On the Observability of Large Fluctuations

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Abstract

The Gallavotti-Cohen fluctuation theorem concerns large deviations in the time evolution of global quantities such as the phase space contraction rate of gaussian thermostatted particle systems, which can be taken as the entropy production rate of the system. The theoretical importance of the Gallavotti-Cohen fluctuation theorem contrasts with the fact that large global fluctuations are not observable in macroscopic systems. In this paper we analyze the probability distribution functions of the fluctuations of the phase space contraction rate of gaussian thermostatted shearing systems, at high shear rates, and we link the large fluctuations thereof to the small fluctuations. This way, the effects of the extremely rare large fluctuations of the Gallavotti-Cohen type become observable, and some evidence of the “axiom-C-like” structure of our dynamics is gathered.

Keywords: Gallavotti-Cohen fluctuation theorem, nonequilibrium molecular dynamics, large deviations

1 Introduction

In molecular dynamics simulations of the steady states of nonequilibrium shearing systems, Evans, Cohen and Morriss [1] discovered a remarkable relation for the fluctuations of the entropy production rate in a steady state. Inspired by these findings, Gallavotti and Cohen derived [2] the same fluctuation relation for a wide class of systems, subject to the following:

Chaotic Hypothesis (CH): A reversible many-particle system in a stationary state can be regarded as a transitive Anosov system for the purpose of computing its macroscopic properties.

The ensuing result is known as the Gallavotti-Cohen Fluctuation Theorem (GCFT), which concerns the fluctuations of the phase space contraction rate associated with the dissipative, time reversible dynamics of systems such as the gaussian thermostatted systems [3]. As discussed in [4], the phase space contraction rate of gaussian systems can be identified with the entropy production rate if the number of particles $N$ is large. Despite the impossibility of observing this kind of fluctuations in real macroscopic fluids the GCFT is important for the consequences that can be derived from it. For instance, near equilibrium, the GCFT implies both the Onsager and Einstein relations [5] and can therefore be interpreted as an extension of such relations to far-from-equilibrium situations. Moreover, a local fluctuation theorem, which concerns local, hence observable, fluctuations, has been proven by Galavotti [6]. This local theorem has the same form of the GCFT, except for boundary terms which are negligible in the limit of large volumes. Nevertheless:

“... in concrete cases not only it is not known whether the system is Anosov but, in fact, it is usually clear that it is not ... Hence the test is necessary to check the CH which says that the failure of the Anosov property should be irrelevant for practical purposes” [7].

\footnote{The numerical experiment of [1] had only 56 particles, a considerable number for numerical simulations of this kind, but certainly not many compared to Avogadro's number.}
For “practical purposes” means that the calculation of quantities of physical interest is not affected by the deviations of the dynamics from the ideal case of an Anosov flow. Therefore, numerical or real experiments are required to test the applicability of the CH, to identify its range of validity and, possibly, to discover new phenomena related to the CH. Several papers have been devoted to this purpose (see, e.g., Refs. [8, 9, 10, 11, 12]), while other papers have investigated the possibility of observing fluctuation relations similar to that of the GCFT in different contexts (see e.g., [12, 13, 14, 15, 16, 17]). These works have shown that —as conjectured in Ref. [2]— the CH effectively works for a wider class of systems than that of topologically mixing Anosov diffeomorphisms or flows. At present such a class includes particle systems with either soft or hard potentials, turbulent fluids, and even one kind of systems whose steady states are periodic (hence have no fluctuations at all) but which verifies a relation similar to the GCFT during its very long and erratic transients [13].

Here, we test the GCFT for gaussian systems made of a few particles subject to high shear rates (hence with high dissipation) described in Section 2. Problems for this test arise at high shear rates, because the negative fluctuations of the phase space contraction rate become less and less frequent as the shear rate grows, till they don’t occur at all in a numerical simulation. This happens despite the time reversibility of the dynamics, and makes impossible a direct test of the validity of the GCFT at moderately high dissipations, because, as explained in section 3, the GCFT concerns the negative fluctuations. The situation worsens for a test of a conjecture of Bonetto, Gallavotti and Garriodo presented in [8], which links the Lyapunov spectrum of a particle system to the properties of the fluctuations of the phase space contraction rate, and is tailored for systems with very high dissipation.

The effect of high shear in a system of a few particles is similar to that of having many particles: the large fluctuations are unobservable. We focus on the fluctuations of high shear few particles systems described in Sect. 2, instead of systems of many particles, because small systems can be controlled better numerically. In Section 3, we describe the form of the probability distribution functions (PDFs) of the fluctuations of the phase space contraction. In Section 4, we introduce two scalings meant to relate to each other the fluctuations averaged over time intervals of different length. In Section 5, we exploit these scalings to link the properties of the rare large fluctuations with those of the frequent fluctuations, so that the presence of non observable fluctuations become indirectly observable. Sections 6. and 7. conclude the paper with a discussion of our results, which can be summarized as follows:

- the first scaling is consistent with our data, suggesting that the CH (hence the GCFT) holds as well;
- the second scaling is consistent with our data, suggesting that the GCFT can be tested indirectly, even in the absence of negative fluctuations, through the test of new fluctuation relations;
- the inferred properties of the large fluctuations are consistent with the axiom C structure of [15].

2 The model

A number of homogeneous algorithms for the simulation of nonequilibrium steady states have been developed by Evans and Morriss [3]. These algorithms are based on deterministic thermostats that maintain a steady state mimicking a continuous removal of heat from the system. Here, we use the gaussian iso-kinetic thermostat that keeps the peculiar kinetic energy a strict constant of the motion. As the temperature at equilibrium is connected to the kinetic energy by the equipartition relation, we use the same relation in nonequilibrium and use it to define a steady state “temperature”. In this sense the effect of the deterministic thermostat is to keep the steady state temperature fixed. It is possible to constrain other system properties, such as the internal energy, to maintain a steady state. In fact, there is a variety of thermostating mechanisms that

\footnote{In this case, the system under consideration has been called “Anosov-like” [2].}
either constrain a property exactly at each timestep, or constrain its average value, and allow the system to reach a steady state [18].

For instance, the color diffusion algorithm considers the system to consist of two types of particles labelled by a color. An external field \( \mathbf{F}_c \) acts on particles depending upon their color. Usually one takes the same number of particles of each color and \( c_i = \pm 1 \) so that the external field does not affect the centre of mass of the system. The corresponding gaussian isokinetic equations of motion are given by

\[
\begin{align*}
\dot{\mathbf{q}}_i &= \frac{\mathbf{p}_i}{m} \\
\dot{\mathbf{p}}_i &= \mathbf{F}_i + c_i \mathbf{F}_c - \alpha \mathbf{p}_i
\end{align*}
\]  

(1)

where the thermostating (Lagrange) multiplier \( \alpha \) is given at each instant by

\[
\alpha = \frac{\sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{p}_i + \mathbf{F}_c \cdot \mathbf{J} \mathbf{V}}{\sum_{i=1}^{N} \mathbf{p}_i^2},
\]

(2)

and the color current \( \mathbf{J} \) is given by

\[
\mathbf{J} = \frac{1}{V} \sum_{i=1}^{N} \frac{\alpha \mathbf{p}_i}{m_i}.
\]

(3)

In this paper, we consider the molecular dynamics of planar Couette flow, which can be simulated using the SLLOD equations of motion and a gaussian isokinetic thermostat [3], for shear flow in periodic boundary conditions. The relevant equations of motion are given by

\[
\begin{align*}
\dot{\mathbf{q}}_i &= \frac{\mathbf{p}_i}{m} + \mathbf{n}_x \gamma y_i \\
\dot{\mathbf{p}}_i &= \mathbf{F}_i - \mathbf{n}_x \gamma p_{yi} - \alpha \mathbf{p}_i
\end{align*}
\]

(4)

where \( \mathbf{n}_x \) is a unit vector in the \( x \) direction, \( \gamma \) is the shear rate, and the gaussian thermostat multiplier is given at each instant of time by

\[
\alpha = \frac{\sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{p}_i - \gamma p_{xi} p_{yi}}{\sum_{i=1}^{N} \mathbf{p}_i^2}.
\]

(5)

In all cases, we define the dynamical temperature \( T \) so that

\[
\frac{1}{2m} \sum_{i=1}^{N} \mathbf{p}_i^2 = (dN - d - 1) \frac{kT}{2}.
\]

(6)

where \( d \) is the spatial dimension and \( dN - d - 1 \) is the number of independent degrees of freedom. To calculate the dissipation for either the iso-kinetic color diffusion or SLLOD algorithms some care must be taken to identify independent variables. We simulate systems with no net external force, so that the centre of mass and the total momentum are conserved. Thus, the quantities \( \mathbf{q}_N = -\sum_{i=1}^{N-1} \mathbf{q}_i \) and \( \mathbf{p}_N = -\sum_{i=1}^{N-1} \mathbf{p}_i \) are not independent variables. Similarly, the constraint of constant total kinetic energy imposed by the thermostat eliminates one further variable but we shall use the constraint explicitly to handle this. The divergence of the equations of motion yields the phase space contraction rate \( \chi \), which can be considered as a measure of the dissipation rate. Then, for color diffusion we have:

\[
\chi = -\sum_{i=1}^{N-1} \frac{\partial}{\partial \mathbf{q}_i} \cdot \dot{\mathbf{q}}_i - \sum_{i=1}^{N-1} \frac{\partial}{\partial \mathbf{p}_i} \cdot \dot{\mathbf{p}}_i = -\sum_{i=1}^{N-1} \frac{\partial}{\partial \mathbf{p}_i} \cdot \dot{\mathbf{p}}_i = d(N - 1) \alpha + \sum_{i=1}^{N-1} \mathbf{p}_i \cdot \frac{\partial \alpha}{\partial \mathbf{p}_i}
\]

(7)

To calculate the last partial derivative on the RHS we rewrite the thermostating multiplier in terms of independent variables as

\[
\alpha = \frac{\sum_{i=1}^{N-1} (\mathbf{F}_i \cdot \mathbf{p}_i + \mathbf{F}_c \cdot c_i \mathbf{p}_i) + \mathbf{F}_N \cdot \mathbf{p}_N + \mathbf{F}_c \cdot c_N \mathbf{p}_N}{\sum_{i=1}^{N-1} \mathbf{p}_i^2 + \mathbf{p}_N^2}.
\]

(8)
Replacing \( q_N \) and \( p_N \) by their expressions in terms of independent variables, the partial derivatives of \( \alpha \) can be calculated, leading to the result that
\[
\sum_{i=1}^{N-1} p_i \cdot \frac{\partial \alpha}{\partial p_i} = -\alpha
\]
(9)
from which it follows that
\[
\chi = (d(N - 1) - 1) \alpha.
\]
(10)
A similar calculation for the iso-kinetic SLLOD algorithm yields:
\[
\chi = (d(N - 1) - 1) \alpha - \frac{\gamma p_{xy}^K V}{(dN - d - 1)kT},
\]
(11)
where \( p_{xy}^K \) is the kinetic part of the shear stress \( P_{xy} \) and \( V \) is the volume. The integral of \( \chi \) along one trajectory segment of duration \( \tau \)
\[
q = \int_\tau^{t+\tau} \chi(s)ds
\]
(12)
is here referred to as the phase space contraction along that trajectory segment.

Our numerical calculations of trajectory segments use a fourth order Gear predictor-corrector algorithm to solve the equations of motion. The interaction potential between particles is the WCA potential, a Lennard-Jones potential cutoff at the potential minimum, and shifted so that the potential is zero at the cutoff distance, and for all large distances [3]. Our shear flow simulations are for a system of 8 particles, at a temperature \( T = 1 \), a density \( \rho = 0.4 \) and shear rates of \( \gamma = 2 \) and \( \gamma = 2.5 \).

The non-vanishing Lyapunov exponents for the SLLOD system have been calculated [20] and are found to satisfy the conjugate pairing rule for all exponents except one, which pairs with one of the vanishing exponents. Our results are reported in Table 1, where the vanishing exponents are omitted. For the iso-kinetic color diffusion algorithm it has been proved that the conjugate pairing rule is satisfied exactly [19], however no corresponding proof for iso-kinetic SLLOD has been obtained.

3 Fluctuations in shearing systems

The fluctuation theorem introduced in [2] for Anosov maps and in [21] for Anosov flows, concerns the fluctuations of the phase space contraction rate \( \chi \) of the dynamical system under investigation. The content of the original version of the GCFT given in [2] can be summarized as follows:

**Theorem (Galavotti and Cohen, 1995):** Let \((\Omega, S)\) be a dynamical system which satisfies the following properties:

A. The average of the phase space contraction rate with respect to the forward time statistics of the motion is positive (dissipation);

B. There is an involution \( i \) on \( \Omega \), which represents time reversal, such that if \( \{S^tx\}_{t \in \mathbb{Z}} \) is one itinerary in \( \mathcal{C} \) under the evolution \( S \), then \( \{iS^{-t}x\}_{t \in \mathbb{Z}} \) is also an itinerary (time reversal invariance);

C. The CH holds and we can treat \((\Omega, S)\) as a transitive Anosov system (chaoticity).

Then, for sufficiently large \( \tau \), the probability \( \pi_\tau(q) \) that the phase space contraction over a trajectory segment of length \( \tau \) equals \( q \) satisfies the large deviation relation
\[
\frac{\pi_\tau(q)}{\pi_\tau(-q)} = e^q
\]
(13)
with an error in the argument of the exponential which can be estimated to be \( q, \tau \) independent.
The physical relevance of the GCFT lies in the identification of the entropy production rate with the phase space contraction rate, which is appropriate at least for gaussian thermostatted systems made of many particles [4].

Let \( \dot{x} = F(x) \) compactly represent the equations of motion of a particle system, and let \( t \to S^t x \) represent the time evolution of an initial condition \( x \) in the phase space \( \Omega \). Run a simulation for a time \( T \) which must be adequately longer than the characteristic time of the fluctuations of \( \chi = -\text{div} \; F \), and assume that relaxation to a steady state takes place in a time \( t_\infty \ll T \). Then, consider the forward time average

\[
\langle \chi \rangle_+ = \frac{1}{T} \int_0^T \chi(S^t x) \, dt,
\]

where we do not indicate the dependence of \( \langle \chi \rangle_+ \) on \( T \) and \( x \), trusting that \( T \) is large enough and the dynamics ergodic enough, that longer simulations and different randomly chosen initial \( x \in \Omega \) do not produce appreciable differences in the result. Divide the time interval \([t_\infty, T]\), in subintervals of length \( \tau \), and consider the quantities

\[
\chi_\tau(i) = \frac{1}{\langle \chi \rangle_+} \int_{t_\infty + (i-1)\tau}^{t_\infty + i\tau} \chi(S^t x) \, dt ; \quad i = 1, \ldots, (T - t_\infty)/\tau.
\]

These values, arranged in a histogram, allow us to construct the PDF \( \pi_\tau \) of the fluctuations of \( \chi_\tau \). Then, a test of the GCFT amounts to verify that, in the limit \( T \to \infty \), the function

\[
C(p; \tau) = \frac{1}{\tau \langle \chi \rangle_+} \sum_{j \in P^+} \log \frac{\pi_\tau(p)}{\pi_\tau(-p)},
\]

obtained for different values of \( \tau \) (in the compact support of \( \pi_\tau \), does converge for growing \( \tau \) to \( C_\infty = 1 \).

In the case of highly dissipative systems, the transitivity property required by part C of the GCFT is likely violated, and \( C_\infty \) is expected to be smaller than 1, as conjectured in [8]. This led Bonetto and Gallavotti [15] to study the properties of a special kind of hyperbolic dynamical systems: systems which verify what these authors called the “axiom C”, i.e. a set of properties from which the conjectures of [8] can be derived. Later, Gallavotti proposed the following relation [14]:

\[
C_\infty = \frac{1}{\langle \chi \rangle_+} \sum_{j \in P^+} (\lambda_j^+ + \lambda_j^-),
\]

generalizing the expression for \( C_\infty \) given in Ref.[8]. Here \( \lambda_j^+ \) and \( \lambda_j^- \) are the largest and the smallest, respectively, Lyapunov exponents of the pair of index \( j \), like in Table 1, and \( P^+ \) is the set of indices of pairs with one nonnegative exponent. For simplicity, we refer to the FT as to the result for which, in the limit of large \( \tau \), \( C(p; \tau) \) converges to the value \( C_\infty \) given in Eq.(17).

As shown in Table 1, where the vanishing \( \lambda_{\infty}^+ \) is not reported, the SLLOD systems are dissipative (\( \langle \chi \rangle_+ > 0 \)). Also, they are time reversal invariant in the sense of the Kawasaki mapping [3], and appear to be chaotic. Therefore, these systems look like good candidates for tests of the CH and of its consequences and, in fact, the fluctuation relation (13) was discovered studying one such system [1]. However, at high dissipations the negative fluctuations of \( \chi_\tau \) are too rare to be observed, especially for large \( \tau \)'s. This makes impossible a direct test of the FT since, in principle, the FT applies to PDFs \( \pi_\tau \) with \( \tau > \tau_c = 1/\lambda_{\text{min}} \), \( \lambda_{\text{min}} \) being the Lyapunov exponent of smallest non vanishing magnitude. Performing our test we observed that, indeed, negative fluctuations become extremely scarce at shear rates \( \gamma \geq 2 \), if \( \tau \geq 1 \), which is rather smaller than \( \lambda_{\text{min}}^{-1} \approx 10^{14} \) (see Table 1). At the same time, we noticed that (cf. Figures 1 and 2):

our data for the quantity \( C(p; \tau) \) defined by Eq.(16) can be properly fitted by a \( p \)-independent function of \( \tau \) only for \( \tau \) greater than a threshold value \( \tau_c > 0 \), where the subscript \( \gamma \) hints at a possible dependence of \( \tau_c \).

\footnote{Note that the \( \gamma \) in Eq.(13) equals \( \tau \langle \chi \rangle_+ \).

\footnote{Besides some technical requirement, cf. [15], in order to verify axiom C a smooth dynamical system must have good hyperbolic properties (i.e. must verify axiom A), and must have unique attracting and repelling basic sets.}
on the shear rate. For each \( \tau \), then, we denote by \( C_{\tau} \) the least square constant fit of the numerical data for \( C(p; \tau) \).

Therefore, there seems to be little room for a test of the FT within our setting. Nevertheless, for \( \gamma = 2 \) and \( \gamma = 2.5 \), a linear fit

\[
C(p; \tau)p = \frac{1}{\tau(x)} \left[ \log(\pi_{\tau}(p)) - \log(\pi_{\tau}(-p)) \right] = C_{\tau}p + \text{err} ,
\]

is appropriate if \( \tau > 0.32 \), since the correction term “err” is then either negligible or within the error bars. Moreover, the least square fit \( C_{\tau} \) of our data for \( C(p; \tau) \) appears to converge to a given value \( C_{\infty} \), with growing \( \tau \) (see Tables 2,3). But, this cannot be tested up to \( \tau \) of order \( O(10^{4}) \), the negative fluctuations being absent at such \( \tau \)'s.\(^5\) The figures 1 and 2, exemplify the situation for \( \gamma = 2 \) and \( \gamma = 2.5 \) respectively.\(^6\) Similarly to previous works [11, 12, 13], the form of our PDFs is not affected by the fact that the trajectory segments are contiguous. Therefore, in order to have better statistics, we did not decorrelate the trajectory segments.

Many kinds of distributions, including gaussian distributions, yield a linear form for \( C(p; \tau)p \). This could lead to the erroneous belief that a trivial connection exists between the FT and the central limit theorem. For instance, the test of the FT presented in Ref.[8] was based on PDFs which couldn’t possibly be gaussian, because it had a cut-off, but could not be numerically distinguished from a gaussian either. In our case, this is not so: our PDFs are clearly non-gaussian. Also, it is interesting to note how the shape of our PDFs changes with \( \tau \) (see Figs. 1,2), undergoing a kind of transition around \( \tau = 0.1 \). At this transition, as \( \tau \) grows, the peak of the distribution draws substantially closer to the mean than at shorter \( \tau \)'s, still preserving a clear lack of symmetry around the peak. The function \( C(p; \tau)p \) changes accordingly, but doesn’t manifest a clear transition like the PDFs do: it only stretches more and more, and around \( \tau = 0.1 \) it is still quite different from a straight line.

4 Scalings of the PDF

Figures 1,2 show that, for sufficiently large \( \tau \), the shape of the PDFs of the fluctuations of \( \chi_{\tau} \) (cf. Eq.(15)) does not change with \( \tau \). Then, there may be a scaling relating the PDFs at different \( \tau \)'s, which transforms one PDF into another one. Something similar is expected also because the function \( \zeta \) defined by

\[
\zeta(p) = \lim_{\tau \to \infty} \frac{1}{\tau} \log \pi_{\tau}(p)
\]

is known to exist and to be analytic and convex for Anosov systems [22]. Therefore, if our models can be considered “Anosov-like”, in the sense that they verify the CH, something like Eq.(19) could hold for these models as well. In particular, we intend to test the following:

**Conjecture 1:** Let \( \pi_{\tau} \) be the PDF of the fluctuations of \( \chi_{\tau} \) defined by Eq.(15). Then, there is a convex function \( \zeta : [-p_{0}, p_{0}] \to \mathbb{R}^{+} \cup \{0\} \) such that

\[
\zeta(p) = \frac{1}{\tau} \log \pi_{\tau}(p) + z_{\tau}(p) ,
\]

where \( z_{\tau}(p) \) is of order \( O(1/\tau) \) uniformly for \( p \in [-p_{0}, p_{0}] \) and \( \tau \to \infty \), and \( p_{0} \) is the largest value of the phase space contraction rate \( \chi \) in the phase space \( \Omega \).

\(^5\)For this reason, the curves in the right panels of Figs.1,2 cover smaller and smaller ranges. This shows that a direct test of the FT is rather difficult, if not impossible, for our systems. Our data, then, are simply not inconsistent with the validity of the FT, in the sense that the curves in Figs.1,2 become more and more like straight lines, as they should, for growing \( \tau \) and in the range where we can observe them.

\(^6\)We do not draw the error bars in these figures since they would merely produce slightly thicker curves. The smoothness of the curves in the figures are a better indication, in this instance, of the (rather low) level of noise in our statistics based on samples of \( O(10^{7}) \) and more data values.
We call Eq. (20) the first scaling. Let us consider two different values of \( \tau \), now, and denote \( \pi_\tau \) and \( z_\tau \) by \( \pi_i \) and \( z_i \), respectively, for \( \tau = \tau_1 \) and \( i = 1,2 \). If Conjecture 1 is verified, we can write
\[
\frac{1}{\tau_1} \log \pi_1(p) = \frac{1}{\tau_2} \log \pi_2(p) + [z_2(p) - z_1(p)] ,
\]
(21)
or
\[
\pi_2(p) = e^{\tau_2 \log [\pi_2(p) / \pi_2]} \pi_1(p)^{\tau_2 / \tau_1} ,
\]
(22)
which allows us to deduce the form of \( \pi_2 \) from that of \( \pi_1 \), and vice versa, if the quantity \( \tau_1 [z_1(p) - z_2(p)] \) is known or can be made small. One may also split the correction term \( z_\tau \) in (20) in two pieces as follows:7
\[
\zeta(p) = \frac{1}{\tau} \log \pi_\tau(p) - \frac{1}{\tau} \phi(p) + \tilde{z}_\tau(p) ,
\]
(23)
where \( \phi \) does not depend on \( \tau \). Then, using Eq. (23) with two different values of \( \tau, \tau_1 \) and \( \tau_2 \), say, one obtains:
\[
\zeta(p) = \frac{1}{\tau_1 - \tau_2} \log \pi_\tau(p) - \log \pi_{\tau_1}(p) + \tau_1 \tilde{z}_{\tau_1}(p) - \tau_2 \tilde{z}_{\tau_2}(p) ,
\]
(24)
\[
\phi(p) = \log \pi_{\tau_1}(p) - \tau_1 \zeta_\tau(p) - \tilde{z}_{\tau_1} , \quad i = 1,2 ,
\]
(25)
This way, part of the correction term \( z_\tau \) of Eq. (20) has been eliminated, and one may expect that the remaining corrections \( \tilde{z}_i \) in Eq. (24) give a smaller contribution to \( \zeta \) than \( z_\tau \) does in Eq. (20). Using Eq. (24), rather than Eq. (20), a more accurate calculation of \( \zeta \) should then be possible.

Note: the convergence of our data to the limiting function \( \zeta \), evidenced in Fig. 3, suggests that the CH is appropriate for our dynamical systems.

Going one step further, one may ask if there are any other scalings that the PDFs of our systems satisfy. The most obvious scaling is the affine transformation which takes a gaussian PDF into the standard one.8 However, our PDFs are non-gaussian, and close examination shows that the slopes \( C_\tau \) should play a role in the scaling as well. Hence, we test the following:

**Conjecture 2:** Let \( \pi_\tau \) be the PDF of \( \chi_\tau \) defined by Eq. (15). Then, there is a function \( \varphi : \mathbb{R} \rightarrow \mathbb{R}^* \cup \{0\} \) such that
\[
\varphi(x) = \frac{\sigma_\tau}{C_\tau} \pi_\tau \left( \frac{\sigma_\tau}{C_\tau} x + 1 \right) + f_\tau(x) ,
\]
(26)
where \( f_\tau \) is of order \( o(1/\tau) \) uniformly in \( x \) for \( \tau \rightarrow \infty \), \( \sigma_\tau \) is the standard deviation of \( \pi_\tau \), and \( C_\tau \) is the slope obtained by the linear least square fit of Eq. (18).

We call Eq. (26) the second scaling. Like the first, the second scaling can be tested considering two different values of \( \tau \), and the following relation:
\[
\frac{\sigma_1}{C_1} \pi_1 \left( \frac{\sigma_1}{C_1} x + 1 \right) = \frac{\sigma_2}{C_2} \pi_2 \left( \frac{\sigma_2}{C_2} x + 1 \right) + [f_2(x) - f_1(x)] .
\]
(27)
Again, for simplicity, the subscripts 1 and 2 stand for \( \pi_1 \) and \( \pi_2 \), respectively. In the limit of large \( \tau \)'s, when \( C_2 \) approaches \( C_1 \), Eq. (27) reduces to
\[
\sigma_1 \pi_1 (\sigma_1 x + 1) = \sigma_2 \pi_2 (\sigma_2 x + 1) + [f_2(x) - f_1(x)] ,
\]
(28)
which is the usual transformation of Gaussian PDF's, but may be applied also to a given class of non-Gaussian PDF's.9 Changing the coordinates from \( x \) to \( p = \pi_\tau x / C_1 + 1 \), Eq. (27) can be put in the form
\[
\pi_1(p) = \frac{C_1 \sigma_2}{C_2 \sigma_1} \pi_2 \left( \frac{C_1 \sigma_2}{C_2 \sigma_1} p + \tilde{p}_{21} \right) + \frac{C_1}{\sigma_1} \left[ f_2 \left( \frac{C_1(p - 1)}{\sigma_1} \right) - f_1 \left( \frac{C_1(p - 1)}{\sigma_1} \right) \right] , \quad \text{with} \quad \tilde{p}_{21} = 1 - \frac{C_1 \sigma_2}{C_2 \sigma_1} ,
\]
(29)
\footnote{We owe this observation to Bonetto, Ciliberto and Gallavotti.}
\footnote{We owe this observation to S. Ruffo.}
\footnote{Our test of the second scaling concerns precisely non-Gaussian PDF's, and is not limited to the small fluctuations around the mean of such PDF's. These small fluctuations verify the Central Limit Theorem for large \( \tau \), so that a small portion of the corresponding PDF's can be approximated by a Gaussian. But the remaining part of these PDF's does not need to be Gaussian, as discussed more in detail below. Therefore, the FT and our present investigation are not restricted to the cases in which the Central Limit Theorem holds.}
By means of the second scaling, the form of $\pi_1$ can be deduced from that of $\pi_2$, if the functions $f_1$ and $f_2$ are known. Alternatively, $\pi_1$ can be obtained with an arbitrarily small error if the value of the square brackets in Eq. (29) goes to zero faster than $\sigma_1$, for growing $\tau$. Combining the first and the second scaling we obtain the following relation:

$$
\pi_2(p) = e^{\tau z_1(p) - z_2(p)} \left\{ \frac{C_1 \sigma_2}{C_2 \sigma_1} \tau_2 \left( \frac{C_1 \sigma_2}{C_2 \sigma_1} p + \hat{p}_2 \right) + \frac{C_1}{\sigma_1} \left[ f_2 \left( \frac{C_1 (p-1)}{\sigma_1} \right) - f_1 \left( \frac{C_1 (p-1)}{\sigma_1} \right) \right] \right\}^{\tau_2/\tau_1} \tag{30}
$$

which can be of some use if the correction terms containing the functions $f_i$ and $z_i$, $i = 1, 2$, rapidly become negligible for growing $\tau_i$.

Some evidence that the first scaling and its modification given in Eq. (23) may indeed hold for our data with $\gamma = 2$ is given in Fig. 3, where the scaled PDFs appear to converge to a given curve, for growing $\tau$, and the expression (23) appears to converge more quickly. The limiting curve is not the logarithm of a Gaussian distribution: it is not symmetric with respect to its maximum. Similarly, Fig. 4 provides evidence that also the second scaling is verified by our data. Again, the limiting curve does not look Gaussian, although the kurtosis of the finite $\tau$ approximants of the limiting distribution appear to converge to zero (see Table 1). Moreover, the comparison of the second scaling with the first, in Fig. 4, suggests that the second scaling converges more rapidly than the first for growing $\tau$, in accord with our conjecture 2.

Clearer evidence of the validity of the two scalings for our systems is found in terms of the moments of the PDFs $\pi_i$. Let $\xi_i$ be the random variable whose PDF is $\pi_i$, for $i = 1, 2$, and let $\mathbb{E}(\xi_i^n)$ be the corresponding $n$-th moment. Then, Eq. (21) yields

$$
\mathbb{E}(\xi_1^n) = \int p^n e^{\tau_z z_1(p) / \tau_1} \pi_1(p) dp \tag{31}
$$

and one can compute the values of

$$
F_{1,2}^{(n)} = \frac{1}{\mathbb{E}(\xi_1^n)} \left[ \mathbb{E}(\xi_1^n) - \int p^n \pi_2(p) dp \right] \tag{32}
$$

in order to assess the decrease of $\tau_1 z_1$ in Eq. (28), with growing $\tau_1$ and $\tau_2$. Similarly, Eq. (29), yields:

$$
\mathbb{E}(\xi_2^n) = \left( \frac{C_2 \sigma_1}{C_1 \sigma_2} \right)^n \mathbb{E}(\xi_2^n) + \frac{C_1}{\sigma_1} \int p^n \left[ f_2 \left( \frac{C_1 (p-1)}{\sigma_1} \right) - f_1 \left( \frac{C_1 (p-1)}{\sigma_1} \right) \right] dp \tag{33}
$$

and one can assess the behaviour of the term $[f_2 - f_1]$ in Eq. (29), by computing the quantity

$$
S_{1,2}^{(n)} = \frac{\mathbb{E}(\xi_2^n) - \left( \frac{C_2 \sigma_1}{C_1 \sigma_2} \right)^n \mathbb{E}(\xi_2^n)}{\mathbb{E}(\xi_1^n)} \int \frac{(\sigma_1 x / C_1 + 1)^n}{\mathbb{E}(\xi_1^n)} [f_2(x) - f_1(x)] dx \tag{34}
$$

The quantities $F_{1,2}^{(n)}$ and $S_{1,2}^{(n)}$ for $\gamma = 2$ and various values of $\tau_1$ and $\tau_2$, are reported in the tables 4 - 7. They represent relative errors, hence their decrease with $\tau_1$ and $\tau_2$ is not merely due to the decrease of $\mathbb{E}(\xi_1^n)$.

The values above the empty diagonal in the tables 4 - 7 refer to $F_{1,1}^{(n)}$, and those below the diagonal refer to $S_{1,2}^{(n)}$. We notice that the absolute values of $F_{1,1}^{(n)}$ do not decrease monotonically, but there is a decreasing trend when either or both of $\tau_1$ and $\tau_2$ are increased. Also, the normalizing factor $\mathbb{E}(\xi_2^n)$ in the definition of $F_{1,2}^{(n)}$ decreases with growing $\tau_2$. Assuming the decreasing trend remains at any $n$, our results appear consistent with the assumption $z_{\tau} \sim \tau^{-1}$ and, perhaps, with a faster decay of $z_{\tau}$. This constitutes further evidence of the validity of the CH for the SLLOD systems at high shear.

The terms $S_{1,2}^{(n)}$ are exactly zero, and the terms $S_{1,2}^{(n)}$ vanish within our numerical accuracy. The magnitude of the terms $S_{1,2}^{(n)}$ for $n = 3, 4$, decreases monotonically along the diagonals of the corresponding tables (where $\tau_1$ and $\tau_2$ increase together), and along the rows ($\tau_1$ is fixed and $\tau_2$ grows). In particular, along the diagonals we have $S_{1,2}^{(n)} \sim \tau^{-\alpha}$ with $\alpha > 2.2$. In principle, higher order moments could be affected by the tails of the PDFs $\pi_i$, leading to unexpected behaviour of these moments with growing $\tau_1$, $\tau_2$. However, for every fixed
order \( n \) this is unlikely to be the case: the tails of \( \pi_\tau \) are bounded by \(-p_0\) and \( p_0\) at all \( \tau \)'s, and the mass of \( \pi_\tau \) draws nearer and nearer to 1 as \( \tau \) grows. Therefore, for every fixed \( n \), the contribution to \( E(\chi_n^2) \) coming from the tails is bound to decrease with \( \tau \), although for fixed \( \tau \) such a contribution may grow with \( n \). Assuming our results remain valid for the moments of any order \( n \), it seems reasonable to conclude that \( f_\tau \) decays faster than \( \tau^{-1} \), as conjectured.

5 The second scaling and its consequences

We now assume that the second scaling holds, and we study some of its consequences. We do not estimate the size of the corrections; we simply use the symbol “err.” for the correction terms derived from the functions \( f_\tau, z_\tau \), assuming that these “err.” terms can be made arbitrarily small. Take \( \tau_1 > \tau_2 \), and \( \tau_2 \) sufficiently large that the data for \( C(p; \tau_2)p \) are well fitted to a straight line, i.e., such that

\[
\pi_\tau(p) = e^{\beta_2 p + \text{err.}} \pi_\tau(-p)
\]

(35)

where \( \beta_2 = C_2(\chi)^+ + \tau_2 \), “err.” is a small correction term, and \( C_2 \) is the slope attributed to \( C(p; \tau_2)p \) by the least square linear fit, Eq.(18), of the data. Performing the change of variables \( x \to p = \sigma_2 x / C_2 + 1 \) in Eq.(27) one obtains

\[
\pi_\tau(p) = C_2 \sigma_1 / C_2 \sigma_2 \pi_1 \left( \hat{p}_{12} + C_2 \sigma_1 / C_1 \sigma_2 p \right) + \text{err.} , \quad \text{with} \quad \hat{p}_{12} = 1 - C_2 \sigma_1 / C_1 \sigma_2 .
\]

(36)

We can then state the following:

The 1-Shift Fluctuation Formula. Assuming that the FT holds, taking \( \tau_1 > \tau_2 \) with \( \tau_2 \) sufficiently large, and combining Eq.(35) with Eq.(36), one obtains

\[
\log \left[ \frac{\pi_1 \left( \hat{p}_{12} + \frac{\sigma_1}{\sigma_2} p \right)}{\pi_1 \left( \hat{p}_{12} - \frac{\sigma_1}{\sigma_2} p \right)} \right] = \beta_2 p + \text{err.} , \quad \text{where the correction term “err.” results from the combination of the “err.” terms of Eqs.(35,36)}.
\]

(37)

This formula, concerns fluctuations around \( \hat{p}_{12} \), and generalizes the fluctuation formula (18), in the sense that Eq.(37) reduces to Eq.(18), with \( C_1 = C_2 = C_1 \), if \( \tau_1 = \tau_2 \). Its interest lies in the fact that, for not too small \( \tau_1 \), \( \hat{p}_{12} \) is greater than zero, hence Eq.(37) concerns fluctuations with better statistics than those of the FT.\(^\text{10}\) Alternatively, one can rewrite Eq.(37) as:

\[
\pi_1 \left( \hat{p}_{12} + \frac{\sigma_1}{\sigma_2} p \right) = e^{\beta_2 p + \text{err.}} \pi_1 \left( \hat{p}_{12} - \frac{\sigma_1}{\sigma_2} p \right)
\]

(38)

and performing the change of variables \( p \to y = \hat{p}_{12} + C_2 \sigma_1 p / C_1 \sigma_2 \), one obtains:

\[
\pi_1(y) = \pi_1 \left( 2 \hat{p}_{12} - y \right) \exp \left( \beta_2 \frac{C_1 \sigma_2}{C_2 \sigma_1} (y - \hat{p}_{12}) + \text{err.} \right) .
\]

(39)

If \( \tau_1 > \tau_2 \), and the FT holds, we have \( \log(\pi_1(y)/\pi_1(-y)) = \beta_1 y + \text{err.} \), where \( \beta_1 = C_1(\chi)^+ \tau_1 \), “err.” is smaller than in Eq.(35), and \( C_1 \) is the slope assigned to \( C(p; \tau_1)p \) in Eq.(18).

The 2-Shift Fluctuation Formula. Assuming that the FT holds, taking \( \tau_1 > \tau_2 \) with \( \tau_2 \) sufficiently large, and combining the 1-Shift formula Eq.(37) with Eq.(39), one obtains

\[
\log \left[ \frac{\pi_1 \left( 2 \hat{p}_{12} + y \right)}{\pi_1 \left( 2 \hat{p}_{12} - y \right)} \right] = \left( 2 \beta_2 \frac{C_1 \sigma_2}{C_2 \sigma_1} - \beta_1 \right) y + \text{err.} ,
\]

(40)

\(^{10}\) Recall that these are not the small fluctuations around the mean of the PDF, hence they do not need to be gaussian.
where the correction term “err.” results from the combination of the “err.” terms present in Eqs. (37,39).

This formula concerns fluctuations around \( \tilde{p}_{12} \), and constitutes another generalization of the fluctuation relation (18), in the same sense as the 1-shift fluctuation formula does. One may continue this way, performing more and more changes of variables, and using the approximate linearity of the quantity \( C(p; \tau)p \), to produce a hierarchy of generalizations of Eq. (18). However, the term “err.” then becomes ever more important, due to the cumulation of “err.” terms. On the other hand, there is no reason to push further this analysis. The purpose of relations such as the 1- or 2-shift formulae rests in the fact that they concern fluctuations around \( \tilde{p}_{12} \) or \( 2\tilde{p}_{12} \), which have better statistics than the fluctuations around 0. Now, \( \tilde{p}_{12} \) can be made arbitrarily close to 1, taking \( \bar{\tau} \) sufficiently larger than \( \tau_1 \), and \( 2\tilde{p}_{12} \) can exceed 1, i.e. can go beyond the values of best statistics. Hence, if Eqs. (37,40) are valid for our data, i.e. if their “err.” terms can be made small, Eqs. (37,40) suffice for an indirect test of the validity of the FT for our systems, through the analysis of the most frequent fluctuations.

Let us now assume that the standard deviation \( \sigma_\tau \) is a continuous decreasing function of \( \tau \), as the data in Tables 2 and 3 indicate. Then, for a given \( \tau_1 \), it is possible to find \( \bar{\tau}_1 \) such that \( 2\tilde{p}_{12} = 1 \) i.e. \( C_1 \sigma_1 / C_1 \sigma_2 = 1/2 \). Using such a \( \bar{\tau}_1 \), and recalling that the small fluctuations of \( \chi_{\tau_1} \) should be gaussian [23] (if the CH holds and \( \bar{\tau}_1 \) is sufficiently large), Eq. (40) yields:

\[
1 = \frac{\pi_1 (1 + y)}{\pi_1 (1 - y)} \text{ err. } = e^{(2 \beta_2 C_2 \sigma_1^2 - \beta_1) y} + \text{ err.} \quad \text{(41)}
\]

for \( y \sim (1/\sqrt{\bar{\tau}_1}) \) or smaller. This means that \( 2 \beta_2 C_2 \sigma_1^2 - \beta_1 = 4 \beta_2 - \beta_1 \) must be approximately zero, i.e. \( C_1 \bar{\tau}_1 \approx 4 C_2 \bar{\tau}_2 \), or

\[
\frac{\sigma_1}{C_1} = \frac{1}{2} \frac{\sigma_2}{C_2} \quad \text{ implies } \quad C_1 \bar{\tau}_1 = 4 C_2 \bar{\tau}_2 + \text{ err.}, \quad \text{(42)}
\]

for sufficiently large \( \bar{\tau}_1 \). We then have:

**The scaling of the variance.** Let \( \pi_0 \) be the PDF of the fluctuations of duration \( \tau_0 > 0 \) (a given reference time). If the 2-shift fluctuation formula Eq. (40) and the FT hold, the standard deviation of \( \pi_\tau \), obeys

\[
\sigma_{\tau} = \left[ \frac{\sigma_0 \sqrt{\tau_0}}{\sqrt{\tau_0}} \right] \sqrt{\frac{C_2}{\tau}} + \text{ err.}, \quad \text{(43)}
\]

where \( \sigma_0 \) is the standard deviation of \( \pi_0 \), \( C_2 \) is the slope of \( C(p; \tau_0)p \), and “err.” results from the “err.” terms of Eqs. (40, 42).

The 1-shift formula can now be used to derive the variance of the gaussian distribution of the small fluctuations. Take \( \bar{\tau}_1, \tau_2 \) and \( p \) so that \( \bar{\tau} = \tilde{p}_{12} - p C_2 \sigma_1 / C_1 \sigma_2 \) be close to 1, i.e. \( \bar{\tau} = 1 - \epsilon \) with a small positive \( \epsilon \). This implies that \( \bar{\tau} = \tilde{p}_{12} + p C_2 \sigma_1 / C_1 \sigma_2 = 1 + \epsilon - 2 C_2 \sigma_1 / C_1 \sigma_2 \) and taking \( \bar{\tau}_1 \) sufficiently large we can make also \( \bar{\tau} \) close to 1. We may then assume that \( \pi_1 \) at \( \bar{\tau} \) is well represented by a gaussian distribution \( g_1(\bar{\tau}) \sim \exp[-(1-\bar{\tau})^2/2\sigma_1^2] \), with unknown variance \( \sigma_1^2 \), and using the 1-shift fluctuation formula we obtain:

\[
\beta_2 p + \text{ err. } = \log \left[ \frac{\pi_1 \left( \tilde{p}_{12} + \frac{C_2 \sigma_1}{C_1 \sigma_2} p \right)}{\pi_1 \left( \tilde{p}_{12} - \frac{C_2 \sigma_1}{C_1 \sigma_2} p \right)} \right] = \frac{2}{\sigma_1^2} \left( \frac{C_2 \sigma_1}{C_1 \sigma_2} \right)^2 p + \text{ err.} \quad \text{(44)}
\]

which yields

\[
\sigma_1^2 = \frac{2}{C_2 (\chi_1 + \tau_2)} \left( \frac{C_2 \sigma_1}{C_1 \sigma_2} \right)^2 . \quad \text{(45)}
\]

In the case that \( C_1 \tau_1 = 4 C_2 \tau_2 \), and that both \( \bar{\tau}_1 \) and \( \bar{\tau}_2 \) are sufficiently large (so that \( C_1 \approx C_2, \tau_1 \approx 4 \tau_2, C_2 \sigma_1 \approx C_1 \sigma_2/2 \), and \( \tilde{p}_{12} \approx 1/2 \)), we get:

\[
\sigma_1^2 = \frac{2}{\tau_1 \chi_1 + C_1} + \text{ err.} \quad \text{(46)}
\]
This expression can be obtained assuming that $\tau_1$ be gaussian, that $s_1^2$ be its variance and that the FT holds, as done in Ref.[8]. However, our PDFs are non-gaussian and $s_1$ differs from the standard deviation $\sigma_1$ of $\pi_1$ (see Tables 2 and 3). Moreover, to consider as small the fluctuations around $\dot{\pi}_1 = 1/2$, we need $1/\sqrt{\pi} \sim 1/2$, i.e. $\tau_1 \sim 4$. Hence, we find no connection between the gaussian distribution of the small fluctuations and the whole PDF, in agreement with the analysis of [8], in which the apparent connection was recognized as a peculiarity of the model investigated there.

We conclude this section considering $\tau_1, \tau_2$ sufficiently large that $C_1 = C_2$ is practically verified. Then, the 1-shift fluctuation formula takes the form:

$$\log \left[ \frac{\pi_1 (\dot{\pi}_1 + \frac{s_1}{\sigma_1} p)}{\pi_1 (\dot{\pi}_1 - \frac{s_1}{\sigma_1} p)} \right] = \beta_2 p + \text{err.}, \quad (47)$$

the 2-Shift fluctuation formula takes the form:

$$\log \left[ \frac{\pi_1 (2\dot{\pi}_1 + y)}{\pi_1 (2\dot{\pi}_1 - y)} \right] = \left( 2\beta_2 \frac{\sigma_2}{\sigma_1} - \beta_1 \right) y + \text{err.}. \quad (48)$$

and the scaling of the variance reduces to:

$$\sigma_\tau = (\sigma_1 \sqrt{\pi}) \tau^{1/2} + \text{err.} \quad (49)$$

6 Results

The second scaling produces a slightly faster convergence of the scaled PDFs to a given asymptotic curve, than Eq.(28) does. This fact is reflected in Fig.5, in which only two scaled $\pi_\tau$’s are drawn for the sake of clarity. Hence, the terms $C_\tau$ in Eq.(26) appear to properly account for the variations of $C_\tau$ in $\pi_\tau$.

For the scaling of the variance, the log-log plots of Fig.6 show that our data are accurately fitted by Eq.(43). In the case $\gamma = 2$ we use $\tau_1 = 6$ and $C_\tau = 1$ for $\tau > 1$, while for $\gamma = 2.5$ we have $\tau_1 = 6$ and $C_\tau = 0.931$ for $\tau > 1$. The values of $C_\tau$ for $\tau \leq 1$ come from Tabs. 2, 3, while those for $\tau > 1$ are obtained from the Lyapunov spectra (Tab. 1), using the axiom C formula (17). A similarly good agreement is obtained using Eq.(49). In this respect, the presence of $C_\tau$ in Eq.(27) does not seem to play an important role.

The test of Eq.(47) presented in Fig.7, instead, shows a systematic difference between the left hand side of the equation, evaluated on our data for $\gamma = 2$, and the straight line $\beta_2 p$. This difference appears to decrease when $\tau_1$ is fixed and $\tau_2$ grows. However, whether this is really the case, or not, cannot be checked directly because $\dot{\pi}_1$ moves to regions of low statistics when $\tau_2$ approaches $\tau_1$ (i.e. $\dot{\pi}_1 \to 0$ as $\tau_2 \to \tau_1$). Therefore, we are unable to reach a definite conclusion on the validity of Eq.(47). The good agreement between our data and the 1-shift fluctuation formula (27), suggests then that the term $C_\tau$ in Eq.(26) is important, at least for not too large $\tau$. The results of our test for $\gamma = 2$ are reported in Fig. 8. There $C_2$ has been taken from Table 2 and $C_1$ has been chosen so that Eq.(37) fits well the data. Changing $\tau_1$, hence $C_2$, the best fit of the numerical data requires slightly different $C_1$’s. The resulting uncertainty on the computed value of $C_1$ can be interpreted as a combination of the uncertainties that our data carry for both $C_1$ and $C_2$. In particular, for the data represented in Fig. 8, we have used $C_1 \in [0.93, 1.03]$, which is consistent with the validity of the second scaling and of the FT (asymptotic slope $C_{\infty} = 1$).

Similar considerations apply to the 2-shift fluctuation formula. Its asymptotic form Eq.(48) shows systematic deviations from our data with small $\tau$’s, while Eq.(40) fits reasonably well the same data. However, the fit is not as good as for the 1-shift fluctuation formula, because of the cumulation of “err” terms. Our results are shown in Fig. 9, where $C_1 \in [0.96, 0.98]$ and $C_2 \in [0.99, 1.02]$, in relative good agreement with the validity of the FT, and with the second scaling.

Figure 10 completes the picture for the 1- and 2-shift formulae for $\gamma = 2.5$. All is similar to the case $\gamma = 2$, except for the not equally good statistics. The test of the 1-shift fluctuation formula (top panels)
has been made with $C_1 \in [0.915, 0.95]$ and $C_2 \in [1.1, 1.19]$, and one should note that the values of $C_1$ are consistent with the axiom C prediction, which yield $C_\infty = 0.931$. The test of the 2-shift formula has been performed with $C_1 \in [0.87, 0.93]$ and $C_2 \in [0.9135, 0.95]$, still close to the axiom C predictions. Similarly to the case $\gamma = 2$, the accord between the 2-shift fluctuation formula and our data is not as good as for the 1-shift fluctuation formula.

The above results can be read in two different ways. If we know the values of $C_\tau$ for each $\tau$, then we have a check of the second scaling and of the FT. Alternatively, trusting that the FT and the second scaling hold, we can compute the otherwise unknown slopes $C_\tau$, by choosing the values which best fit Eqs.(37,40,43) to the data. Viewing our results this second way, the second scaling becomes a tool to compute the slopes of the fluctuation formula, even in the absence of negative fluctuations, and we gather some evidence of the “axiom-C-like” structure of the SLLOD systems.

7 Observability of large fluctuations

1. We have tested the validity of two scalings for the PDFs $\pi_\tau$ of the fluctuations of the phase space contraction $\chi_\tau$ of SLLOD particle systems at high shear rate. The evidence reported in Sect. 4 is in favor of the validity of both scalings. The validity of the first scaling suggests that the dissipative, time reversal invariant SLLOD systems can be considered as Anosov- or axiom-C-like. One then expects the FT to hold, without implying that the relevant fluctuations be actually observable, or that the asymptotic slope $C_\infty$ is given by Eq.(17).

2. The validity of the second scaling, instead, has no immediate theoretical relevance: it is an empirical observation for which we lack a theoretical explanation at present. Nevertheless, being solely based on the shape of the PDFs, this scaling would equally be valid (in fact should be better verified) in those systems whose $\pi_\tau$’s are indistinguishable from gaussian distributions, and scale with $\tau$ so that $\sigma_\tau^2 \equiv \text{const}$. This is the case of Ref.[8], for instance. Moreover, the present study was motivated by unpublished results of Lepri, Livi, Politi and Ruffo, who found that (28) holds for the PDF’s of Ref.[10]. Therefore, the second scaling seems to hold in a rather interesting variety of physical models, and further study of it seems worthwhile.

3. If one assumes that the second scaling holds, the large fluctuations of the FT, which are practically unobservable in large systems or in strongly dissipative systems, may become indirectly observable through their influence on the scaling of the variance, and on the 1- and 2-shift formulae. Indeed, the results discussed in Sect. 6 suggest that the second scaling is appropriate to characterize our data. Hence, the validity of Eqs.(37,40,43) becomes indirect evidence of the validity of the FT for the SLLOD systems. Moreover, the asymptotic slope $C_\infty$, obtained from the Lyapunov spectrum of the case $\gamma = 2.5$ is within the range of values for which the scaling of the variance and the 1- and 2-shift formulae hold. These results, then, indicate that the SLLOD systems can be considered as “axiom-C-like”, and verify the FT with the slope given in Eq.(17), despite the fact that the fluctuations of $\chi_\tau$ are not directly observable.

4. The form of the second scaling may appear to be derived from the Central Limit Theorem, however that is not the case. In the first place the Central Limit Theorem applies to the small fluctuations around the mean of the relevant PDF, while we use the second scaling over the whole support of our (non-Gaussian) PDFs. Moreover, if the second scaling holds and $\pi_1$ in Eq.(27) is not gaussian, then also $\pi_2$ will not be Gaussian, for all $\tau_2 > \tau_1$, as our results indicate.

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Figure 1: The PDFs $\pi_+ (p)$ (left panels), and the functions $C(p; \tau)p$ (right panels) for $\gamma = 2$ and various values of $\tau$. The functions $C(p; \tau)p$ are well fitted to straight lines only for $\tau \geq 0.32$. As $\tau$ grows, a linear shape is better and better approximated by $C(p; \tau)p$, and $C_+$ appears to converge to $C_{\infty} = 1$ (see table 2). This is not sufficient for a direct test of the FT, since the PDFs with large $\tau$ have no negative tails.


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$\langle \chi \rangle_+$ 20.62000 0.35460 31.74600 0.93766

sum -20.62000 0.35460 -31.31200 0.91535

Table 1: Tables of Lyapunov exponents for iso-kinetic SLLOD. The vanishing exponents are not reported. The quantity called sum is the sum of the exponents, and should be $-\langle \chi \rangle_+$. Note that the number of positive Lyapunov exponents for $\gamma = 2.5$ is one less than for $\gamma = 2$. 
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<th>$E(\chi^4)$</th>
<th>$E(\chi^5)$</th>
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Table 2: Slope $C_\tau$ of the line approximating the function $C(p; \tau)p$, and moments of the PDFs $\pi_\tau$ up to fifth order, for $\gamma = 2$ and various values of $\tau$. The values of $C_\tau$ are not very meaningful for $\tau < 0.32$, being the result of a least square fit of points in a non-linear curve, while they cannot be computed directly for $\tau > 1.8$, due to low statistics of the negative fluctuations. The moments of order 0 and 1 are not reported because they equal 1 for all $\tau$'s, while $\kappa = [(p-1)^4]/\sigma^4 - 3$ is the kurtosis of the PDFs, which vanishes for gaussian densities. The variance $\sigma^2$ scales as $\tau^{-1}$. 
Figure 2: The PDF’s $\pi_r(p)$ (left panels), and the functions $C(p; \tau)p$ (right panels) for $\gamma = 2.5$ and various values of $\tau$. The functions $C(p; \tau)p$ are well fitted to straight lines only for $\tau \geq 0.32$. As $\tau$ grows, a linear shape is better and better approximated by $C(p; \tau)p$, and $C_r$ seems to converge to $C_{\infty} < 1$, (see table 3). Negative fluctuations become unobservable as $\tau$ grows, as in Fig. 1.

Figure 3: Function $(1/\tau)\log \pi_r(p)$, and function $(1/(\tau_1 - \tau_2))\log(\pi_1(p)/\pi_2(p))$ (right panel), for $\gamma = 2$. On the left we have $\tau = 3, 5, 9, 12, 16$. On the right we have $\tau_1 = 16$ and $\tau_2 = 2, 3, 5, 9, 12$. The direction of growing $\tau$ is indicated by the arrows.
<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$C_\tau$</th>
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<th>$\kappa$</th>
<th>$IE(\chi^4)$</th>
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Table 3: Slope $C_\tau$ of the line approximating the function $C(p; \tau)p$ and moments of the PDFs, as in Table 1, for $\gamma = 2.5$. The variance $\sigma^2$ scales approximately as $\tau^{-0.9}$.

Figure 4: Function $(1/(\tau_1 - \tau_2)) \log(\pi_1(p)/\pi_2(p))$ (left panel), and function $\log(\pi_1/(C_\tau)\kappa_\tau((\pi_1/C_\tau)x + 1))$, for $\gamma = 2$. In the left panel the values of $\tau_1$ and $\tau_2$ are the same as in Fig.3, while in the right panel we have $\tau = 3, 5, 9, 12, 16$. 

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Table 4: Values of $F_{x_1}^{(1)}$ (above the diagonal), and of $S_{x_1}^{(1)}$ (under the diagonal), for $\gamma = 2$.

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Table 5: Values of $F_{x_1}^{(2)}$ (above the diagonal), and of $10^5 \cdot S_{x_1}^{(2)}$ (under the diagonal), for $\gamma = 2$.

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Table 6: Values of $F_{x_1}^{(3)}$ (above the diagonal), and of $S_{x_1}^{(3)}$ (under the diagonal), for $\gamma = 2$.

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Table 7: Values of $F_{x_1}^{(4)}$ (above the diagonal), and of $S_{x_1}^{(4)}$ (under the diagonal), for $\gamma = 2$.

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Figure 5: Functions \( \sigma_{\tau}(\sigma_{\tau}x + 1) \) (left panel), and functions \( (\sigma_{\tau}/C_{\tau})\pi_{\tau}((\sigma_{\tau}/C_{\tau})x + 1) \) (right panel), for \( \gamma = 2 \), and \( \tau = 0.64, 6 \). The presence of \( C_{\tau} \) in Eq.(26) improves the convergence to a limit distribution \( \varphi \).

Figure 6: Log-log plots of the standard deviation \( \sigma_{\tau} \) (crosses), and of \( \sigma_0 \sqrt{\tau_0/C_0} \sqrt{\tau/C_{\tau}} \) (cf. Eq.(13)) as functions of \( \tau \). In the case \( \gamma = 2 \), we chose \( \tau_0 = 6 \), and \( C_{\tau} = 1 \), for \( \tau > 1 \), and the largest \( \tau \) is 16. In the case \( \gamma = 2.5 \), we chose \( \tau_0 = 6 \), and \( C_{\tau} = 0.931 \) for \( \tau > 1 \), and the largest \( \tau \) is 13.
Figure 7: Test of Eq. (47). For each choice of $\tau_1$ and $\tau_2$, the left hand side of Eq. (47) is represented by the noisy curves, as functions of $p$, while the dashed straight lines represent the term $\beta z p$. In each panel, the curves have fixed $\tau_1$, while $\tau_2$ decreases in the clockwise direction.

Figure 8: Plots of the left hand side and of the right hand side of Eq. (37) as functions of $p$, for $\gamma = 2$. The values of the slope $C_2$ are taken from table 2, while $C_1$ is chosen in such a way that Eq. (37) fits well the numerical data. In particular, $C_1 \in [0.96, 1.03]$ for $\tau_1 = 3$, $C_1 \in [0.95, 1]$ for $\tau_1 = 4$, $C_1 \in [0.94, 1]$ for $\tau_1 = 5$, and $C_1 \in [0.93, 1]$ for $\tau_1 = 6.$
Figure 9: Plots of the left hand side and of the right hand side of Eq.(48) as functions of $p$, for $\gamma = 2$. Here, $C_1 \in [0.96, 0.98]$ and $C_2 \in [0.99, 1.02]$, so that Eq.(40) fits well the numerical data.

Figure 10: Plots of the left hand side and of the right hand side of Eq.(37) (upper panels) and of Eq.(40) (lower panels) as functions of $p$, for $\gamma = 2.5$. In the top panels we have $C_1 \in [0.915, 0.95]$ and $C_2 \in [1.1, 1.19]$, while in the bottom panels we have $C_1 \in [0.87, 0.93]$ and $C_2 \in [0.913, 0.954]$. 