Action of Overalgebra in Plancherel Decomposition and Shift Operators in Imaginary Direction

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In last 50 years, there were solved many problems of explicit spectral decompositions for restrictions of unitary representations to subgroups (see the bibliography in [22]).

This class of problems also contains the following problems, which are formulated in other terms.

1. Decomposition of the tensor product of representations $\rho_1$, $\rho_2$ of a group $G$. Indeed, this is exactly the problem of restriction of representations of $G \times G$ to the diagonal subgroup $G \subset G \times G$.

2. Decomposition of $L^2$ on a pseudo-Riemannian symmetric space $G/H$. As was observed in [23], for each classical pseudo-Riemannian symmetric space $G/H$, there exists a canonical classical group $G \supset G$ and a representation $\rho$ of the group $\tilde{G}$ of degenerated principal series satisfying one of two following properties (usually the first variant is realized):
   
   — the restriction of $\rho$ to $G$ is $L^2(G/H)$
   — the restriction of $\rho$ to $G$ is the direct sum of the spaces $L^2(G/H_j)$, there $G/H_j$ is a finite collection of symmetric spaces, and $G/H$ is one of the spaces $G/H_j$.

Hence the decomposition of $L^2$ on a classical Riemannian symmetric space can be considered as a restriction problem.

Description of the spectral type (without explicit Plancherel formula) for all the pseudo-Riemannian symmetric spaces was recently obtained in the works of van den Ban, Schlichtkrull, Delorme, and Oshima (the proof is contained in the union of a large collection of papers, for references, see [2], [7]). It seems that for classical symmetric spaces the problem of evaluation of the Plancherel measure is near the final solution.

For some cases, the explicit Plancherel measure is known; in particular, for $L^2$ on semisimple groups ([9], [14]), on Riemannian symmetric spaces ([9], [10], [11], see also [15]), on rank 1 spaces ([20], [21]) and on the spaces $G_C/G_{\mathbb{R}}$, where $G_C$ is a complex group, and $G_{\mathbb{R}}$ is its real form ([12], [13]).

3. Berezin kernel representations (deformations of $L^2$ on Riemannian non-compact symmetric space $G/K$) also can be obtained by the restriction from some overgroup $G^\ast \supset G$, see [24], [26] and references in [26].

Usually the problem of the noncommutative harmonic analysis is formulated as the problem about the spectrum of a representation or as the more complicated problem of an explicit decomposition of a representation into a direct integral of irreducible representations; the last question includes the explicit

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2This phenomenon could be easily observed from the Makarevich paper [18] published in 1973, in a strange way it remained nonformulated for a long time.
evaluation of the spectral measure (the so-called Plancherel measure). Recent works of the author [25], [26] contain an attempt of an investigation of "the analysis after the Plancherel formula".

This work is a continuation of [25]. Here we are trying to understand the answer to following question.

**Question.** Assume that we know the explicit Plancherel formula for the restriction of a unitary representation $\rho$ of a group $G$ to a subgroup $H$. Is it possible to write the action of Lie algebra of $G$ in the direct integral of representations of $H$?

We obtain the positive solution of this problem for one of the simplest possible examples, precisely, for the tensor product of a representation of $\text{SL}(2, \mathbb{R})$ with a highest weight and the conjugate representation with a lowest weight. This tensor product and its decomposition were widely discussed in the literature on representation theory and special functions in last 40 years (some references: [27], [5], [29], [19], [8]). Nevertheless, the formula for the action of $\text{sl}_2 \oplus \text{sl}_2$ in tensor product were not appear. The reason is the unusual for representation theory form of these Lie algebra operators.

It turns out that the operators (12)-(14) of Lie algebra $\text{sl}_2 \oplus \text{sl}_2$ are the second order differential operators with respect to one variable and the second order difference operators with respect to the another variable; moreover, it turns out that the difference operators are defined in the terms of a shift in the imaginary direction, i.e., in our formula, there appear the operators of the form

$$T f(x) = f(x + i) \quad \text{for} \quad f \in L^2(\mathbb{R})$$

$(i^2 = -1)$. There is no self-contradiction in this expression, the shift operators are well defined on functions admitting the holomorphic continuation to an appropriate strip. Operators $f \mapsto x f$ and $f \mapsto \frac{d}{dx}$ also are not defined on the whole $L^2(\mathbb{R})$.

First, these Lie algebra operators were obtained by author using theorems on the operational calculus for the index hypergeometric transform (17) from [25]. But the final formulas are elementary and admit a direct verification by elementary tools.

Since the approximate structure of formulas for action of overalgebra now became more clear, it is natural to formulate the general problem given above.

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1. **Group $\text{SL}(2, \mathbb{R})$.** We realize the group $\text{SL}(2, \mathbb{R})$ as the group of complex $2 \times 2$ matrices having the form

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}, \quad \text{where} \quad |a|^2 - |b|^2 = 1.$$

By $D$ we denote the disk $|z| < 1$ on the complex plane $\mathbb{C}$, by $S^1$ we denote the circle $|z| = 1$; we represent the points of the circle in the form $z = e^{i\varphi}$. 

2
The group $\text{SL}(2, \mathbb{R})$ acts on the disk by the Möbius transformations
\[
\begin{pmatrix} a & b \\ b & d \end{pmatrix} : \quad z \mapsto \frac{az + b}{\overline{bz} + \overline{d}}.
\] (1)

If $z \in S^1$, then its image under (1) also is contained in $S^1$.

2. **Highest weight representations of $\text{SL}(2, \mathbb{R})$.** Fix $\alpha > 1$. Consider the space $H_\alpha$ of holomorphic functions $f$ on the circle $D$ satisfying the condition
\[
\int_{D} |f(z)|^2 (1 - |z|^2)^{\alpha - 2} \{dz\} < \infty,
\] (2)
where $\{dz\}$ denotes the Lebesgue measure on $D$. Define the inner product in the space $H_\alpha$ by
\[
\langle f, g \rangle_\alpha = \frac{\alpha - 1}{\pi} \int_{D} f(z)\overline{g(z)}(1 - |z|^2)^{\alpha - 2} \{dz\};
\] (3)
the pre-integral factor provides the identity $\langle 1, 1 \rangle_\alpha = 1$. In terms of the Taylor coefficients, the inner product is given by
\[
\langle \sum c_k z^k, \sum c'_k z^k \rangle_\alpha = \sum_{k \geq 0} c_k \overline{c'_k} \frac{k!}{\alpha(\alpha + 1)\ldots(\alpha + k - 1)},
\]
and condition (2) for $f(z) = \sum c_k z^k$ has the form
\[
\sum |c_k|^2 k^{-(\alpha - 1)} < \infty.
\] (4)

It can easily be checked that the space $H_\alpha$ is complete with respect to the norm defined by this inner product, i.e., $H_\alpha$ is a Hilbert space.

The group $\text{SL}(2, \mathbb{R})$ acts in the space $H_\alpha$ by the unitary operators
\[
T_\alpha \begin{pmatrix} a & b \\ b & d \end{pmatrix} (z) = f \left( \frac{az + b}{\overline{bz} + \overline{d}} \right) \left( \frac{\overline{bz} + \overline{d}}{\overline{bz} + \overline{d}} \right)^{-\alpha}.
\] (5)

First, consider the case, when $\alpha$ is an integer. The factor $(\overline{bz} + \overline{d})^{-\alpha}$ is a degree of the derivative of the function (1), this implies that $T_\alpha$ is a representation:
\[
T_\alpha(g_1)T_\alpha(g_2) = T_\alpha(g_1g_2).
\] (6)
The operators $T_\alpha(g)$ are unitary, i.e.,
\[
\langle T_\alpha(g)f_1, T_\alpha(g)f_2 \rangle_\alpha = \langle f_1, f_2 \rangle_\alpha,
\]
this can be easily checked by a change of the variable.
For a noninteger \( \alpha \), \( T_\alpha(g) \) also is an unitary representation; we only must explain the meaning of the expression

\[
(\overline{b}z + \overline{\alpha})^{-\alpha} = (1 + b\overline{\alpha}^{-1}z)^{-\alpha} = (1 + b\overline{\alpha}^{-1}z)^{-\alpha} e^{-\alpha \text{ln} a + 2\pi i k}. \tag{7}
\]

Obviously, \(|b\overline{\alpha}^{-1}| < 1\). Hence the function

\[
(1 + b\overline{\alpha}^{-1}z)^{-\alpha} := 1 + \frac{a}{1!}b\overline{\alpha}^{-1}z + \frac{a(a-1)}{2!}(b\overline{\alpha}^{-1})^2 z^2 + \ldots
\]

is well defined.

Therefore the operator \((5)\) is defined up to the factor \(e^{-2\pi i k\alpha}\), the absolute value of this factor is 1. Hence equality \((6)\) is replaced by

\[
T_\alpha(g_1)T_\alpha(g_2) = \theta \cdot T_\alpha(g_1 g_2),
\]

where \(|\theta| = 1\). Thus, \(T_\alpha(g)\) is an unitary projective representation of the group \(\text{SL}(2, \mathbb{R})\).

**Remark.** Clearly, \(T_\alpha(g)\) can be also considered as a linear representation of the universal covering group of \(\text{SL}(2, \mathbb{R})\).

3. **Tensor product.** By \(\overline{T}_\alpha(g)\) we denote the representation complex conjugate to \(T_\alpha(g)\); it acts in the space \(\overline{H}_\alpha\) of antiholomorphic functions in the circle \(|u| < 1\) by the formula

\[
T_\alpha \left( \begin{array}{c} a \\ b \\ \overline{\alpha} \\ \overline{\pi} \end{array} \right) f(z) = f \left( \frac{\overline{\alpha} z + \overline{b}}{b z + a} \right) (b\overline{z} + a)^{-\alpha}.
\]

The inner product in \(\overline{H}_\alpha\) is given by \((3)\).

Consider the space \(H_\alpha \otimes \overline{H}_\alpha\). It consists of functions \(f(z, \overline{\pi})\) on the bidisk \(\mathbb{D} \times \mathbb{D}\), holomorphic with respect to the variable \(z\) and antiholomorphic in \(u\); the inner product in \(H_\alpha \otimes \overline{H}_\alpha\) is given by

\[
\langle f_1, f_2 \rangle = \frac{(a - 1)^2}{\pi^2} \iint_{\mathbb{D} \times \mathbb{D}} f_1(z, \overline{\pi}) f_2(z, \overline{\pi}) (1 - z \overline{\pi})^{a-1} (1 - \overline{\pi} z)^{a-1} |dz| \{du\}.
\]

The group \(\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})\) acts in \(H_\alpha \otimes \overline{H}_\alpha\) by the operators \(T_\alpha(g_1) \otimes \overline{T}_\alpha(g_2)\) given by

\[
T_\alpha \left( \begin{array}{c} a_1 \\ b_1 \\ \overline{\alpha}_1 \\ \overline{\pi}_1 \end{array} \right) \otimes \overline{T}_\alpha \left( \begin{array}{c} a_2 \\ b_2 \\ \overline{\alpha}_2 \\ \overline{\pi}_2 \end{array} \right) f(z, u) = \nonumber
\]

\[
= f \left( \frac{a_1 z + b_1}{b_1 z + a_1}, \frac{\overline{\alpha}_1 \overline{\pi} + \overline{b}_1}{b_1 \overline{\pi} + a_1} \right) (a_1 \overline{\pi} + a_2)^{-\alpha} (b_2 \overline{\pi} + a_2)^{-\alpha}.
\]

We restrict this representation to the diagonal subgroup \(\text{SL}_2(\mathbb{R}) \subset \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})\), this corresponds to the substitution \(a_1 = a_2 = a, b_1 = b_2 = b\) to the
last formula. This representation of $\text{SL}_2(\mathbb{R})$ is a linear (nonprojective) representation. Indeed,

$$[\bar{z} + \varpi]^{-\alpha}(b\varpi + a)^{-\alpha} = (1 + \varpi^{-1}\bar{z})^{-\alpha}(1 + a^{-1}b\varpi)^{-\alpha}(a\varpi)^{-\alpha},$$

and this expression is a single-valued function for $|z| < 1, |u| < 1$.

4. Principal series of representations. Fix $s \in \mathbb{R}$. Consider the representation $\rho_s$ of the group $\text{SL}_2(\mathbb{R})$ in $L^2$ on the circle $S^1$ given by

$$\rho_s \left( \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \right) f(e^{i\varphi}) = f \left( \frac{ae^{i\varphi} + b}{\bar{b}e^{i\varphi} + \bar{a}} \right) e^{is\varphi} + \bar{a}|^{-1-2is}. $$

These unitary representations are the so-called representations of the principal series. Recall, that the representation $T_x$ is equivalent to $T_{-x}$ (see [3]).

5. Spectral decomposition. Let $\alpha$ be the same as in pp.2–3. Consider the kernel

$$K_\alpha(\varphi, s; z, u) := \frac{(1 - z e^{i\varphi})^{-1/2-\alpha/2}(1 - u e^{-i\varphi})^{-1/2-\alpha/2}}{(1 - z u)^{-\alpha-1/2-\alpha/2}},$$

where $|z| < 1, |u| < 1, \varphi \in [0, 2\pi], s \in \mathbb{R}$. Consider the integral operator $J_\alpha$, that takes a function $f \in H_\alpha \otimes \overline{H_\alpha}$ to the function $F(\varphi, s)$ given by

$$F(\varphi, s) = \int_{D \times D} K_\alpha(\varphi, s; z, u) f(z, \varpi)(1 - z \varpi)^{-\alpha/2}(1 - u \varpi)^{-\alpha/2} \{dz\} \{du\}. \quad (8)$$

Consider the action of the group $\text{SL}_2(\mathbb{R})$ in space of functions of the variables $(\varphi, s)$ defined by

$$R \left( \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \right) F(e^{i\varphi}, s) = F \left( \frac{ae^{i\varphi} + b}{\bar{b}e^{i\varphi} + \bar{a}} \right) e^{is\varphi} + \bar{a}|^{-1-2is}. $$

Note, that for a fixed $s \in \mathbb{R}$ the function $F(e^{i\varphi}, s)$ (as a function in $\varphi$) transforms by the formula for principal series representations.

Simple calculation show, that the operator $J_\alpha$ intertwines the representations $T_\alpha(g) \otimes T_\alpha(g)$ and $R$ of $\text{SL}_2(\mathbb{R})$:

$$J_\alpha \cdot (T_\alpha(g) \otimes T_\alpha(g)) = R(g) \cdot J_\alpha \quad (9)$$

Theorem 1. Operator $J_\alpha$ is a unitary operator from $H_\alpha \otimes \overline{H_\alpha}$ to the space $L^2$ on $\varphi \in [0, 2\pi], s \geq 0$ with respect to the measure

$$\frac{|\Gamma(\alpha - 1/2 + is)|^2}{\Gamma(\alpha)^2} \frac{\text{sh}(\pi s)}{\text{ch}(\pi s)} ds \, d\varphi = \left| \frac{\Gamma(\alpha - 1/2 + is)\Gamma(1/2 + is)}{\Gamma(\alpha)\Gamma(is)} \right|^2 \, ds \, d\varphi. \quad (10)$$
Thus $T_\alpha \otimes \overline{T_\alpha}$ is a multiplicity free integral over principal series representations (this fact was obtained by Pukanszky [27]). Various ways for obtaining the Plancherel measure (9) are contained in [29], [19], [8], [25].

6. Holomorphic continuation of $J_\alpha f(\varphi, s)$. Denote by $W$ the space of all $f \in H_\alpha \otimes \overline{H_\alpha}$ that are smooth up to the boundary of the bidisk $|z| < 1$, $|u| < 1$.

For $f \in W$, consider its Taylor series
\[ f(z, \pi) = \sum_{k,l} c_{k,l} z^k \pi^l. \]
Obviously, its coefficients $c_{k,l}$ rapidly decrease for $k + l \to \infty$, i.e.
\[ |c_{k,l}| = o(k + l)^{-N} \quad \text{for all } N. \]

**Lemma.** Fix $\varphi$. For $f \in W$, the function $J_\alpha f(\varphi, s)$ can be extended holomorphically to the whole complex plane $s \in \mathbb{C}$.

**Proof.** Fix $s \in \mathbb{C}$, $\varphi \in [0, 2\pi]$. The function $K_\alpha(\varphi, s; z, u)$ as a function in the variables $z$, $u$ has a polynomial growth near the boundary of the bidisk $D \times D$:
\[ |K_\alpha(\varphi, s; z, u)| \leq \exp\{4\pi (1 + |\text{Im } s|) (1 - |z|)^{1+\alpha+2|\text{Re } s|} (1 - |u|)^{1+\alpha+2|\text{Re } s|}\}. \]

Hence, for fixed $s, \varphi$, the coefficients $a_{k,l} = a_{k,l}(\varphi, s)$ of the series
\[ K_\alpha(\varphi, s; z, u) = \sum a_{k,l}(s, \varphi) z^k u^l \]
have the polynomial growth as $k, l \to +\infty$.

Indeed, let $q(z, u)$ be a function in the bidisk satisfying
\[ |q(z, u)| \leq C \cdot \delta^{-h} \quad \text{for } |z| \leq 1 - \delta, |u| \leq 1 - \delta. \]
The Taylor coefficients $b_{k,l}$ of $q(z, u)$ are given by the formula
\[ b_{k,l} = \frac{1}{(2\pi)^2} \iint_{|z| = 1 - \delta, |u| = 1 - \delta} q(z) dz du. \]
Hence,
\[ |b_{k,l}| \leq \text{const} \cdot \delta^{-h} (1 - \delta)^{-k-l-2}. \]
for all $\delta$. We choose $\delta = h/(h + k + l + 2)$ and obtain polynomial growth for $b_{k,l}$.

For the kernel $K_\alpha(z, u)$, we obtain in this way the uniform estimates of the form $|a_{k,l}| \leq A \cdot (1 + k + l)^7$ in each rectangle
\[ |\text{Re } s| \leq M, \quad |\text{Im } s| \leq N. \tag{11} \]
Since the Taylor coefficients $c_{kl}$ for $f \in W$ rapidly decrease, the series

$$J_{\alpha} f(\varphi, s) = \sum c_{k,l} a_{k,l}(s, \varphi) \cdot \frac{k!}{(\alpha)^k}$$

is absolutely convergent (see, (4)), the summands are holomorphic with respect to $s$, and the series is uniformly convergent on the rectangles (10).

7. Correspondence of differential operators. The operators of the Lie algebra $sl(2) \oplus sl(2)$ in the space $H_\alpha \otimes H_\alpha$ have the form

$$L_0^{(z)} = z \frac{\partial}{\partial z} + \frac{\alpha}{2}; \quad L_1^{(z)} = z^2 \frac{\partial}{\partial z} + \alpha z; \quad L_{-1}^{(z)} = \frac{\partial}{\partial z}$$

$$L_0^{(u)} = \bar{\alpha} \frac{\partial}{\partial \bar{u}} + \frac{\alpha}{2}; \quad L_1^{(u)} = \bar{\alpha} \frac{\partial}{\partial \bar{u}}; \quad L_{-1}^{(u)} = \bar{\alpha} \frac{\partial}{\partial \bar{u}}$$

For us it will be more convenient the following collection of the operators

$$L_0 := L_0^{(z)} - L_0^{(u)}; \quad L_{-1} := L_1^{(u)} - L_{-1}^{(z)}; \quad L_1 := L_1^{(z)} - L_{-1}^{(u)};$$

$$M_0 := L_0^{(z)} + L_0^{(u)}; \quad M_{-1} := L_1^{(u)} + L_{-1}^{(z)}; \quad M_1 := L_1^{(z)} + L_{-1}^{(u)};$$

The operators $L_0, L_1, L_{-1}$ span the diagonal subalgebra $sl_2$ in $sl_2 \oplus sl_2$. Their images under the operator $J_{\alpha}$ are defined by the formulas

$$J_{\alpha} \left[ z \frac{\partial}{\partial z} - \bar{\alpha} \frac{\partial}{\partial \bar{u}} \right] f(\varphi, s) = \left[ \frac{\partial}{i \partial \varphi} \circ J_{\alpha} \right] f(\varphi, s)$$

$$J_{\alpha} \left[ \bar{\alpha} \frac{\partial}{\partial \bar{u}} + \frac{\alpha}{2} \frac{\partial}{\partial \bar{u}} \right] f(\varphi, s) = - \left[ \frac{\partial}{i \partial \varphi} + \frac{1}{2} \frac{\partial}{\partial \varphi} \right] J_{\alpha} f(\varphi, s)$$

$$J_{\alpha} \left[ \frac{\alpha}{2} \frac{\partial}{\partial \bar{u}} + \alpha z - \frac{\partial}{\partial \bar{u}} \right] = \left[ \frac{\partial}{i \partial \varphi} - \frac{1}{2} \frac{\partial}{\partial \varphi} \right] J_{\alpha} f(\varphi, s)$$

These three formulas easily follow from (8).

Theorem 2. The unitary operator $J_{\alpha}$ transform the operator $M_0$ to the operator

$$Q_0 f(\varphi, s) = \frac{(-\frac{1}{2} + is)(-\frac{1}{2} + is)}{2is} f(\varphi, s + i) +$$

$$+ \frac{(\frac{1}{2} + is)(\frac{1}{2} + is)}{2is} f(\varphi, s - i) = \frac{-\alpha + \frac{1}{2} + is}{2is(-\frac{1}{2} + is)} \frac{\partial^2 f(\varphi, s + i)}{\partial \varphi^2}, \quad (12)$$

i.e., for any $f \in W$ (see our Section 6 ),

$$Q_0 J_{\alpha} f = J_{\alpha} M_0 f.$$
The operator $M_1$ transforms to

\[ Q_1 f(\varphi, s) =
= e^{is} \left[ \frac{(-z^2 + is)(1/2 + i/2 + i/2) + 1/2 - i/2 + i/2}{2is} f(\varphi, s + i) + \frac{(-z^2 + is)(-z^2 + i/2 + i/2) + 1/2 - i/2 + i/2}{2is} f(\varphi, s - i) - \frac{-z^2 + is}{2is(-z^2 + i/2 + i/2) d\varphi^2} f(\varphi, s + i) + \frac{-z^2 + is}{2is(-z^2 + i/2 + i/2) d\varphi^2} f(\varphi, s - i) \right] \right], \quad (13) \]

and the operator $M_{-1}$ transforms to

\[ Q_{-1} f(\varphi, s) =
= e^{-is} \left[ \frac{(-z^2 + is)(1/2 + i/2 + i/2) + 1/2 - i/2 + i/2}{2is} f(\varphi, s + i) + \frac{(-z^2 + is)(-z^2 + i/2 + i/2) + 1/2 - i/2 + i/2}{2is} f(\varphi, s - i) - \frac{-z^2 + is}{2is(-z^2 + i/2 + i/2) d\varphi^2} f(\varphi, s + i) + \frac{-z^2 + is}{2is(-z^2 + i/2 + i/2) d\varphi^2} f(\varphi, s - i) \right] \right], \quad (14) \]

**Proof.** These formulas can be checked by direct calculations. For instance, let us consider $M_0$.

Obviously, the operator $M_0$ is selfadjoint in $H_\alpha \otimes \Pi_\alpha$. Hence

\[ J_\alpha M_0 f(\varphi, s) =
\int_D \int_D K(\varphi, s, z, u) \left[ \left( \frac{\partial}{\partial \varphi} + \Pi \frac{\partial}{\partial \varphi} + \alpha \right) f(z, u) \cdot (1 - z\varphi)^{\alpha - 2} (1 - u\varphi)^{\alpha - 2} \right] \{dz\} \{du\} =
\int_D \int_D \left[ \frac{\partial}{\partial \varphi} + \Pi \frac{\partial}{\partial \varphi} + \alpha \right] K(\varphi, s, z, u) f(z, u) (1 - z\varphi)^{\alpha - 2} (1 - u\varphi)^{\alpha - 2} \{dz\} \{du\} \]

Thus the first statement of the theorem is equivalent to the identity

\[ \left( \frac{\partial}{\partial u} + \frac{\partial}{\partial \varphi} + \alpha \right) K(\varphi, s; \varphi, u) - Q_0 [K(\varphi, s; \varphi, u)] = 0 \quad (15) \]

After division by $K$, this identity transforms to the form

\[ K^{-1} \left( \frac{\partial}{\partial u} + \frac{\partial}{\partial \varphi} + \alpha \right) K + K^{-1} \left( \frac{-\alpha + 1/2 + i/2 + i/2}{2is} \right) K(s + i) -
\]

\[ K^{-1} \left( \frac{\alpha - 1/2 + i/2}{2is} \right) K(s - i) + K^{-1} \left( \frac{-\alpha + 1/2 + i/2}{2is(-z^2 + i/2 + i/2) d\varphi^2} \right) K(s + i) \quad (16) \]

The function in the left side is a long rational expression in $z, \varphi, e^{i\varphi}, s$; the identity can be easily verified by MAPLE.
Let us explain how to verify the identity (15) "by hands". Each summand of (15) can be represented as a linear combination

\[
a(s) + b(s) \cdot \frac{1}{1 - z^u} + c(s) \cdot \left[ \frac{1}{1 - z e^{i\phi}} + \frac{1}{1 - \bar{z} e^{-i\phi}} \right] + d(s) \cdot \frac{(1 - z e^{i\phi})(1 - \bar{z} e^{-i\phi})}{1 - z^u}\]

After this, it remains to sum the coefficients.

Remark. Let us apply the operator \(Q_0\) to the functions \(f(\phi, s) = g(s)\). Then the equations

\[
Q_0 g(s) = (k + a) g(s)
\]

coincide with a partial case of the difference equations for the continuous dual Hahn polynomials (see [1], (6.10.9)).

8. Some remarks. In the work of the author [25], there were obtained some elements of an operational calculus for the index hypergeometric transform (it is called also by Olevsky transform or Jacobi transform, see [30], [17])

\[
g(x) \mapsto \hat{g}(s) = \frac{1}{\Gamma(b + c)} \int_0^\infty g(x) \, {}_2F_1(b + is, b - is; b + c; -x) x^{b+c-1} (1 + x)^b - x \, dx,
\]

In [25], it was shown that the index hypergeometric transform maps the differential operators

\[
A g(x) = x g(x); \quad B g(x) = x \frac{\partial}{\partial x} g(x)
\]

(and hence all the operators admitting polynomial expression in \(x, x \frac{\partial}{\partial x}\)) to difference operators in imaginary direction; see also the work of Cherednik [6] containing some similar statements for symmetric functions in multidimensional case. Existence of the formulas (11)-(13) more or less follows from these results, but this way for obtaining the expressions (11)-(13) also is not very simple.

There arises the following question.

Question. Is it possible to write explicitly operators of the overalgebra for the case of \(L^2\) on a pseudo-Riemannian symmetric space and for the kernel representations?

Is it possible to do this at least for rank 1 symmetric spaces?

References


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