + iA\|_{E^p_{\Omega}(\Lambda)}^p + C \|\nabla + iA\|_{E^p_{\Omega(\Gamma)}}^p \geq \left( \frac{1}{S^p_{\delta} + \varepsilon + E^p_A} \right) \|f\|_{E^p_{\Omega(\Omega)}}^p,$$  \hspace{1cm} (42)

where $r = \max\{1, qn/(n+q)\}$ if $1 < q < \infty$, and $S^p_\delta$ is the optimal constant in (41). For $q = \infty$ (42) holds for any $n > 2$, and $C$ will also depend on $r$.

We remark that for $n/(n-2) \leq q < \infty$ the $r$ appearing in (42) is optimal, i.e., it cannot be decreased (compare with Remark 4 after Theorem 2).

**Appendix: Spectrum of $(\nabla + iA)^2$**

Here we prove two facts about $(\nabla + iA)^2$, which were used in the text. As before, we have a bounded, connected domain $\Omega$ in $\mathbb{R}^n$ with the cone property. Connectedness is not really necessary here, but the number of connected components should be finite. The vector field $A$ is bounded and measurable. The boundedness is not crucial but we assume it for simplicity.

We define the eigenvalues (spectrum) $E_k$ and eigenfunctions $\phi_k$ of $-(\nabla + iA)^2$ in $L^2(\Omega)$ by means of quadratic forms as in [1], i.e., $E_{k+1}$ is defined by

$$E_{k+1} = \inf \left\{ \|\nabla + iA\|^2_{L^2(\Omega)} : \|f\|^2_{L^2(\Omega)} = 1, \int_\Omega f\phi_j = 0 \text{ for } j = 1, \ldots, k \right\}.$$

Then, by standard methods (using the Rellich-Kondrashov theorem), one shows that there is a minimizer for $E_{k+1}$, which is called $\phi_{k+1}$. $E_1$ was called $E_A$ in the text.

The two facts are the following.

**Lemma 2** (Spectrum of $-(\nabla + iA)^2$).

A. The spectrum is discrete. i.e., the number of eigenvalues less than any number $E$ is finite.

B. The multiplicity of $E_1 = E_A$ can be greater than one.

**Proof.** To prove A we suppose that there are infinitely many eigenvalues below $E$. If $\psi \in W^{1,2}(\Omega)$, with $\|\psi\|_{L^2(\Omega)} = 1$ and $\|\nabla + iA\psi\|_{L^2(\Omega)} \leq E^{1/2}$, then, since $A$ is bounded, $\|\nabla \psi\|_{L^2(\Omega)} \leq E^{1/2} + \|A\|_{L^\infty(\Omega)}$. Thus, the infinite sequence of functions $\phi_1, \phi_2, \ldots$ is bounded in $W^{1,2}(\Omega)$. By the Rellich-Kondrashov theorem this sequence has a subsequence that converges strongly in $L^2(\Omega)$. This is impossible since the $\phi_i$'s are orthonormal (and hence $\|\phi_i - \phi_j\|_{L^2(\Omega)} = \sqrt{2}$ for $i \neq j$).
To prove B, consider the case in which $\Omega \subset \mathbb{R}^2$ is an annulus centered at the origin (or a cylinder in $\mathbb{R}^n$ based on an annulus in $\mathbb{R}^2$). We take $A$ to be (in polar coordinates) $A(r, \theta) = \lambda r^{-1} \hat{e}_\theta$, where $\hat{e}_\theta$ is the unit vector in the $\theta$ direction. We shall show that for suitable values of $\lambda$ the multiplicity of $E_A$ is two.

The Hilbert space $W^{1,2}(\Omega)$ is the direct sum of Hilbert spaces $W^{1,2}_l(\Omega)$, $l \in \mathbb{Z}$, consisting of functions of the form $\exp(-i\lambda \theta) g(r)$, and these subspaces continue to be orthogonal under the action of $\nabla + iA$. Thus, the eigenvectors in our case can be chosen to belong to exactly one of these subspaces. Thus, we can define $E_A(l)$ to be the lowest eigenvalue in $W^{1,2}_l(\Omega)$, and $E_A$ is then the minimum among the numbers $E_A(l)$. Since

$$-(\nabla + iA)^2 = -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \left( \frac{\partial}{\partial \theta} + i\lambda \right)^2,$$

$E_A = E_A(l_0)$, where $l_0$ is the integer closest to $\lambda$. Therefore, for $\lambda \in \mathbb{Z} + \frac{1}{2}$, there are two eigenfunctions with the same eigenvalue $E_A = E_A(\lambda - \frac{1}{2}) = E_A(\lambda + \frac{1}{2})$.

The reader is invited to construct examples with arbitrarily large multiplicities of the lowest eigenvalue $E_A$.

References


