Homoclinic Points and Isomorphism Rigidity of Algebraic Z\(^{\pi}\)–Actions on Zero–Dimensional Compact Abelian Groups

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Vienna, Preprint ESI 1127 (2002)  
February 8, 2002

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at
HOMOCLINIC POINTS AND ISOMORPHISM RIGIDITY
OF ALGEBRAIC $\mathbb{Z}^d$-ACTIONS ON ZERO-DIMENSIONAL
COMPACT ABELIAN GROUPS

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Abstract. Let $d > 1$, and let $\alpha$ and $\beta$ be mixing $\mathbb{Z}^d$-actions by auto-
morphisms of zero-dimensional compact abelian groups $X$ and $Y$, respectively. By analyzing the homoclinic groups of certain sub-actions of $\alpha$ and $\beta$ we prove that, if the restriction of $\alpha$ to some subgroup $\Gamma \subset \mathbb{Z}^d$ of infinite index is expansive and has completely positive entropy, then every measurable factor map $\phi: (X, \alpha) \rightarrow (Y, \beta)$ is almost everywhere equal to an affine map. The hypotheses of this result are automatically satisfied if the action $\alpha$ contains an expansive automorphism $\alpha^n$, $n \in \mathbb{Z}^d$, or if $\alpha$ is arises from a nonzero prime ideal in the ring of Laurent polynomials in $d$ variables with coefficients in a finite prime field. Both these corollaries generalize the main theorem in [8]. In several examples we show that this kind of isomorphism rigidity breaks down if our hypothesis are weakened.

1. Introduction

Throughout this paper the term compact abelian group will denote an infinite compact metrizable abelian group.

Let $X$ be an additive compact abelian group with identity element $0_X$, normalized Haar measure $\lambda_X$ and additive dual group $\hat{X}$. For every $x \in X$ and $a \in \hat{X}$ we denote by $\langle a, x \rangle \in \mathbb{S} = \{z \in \mathbb{C}: |z| = 1\}$ the value of the character $a \in \hat{X}$ at the point $x \in X$. An algebraic action $\alpha$ of a countable group $\Gamma$ on $X$ is a homomorphism $\alpha: \gamma \mapsto \alpha^\gamma$ from $\Gamma$ into the group $\text{Aut}(X)$ of continuous automorphisms of $X$. An algebraic $\Gamma$-action $\alpha$ on a compact abelian group $X$ is expansive if there exists an open set $O \subset X$ with

$$\bigcap_{\gamma \in \Gamma} \alpha^\gamma(O) = \{0_X\},$$

and mixing if there exists, for all nonempty open subsets $O_1, O_2 \subset X$, a finite set $F \subset \Gamma$ with

$$O_1 \cap \alpha^\gamma(O_2) \neq \emptyset$$

for every $\gamma \in \Gamma \setminus F$. 

2000 Mathematics Subject Classification. 13E05, 37A35, 37B05, 37B50, 37C29, 37C85.

Key words and phrases. Algebraic actions of higher-rank abelian groups, Expansiveness, Homoclinic points, Isomorphism rigidity.

The first author would like to thank the FWF research project P14379-MAT for financial support and the Erwin Schrödinger Institute, Vienna, for hospitality while this work was done.
Let $\alpha$ and $\beta$ be algebraic $\Gamma$-actions on compact abelian groups $X$ and $Y$, respectively. A Borel map $\phi: X \rightarrow Y$ is \textit{equivariant} if
\[
\phi \circ \alpha^\gamma = \beta^\gamma \circ \phi \text{ for every } \gamma \in \Gamma.
\]  
(1.1)

A surjective equivariant Borel map $\phi: X \rightarrow Y$ in (1.1) with $\lambda_Y = \lambda_X \phi^{-1}$ is called a \textit{measurable factor map}
\[
\phi: (X, \alpha) \rightarrow (Y, \beta).
\]  
(1.2)

If there exists a measurable (or continuous) factor map $\phi: (X, \alpha) \rightarrow (Y, \beta)$ then $(Y, \beta)$ is a \textit{measurable (or topological) factor of $(X, \alpha)$}. If the factor map $\phi$ in (1.2) is a continuous surjective group homomorphism then $(Y, \beta)$ is an \textit{algebraic factor of $(X, \alpha)$} and $\phi$ is an \textit{algebraic factor map}. The actions $\alpha$ and $\beta$ are \textit{measurably, topologically or algebraically conjugate} if the map $\phi$ in (1.2) can be chosen to be a Borel isomorphism, a homeomorphism or a continuous group isomorphism (in which case $\phi$ is called a \textit{measurable, topological or algebraic conjugacy of $(X, \alpha)$ and $(Y, \beta)$}).

A map $\psi: X \rightarrow Y$ is \textit{affine} if there exist a continuous group homomorphism $\psi': X \rightarrow Y$ and an element $y \in Y$ with
\[
\psi(x) = \psi'(x) + y
\]  
for every $x \in X$. If there exists an affine factor map $\psi: (X, \alpha) \rightarrow (Y, \beta)$ then $(Y, \beta)$ is obviously an algebraic factor of $(X, \alpha)$.

For $d = 1$, any algebraic $\mathbb{Z}$-action is determined by the powers of a single group automorphism $\alpha$. If $\alpha$ is ergodic, then it is Bernoulli (cf. e.g. [1]), which implies that two such actions with equal entropy are measurably conjugate even if they are algebraically non-conjugate.

If $d > 1$ and $\alpha_1, \alpha_2$ are algebraic $\mathbb{Z}^d$-actions with completely positive entropy with respect to Haar measure, then they are Bernoulli by [10] and can thus again be measurably conjugate without being algebraically conjugate. However, if these actions are mixing with zero entropy, then measurable conjugacy implies — under certain additional conditions — not only algebraic conjugacy, but also that every measurable conjugacy between such actions is (almost everywhere equal to) an affine map. For irreducible\footnote{An algebraic $\mathbb{Z}^d$-action $\alpha$ on a compact abelian group $X$ is \textit{irreducible} if every closed $\alpha$-invariant subgroup $Y \subseteq X$ is finite.} and mixing algebraic $\mathbb{Z}^d$-actions with $d > 1$ this kind of strong isomorphism rigidity was proved in [7]–[8], and in [12] the (cautious) conjecture was formulated that every measurably conjugate pair of expansive and mixing zero-entropy algebraic $\mathbb{Z}^d$-actions with $d > 1$ is algebraically conjugate, and that every measurable conjugacy between such actions is affine.

In [2], the first author presented a counterexample to this conjecture: there exist two measurably conjugate expansive and mixing zero-entropy algebraic $\mathbb{Z}^d$-actions $\alpha_1$ and $\alpha_2$ on non-isomorphic zero-dimensional compact abelian groups $X_1$ and $X_2$, respectively. On the positive side it was shown in [3] that, for $d > 1$, every measurable conjugacy between expansive and mixing zero-entropy algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups is (almost everywhere equal to) a continuous map with certain additional algebraic properties.
In this paper we present further counterexamples to the rigidity conjecture in [12], including two measurably conjugate, but algebraically non-conjugate, expansive and mixing zero-entropy $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups. However, if $d > 1$, and if $\alpha_1$ and $\alpha_2$ are mixing algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups $X_1$ and $X_2$ such that the restriction of $\alpha_1$ to some subgroup $\Gamma \subset \mathbb{Z}^d$ of infinite index is expansive and has completely positive entropy, then every measurable factor map between $\alpha_1$ and $\alpha_2$ is affine (Theorem 4.1). Since this condition is automatically satisfied if $\alpha_1$ is an expansive $\mathbb{Z}^d$-action with zero entropy (or, more generally, if $\alpha_1$ contains an expansive element $\alpha_1^n$), all expansive and mixing zero-entropy algebraic $\mathbb{Z}^d$-actions (or all mixing algebraic $\mathbb{Z}^d$-actions containing an expansive element) on zero-dimensional compact abelian groups exhibit strong isomorphism rigidity (Corollary 4.2). In a second corollary (Corollary 4.3) we show that any measurable conjugacy between two mixing algebraic $\mathbb{Z}^d$-actions $\alpha_1, \alpha_2$ arising from nonzero prime ideals in the ring $R^{(p)}_d$ of Laurent polynomials in $d$ variables with coefficients in a finite prime field $F_p$ via the construction (2.10) is affine.

The key tools for the proof of Theorem 4.1 are the continuity of measurable equivariant maps proved in [3] and a detailed investigation of the homoclinic groups of certain sub-actions of the $\mathbb{Z}^d$-actions $\alpha_1$ and $\alpha_2$ in Proposition 3.5.

We should mention that Manfred Einsiedler has indicated to the authors a proof of Theorem 4.1 by a different method based on relative entropy considerations in the sense of [6].

2. Algebraic $\mathbb{Z}^d$-actions on zero-dimensional groups

Let $\alpha$ be an algebraic $\Gamma$-action on a compact abelian group $X$. For every subgroup $\Gamma' \subset \Gamma$ we denote by $\alpha^{\Gamma'}$ the restriction of $\alpha$ to $\Gamma'$. If $Z \subset X$ is a closed $\alpha$-invariant subgroup we write $\alpha_Z$ and $\alpha_{X/Z}$ for the algebraic $\mathbb{Z}^d$-actions induced by $\alpha$ on $Z$ and $X/Z$, respectively.

We denote by $R_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ the ring of Laurent polynomials with integral coefficients in the variables $u_1, \ldots, u_d$ and write the elements $f \in R_d$ as
\begin{equation}
    f = \sum_{n \in \mathbb{Z}^d} f_n u^n
\end{equation}
with $u^n = u_1^{n_1} \cdots u_d^{n_d}$ and $f_n \in \mathbb{Z}$ for all $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, where $f_n = 0$ for all but finitely many $n \in \mathbb{Z}^d$.

If $\alpha$ is an algebraic $\mathbb{Z}^d$-action on a compact abelian group $X$, then the additively-written dual group $M = \hat{X}$ is a module over the ring $R_d$ with respect to the operation
\begin{equation}
    f \cdot a = f(\hat{\alpha})(a) = \sum_{n \in \mathbb{Z}^d} f_n \hat{\alpha}^n(a)
\end{equation}
for $f \in R_d$ and $a \in M$, where $\hat{\alpha}^n$ denotes the automorphism of $\hat{X}$ dual to $\alpha^n$. The module $M = \hat{X}$ is called the dual module of $\alpha$. 
Conversely, if $M$ is a module over $R_d$, then we obtain an algebraic $\mathbb{Z}^d$-action $\alpha_M$ on $X_M = \hat{M}$ by setting
\[
\alpha_M^n(a) = a^n \cdot a
\]
for every $n \in \mathbb{Z}^d$ and $a \in M$. Clearly, $M$ is the dual module of $\alpha_M$.

Let $\alpha$ be an algebraic $\mathbb{Z}^d$-action on a compact abelian group $X$ with dual module $M = \hat{X}$. For every $f = \sum_{n \in \mathbb{Z}^d} f_n n \in R_d$ we define a continuous group homomorphism $f(\alpha) : X \to \hat{X}$ by setting, for every $x \in X$,
\[
f(\alpha)(x) = \sum_{n \in \mathbb{Z}^d} f_n a^n x.
\]

Note that $f(\alpha)$ is dual to multiplication by $f$ on $M = \hat{X}$ (or, equivalently, that $\overline{f(\alpha)} = f(\hat{\alpha})$ in (2.2)). Hence $f(\alpha)$ is surjective if and only if $f$ does not lie in any prime ideal associated\(^2\) with $M$. For details we refer to [11].

In this paper we restrict our attention to algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups. We recall the following results (cf. [11, Propositions 6.6 and 6.9]).

**Lemma 2.1.** Let $\alpha$ be an algebraic $\mathbb{Z}^d$-action on a compact abelian group $X$. Then the group $X$ is zero-dimensional if and only if every prime ideal $p$ associated with the dual module $M = \hat{X}$ of $\alpha$ contains a rational prime constant $p(p) > 1$.

**Lemma 2.2.** Let $\alpha$ be an algebraic $\mathbb{Z}^d$-action on a zero-dimensional compact abelian group $X$ with dual module $M = \hat{X}$.

1. The following conditions are equivalent.
   (a) $\alpha$ is expansive;
   (b) The module $M$ is Noetherian.

2. The following conditions are equivalent.
   (a) $\alpha_M$ is mixing;
   (b) $\alpha_{R_d/p}$ is mixing for every $p \in \text{Asc}(M)$;
   (c) $p \cap \{a^n - 1 : 0 \neq n \in \mathbb{Z}^d\} = \emptyset$ for every $p \in \text{Asc}(M)$.

3. The following conditions are equivalent.
   (a) $\alpha_M$ has positive entropy (with respect to the normalized Haar measure $\lambda_X$ of $X$);
   (b) $\alpha_{R_d/p}$ has positive entropy for some $p \in \text{Asc}(M)$;
   (c) Some $p \in \text{Asc}(M)$ is principal (and hence of the form $p = (p) = pR_d$ for some rational prime constant $p > 1$).

4. The following conditions are equivalent.
   (a) $\alpha_M$ has completely positive entropy (with respect to $\lambda_X$);
   (b) $\alpha_{R_d/p}$ has positive entropy for every $p \in \text{Asc}(M)$;
   (c) Every $p \in \text{Asc}(M)$ of the form $p = (p) = pR_d$ for some rational prime constant $p(p) > 1$.

\(^2\)A prime ideal $p \subset R_d$ is associated with an $R_d$-module $M$ if $p = \text{ann}(a) = \{f \in R_d : f \cdot a = 0\}$ for some $a \in M$, and the module $M$ is associated with a prime ideal $p \subset R_d$ if $p$ is the only prime ideal associated with $M$. The set of prime ideals associated with a Noetherian $R_d$-module $M$ is finite and denoted by $\text{Asc}(M)$. 
Lemma 2.3. Let \( \alpha \) be an expansive algebraic \( \mathbb{Z}^d \)-action on a zero-dimensional compact abelian group \( X \) with dual module \( M = \hat{X} \). If \( \text{Asc}(M) = \{p_1, \ldots, p_m\} \), then there exist Noetherian \( R_d \)-modules \( N \supseteq M \supseteq N' \) with the following properties.

1. \( N = N(1) \oplus \cdots \oplus N(m) \), where each of the modules \( N(j) \) has a finite sequence of submodules \( N(j) = N(j)_1 \supset \cdots \supset N(j)_0 = \{0\} \) with \( N(j)_k / N(j)_{k-1} \cong R_d / p_j \) for \( k = 1, \ldots, s_j \);

2. \( N \) and \( N' \) are isomorphic as \( R_d \)-modules.

In view of the Lemmas 2.1–2.3 it is useful to have an explicit realization of \( \mathbb{Z}^d \)-actions of the form \( \alpha_{R_d/p} \), where \( p \subset R_d \) is a prime ideal containing a rational prime constant \( p > 1 \).

Denote by \( R_d^{(p)} = F_p[u_1^{\pm 1}, \ldots, u_d^{\pm 1}] \) the ring of Laurent polynomials in the variables \( u_1, \ldots, u_d \) with coefficients in the prime field \( F_p = \mathbb{Z} / p \mathbb{Z} \) and define a ring homomorphism \( f \mapsto f_{/p} \) from \( R_d \) to \( R_d^{(p)} \) by reducing each coefficient of \( f \) modulo \( p \). As in (2.1) we write every \( h \in R_d^{(p)} \) as \( h = \sum_{n \in \mathbb{Z}^d} \hat{h}_n u^n \) with \( \hat{h}_n \in F_p \) for every \( n \in \mathbb{Z}^d \). The set

\[
S(h) = \{ n \in \mathbb{Z}^d : c_h(n) \neq 0 \}
\]

is called the support of \( h \in R_d^{(p)} \).

If \( p \subset R_d \) is a prime ideal containing the constant \( p \), then

\[
\overline{p} = \{ f_{/p} : f \in p \} \subset R_d^{(p)}
\]

is again a prime ideal, and the map \( f \mapsto f_{/p} \) induces an \( R_d \)-module isomorphism

\[
R_d / p \cong R_d^{(p)} / \overline{p}.
\]

Let \( \Omega = F_p^\mathbb{Z}^d \), furnished with the product topology and component-wise addition. We write every \( \omega \in \Omega \) as \( \omega = (\omega_n) \) with \( \omega_n \in F_p \) for every \( n \in \mathbb{Z}^d \) and define the shift-action \( \sigma \) of \( \mathbb{Z}^d \) on \( \Omega \) by

\[
(\sigma^m \omega)_n = \omega_{m+n}
\]

for every \( m \in \mathbb{Z}^d \) and \( \omega = (\omega_n) \in \Omega \). For every \( h = \sum_{n \in \mathbb{Z}^d} \hat{h}_n u^n \in R_d^{(p)} \) we define a continuous group homomorphism \( h(\sigma) : \Omega \to \Omega \) as in (2.4) by

\[
h(\sigma) = \sum_{n \in \mathbb{Z}^d} h_n \sigma^n.
\]

The additive group \( R_d^{(p)} \) can be identified with the dual group \( \hat{\Omega} \) of \( \Omega \) by setting

\[
\langle h, \omega \rangle = e^{2\pi i (\sum_{n \in \mathbb{Z}^d} h_n \omega_n) / p}
\]

for every \( h \in R_d^{(p)} \) and \( \omega \in \Omega \). With this identification the shift \( \sigma^m : \Omega \to \Omega \) is dual to multiplication by \( u^m \) on \( \hat{\Omega} = R_d^{(p)} \), and \( h(\sigma) \) is dual to multiplication by \( h \) on \( R_d^{(p)} \) for every \( h \in R_d^{(p)} \).
If \( q \subset R_d^{(p)} \) is an ideal with generators \( \{ h^{(1)}, \ldots, h^{(k)} \} \), then
\[
q^\perp = \overline{R_d^{(p)}}/q = X_{R_d^{(p)}/q} = \{ \omega \in \Omega : \langle h, \omega \rangle = 1 \text{ for every } h \in q \}
\]
\[
= \bigcap_{h \in q} \ker(h(\sigma)) = \bigcap_{i=1}^{k} \ker(h^{(i)}(\sigma)).
\]
(2.10)
is a closed, shift-invariant subgroup of \( \Omega \), and
\[
\alpha_{R_d^{(p)}/q} = \sigma_{X_{R_d^{(p)}/q}}
\]
(2.11)
is the restriction of the shift-action \( \sigma \) to \( X_{R_d^{(p)}/q} \subset \Omega \).

We will use the following result from [3] on measurable equivariant maps between algebraic \( \mathbb{Z}^d \)-actions on zero-dimensional groups (cf. [3, Corollary 1.2]).

**Lemma 2.4.** Let \( d > 1 \), and let \( \alpha \) and \( \beta \) be mixing zero-entropy algebraic \( \mathbb{Z}^d \)-actions on compact abelian groups \( X \) and \( Y \), respectively. Then there exists, for every measurable \( \mathbb{Z}^d \)-equivariant map \( \phi : (X, \alpha) \longrightarrow (Y, \beta) \), a continuous \( \mathbb{Z}^d \)-equivariant map \( \phi' : (X, \alpha) \longrightarrow (Y, \beta) \) such that \( \phi = \phi' \lambda_X \text{-a.e.} \)

3. **Homoclinic points**

**Definition 3.1.** Let \( \alpha \) be an algebraic \( \mathbb{Z}^d \)-action on a compact abelian group \( X \), and let \( \Gamma \subset \mathbb{Z}^d \) be a subgroup. An element \( x \in X \) is \( (\alpha, \Gamma) \)-homoclinic (to the identity element \( 0_X \) of \( X \)), if
\[
\lim_{n \to \infty} \alpha^n x = 0_X.
\]
The \( \alpha \)-invariant subgroup \( \Delta_{(\alpha, \Gamma)}(X) \subset X \) of all \( (\alpha, \Gamma) \)-homoclinic points is an \( R_d \)-module under the operation
\[
f \cdot x = f(\alpha)(x)
\]
for every \( f \in R_d \) and \( x \in \Delta_{(\alpha, \Gamma)}(X) \) (cf. (2.4)), and is called the \( \Gamma \)-homoclinic module of \( \alpha \) (cf. [9]).

**Proposition 3.2.** Let \( \alpha \) be an expansive algebraic \( \mathbb{Z}^d \)-action on a compact abelian group \( X \), and let \( \Gamma \subset \mathbb{Z}^d \) be a subgroup. Then \( \Delta_{(\alpha, \Gamma)} \neq \{0_X\} \) if and only if the entropy \( h(\alpha^\Gamma) \) of the algebraic \( \Gamma \)-action \( \alpha^\Gamma \) on \( X \) is positive, and \( \Delta_{(\alpha, \Gamma)} \) is dense in \( X \) if and only if \( \alpha^\Gamma \) has completely positive entropy (where entropy is always taken with respect to Haar measure).

**Proof.** This is [9, Theorems 4.1 and 4.2].

If an expansive and mixing algebraic \( \mathbb{Z}^d \)-action \( \alpha \) on a compact abelian group \( X \) has zero entropy, then the homoclinic group \( \Delta_{\alpha}(X) \) of this \( \mathbb{Z}^d \)-action is trivial by Proposition 3.2, but \( \Delta_{(\alpha, \Gamma)} \) will be dense in \( X \) for appropriate subgroups \( \Gamma \subset \mathbb{Z}^d \). We investigate this phenomenon in the special case where \( p > 1 \) is a rational prime, \( f \in R_d^{(p)} \) an irreducible Laurent polynomial such that the convex hull \( \mathcal{C}(f) \subset \mathbb{R}^d \) of the support \( S(f) \subset \mathbb{Z}^d \) of \( f \) contains an interior point (cf. (2.5)), and where \( \alpha = \alpha_{R_d^{(p)}/(f)} \) is the shift-action of \( \mathbb{Z}^d \) on the compact abelian group \( X = X_{R_d^{(p)}/(f)} \subset F_{\mathbb{Z}^d}^{(p)} \) defined in (2.10)-(2.11),
We write $[\ldots]$ and $\|\cdot\|$ for the Euclidean inner product and norm on $\mathbb{R}^d$, and set, for every nonzero element $\mathbf{m} \in \mathbb{Z}^d$,

$$\Gamma_{\mathbf{m}} = \{ \mathbf{n} \in \mathbb{Z}^d : [\mathbf{m}, \mathbf{n}] = 0 \}. \quad (3.1)$$

Let

$$S_{d-1} = \{ \mathbf{v} \in \mathbb{R}^d : \|\mathbf{v}\| = 1 \}$$

be the unit sphere in $\mathbb{R}^d$ and put, for every $\mathbf{v} \in S_{d-1}$,

$$H_{\mathbf{v}} = \{ \mathbf{w} \in \mathbb{Z}^d : [\mathbf{v}, \mathbf{w}] \leq 0 \},$$

$$X_{\mathbf{v}} = \{ x \in X : x_n = 0 \text{ for every } \mathbf{n} \in H_{\mathbf{v}} \}.$$

Following [5] we observe that the set

$$N(\alpha) = \{ \mathbf{v} \in S_{d-1} : X_{\mathbf{v}} \neq \{0_X\} \}$$

consists of all $\mathbf{v} \in S_{d-1}$ such that

$$\{ \mathbf{w} \in \mathcal{C}(f) : [\mathbf{w}, \mathbf{v}] = \max_{\mathbf{w}' \in \mathcal{C}(f)} [\mathbf{w}', \mathbf{v}] \}$$

contains a (one-dimensional) edge of $\mathcal{C}(f)$. The complement

$$E(\alpha) = S_{d-1} \setminus N(\alpha) \quad (3.2)$$

of $N(\alpha)$ is dense, open, and consists of finitely many connected components. Hence the set

$$E^*(\alpha) = E(\alpha) \cap (-E(\alpha)) = S_{d-1} \setminus (N(\alpha) \cup (-N(\alpha))) \quad (3.3)$$

is again dense, open, and has finitely many connected components, called the Weyl chambers of $\alpha$. For every nonzero $\mathbf{m} \in \mathbb{Z}^d$ with

$$\mathbf{m}^* = \frac{\mathbf{m}}{\|\mathbf{m}\|} \in E^*(\alpha) \quad (3.4)$$

we denote by $W(\mathbf{m})$ the connected component of $E(\alpha)$ containing $\mathbf{m}^*$ and write $W^*(\mathbf{m}) = W(\mathbf{m}) \cap W(-\mathbf{m})$ for the Weyl chamber of $E^*(\alpha)$ containing $\mathbf{m}^*$. In this notation we have the following lemma.

**Lemma 3.3.** Let $f \in R_d^{(p)}$ be an irreducible Laurent polynomial such that the convex hull $\mathcal{C}(f) \subset \mathbb{R}^d$ of the support $S(f) \subset \mathbb{Z}^d$ of $f$ contains an interior point, and let $\alpha = \alpha_{R_d^{(p)}} / f$ be the shift-action of $\mathbb{Z}^d$ on the compact abelian group $X = X_{R_d^{(p)}} / \langle f \rangle \subset F_{\mathbb{Z}^d}$ defined in (2.10)-(2.11).

1. For every nonzero element $\mathbf{m} \in \mathbb{Z}^d$, the action $\alpha^{\Gamma_{\mathbf{m}}}$ is expansive if and only if $\mathbf{m}$ satisfies (3.4);
2. If $\mathbf{m}$ satisfies (3.4) then $\Delta_{(\alpha, \Gamma_{\mathbf{m}})}$ is dense in $X$ and there exists a fundamental homoclinic point $x^\Delta \in \Delta_{(\alpha, \Gamma_{\mathbf{m}})}$ such that

$$\{ h(\alpha)(x^\Delta) : h \in R_d^{(p)} \} = \Delta_{(\alpha, \Gamma_{\mathbf{m}})} \quad (3.5)$$

and

$$h(\alpha)(x^\Delta) = 0_X \text{ if and only if } h \in \langle f \rangle. \quad (3.6)$$

3. If $\mathbf{n} \in \mathbb{Z}^d$ is a second nonzero element satisfying (3.4), then $\Delta_{(\alpha, \Gamma_{\mathbf{m}})} = \Delta_{(\alpha, \Gamma_{\mathbf{n}})}$ whenever $W^*(\mathbf{m}) = W^*(\mathbf{n})$. 

Proof. The assertion (1) follows from [4], [5] or an elementary direct argument. In order to prove the existence of a fundamental homoclinic point $x^\Delta$ in (2) we choose an element $m' \in \mathbb{Z}^d$ with $\mathbb{Z}^d = \Gamma_m + \{km' : k \in \mathbb{Z}\}$ and write $f$ as $f = \sum_{k=k_1}^{k_2} u^k m' g(k)$ for appropriate integers $k_1 < k_2$, where $S(g(k)) \subseteq \Gamma_m$ for every $k = k_1, \ldots, k_2$, and where $g(k)$ and $g(k_2)$ each have a single nonzero entry. As $X = \ker(f(\sigma))$ by (2.10), every $x \in X$ is determined completely by its coordinates in the subset $S = \Gamma_m + \{k_1 m', \ldots, (k_2 - 1)m'\} \subseteq \mathbb{Z}^d$. Furthermore, the projection $\pi_S : X \rightarrow F^S_p$ onto the coordinates in $S$ is bijective and

$$\Delta_{(\alpha, \Gamma_m)} = \{x = (x_k) \in X : x_k \neq 0 \text{ for only finitely many } k \in \mathbb{S}\}.$$ 

The point $x^\Delta \in X$ with $x_k^\Delta m' = 1$ and $x_k^\Delta m' = 0$ for every $k \in S \setminus \{k_1 m'\}$ will satisfy (3.5)-(3.6). Note that we have proved in passing that $\alpha^{\Gamma_m}$ is the shift-action of $\Gamma_m$ on $A^{\Gamma_m}$ for some finite abelian group $A$, and that $\Delta_{(\alpha, \Gamma_m)}$ is dense in $X$.

For (3) we consider the convex cone

$$C'(m) = \{v \in \mathbb{R}^d \setminus \{0\} : v^* \in W(m)\}$$

with dual cone

$$C(m) = \{w \in \mathbb{R}^d : [w, v] \leq 0 \text{ for every } v \in C'(m)\}. \quad (3.7)$$

If $l \in C(f)$ is the unique vertex with

$$[l, m] = \max \{[k, m] : k \in S(f)\},$$

then $C(m)$ is the smallest cone in $\mathbb{R}^d$ containing $S(f) - 1 = S(n^{-1} f)$. Furthermore, if $n \in \mathbb{Z}^d \setminus \{0\}$ and $n^* \in E^*(\alpha)$, then

$$C(m) = C(n) \quad \text{if and only if} \quad W(m) = W(n) \quad (3.8)$$

(cf. (3.7)), but the interiors of $C(m)$ and $C(n)$ may obviously have nonempty intersection even if $W(m) \neq W(n)$.

For every homoclinic point $x \in \Delta_{(\alpha, \Gamma_m)}(X)$ we set

$$S(x) = \{n \in \mathbb{Z}^d : x_n \neq 0\}$$

and note that there exist elements $k^+ \in \mathbb{Z}^d$ with

$$S(x) \subset (k^+ - C(m)) \cup (k^- - C(m)). \quad (3.9)$$

This shows that $x$ is homoclinic for every $\alpha^m$ with $n^* \in W^*(m)$. Since $x \in \Delta_{(\alpha, \Gamma_m)}(X)$ was arbitrary, and since the situation is symmetric in $m$ and $n$, this proves (3).

\textbf{Lemma 3.4.} Let $d > 1$, $p > 1$ a rational prime, and let $f \in R_+^{(p)}$ be an irreducible Laurent polynomial such that the convex hull $C(f) \subseteq \mathbb{R}^d$ of the support $S(f) \subseteq \mathbb{Z}^d$ contains an interior point. Let $\alpha = \alpha^{R_+^{(p)}}(f)$ be the shift-action of $\mathbb{Z}^d$ on the compact abelian group $X = X_{R_+^{(p)}}(f) \subseteq F_p^{\mathbb{Z}^d}$ defined in (2.10)-(2.11), and let $z \in X$ be a point with the following property: there exist an integer $k \geq 1$ and elements $n_i$, $i = 1, \ldots, k$, in $\mathbb{Z}^d \setminus \{0\}$ such that

$$S(z) = \{n \in \mathbb{Z}^d : n \neq 0\} \subset \left( \bigcup_{i=1}^{k} \Gamma_{n_i} \right) + Q(N) \quad (3.10)$$
for some integer $N \geq 0$, where
\[ Q(M) = \{-M, \ldots, M\}^d \subset \mathbb{Z}^d \]
for every $M \geq 0$. Then there exists a Laurent polynomial $g \in R_d^{(p)} \setminus (f)$ with $g(\alpha)(z) = 0_X$.

**Proof.** We write $f$ in the form (2.1), assume without loss in generality (by multiplying $f$ by a monomial $u^k$, if necessary) that
\[ S = S(f) \cap \Gamma_n \neq \emptyset, \]
and set
\[ h_k = \sum_{n \in \Gamma_n} f_n u^n. \]
Since the convex hull of the support of $h_k$ has no interior point, $h_k \not\in (f)$.

Choose $M \geq 1$ with $S(f) \subset Q(M)$ (cf. (3.11)), and let $r \geq 1$ be an integer with $p^r > 2dN$. For every $k \in \mathbb{Z}^d$ with
\[ k \not\in \left( \bigcup_{i=1}^{k-1} \Gamma_{n_i} \right) + Q(p^r M + N), \]
the support of the Laurent polynomial $u^k f^{p^r}$ does not intersect
\[ \left( \bigcup_{i=1}^{k-1} \Gamma_{n_i} \right) + Q(N). \]
Furthermore, if
\[ S(u^k h_k^{p^r}) \cap (\Gamma_n + Q(N)) = S(u^k h_k^{p^r}) \cap \left( \bigcup_{i=1}^{k} \Gamma_{n_i} \right) + Q(N) \neq \emptyset, \]
then
\[ S(u^k f^{p^r}) \cap (\Gamma_n + Q(N)) = S(u^k h_k^{p^r}) \cap \left( \bigcup_{i=1}^{k} \Gamma_{n_i} \right) + Q(N) \]
\[ = S(u^k h_k^{p^r}) \cap (\Gamma_n + Q(N)). \]
According to the definition of $X$ in (2.11), $f^{\alpha M}(\alpha)(z) = 0_X$, and hence
\[ 0 = f^{p^r}(\alpha)(z) - k = (u^k f^{p^r})(\alpha)(z)_0 = \sum_{n \in S(f)} f_n z_{k+p^r n} \]
\[ = \sum_{n \in S(h_k)} f_n z_{k+p^r n} = (u^k h_k^{p^r})(\alpha)(z)_0 = h_k^{p^r}(\alpha)(z) - k, \]
where the identity marked with $\ast$ follows from (3.12). The Laurent polynomial $h_k^{p^r} \neq (f)$ thus has the property that
\[ S(h_k^{p^r}(\alpha)(z)) \subset \left( \bigcup_{i=1}^{k-1} \Gamma_{n_i} \right) + Q(N') \]
for some integer $N' \geq 1$.

We repeat the argument with $k, \varepsilon$ and $N \geq 0$ replaced by $k - 1, h_k^\varepsilon(\alpha)(z)$ and $N'$, respectively. After $k$ steps we obtain Laurent polynomials $h_1^\varepsilon, \ldots, h_k^\varepsilon$ in $R_d^{(p)}$ such that $g = \prod_{i=1}^{k} h_i^\varepsilon \not\in (f)$ and $S(g(\alpha)(z))$ is finite. In other words, the point $g(\alpha)(z)$ is homoclinic and hence, since $\alpha$ has entropy zero, equal to $0_X$ by Proposition 3.2.

Now we can state the main results of this section.
Proposition 3.5. Let $f \in R_d^{(p)}$ be an irreducible Laurent polynomial such that the convex hull $\mathcal{C}(f) \subset \mathbb{R}^d$ of the support $S(f) \subset \mathbb{Z}^d$ of $f$ contains an interior point, and let $\alpha = \alpha_{R_d^{(p)}/(f)}$ be the shift-action of $\mathbb{Z}^d$ on the compact abelian group $X = X_{R_d^{(p)}/(f)} \subset F_p^d$ defined in (2.10)-(2.11). Then there exists, for every Weyl chamber $W_1^*$ of $\alpha$, a Weyl chamber $W_2^*$ of $\alpha$ such that the following properties are satisfied for all nonzero $m, n \in \mathbb{Z}^d$ with $m^* \in W_1^*$ and $n^* \in W_2^*$.

(1) The homoclinic groups $\Delta_{(\alpha, \Gamma_m)}(X)$ and $\Delta_{(\alpha, \Gamma_n)}(X)$ are dense in $X$;
(2) $\Delta_{(\alpha, \Gamma_m)}(X) \cap \Delta_{(\alpha, \Gamma_n)}(X) = \{0_X\}$.

Proof. We fix a nonzero element $m \in \mathbb{Z}^d$ with $m^* \in W_1^*$. Then the homoclinic group $\Delta_{(\alpha, \Gamma_m)}$ is dense in $X$ and isomorphic to $R_d^{(p)}/(f)$ by Lemma 3.3.

Suppose that $\Delta_{(\alpha, \Gamma_m)} \cap \Delta_{(\alpha, \Gamma_n)} \neq \{0_X\}$ for every nonzero $n \in \mathbb{Z}^d$ satisfying (3.4) (with $n$ replacing $m$). Under this hypothesis we shall prove the existence of a Laurent polynomial $g \in R_d^{(p)} \setminus (f)$ such that $g(\alpha)(X) = \{0_X\}$.

By duality, $(g) = gR_d^{(p)} \subset (f)$, and this contradiction will prove the proposition.

In order to construct such a Laurent polynomial $g$ we choose an enumeration $W_1^*, \ldots, W_k^*$ of the Weyl chambers of $\alpha$, set $n_1 = m$, and choose elements $n_i, i = 2, \ldots, k$, such that $n_i^* \in W_i^*$ for $i = 2, \ldots, k$. By hypothesis, $\Delta_{(\alpha, \Gamma_m)} \cap \Delta_{(\alpha, \Gamma_n_i)} \neq \{0_X\}$ for $i = 2, \ldots, k$, and (3.5)-(3.6) allows us to find Laurent polynomials $h^{(i)} \in R_d^{(p)} \setminus (f)$ with $h^{(i)}(\alpha)(x^\Delta) \in \Delta_{(\alpha, \Gamma_n_i)} \setminus \{0_X\}$ for $i = 2, \ldots, k$. The Laurent polynomial $h = \prod_{i=2}^k h^{(i)} \in R_d^{(p)} \setminus (f)$ has the property that
\begin{equation}
0_X \neq y^\Delta = h(\alpha)(x^\Delta) \in \Delta_{(\alpha, \Gamma_n_i)}
\end{equation}
for $i = 1, \ldots, m$. It follows that $y^\Delta \in \Delta_{(\alpha, \Gamma_n_i)}$ and hence that
\begin{equation}
\lim_{k \to \infty} a^k y^\Delta = 0_X
\end{equation}
for every nonzero $n \in \mathbb{Z}^d$ for which $\alpha^{\Gamma n}$ is expansive.

From (3.9) we conclude that there exist elements $k_i^\pm \in \mathbb{Z}^d, i = 1, \ldots, k$, with
\begin{equation}
S(y^\Delta) \subset \bigcap_{i=1}^k \left( (k_i^+ - C(n_i)) \cup (k_i^- - C(-n_i)) \right).
\end{equation}

We write $\mathcal{F}(f)$ for the set of $((d-1)$-dimensional) faces of the convex polyhedron $\mathcal{C}(f)$, choose, for every face $F \in \mathcal{F}(f)$, an element $v_F \in N(\alpha)$ orthogonal to $F$, and set
\begin{equation}
\Gamma(F) = \Gamma_{v_F}
\end{equation}
as in (3.1). From (3.15) and the definition of $X = X_{R_d^{(p)}/(f)}$ in (2.10) we conclude that there exists an integer $N \geq 0$ with
\begin{equation}
S(y^\Delta) \subset \left( \bigcup_{F \in \mathcal{F}(f)} \Gamma(F) \right) + Q(N).
\end{equation}
Lemma 3.4 implies the existence of a Laurent polynomial \( g \in \hat{R}_d^{(p)} \setminus \langle f \rangle \) with
\[
g(\alpha)(y^\Delta) = (gh)(\alpha)(x^\Delta) = 0_X.
\]
As explained above, this completes the proof of the proposition. \( \square \)

**Proposition 3.6.** Let \( d > 1, p > 1 \) a rational prime, \( f \in \hat{R}_d^{(p)} \) an irreducible Laurent polynomial such that the shift-action \( \alpha = \alpha_{R_d^{(p)}/(f)} \) of \( \mathbb{Z}^d \) on the compact abelian group \( X = X_{R_d^{(p)}/(f)} \subset \mathbb{F}_p^\mathbb{Z}^d \) in (2.10)-(2.11) is mixing, and let \( m \in \mathbb{Z}^d \) be a nonzero element such that the restriction \( \alpha^{m} \) of \( \alpha \) to the subgroup \( \Gamma_m \) in (3.1) is expansive. Then the homoclinic group \( \Delta(\alpha, \Gamma_m) \) is dense in \( X \). Furthermore there exists an open subset \( W \subset S_{d-1} \) such that every nonzero element \( n \in \mathbb{Z}^d \) with \( n^* \in S_{d-1} \) has the following properties.

1. \( \Delta(\alpha, \Gamma_m)(X) \) is dense in \( X \);
2. \( \Delta(\alpha, \Gamma_m)(X) \cap \Delta(\alpha, \Gamma_n)(X) = \{0_X\} \).

**Proof.** If the convex hull \( \mathcal{C}(f) \subset \mathbb{R}^{d} \) of the support \( S(f) \subset \mathbb{Z}^d \) of \( f \) contains an interior point then Proposition 3.6 is essentially a re-statement of Proposition 3.5.

If \( \mathcal{C}(f) \) does not contain an interior point, then we may assume without loss in generality (after multiplying \( f \) by a monomial \( a^m \), if necessary) that \( S(f) \) is contained in some subspace \( V \subset \mathbb{R}^d \) of dimension \( d' < d \), where we assume that \( d' \) is minimal (i.e., that there does not exist a \( n \in \mathbb{Z}^d \) such that \( S(a^m f) \) is contained in a subspace of lower dimension). Since \( \alpha \) is mixing, Lemma 2.2 (2) implies that \( d' \geq 2 \).

Put \( \Gamma = V \cap \mathbb{Z}^d \cong \mathbb{Z}^{d'} \) and choose a subgroup \( \Gamma' \subset \mathbb{Z}^d \) with \( \Gamma \cap \Gamma' = \{0\} \) and \( \Gamma + \Gamma' = \mathbb{Z}^d \). We identify \( \Gamma \) with \( \mathbb{Z}^{d'} \), regard \( f \) as an element of \( R_{d'}^{(p)} \), and apply Proposition 3.5 to the \( \mathbb{Z}^{d'} \)-action \( \alpha_{R_{d'}^{(p)}/(f)} \) on \( X_{R_{d'}^{(p)}/(f)} \) to find, for every \( m \in \Gamma \) such that the restriction of \( \alpha_{R_{d'}^{(p)}/(f)} \) to the group \( \Gamma_m = \{ n \in \Gamma : [n, m] = 0 \} \) is expansive, a Weyl chamber \( W_2 \) of the \( \mathbb{Z}^{d'} \)-action \( \alpha_{R_{d'}^{(p)}/(f)} \) such that, for every nonzero \( n \in \mathbb{Z}^{d'} \) with \( n^* \in W_2 \), the restriction of \( \alpha_{R_{d'}^{(p)}/(f)} \) to \( \Gamma_n \) is again expansive and the homoclinic groups \( \Delta(\alpha_{R_{d'}^{(p)}/(f)}, \Gamma_n)(X_{R_{d'}^{(p)}/(f)}) \) and \( \Delta(\alpha_{R_{d'}^{(p)}/(f)}, \Gamma_m)(X_{R_{d'}^{(p)}/(f)}) \) have trivial intersection.

Since the restriction \( \alpha \) of \( \alpha \) to \( \Gamma \) is algebraically conjugate to the product action of \( \Gamma \) on \( X \cong (X_{R_{d'}^{(p)}/(f)})^{\Gamma'} \), we obtain that the restrictions of \( \alpha \) to the groups \( \Gamma_m + \Gamma' \) and \( \Gamma_n + \Gamma' \) are expansive, and that the homoclinic groups \( \Delta(\alpha, \Gamma_n + \Gamma')(X) \) and \( \Delta(\alpha, \Gamma_m + \Gamma')(X) \) have trivial intersection. It is easy to see that this implies the statement of the proposition in the case where \( \mathcal{C}(f) \) does not have an interior point (in fact, the open set \( W \subset S_{d-1} \) can again be interpreted as a Weyl chamber of \( \alpha \)). \( \square \)
4. Isomorphism Rigidity

In this section we prove the following rigidity result for measurable factor maps between algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups.

**Theorem 4.1.** Let $d > 1$, and let $\alpha$ and $\beta$ be mixing algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups $X$ and $Y$, respectively. Suppose that there exists a subgroup $\Gamma \subset \mathbb{Z}^d$ of of infinite index such that the restriction $\alpha^\Gamma$ of $\alpha$ to $\Gamma$ is expansive and has completely positive entropy. Then every measurable factor map $\phi: (X, \alpha) \to (Y, \beta)$ is $\lambda_X$-a.e. equal to an affine map.

Before turning to the proof of this result we mention a couple of corollaries which generalize the main result in [8] in different directions.

**Corollary 4.2.** Let $d > 1$, and let $\alpha$ and $\beta$ be mixing algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups $X$ and $Y$, respectively. Suppose that there exists a non-zero element $n \in \mathbb{Z}^d$ such that the automorphism $\alpha^n$ is expansive. Then every measurable factor map $\phi: (X, \alpha) \to (Y, \beta)$ is $\lambda_X$-a.e. equal to an affine map.

**Proof.** Since every mixing (= ergodic) group automorphism has completely positive entropy, this is Theorem 4.1 with $\Gamma$ of rank one. \hfill \Box

**Corollary 4.3.** Let $d > 1$, $p$ a rational prime, and $p, q \in R_d^{(p)}$ nonzero prime ideals such that the $\mathbb{Z}^d$-actions $\alpha = \alpha_{R_d^{(p)}}/p$ and $\beta = \alpha_{R_d^{(p)}}/q$ on the compact zero dimensional groups $X = X_{R_d^{(p)}}/p$ and $Y = X_{R_d^{(p)}}/q$ in (2.10)-(2.11) are mixing. Then $\alpha$ and $\beta$ are measurably conjugate if and only if they are algebraically conjugate, and hence if and only if $p = q$. Furthermore, every measurable conjugacy $\phi: (X, \alpha) \to (Y, \beta)$ is $\lambda_X$-a.e. equal to an affine map.

**Proof.** The existence of a subgroup $\Gamma \subset \mathbb{Z}^d$ of infinite index with the properties required by Theorem 4.1 is proved in [5] (the rank of $\Gamma$ is the maximal number of algebraically independent elements in the set $\{u^n + p : n \in \mathbb{Z}^d\} \subset R_d^{(p)}/p$). Let $\phi: (X, \alpha) \to (Y, \beta)$ be a measurable conjugacy. By Theorem 4.1, there exist $y \in Y$ and a continuous homomorphism $\theta: X \to Y$ such that $\phi(x) = y + \theta(x)$ for $\lambda_X$-a.e. $x \in X$. It is easy to verify that $\theta$ is an algebraic conjugacy of $(X, \alpha)$ and $(Y, \beta)$.

In order to see that algebraic conjugacy implies that $p = q$ we note that, for every $f \in R_d^{(p)}$, the maps $f(\alpha)$ and $f(\beta)$ in (2.4) are surjective if and only if $f \notin p$ (resp. $f \notin q$). \hfill \Box

We begin the proof of Theorem 4.1 with a lemma.

**Lemma 4.4.** For $i = 1, 2, 3$, let $\alpha_i$ be a mixing algebraic $\mathbb{Z}^d$-action on a compact abelian group $X_i$, and let $\phi: (X_1 \times X_2, \alpha_1 \times \alpha_2) \to (X_3, \alpha_3)$ be a continuous factor map such that $\phi(x_1, x_2) = 0_X$ whenever $x_1 = 0_X$ or $x_2 = 0_X$. Suppose furthermore that there exist subgroups $\Gamma_1, \Gamma_2$ in $\mathbb{Z}^d$ such that the homoclinic groups $\Delta_{(\alpha_3, \Gamma_1)}(X_1)$ are dense in $X_i$ for $i = 1, 2$, and that $\Delta_{(\alpha_3, \Gamma_1)}(X_3) \cap \Delta_{(\alpha_3, \Gamma_2)}(X_3) = \{0_X\}$. Then $\phi(X_1 \times X_2) = \{0_X\}$. 

Proof. Since $\phi$ is a continuous factor map,
\[
\lim_{m \to \infty} a_n^m \phi(x_1, x_2) = \lim_{m \to \infty} \phi(a_n^1 x_1, a_n^2 x_2) = 0_{X_3}
\]
for every $x_i \in \Delta_{(a_i, \Gamma_i)}(X_i)$, $i = 1, 2$. Hence
\[
\phi(x_1, x_2) \in \Delta_{(a_3, \Gamma_3)}(X_3) \cap \Delta_{(a_2, \Gamma_2)}(X_3) = \{0_{X_3}\}.
\]
As $\Delta_{(a_i, \Gamma_i)}(X_i) \subset X_i$ is dense for $i = 1, 2$ and $\phi$ is continuous this implies our assertion. 

Proof of Theorem 4.1. We assume without loss in generality that $\mathbb{Z}^d/\Gamma$ is a torsion-free abelian group and that $\Gamma \cong \mathbb{Z}^d'$ with $d' < d$. Choose a primitive\(^3\) element $n \in \mathbb{Z}^d$, $\Gamma$ and set $\Gamma' = \Gamma + \{kn : k \in \mathbb{Z}\} \cong \mathbb{Z}^{d'+1}$. Since $\alpha$ is mixing, the same is true for $\alpha' = \alpha^{\Gamma'}$, and the expansiveness of $\alpha^{\Gamma}$ implies that of $\alpha^{\Gamma'}$. Furthermore, since $\alpha^{\Gamma'}$ is expansive, the $\Gamma'$-action $\alpha^{\Gamma'}$ has finite entropy and hence $\alpha^{\Gamma'}$ has zero entropy. We restrict $\alpha$ and $\beta$ to $\Gamma'$ and assert that $d = d' + 1$, that $\alpha$ is an expansive and mixing $\mathbb{Z}^d$-action, and that $\Gamma' \subset \mathbb{Z}^d$ is a subgroup of rank $d - 1$ such that $\alpha^{\Gamma'}$ is expansive and has completely positive entropy. Since the restriction to subgroups $\Gamma'' \subset \Gamma$ of finite index changes neither expansiveness nor completely positive entropy we shall assume for simplicity that
\[
\Gamma = \{n = (n_1, \ldots, n_d) \in \mathbb{Z}^d : n_d = 0\} = \mathbb{Z}^{d-1}.
\]
As the $\mathbb{Z}^{d-1}$-action $\alpha^{\Gamma}$ has finite and completely positive entropy, the same is true for $\beta^{\Gamma'}$, and Lemma 2.2 shows that every prime ideal $q \subset R_{d-1}$ associated with the dual module $N' = \hat{\mathbb{Y}}$ of the $\mathbb{Z}^{d-1}$-action $\beta^{\Gamma'}$ is of the form $q = p(q) > 1$. The existence of the filtrations described in Lemma 2.3 guarantees that $N'$ is Noetherian as a module over $R_{d-1}$ and hence that $\beta^{\Gamma'}$ is expansive. It follows that $\beta$ is expansive, that the dual module $N = \hat{\mathbb{Y}}$ of the $\mathbb{Z}^d$-action $\beta$ is Noetherian, and that every prime ideal $p \subset R_d$ associated with $N$ is of the form $p = (p, f) = pR_d + \{f\}$ for some rational prime $p \geq 2$ and some Laurent polynomial $f \in R_d$ whose reduction $f_{/p}$ modulo $p$ is nonzero (otherwise $\beta$ would have positive entropy by Lemma 2.2).

We apply Lemma 2.3 and choose isomorphic $R_d$-modules $L \supseteq N \supseteq L'$ with the properties mentioned there. As $L$ and $L'$ are isomorphic, the restrictions to $\Gamma$ of the $\mathbb{Z}^d$-actions $\alpha_L$, $\beta$, $\beta' = \alpha_L$, all have the same entropy. The inclusion $L' \subset N$ induces a dual algebraic factor map $\psi: (Y, \beta) \to (X_L, \beta')$, and the filtration of $L'$ described in Lemma 2.3 induces a filtration $Y_0 = X_L \supseteq \ldots \supseteq Y_0 = \{0\}$ of $Y$ by $\beta'$-invariant subgroups such that each $(Y_j/Y_{j-1}, \beta_{Y_j/Y_{j-1}})$ is algebraically conjugate to $(X_L^{(p)/\pi_j}, \alpha_L^{(p)/\pi_j})$ for some rational prime $p \geq 2$ and some nonzero element $f \in R_d^{(p)}$ such that $\alpha^{(p)/\pi_j}$ is mixing. For every $j = 0, \ldots, k$ we denote by $\pi_j : Y_k \to Y_k/Y_j$ the quotient map.

\[^3\]A nonzero element $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ is \textit{primitive} if $\gcd(n_1, \ldots, n_d) = 1$. 
Suppose that \( \phi: (X, \alpha) \to (Y, \beta) \) is a measurable factor map. Lemma 2.4 allows us to assume that \( \phi \) is continuous, and we set \( \phi_j = \pi_j \circ \psi \circ \phi: X \to Y_k/Y_j \) for \( j = 0, \ldots, k - 1 \).

We set \( j = k - 1 \), \( Y'' = Y_k/Y_{k-1} \), and write \( \beta'' = \beta''_{Y''} \) for the \( \mathbb{Z}^d \)-action induced by \( \beta \) on \( Y'' \). Then the restriction \( \beta''_{Y''} \) of \( \beta'' \) to \( \Gamma \) is expansive, and Proposition 3.6 and Lemma 3.3 (1) allow us to find a nonzero element \( n \in \mathbb{Z}^d \) such that the restrictions \( \alpha^{\Gamma'_{n}} \) and \( \beta''_{Y''} \) of \( \alpha^{\Gamma'} \) to \( \Gamma'_{n} \) are expansive, the homoclinic group \( \Delta_{(\alpha^{\Gamma'}, \Gamma')}(X) \) is dense\(^4\) in \( X \), and the homoclinic groups \( \Delta_{(\beta''_{Y''}, \Gamma)}(Y'') \) and \( \Delta_{(\beta''_{Y''}, \Gamma')}^n(Y'') \) have trivial intersections. We write \( \Phi: X \times X \to Y'' \) for the map

\[
\Phi(x_1, x_2) = \phi_{k-1}(x_1 + x_2) - \phi_{k-1}(x_1) - \phi_{k-1}(x_2) + \phi_{k-1}(0_X)
\]

and obtain from Lemma 4.4 that \( \Phi \equiv 0_{Y''} \) or, equivalently, that

\[
\psi \circ \phi(x_1 + x_2) - \psi \circ \phi(x_1) - \psi \circ \phi(x_2) + \psi \circ \phi(0_X) \in Y_{k-1}
\]

for every \( x_1, x_2 \in X \). By repeating this argument we obtain inductively that

\[
\psi \circ \phi(x_1 + x_2) - \psi \circ \phi(x_1) - \psi \circ \phi(x_2) + \psi \circ \phi(0_X) \in Y_j
\]

for every \( j = k - 1, \ldots, 0 \), which implies that

\[
\phi(x_1 + x_2) - \phi(x_1) - \phi(x_2) + \phi(0_X) \in \ker(\psi)
\]

for every \( x_1, x_2 \in X \). From Lemma 2.3 we know that the \( \Gamma \)-action induced by \( \beta \) on \( Y_k = X_k \) has the same entropy as \( \beta_{Y_k}^\Gamma \), and hence that the restriction \( \beta_{\ker(\psi)}^\Gamma \) of \( \beta_{Y_k}^\Gamma \) to \( \ker(\psi) \) has zero entropy. Since the map

\[
(x_1, x_2) \mapsto \phi(x_1 + x_2) - \phi(x_1) - \phi(x_2) + \phi(0_X)
\]

is a measurable factor map from \( (X \times X, \alpha^{\Gamma} \times \alpha^{\Gamma}) \) to \( (\ker(\psi), \beta_{\ker(\psi)}^\Gamma) \), and since the first of these \( \Gamma \)-actions has completely positive entropy by assumption and the second one zero entropy, it follows that

\[
\phi(x_1 + x_2) - \phi(x_1) - \phi(x_2) + \phi(0_X) = 0_Y
\]

for every \( x_1, x_2 \in X \), i.e. that \( \phi \) is affine. \( \square \)

The following examples show that Theorem 4.1 and Corollary 4.3 do not hold if any of the assumptions is dropped. Our first example implies that the surjectivity of \( \phi \) is necessary in Corollary 4.3 (and hence in Theorem 4.1).

**Example 4.5.** Let \( d = 3 \), \( p = 2 \), and consider the polynomials \( f_1, f_2 \in R_3^{(2)} \) defined by \( f_1 = 1 + u_1 + u_2, f_2 = 1 + u_1 + u_2 + u_1^2 + u_1 u_2 + u_2^2 + u_3 \). Let \( p \subset R_3^{(2)} \) denote the ideal generated by \( f_1 \) and \( f_2 \), and let \( q \subset R_3^{(2)} \) denote the principal ideal generated by \( f_2 \). It is easy to see that \( p \) and \( q \) are prime ideals. We define the shift-actions \( \alpha_1 = \alpha_{R_3^{(2)}} / p \) and \( \alpha_2 = \alpha_{R_3^{(2)}} / q \) on \( X_1 = X_{R_3^{(2)}} / p \subset F_3^{\mathbb{Z}^2} \) and \( X_2 = X_{R_3^{(2)}} / q \subset F_3^{\mathbb{Z}^2} \), respectively, by (2.10)–(2.11). From Lemma 2.2 it is clear that \( \alpha_1 \) and \( \alpha_2 \) are mixing and have zero entropy.

\(^4\)The density of the homoclinic group \( \Delta_{(\alpha^{\Gamma'}, \Gamma)}(X) \) in \( X \) is clear from Proposition 3.2, since \( \alpha^{\Gamma'} \) is expansive and has completely positive entropy.
We write $\ast$ for the component-wise multiplication $(z \ast z')_n = z_n z'_n$ in $F_{n}^{3}$ and observe that
$$\sigma^n (z \ast z') = (\sigma^n z) \ast (\sigma^n z')$$
for every $z, z' \in F_{n}^{3}$ and $n \in \mathbb{Z}^3$ (cf. (2.8)). An elementary direct calculation shows that $x \ast x' \in X_2$ for every $x, x' \in X_1$. We choose a non-zero $m \in \mathbb{Z}^3$ such that $\alpha^m_1 x = z$ for some non-zero $z \in X_1$ and define $\phi : X_1 \longrightarrow X_2$ by $\phi(x) = x \ast \alpha^m_1 x$. Clearly $\phi$ is a $\mathbb{Z}^3$-equivariant map from $(X_1, \alpha_1)$ to $(X_2, \alpha_2)$. We choose $y \in X_1$ such that $z \ast (\alpha^m y - y) \neq 0_{X_1}$. Since $\phi(0_{X_1}) = 0_{X_2}$ and $\phi(z + y) - \phi(z) - \phi(y) = z \ast (\alpha^m y - y) \neq 0_{X_2}$, the map $\phi$ is not affine.

In the next example we construct a non-affine factor map $\psi : (X, \alpha) \longrightarrow (X', \alpha')$ between expansive and mixing zero-entropy algebraic $\mathbb{Z}^3$-actions, where $\alpha'$ has an expansive $\mathbb{Z}^2$-sub-action with completely positive entropy.

**Example 4.6.** We use the same notation as in the previous example. Let $r = pq = (f_1 f_2, f_2 f_2) \subseteq H_3^{(2)}$ be the ideal generated by $f_1 f_2$ and $f_2 f_2$ and let $\beta$ denote the algebraic $\mathbb{Z}^3$-action $\alpha_{H_3^{(2)}}$ on $Y = X_{a_{H_3^{(2)}}} \subseteq F_{n}^{3}$. From Lemma 2.2 it follows that the action $(Y, \beta)$ is mixing and has zero entropy. We define continuous group homomorphisms $\theta_1 : Y \longrightarrow X_1$ and $\theta_2 : Y \longrightarrow X_2$ by
$$\theta_1(y) = f_2(\sigma)(y), \quad \theta_2(y) = f_1(\sigma)(y).$$
It is easy to verify that for $i = 1, 2$, $\theta_i : (Y, \beta) \longrightarrow (X_i, \alpha_i)$ is an algebraic factor map. Let $\psi : (Y, \beta) \longrightarrow (X_2, \alpha_2)$ be the $\mathbb{Z}^3$-equivariant continuous map defined by
$$\psi(x) = \theta_1(x) + \phi \circ \theta_2(x),$$
where $\phi : X_1 \longrightarrow X_2$ is as in the previous example. Since $\theta_2$ is a surjective homomorphism and $\phi$ is non-affine, it follows that $\phi \circ \theta_2$ is non-affine, i.e., that $\psi$ is a non-affine map. It is easy to see that the restriction of $\theta_1$ to $X_2$ is a surjective map from $X_2$ to itself. Since $\theta_2(x) = 0$ for all $x$ in $X_2$, this shows that $\psi$ is a non-affine factor map from $(Y, \beta)$ to $(X_2, \alpha_2)$ (in fact, it can be shown that $\tau \circ \psi$ is non-affine for every surjective $\alpha_2$-equivariant group homomorphism $\tau : X_2 \longrightarrow X_2$).

Our final example shows that there exist measurably conjugate expansive and mixing zero-entropy algebraic $\mathbb{Z}^3$-actions on non-isomorphic compact zero-dimensional abelian groups.

**Example 4.7.** Let $(X_1, \alpha_1)$ and $(X_2, \alpha_2)$ be as in Example 4.5, and let $(X, \alpha)$ denote the product action $(X_1, \alpha_1) \times (X_2, \alpha_2)$. Following [2] we define a zero-dimensional compact abelian group $Y$ and an algebraic $\mathbb{Z}^3$-action $\beta$ on $Y$ by setting $Y = X_1 \times X_2$ with composition
$$(x, y) \circ (x', y') = (x + x', x \ast x' + y + y')$$
for every $(x, x'), (y, y') \in Y$, and by letting
$$\beta^n(x, y) = (\alpha_1^n x, \alpha_2^n y)$$
for every $(x, y) \in Y$ and $n \in \mathbb{Z}^3$. The \textquoteleft identity' map $\phi : X \longrightarrow Y$, defined by
$$\phi(x, y) = (x, y)$$
for every $(x, y) \in X$, is obviously a topological conjugacy of $(X, \alpha)$ and $(Y, \beta)$ with $\lambda_X \phi^{-1} = \lambda_Y$ (by Fubini's theorem). However, $\phi$ is not a group.
isomorphism; in fact, the groups $X$ and $Y$ are not isomorphic, since every element in $X$ has order 2, whereas $(0_{X_1}, y) \odot (0_{X_1}, y) = (y, 0_{X_2}) \neq 0_Y$ for every nonzero $y \in X_2$.

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