Dynamical Properties of Space of Space–Times

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We have many phase planes for cosmological models summarized in Wainwright and Ellis [1] and subsequent papers [2]. These represent partial sections of the space-of-space-times. The aim is to put these together and find properties of the space of cosmological space-times as whole, regarded as an infinite dimensional dynamical system, thus understanding the generic behaviour of families of spacetimes (asymptotics at late times, at early times, 'close to FL dynamics).

In particular: it is worth investigating dynamical systems approach to chaotic inflation [3] to confirm if its behaviour is indeed generic as claimed (NB most studies of inflation are done in FRW context, and indeed in flat FRW context; that cannot decide these issues). This demands representation of dynamic scalar field if done properly.

1 Definition of Space of Space-Times

A: Description of Space Of Spacetimes $S$: state space, each point a space-time described via
1. Local coordinates (identifying points in each space-time),
2. Tetrad components,
3. Rotation coefficients,

Matter Description: what kind of matter? in the cosmological context.
(i) Fluid with well-defined equation of state, and $\mu + p \neq 0$,
(ii) Dynamic scalar field (required in order to deal with inflation adequately).

NB: this means there will be a unique 4-velocity $u^a$ defined as eigenvector of the Ricci tensor, and so breaks boost-invariance of equations and leads to 1+3 decomposition of variables. Furthermore at some stage of evolution of the model, it should contain a region where (ii) the fluid expansion (defined by this vector field) is positive and nearly isotropic in some open domain. We take these as requirements for a cosmological model, and so for the space of cosmological space-times.

Equations: These 1+3 decomposed variables are then related by first-order identities and by the field equations, see van Elst and Ugglal [4] or Ellis and van
Elst [5] for summary. These can be brought to FOSH form [6] in general, but not necessarily if normalized. An important issue is if this is the optimal thing to do.

**Optimal Description:** Issues regarding these descriptions arise as follows:

*Question 1:* Include Weyl tensor components $E$, $H$? Redundant variables, but give important covariant information (cf Newman and Penrose [7], [8]).

*Question 2:* Normalise to get dimensionless variables? If so, how? (by fluid expansion or other quantity?)

*Question 3:* Gauge choices - some of the variables are gauge variables that fix the description uniquely. We do not have to allow all choices, rather can fix the gauge so as to reduce number of variables and fit specific situations. Associated with this is choice of a dynamical systems time variable that simplifies the equations.

*Issue:* Should we go for one in all cases, or many, to suit different situations [9]? Probably the latter, giving a regional description of $S$, but then how do we stitch them all together to give the overall space of space-times? [like a coordinate atlas]. Note that the 'natural' choice is a comoving gauge, but this will not be adequate for all studies, for example relating the models to Newtonian models. Presumably there is a dynamical system local equivalence under changes of gauge.

*Question 4:* Local or global description? Because the topologies can be wildly different, latter can be very difficult. To start with, go for former, but we need to note carefully when models are necessarily spatially closed (e.g. FRW $k = +1$, Bianchi IX, Gowdy vacuum) and when this is not necessarily the case.

*Question 5:* Higher Symmetry Spaces: how do they fit in to this space, and generate sub-spaces that can be described by fewer functions? (a) space-times invariant under Lie Groups of Killing vectors (simply and multiply transitive) (b) self-similar space times? [10]. Problem: what appears to be self-similar may be gauge dependent. Need to characterize optimal basis, and then appearance in other bases. Issue: How do we characterize how generic subspaces are? e.g. by counting of functions?

*Question 6:* FOSH choice? are combinations of the obvious variables desirable, e.g. in order to get a FOSH representation? Potential problem: this is really a 2+2 rather than 1+3 decomposition. However it may relate the analysis in a strong way to existing mathematical theorems, particularly re existence and uniqueness of solutions and the nature of their characteristics, and that may be very useful.

**B: Model evolution:** Each model will track some evolution history through
state space, and so will appear quite different at different stages of its history; nevertheless each of these states is the same model but viewed at different times. Can only tell its the same model by evolving the equations, usually.

*Question 6*: what **invariants** are there that will characterize a model as being the same model but at different stages of its evolution? [there may easily be invariants characterizing it as belonging to the same class of models, but not necessarily as representing the same model at different evolutionary stages].

**C: Equivalence problem**: There will be multiple appearances of the same model, if we represent it in different bases in the same gauge.

*Recommendation*: do not factor out equivalent spaces, so that there is only one representation of each space-time. Rather identify where the same space-time occurs, given the chosen gauge - i.e. at least implicitly, solve the equivalence problem for cosmological solutions. This may help identify suitable gauges. The issue then is, when are space-times the same? This arises re homogeneity and isotropy.

**Homogeneity**: determine corresponding points by use of invariants (scalars, or unique tetrad components). When is this not unique? When there are isometries on submanifolds, and so specialised tetrads related to that homogeneity.

*Issue*: show how to handle the question of possible isometric point identification when there are isometries.

**Isotropy**: determine corresponding properties at the corresponding points by use of tetrad components. This is unique when there is a unique tetrad (indeed then tetrad components are scalar invariants), but not otherwise.

*Query 1*: when is there a unique tetrad? Answer: when not LRS [11].

*Query 2*: If not LRS, when is that tetrad not determinable algebraically by (i) shear $\sigma_{ab}$, (ii) $E_{ab}$? Answer: when a PLRS space-time [12]. Thus we can always find unique tetrad choices to use in equivalence comparison except of LRS, when not possible, and PLRS, when need higher-order choices. However both classes are determinable, at least in principle.

Given answers to these two questions, we can in principle solve the equivalence problem for particular gauge choices.

**D: Distance between models**: the inverse of the equivalence problem is defining a distance between models. In principle we can do this by standard functional techniques once we have chosen an algorithm for point comparison and tetrad choice.

*Issue*: show how to define such a distance, and relate it to the equivalence problem.

2 Involutive Subsets

Consistent evolutions will define involutive subsets. We need to understand the consistency conditions and best representations that arise for particular families
of models, e.g.

1. **Kinematic constraints**: Zero vorticity, zero shear, zero acceleration, and combinations of these quantities;

2. **Weyl tensor constraints**: Zero $E$, $H$, $\text{div } E$, $\text{div } H$, $\text{curl } E$, $\text{curl } H$, silent universes [14].

3. **Symmetry constraints**: Models with symmetries
   - isometry groups: LRS (G6, G4, and G3), Bianchi G3 (SH), Abelian G2, non-Abelian G2
   - self-similar/homothetic motions (SS)

4. **Perturbation equations** in various gauges, e.g.
   (i) Newtonian gauge $\sigma = 0$; [13]
   (ii) constant density gauge $X_a = 0 \Leftrightarrow \nabla_a \rho = 0$;
   (iii) constant expansion gauge $Z_a = 0 \Leftrightarrow \nabla_a \Theta = 0$;
   (iv) constant curvature gauge $R_a = 0 \Leftrightarrow \nabla_a^3 R = 0$.

This relates to finding invariants for sets of solutions, see above. Key issue is consistency of all equations, and in particular that constraints are preserved by time evolution. Note also that constraint equations may prevent modes that otherwise look possible (see FOSH case in particular [6]).

3 **Dynamic Behaviour**

Characterize behaviour of families of general models and higher symmetry models in terms of dynamical systems concepts:
- attractors, basins of attraction, limit points;
- saddle points, intermediate behaviour.

Can we find generic properties, independent of symmetries? [cf the singularity theorems, vorticity conservation theorems, etc]. How do generic properties of symmetric subspaces relate to generic properties of the whole?

**Question 1**: **The Past Attractor. BKL conjecture**: ‘matter does not matter at early times’, and models are steered to Kasner set [15]. Considerable evidence for this; key factor is the “unphysical boundary”, the invariant manifold on which the spatial differential operator is zero. But
- is this still true if there is a timelike singularity? [these may be generic]
- a ‘whimper’ singularity? [but these are not generic, see Siklos paper]
- a Gowdy model with spikes? etc.

Is it true for a generic family of models, with other generic families where it is not true; or is it true almost always??
Question 2: **Intermediate evolution** including oscillatory and/or intermittent behaviour, and **Isotropisation Properties** at early times and late times including studies of the **Future Attractor**. In particular, how successful is inflation in causing isotropy?, see Wald [16], for a strong indication that it can do this in many cases; but,

- when can inflation start? see e.g. [17]; it can be suppressed by inhomogeneity and anisotropy.
- do velocities die away?? maybe not (Goliath and Ellis [18]) [they do not die away in some frames; are there other frames where they do indeed do so?]
- is late time attractor self-similar?

*Note 1:* analyses of late time attractor are sometimes used to analyse intermediate properties, e.g. Wald’s inflationary study is meant to characterise what happens in inflation, not in the far distant universe.

*Note 2:* **Isotropisation at intermediate times** occurs and is of interest (Wainwright et al [2]). How generic is this??

**Question 3:** **Guiding Skeleton:** does a skeleton of higher symmetry models acts as a guide for the development of lower symmetry models [1]? What is clear is that the higher symmetry models are consistent solutions of the EFE and so are indeed involutive subsets of the full space. They will be fixed points in some representations. They intersect in a skeleton-like structure, because higher symmetry spaces are subspaces of lower symmetry spaces, which in appropriate gauges will appear to be lower dimensional.

**Issue:** can they be isolated, or must the higher symmetry spaces always join to lower symmetry ones? Guess: the latter is the case (unless one runs into a boundary), because there should always be directions in which some of the lower symmetry sub-groups of the higher symmetry model are preserved. NB: the key issue here is that all the Bianchi models except type IX and all the FRW models do indeed have lower dimensional subgroups. The exceptional cases are Bianchi IX, Kantowski-Sachs, and Tolman-Bondi inhomogeneous (positive curvature 2-spaces). Query: can these appear as isolated models? (Note that in each case there is a negative curvature analogue that does indeed have subgroups).

Specific cases show how higher symmetry spaces act as lower dimensional subspaces of $S$ that are attractors (e.g. the Kasner ring) and as saddle points (e.g. FRW models). Whether they are represented as fixed points or nor, one can linearise about them and find in what directions they act as attractors and in what directions they act as repellors, with respect to general perturbations. Query: what more is needed? [One can linearise about any solution, whether higher symmetry or not - what is special about these solutions that makes them stand out in the whole space, apart from the fact that we can explicitly solve for them in many cases?]

**Question 4:** **Self-Similar Models:** do self-similar models play a major role in determining evolution of others? They certainly do in the spatially homogeneous case, and in the case of collapse. [19], [20]. How generic is this? (Carr conjectures it is always true in collapse).
Question 5: Stability/Fragility of particular solutions and families of models [21].

- do they have stable behaviour (to the future/to the past)?

Attractors are stable by definition (although they can represent oscillatory behaviour), and repellors/saddle points unstable, so the need is to determine these features and their associated basins of attraction. But

- solutions can hover near saddle points for a long time, even though unstable; this leads to intermittent behaviour and spikes, and even to “chaos” in some cases;

- bifurcations can take place as parameters vary. How is this represented in \( S \)? When do they occur (in terms of geometry; of equations of state? How common are they? [Simple case: FRW models as spatial curvature changes from +1 to −1. The \( k = 0 \) model is unstable for normal equations of state].

NB major issue: many analyses of dynamical systems demand a compact state space [22]. But often the state space here is not compact. Question: can we either compactify it, or obtain results even though not compact?

4 Structure Formation and Entropy

Here we linearise about higher symmetry models, and particularly FRW models.

1. Linearised solutions and gauge issue: these show how other models relate to higher symmetry models, determining evolution in a small neighbourhood of the higher symmetry models.

Question 1: Measures of inhomogeneity of gravitational field?

Fourier analysis and associated power spectra (wave number \( k \)) as measures of inhomogeneity: Note use of CGI variables and Bardeen variables: else apparent inhomogeneity is pure gauge. Note also different scaling with time of inhomogeneity to normalised variables: inhomogeneity wavelengths \( L/k \) are comoving, and so scaled by \( a(t) \), whereas dynamical variables are scaled by \( 1/H(t) \), hence need to track \( k/(aH) \). By definitions,

\[
\partial_q(lH) = -q(lH)
\]

(Wainwright), where \( l \) is the volume average length scale defined by \( \Theta \), and \( q \) the associated deceleration parameter; hence inflation \((q < 0) \) makes \((lH) \) grow, non-inflationary phase \((q < 0) \) makes it decay.

Linearisation and almost-FRW models: what do we mean by close to FRW?

\[
\sigma_{ab}/H^2, E_{ab}/H^2, E_{ab}/H^3
\]

as measures of distance form FRW [26]. Does this necessarily imply perturbed FRW? cf [27] for queries in this regard, but see [28] for a local demonstratin it is Ok (but: issue of averaging scales is unresolved ...)

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Question 2: Linearisation stability: when does linearised solution represent well behaviour of full solutions? [29] When is this not true? [Issue of true domain of dependence for structure formation (characteristic as limits, but real domains of influence). Particle horizon as compared to influence domain.]

Question 3: Different kinds of inhomogeneity (scalar, vector, tensor): extension to non-linear models? What are the subspaces of $S$ representing these models? Also entropy/adiabatic perturbations, and their relative growth/interaction? Show what subspaces of $S$ these each represent.

2. Coarse-graining: maps in $S$ between same solution represented on different averaging scales,
   - averaging problem for dynamics: what is effective equation of state for averaged model, and how does it relate to standard EFE for that model? (the “averaging problem” for GR). NB: this also affects observations.
   - Loss of information associated with coarse graining: how to characterize that?

3. Measures of inhomogeneity and entropy of gravitational field? Relate entropy measure to growth of inhomogeneity and spontaneous structure formation. What is the subset of models where spontaneous structure formation takes place?

4: Arrow of time. Underlying this is the arrow of time problem. Structures should form in both directions of time (EFE are time symmetric). Where does the one-way effect come from? Presumably from expansion of universe - but how is this related to the various other arrows of time? What is family of models for which an arrow of time emerges as in standard macro-physics? [30]

5 Ensembles and Probabilities

The previous section looks at what is possible. But the further issue is, is it probable? In particular, this arises as regards Inflationary universes: are they in fact probable, (a) as regards initial conditions, (b) as regards later development?

Thus we need to look at

Question 1: A geometric/kinematic measure on $S$ to determine how probable models are, in terms of their intrinsic geometry:
   - Generality of families of models: how to characterize probabilities, and generality of subspaces? For example:
     - given anisotropic homogeneous modes that grow in the future and past [1], generic models presumably have these modes non-zero.
     - Probably needs Bayesian statistical approach, see e.g. [23]
Question 2: Relation of measure to dynamics [24]. What is effect of
dynamics on families of models? [Whole idea of chaotic inflation: even if
probability to start is very small, they will come to dominate universe in terms
of volume behaviour and so of probabilities at late times [3], [25]]
do they remain general if they start out so? do they become special from
generic initial conditions?

Question 3: Description of ensembles: Need here a proper description
of an ensemble of models and of its properties, and hence a possibility of seeing
dynamical evolution of the ensemble and defining associated probabilities. This
can be asked in general, or for particular families of models characterized by
specific symmetries or matter behaviour. Nb issue arises in various contexts:
spatial separation, time separation, completely disconnected.

6 Observations and Horizons

Which classes in $S$ are possible models of real universe? The have to correspond
to astronomical observations. Major issues:

1. Basic observations
   A: Isotropy of expansion, at some epoch
   B: Nucleosynthesis - correct expansion rates at early times
   C: How does CBR anisotropy behave? - suggests almost FRW state (see
   almost EGS theorem); but
   - Bianchi model examples
   - What kinds of counter examples are there to almost EGS theorem? (Wain-
   Wright et al, Clarkson)

2. Further observational issues
   D: Structure formation - to look like what we see requires conditions where
   gravitational instability can spontaneously form structures that do not immediately
collapse to black holes
   E: Existence of life - suitable conditions for life to exist, the Anthropic
   principle, see e.g. Weinberg [31]. Related to this are (i) conditions where quantum
   theory leads to classical behaviour, and (ii) conditions where GR leads to
   Newtonian-like behaviour, with “an isolated system” being a viable concept.
   F: How is testing limited by horizons?
   - How global need our models be, given that we can only test them “lo-
   cally”??

Together, these requirements define a viable subspace of $S$ that can be used
to represent the real universe. The major issue is
   - how large this is,
   - where it can come from: what is its basin of origin?
   - where can it go to - what is the future?
3: Issue of prior assumptions: a major issue: make clear what ‘priors’ are assumed. That is, there is
- one set of probabilities based purely on generic initial conditions and generic dynamics;
- a second set when we modify those probabilities by using observations of what we see around us. Both give valid approaches, but answer different questions (what is likely a priori, given our understanding of physics; what is likely when that is modified by ‘prior assumptions’ to take into account what has actually happened).

We need clarity on which we are asking. In effect, the one works forward from arbitrary initial conditions to determine likely outcomes; the second backwards from present day observations to determine likely past histories.

References


[27] Lim W C Seminar Waterloo.

