Classification of the Measurable Functions of Several Arguments and Invariant Measures on the Space of Matrices and Tensors, 1. Classification.

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Supported by Federal Ministry of Science and Transport, Austria
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Partially supported by RFBR grant 99-01-00098 and by the special ESI-program

8.12.2001

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1 AMS Classification Primary 46T12, 28D05; Secondary 37L4
1 Introduction: list of problems.

This is the first part of the article devoted to the several topics which includes classification of the measurable functions and random processes, invariant measures on the space of tensors an matrices, actions of the powers oof infinite symmetric groups, random matrices, metric spaces with measures (mm-spaces), universality, Urysohn space and random metrics on the naturals.

In this part we want to discuss only the problem of the classification of measurable functions of $k$ arguments (Problem F) and a link with the problem of the description of $(S_N)^k$-invariant measures on the set of tensors (Problem G). The main case $k = 2$ includes very important partial problem - classification of the metric spaces with measure (Problem M). We will formulate all the problems for $k = 2$ and then will make remarks about the general case. So we consider first of all

Problem F

To classify measurable (or continuous, smooth etc.) functions of two or more arguments on Lebesgue space (which is standard measure space with continuous measure) upto metric isomorphism e.g. upto direct product measure preserving groups of each variables.

More exactly: two real-valued measurable functions $f_1, f_2$ defined on the Lebesgue spaces $(X_1 \times Y_1, \mu_1 \times \nu_1), (X_2 \times Y_2, \mu_2 \times \nu_2)$ with continuous measures are metrically isomorphic iff

$$f_2(x, y) = f_1(Sx, Ty)$$

where $T, S$ are the arbitrary invertible measure preserving transformations - $S : (X_2, \mu_2) \rightarrow (X_1, \mu_1), T : (Y_2, \nu_2) \rightarrow (Y_1, \nu_1)$, and for symmetric case: two real-valued measurable symmetric $(f(x_1, x_2) = f(x_2, x_1))$ functions $f_1, f_2$ defined on the Lebesgue spaces $(X_1 \times X_1, \mu_1 \times \mu_1)$ and $(X_2 \times X_2, \mu_2 \times \mu_2)$ are symmetrically isomorphic iff

$$f_2(x_1, x_2) = f_1(Sx_1, Sx_2),$$

where $S$ is a measure preserving isomorphism of the spaces $S : (X_2, \mu_2) \rightarrow (X_1, \mu_1)$.

The changes which we must put for the case of more than two arguments are evident. It is possible to consider various conditions of symmetry (not only symmetry or skew symmetry). We will see that the problem F for several variables could be considered in the same way as for two, and will return to this later.

The case of functions of one variable was solved many years ago by V.A.Rokhlin [10], we will recall his result, the case of the functions of several arguments is much more complicate. The problem about metric classification for specific classes of functions, say for smooth functions on the torus etc. looks as an improtant problem. Our main result about invariants
of metric classification of the functions allows to reduce this question to the question about the special properties of that invariant for given classes of functions.

The probabilistic interpretation of the problem $F$ is very interesting for the theory of random processes: let us consider space $X$ as “time” with probability measure $\mu$ and $(Y(\equiv \Omega), \nu)$ as some abstract probability (Lebesgue) space - the domain of the random variables. Then a function $f$ could be viewed as random field (process) with “time” $X$ and measurable w.r.t $(X, \mu)$ realizations $x \to f(x, \cdot)$ - a random variable at the “moment” $x$. Our problem now is a classification of the random process with measurable time up to arbitrary measure preserving changes of time. This point of view helps to interpret our main construction. For example, metric classification of the gaussian processes with “time” $(X, \mu)$ immediately reduced to the metric classification of symmetric measurable correlation functions of that process $- B(\cdot, \cdot)$ on $X \times X$. It is interesting that in this setting both spaces - “time-space $(X)$” and probability space $- Y \equiv \Omega$ play symmetric roles.

A very important partial case of this problem is the following

**Problem M. To classify Polish (=separable complete metric) spaces $(X, \rho, \mu)$ with metric $\rho$ and probability borel measure $\mu$ up to measure preserving isometry.**

Such triples M.Gromov called “mm-space”[4], I used in [11] term “Gromov triple” or “metric triples”. It is evident that that classification of the metric spaces is the same as classification of metrics as measurable functions of two variables on the square of the space, and this is partial case of the problem F. The studying such triples as well as classification problem are very useful in various areas and well-known problem (see [4]).

Suppose now that $R$ is arbitrary polish space f.e. real numbers $R = \mathbb{R}$ and $M_N R = \{ r = \{ r_{i,j} \}, i, j = 1 \ldots \in R \}$ - the space of all infinite matrices with entries in $R$, provided with weak topology, and $M^\text{Symm}_N (R)$ is subset of symmetric matrices. Let $S_N$ is infinite symmetric group of finite permutations of naturals $\mathbb{N}$, so we can define an action of $S_N \times S_N$ on $M_N (R)$ as follow

$$(grh^{-1})_{i,j} = r_{g(i),h(j)}$$

$(g, h) \in S_N -$ (this is a “separate action” of $S_N$), and action of $S_N$ on $M^\text{Symm}_N (R)$:

$$(grg^{-1})_{i,j} = r_{g(i),g(j)}$$

- “joint action” of $S_N$.

For the case of tensors of higher order we can consider an action of $(S_N)^k$ as follow

$$\{ r_{i_1 \ldots i_k} \} \to \{ r_{g_1(i_1) \ldots g_k(i_k)} \}$$

where $(g_1, \ldots g_k) \in (S_N)^k$ - this is the action on the covariant tensors (or on the space of multilinear functionals). It is possible to restrict this action onto some subgroups (various diagonals) and subspaces of the tensors with given type of symmetries.
For the space of real - $M_N(R)$, or complex - $M_N(C)$ matrices we can consider an action of infinite orthogonal - $O(N)$, or infinite unitary - $U(N)$. The similar actions we can considered on the space of tensors of higher order with or without the symmetries. We will discuss now the case of matrices and infinite symmetric group only.

**Problem G** To describe all borel measures on $M_N(R)$ (or on the space $M_N^{Symm}(R)$ or on the space of tensors etc.) which are invariant under the actions of mentioned groups. It is enough to describe only ergodic measures.

We will consider here only the case of the separate action of $S_N \times S_N$ on the space of matrices $M_N(R)$, other examples and the general cases for various groups will be considered somewhere.

One class of the examples of such measures is evident - all entries of the matrix are i.i.d. - independent identically distributed in $R$, we call it as the measures of product-type - but there are a lot of the degenerated $S_N \times S_N$-invariant ergodic measures in both cases which are naturally appeared just as invariants of the measurable functions. These examples are very interesting from point of view of the theory of dynamical systems as well as representation theory: each such measure define the measure preserving actions of various groups and also representations of its. The examples of degenerated measures give a new interesting factor representations of the infinite symmetric group.

Problem (G) was considered from probabilistic point of view by J.Aldous.[1, 2] who actually gave a remarkable reduction of the Problem (G) to the concrete version of the Problem (F) and later in the series of papers by O.Kallenberg [5, 6, 7] with detailed and interesting analysis and links (see also references in those papers). As to problem (F) I do not know if anybody had considered it itself, - it had been arised in measure theory and ergodic theory but there were no good methods to solve it.

The problem (M) was discussed by Gromov [4] who suggested in some form the reduction of it to the problem (G). In the paper by author [11] the precise form of this reduction was done. In this paper we give a reduction of Problem (F) to the special case of (G) for the general measurable functions, the case of metric much simpler because it is easy to restore whole space by one countable set.

Anyway, in a sense we have only mutual reductions of each problems to another which was done by different group of mathematicians! All reductions are not trivial and already gave some important corollaries for each problem. Moreover, the link between (F) and (G) is useful in both directions, sometimes for the studying of invariant measures on the space of matrices which is generated by a measurable function it is useful to return back to that functions in order to obtain the information about the measure.

What is very important and not evident from the beginning - that all those problems (F), (M) and (G) are “smooth” problems in the sense of classification theory which means that
the classification problem has a good parametric answer or normal forms. Nevertheless as usual, indeterminacy of all classification problems as well as problem about the description of the some class of objects is in indeterminacy of the terms in which we try to give an answer. So there are many variants of the answer depending of the choice of parameter (choice on normal form). So the question is how to optimize the parametric set for our case. It is not clear if we already have the best terms for this.

The methods and statements of the problems in this paper are in the spirit of ergodic and measure theory like in [12] more than probabilistic or functional analytic approches. In my paper [12] I have posed the problem (G) for symmettric and unitary groups and suggested so called “ergodic method” for solution of such kind of problems. It was very efficient for “one dimensional” problems - for example for de Finetti’s theorem (about exchangelability) and Schonberg’ theorem - about spherical invariant measures in Hilbert space and so on. Unfortunately the list of answers for matrix case was incomplete in that paper - all degenrated measures were lost. Here we will use the same “ergodic method” for all kind of such problems and show that it fits in these case also. In the same time there are the alternating approaches to the problem - see [1, 2, 5, 6, 7] and references there. For the case of unitary and orthogonal groups there are also several methods. In the papers with G.Olshansky [8] the problem for unitary group was solved by the method of the theory of symmetric functions together with some fomr of ergodic method, another approach had been considered before in the paper by D.Picrell [9].

The main results of this paper are the following:

1. Complete invariant of the isomorphism of measurable functions of two and more arguments on the Lebesgue space with values in the arbitrary standard borel space $R$. This called matrix distribution and this is a $S_N \times S_N$-invariant measure on the space of matrices $M_N(R)$; matrix distribution is a right generalization of the notion of distribution of measurable function of one variables; in a sense this is a solution of the problem (F) - see theorem 2; - or more exactly - reduction of problem (F) to the problem (G) (section 3).

2. Characteristic properties of matrix distribution as proper subclass of the class of all ergodic $S_N \times S_N$-invariant measures on $M_N(R)$ - ‘simple’ measures. Canonical model of the measurable function with given matrix distribution - reconstruction of measurable functions by countable set of values (section 4).

We will continue the discussion of the problem in the further papers. In particular we will give an alternative answer (with comparison with [1]) on the Problem (G) - about the description of the structure of all ergodic $S_N \times S_N$-invariant measures on $M_N(R)$. The examples - metric with measure, Urysohn space and other will be presented as well as a new way of description of $U(\infty)$-invariant measures on the symmetric tensors from the point of view of metric classification of the bilinear forms on gaussian measure spaces.
2 Classification of measurable functions: one variable and pure functions

2.1 Classical case - functions of one variable

First of all we recall Rokhlin’s answer on the problem of classification of the measurable functions of one variable with values in standard Borel space in slightly different form than in [10]

Let $f : X \to R$ a measurable function on the Lebesgue space $(X, \mathcal{B}, \mu)$ where $\mu$ is continuous measure which is defined in the sigma-field $\mathcal{B}$. (usually we will omit the notation of the sigma-field) and with target space $R$ as arbitrary standard borel space (i.e. $\mathbb{R}$, or $[0,1]$ etc). The group $\mathcal{A}(X, \mu)$ of all measure preserving transformations of the space $(X, \mu)$ acts on the space $S_\mu(X; R)$ of all measurable functions. More exactly and it is very important, to say that the group of classes mod0 transformations acts on space of classes mod0 functions but we will omit an evident verifications of correctness (mod0) of all definitions and claims. It is clear that pushforward measure $f \cdot \mu$ (or distribution of $f$) on the space $R$ is metric invariant of the function. But it is not completely invariant, because it does not take into account the multiplicities of the values. Define a measurable partition $\zeta_f^X$ of $(X, \mu)$ on the preimages of the points - the element of this partition is a level of constancy of the function $f$. This is correct definition of a measurable partition. As all measurable partitions our partitions have the system of conditional (canonical) measures, and almost all element of partitions equipped with that conditional measure becomes a Lebesgue space. Metric type of each Lebesgue space is a point of infinite dimensional simplex, namely the sequence of measures of atoms.

\[ m^1 \geq m^2 \geq \cdots \geq 0, \quad \sum_{i=1}^{\infty} m^i \leq 1 \]

where $m^i$ is a measure of $i$-th atom of the space. (if $m^1 \equiv 0$ then we have space with continuous measure, if $m^1 \equiv 1$ - this is the one-point space etc.) Now let us denote \( \{m^i(r)\}_{i=1}^{\infty} \) a metric type of the element of partition $\zeta_f$ which is level on constancy of the value $r$: \( \{x : f(x) = r\} \). So we have defined correctly (mod0) a new Lebesgue space $(R, f \cdot \mu)$ and the measurable function $m_f$ on it with values in the simplex $\Sigma$ of positive series with the sum less or equal to 1: $m_f(r) = \{m^i(r)\}_{i=1}^{\infty}$.
Theorem 1 (see Rokhlin [10]). The complete system of metric invariants of the measurable functions on Lebesgue space with respect to group of measure preserving (invertable) transformations of the argument is the

1) measure $f \mu$ on the target space $R$ of values or “distribution of function $f$ ’, and
2) the measurable function $m_f$ on target measure space $(R, f \mu)$ - “multiplicity function” - with values in the infinite dimensional simplex $\Sigma$.

For univalent functions (when $m^1(.) \equiv 1$) the claim is trivial - distribution is unique invariant of the function and the normal form is “monotonic perestroika” of $f$. The general case in which we have multiplicities is based on the Rokhlin’s classification of measurable partitions and is equivalent to that theorem. So, the theorem exhausts the description of the orbits of action of the group $A(X, \mu)$ on the space $S_{\mu}(X; R)$: necessary and sufficient conditions for metric isomorphism of two measurable functions (to belong to the same orbit) are coincideness of triples $(R, f \cdot \mu, m_f)$, where measure on $R - f \cdot \mu$ is the distribution of function $f$ and $m_f$ is a function on $R$ with values in $\Sigma$.

One problem of the classification of functions $f(\ldots)$ of two arguments on the space $(X \times Y, \mu \times \nu)$ could be solved in the similar way; this is the classification with respect to the group of all measure preserving automorphisms which are cross-product - $(T, \{S_x\})$ where $T$ is automorphism of $(X, \mu)$ and $\{S_x\}$ is measurable set with parameter $x \in X$ of the automorphisms of space $(Y, \nu)$. In this case the group (and orbits) is too large. But the problem (F) which we put above - about the classification the functions of several arguments with respect to direct product of measure preserving groups of each arguments is more difficult and demands in new ideas.

### 2.2 Pure functions

Consider the case of function of two variables on the space $(X \times Y, B_X \times B_Y, \mu \times \nu)$. The group $A(X, \mu) \times A(Y, \nu)$ acts (separately) on both arguments. For $(X, \mu) = (Y, \nu)$ we will consider also a symmetric case which means diagonal action of $A(X, \mu)$ on the space of symmetric functions $f(x, y) = f(y, x)$. It happened that this case is not very different with classifcation of the separate action. We postpone the discussion on this as well as on the case of many variables).

First of all we will exclude “multiplicities”.

Let us define on the space $(X, \mu)$ ($(Y, \nu)$) a partition $\zeta^X$ ($\zeta^Y$) as follow; two points $x_1, x_2$ of $X$ ($y_1, y_2$ of $Y$) are in the in the same element of $\zeta^X$ ($\zeta^Y$) if

$$f(x_1, y) \equiv f(x_2, y)$$
for \( \nu \text{-a.e. } y, \)

\[ f(x, y_1) \equiv f(x, y_2) \]

for \( \mu \text{-a.e. } x. \)

These are the measurable partitions because of measurability of the function \( f. \)

**Definition 1** The function \( f(\ldots) \) called pure with respect to the first (second) argument if partition \( \zeta^X \) (\( \zeta^Y \)) is a partition on the separate points (mod0) of the space \( (X, \mu) \) (resp. \( Y, \nu \)); and simply pure if both partitions are partitions on the separate points mod0.

Pure functions play the role of univalent functions in the case of one variable. Classification problem immediately reduced to the case of pure functions.

Suppose \( f(\ldots) \) is an arbitrary measurable function on the space \( (X \times Y, \mu \times \nu) \) and suppose that \( \zeta^X, \zeta^Y \) are the measurable partitions defined above. Let us factorize the space \( X \) over partition \( \zeta^X \) and the space \( Y \) over partition \( \zeta^Y. \) We can correctly define on the quotient space

\[
\left( X/\zeta^X \times Y/\zeta^Y, \mu_{\zeta^X} \times \nu_{\zeta^Y} \right)
\]

where \( \mu_{\zeta^X}, \nu_{\zeta^Y} \) are quotient measures, and a new function \( \tilde{f} \) on the quotient space \( (X/\zeta^X \times Y/\zeta^Y) \) with new target space \( \tilde{R} = R \times \Sigma \times \Sigma \) as follow:

the value of the function \( \tilde{f} \) at the point \((\tilde{x}, \tilde{y}) \in (X/\zeta \times Y/\zeta) \) is a set

\[
(f(x, y), \{m^i(\tilde{x})\}_{i=1}^{\infty}, \{m^i(\tilde{y})\}_{i=1}^{\infty}) \in (R \times \Sigma \times \Sigma)
\]

where \( \tilde{x} \) (\( \tilde{y} \)) is an element of partition \( \zeta^X \) (\( \zeta^Y \)) which contains point \( x \) (\( y \)), and sequence \( \{m^i(\tilde{x})\}_{i=1}^{\infty} \) (\( \{m^i(\tilde{y})\}_{i=1}^{\infty} \)) is a metric types of the conditional measure of the element \( \tilde{x}(\tilde{y}) \) of the partitions \( \zeta^X \) (\( \zeta^Y \)) of the space \( (X, \mu) \) (resp. of the space \( (Y, \nu) \)). Correctness follows from the definition of the partitions \( \zeta \) and from definition of the function \( \tilde{f}. \) The next lemma shows that enlarging target space we can reduce our problem to the most interesting case of pure functions.

**Lemma 1** 1) For any measurable function \( f \) the function \( \tilde{f} \) defined above is a pure function (with new target space).

2) Complete system of the metric invariants of function \( f \) coincides with complete systems of metric invariants of pure function \( \tilde{f}. \) In another words -two measurable functions \( f_1, f_2 \) are isomorphic if the same is true for corresponding pure functions \( \tilde{f}_1, \tilde{f}_2. \)

**Proof.** The item 1) follows from the definition; the proof of the item 2) is the same as in Rokhlin theorem; values \( \{m^i(x), m^i(y)\}_{i=1}^{\infty} \) play the same role as multiplicities in his theorem about the case of one variable (see above) \( \Box \)

The analogy between univalent function of one variable and pure function of many variables will be more clear later.

3 Complete metric invariant of the measurable functions of several arguments: matrix distribution

3.1 Definition of matrix distribution

In the following considerations we will study the case of pure functions only. Because each standard Borel space is isomorphic to the interval $[0,1]$ as a borel space without of diminishing of the generality we can restrict ourself with the case when target space is the borel space $[0,1]$ or $\mathbb{R}$. For the same reason we can assume that the domain of the functions of two variables could be considered as unit square $(X \times Y, \mu \times \nu) = (I \times I, m \times m)$, where $m$ is Lebesgue measure on unit interval $I$, but we do not use this. In order to apply ergodic theorem we need to use a linear stucture in the target space so it is more convinient to use sometime real numbers $\mathbb{R}$ as a target space.

Let $f$ is a real measurable function of two variables on the space $(X \times Y, \mu \times \nu)$ with values in some standard borel space $R$ and $((X \times Y)^N, (\mu \times \nu)^N)$ is infinite direct product of domain spaces. Let $M_N(R)$ is the space of all matrices with entries from $R$.

Define a map:

$$F_f : (X \times Y)^N \to M_N(R)$$

by formula

$$F_f(x_1, \ldots, y_1 \ldots) = \{f(x_i, y_j)\}_{i,j=1}^\infty$$

Definition 2 A pushforward measure $F_f(\mu \times \nu)^N$ on the space of matrices $M_N(R)$ (image of the measure $(\mu \times \nu)^N$ under the map $F_f$) we will call matrix distribution of measurable function $f$ and denote $D_f$. For symmetric case we define instead the map $F_f : X^N \to M_N(R)$ as $F_f(x_1 \ldots) = \{f(x_i, x_j), i, j = 1 \ldots\}$ matrix distribution is a measure and $D_f = F_f(\mu^N)$.

So a measure $D_f$ on $M_N(R)$ is the distribution of the random matrices

$$\{f(x_i, y_j)\}_{i,j=1}^\infty$$

when $\{x_i\}, \{y_j\}$ are sequences of the independent identically distributed random points of $X, Y$ with distribution $\mu, \nu$. The set of all finitedimensional distributions of the matrices $\{f(x_i, y_j)\}_{i,j=1}^n$ of order $n$ when $n$ runs over $\mathbb{N}$ uniquely defined $D_f$; for example ordinary distribution of the function $f$ as function of one argument $(x, y)$ on the measure space $(X \times Y, \mu \times \nu)$ is nothing that distribution of one entries of our matrix, f.e. of the $r_{1,1}$ (because the distribution of all entries coincide). In [4] (for the case of metric on Polish
spaces with measure) was used languish of finitedimensional distributions but it is natural to use the infinite matrices.

Remark that a map \( F_j \) could be an isomorphism of the measure spaces \( ((X \times Y)^N, (\mu \times \nu)^N) \) and \( (M_N(R), D_f) \) as well as homomorphism - see remark after the theorem. In order to explain the term “matrix distribution” consider the case of function of one variable in the same framework. Let \( f : (X, \mu) \rightarrow (R, d_f) \) where measure \( d_f \) in \( R \) is distribution measure of the function \( f \) and let \( \{x_i\}_{i=1}^N \) an element of \( X^N \), then the map \( F_j : X^N \rightarrow R^N \) which is defined in the same way as in the case above: \( F_j(\{x_i\}) = \{f(x_i)\} \in R^N \) brings the measure \( \mu^N \) in \( X^N \) to the product measure \( d^N_j \) in \( R^N \). Evidently measure \( d^N_j \) is \( S_N \)-invariant and ergodic. So matrix distribution \( D_f \) plays the same role as product the \( d^N_j \) for one variable but the structure of matrix distribution \( D_f \) is much more complicate than the product-measure \( d^N_j \) on \( R^N \).

Now we will put in account the group which will play a crucial role - the direct product of the infinite symmetric group \( - S_N \times S_N \). This group acts in natural way by the separate permutations of rows and columns on the space \( M_N(R) \).

Lemma 2 For each measurable function \( f \) the metric distribution \( D_f \) is an \( S_N \times S_N \)-invariant ergodic measure on the space of matrices \( M_N(R) \).

Proof. The measure \( (\mu \times \nu)^N \) in the space \((X \times Y)^N \) is evidently invariant and ergodic respectively to that group, but the space of matrices is an image-space under the map \( F_f \) which commutes with actions of the groups in the image and preimage. \( \square \) Another evident property of the matrix distributions as well as all \( S_N \times S_N \)-invariant ergodic measures:

Remark 1 Two entries of matrix with disjoint indeces \( (i, j) \) and \( (i_1, j_1) \) where \( i \neq i_1, j \neq j_1 \) are independent as the functions on \( (M_N(R), D_f) \); more over, two disjoint arrays \( \{r_{i,j}\}_{i,j<N} \) and \( \{r_{i,j}\}_{i,j \geq N} \) are independent for all natural \( N \).

This is evident for matrix distribution by definition but it is also follows from de Finetti’s theorem for all \( S_N \times S_N \)-invariant ergodic measures.

3.2 Complete invariant in the Probelm (F).

Now we formulate one the main results of this section.

Theorem 2 (Classification of pure measurable functions of two variables) Suppose \( f \) is pure measurable function on the space \((X \times Y, \mu \times \nu) \) with values in a standard borel space \( R \), then the matrix distribution that is a measure \( D_f \) in \( M_N(R) \) is complete invariant of the function...
\[ f \text{ under the action of the direct product of the group of measure preserving transformations } \mathcal{A}(X, \mu) \times \mathcal{A}(Y, \nu). \text{ In another words}
\]

1) If two (not necessary pure) real functions \( f_1 \) and \( f_2 \) are isomorphic then \( D_{f_1} = D_{f_2} \); and

2) If for two pure measurable functions \( f_1, f_2 \) which are defined in the spaces \((X_1 \times Y_1, \mu_1 \times \nu_1)\) and \((X_2 \times Y_2, \mu_2 \times \nu_2)\) correspondingly and have the same measures \( D_{f_1} = D_{f_2} \) on the space \( M_N(\mathbb{R}) \) then they are isomorphic e.g. there exist measure preserving automorphisms \( S_1, S_2 \), where \( S_1: (X_1, \mu_1) \to (X_2, \mu_2) \) and \( S_2: (Y_1, \nu_1) \to (Y_2, \nu_2) \) such that \( f_2(x, y) = f_1(S_1 x, S_2 y) \) for almost all \((x, y) \in (X_2 \times Y_2)\).

**Proof.** First we prove claim 1 which is obvious. Indeed if we have two isomorphic functions on the different spaces \((X_1 \times Y_1, \mu_1 \times \nu_1)\) and \((X_2 \times Y_2, \mu_2 \times \nu_2)\) and if there are two invertible measure preserving homomorphisms \( S_1: X_1 \to X_2 \) and \( S_2: Y_1 \to Y_2 \) such that \( f_2(., .) = f_1(T_., .) \) then the transformation \((S_1 \times S_2)^N\) brings \((X_1 \times Y_1)^N\) to \((X_2 \times Y_2)^N\) and measure \((\mu_1 \times \nu_1)\) to measure \((\mu_2 \times \nu_2)\) so measure preserving transformation \((S_1 \times S_2)^N\) gives the equivalence of the functions \( F_{f_1}: (X_1 \times Y_1) \to M_N(\mathbb{R}) \) and \( F_{f_2}: (X_2 \times Y_2) \to M_N(\mathbb{R}) \) (as a function of “one” argument) So, these functions have the same pushforward measures \( D_{f_1} \) and \( D_{f_2} \) in \( M_N(\mathbb{R}). \)

As we already have remarked a map \( F_f: (X \times Y)^N \to M_N(\mathbb{R}) \) could be not univalent in measure sense - e.g could have the multiplicities of the values, so the measures \( D_f \) could be not complete invariant of the map \( F_f \) as a measurable function of one variable, but because of the special features of functions \( F_f \) the absence of multiplicities depends on measure \( D_f \) itself. In another words the map \( F_f \) could be an isomorphism or not isomorphism depending on the image \( D_f \). For example, if \( f \) is metric in the triple \((X, \rho, \mu)\) where \((X, \rho)\) is a Polish space with metric \( \rho \) and with some borel measure \( \mu \) then \( F_f \) is isomorphism between \((X \times X, \mu \times \mu)^N\) and \((M_N(\mathbb{R}^+), D_f)\) iff the group of measure preserving isometries is trivial.

For functions of \( k \) variables \( f(x, y, \ldots, z) \) the corresponding map \( F_f \) is defined by similar rule: \( F_f: (X \times Y \times \ldots \times Z) \to Tens_k^N(\mathbb{R}) \) (remark that \( M_N(\mathbb{R}) = Tens_k^N(\mathbb{R}) \)) as \( F(\{x_i\}, \{y_j\}, \ldots, \{z_l\}) = \{f(x_i, y_j, \ldots, z_l)\}_{i,j,\ldots,l=1}^\infty \). The assertion of theorem is also true in this case as well as for symmetric case.

For proving of the claim 2) of theorem 2 we need in the following lemma.

**Lemma 3** Let \( f \) is a pure measurable function with respect to the second (first) argument, and \( \{C_{n,k}\}_{k=1}^\infty \) a countable basis of the borel sets in the space \( R^n \), \( n = 1, \ldots, \) then for almost all with respect to measure \( \mu^N \) sequences \( \{x_i\}_{i=1}^\infty \) (corresp \( \{y_j\}_{j=1}^\infty \) ) the system of the measurable subsets in \( Y \) which is defined as \( B_{n,k}^2 = \{y: \{f(x_i, y)\}_{i=1}^n \in C_{n,k}\} \) (corresp. \( B_{n,p}^1 = \{x: \{f(x, y_j)\}_{j=1}^n \in C_{n,p}\} \}) generates a countable basis of complete sigma-field \( B_Y \) in \((Y, \nu)\) (corresp. \( B_X \) in \((X, \mu)\).
Proof. Let us define subsigma-field $\mathcal{B}_n$ generated by all the sets $B_{n,k}$ defined above for fixed $n$ and a subspace of measurable functions on $(Y, \nu)$ which are measurable with respect to $\mathcal{B}_n$; denote this subspace as $\mathcal{F}_n$. Suppose the family of all functions from the union $\mathcal{F} = \bigcup_n \mathcal{F}_n$ does not distinguish the points of the space $(Y, \nu)$. Then the measurable partition in $(Y, \nu)$ on the level of constancy of all functions from $\mathcal{F}$ is not trivial and we can conclude (using the definition of the sets $B_{n,k}$ and the fact that $\{C_{n,k}\}_{k=1}^\infty$ is a basis of borel sets in $\mathbb{R}^n$) that

$$f(x_i, y_i) = f(x_i, y_2)$$

for all $i = 1, \ldots$ Consequently for $\mu^N$-almost all sequences $x_i$ this implies the equalities $f(x, y_i) = f(x, y_2)$ almost everywhere on $x$. Indeed if $f(x, y_i) \neq f(x, y_2)$ on the set of positive measure in $(X, \mu)$ then by ergodic theorem it must be true for some $x_i$ with $i$ from the set of positive density in $\mathbb{N}$ and for $\mu^N$-almost all sequences $x_i$. But equality $f(x, y_i) = f(x, y_2)$ for pairs $(y_1, y_2)$ from nontrivial partition contradicts to the fact that function $f$ is pure. So the set of all functions from $\mathcal{F}$ distinguish the points in $Y$ and consequently the system of measurable sets $B_{n,k}$ is the basis of sigma-field in the space $(Y, \nu)$. The same arguments are valid for the space $(X, \mu)$. ■

Corollary 1 If $f$ is a pure function then for almost all $\{x_i\}$ ($\{y_j\}$) restriction of the map $F_f$ on the subspace (more exactly - on the element of partition) $\{x_i\} \times Y^N$ ($X^N \times \{y_j\}$) is one-to-one mod 0. Nevertheless the map $F_f$ could be not univalent on the whole space - see subsection 3.3).

Now we will finish the proof of the claim 2) and prove that $D_f$ is complete invariant.

Proof. We will show how to reconstruct (interpolate) a measurable function of two variables using its values on the "generic" countable set $\{x_i, y_j\}_{i,j=1}^\infty$.

Suppose we have a pure measurable function $f$ with real values $\mathbb{R}$ defined on the space $(X \times Y, \mu \times \nu)$. We will assume that $f$ is integrable, this is not restriction because as it was mentioned before the target space in our considerations could be arbitrary - we even can suppose that it is interval $[0, 1]$. We need in assumption $f \in L^1$ in order to apply ergodic theorem. Let $D_f$ is that measure on the space of matrices $M_N(R)$ Choose some matrix $\{r_{i,j}\}_{i,j=1}^\infty$ where $r_{i,j} = f(x_i, y_j)$ for some sequences $\{x_i\}$ from $(X, \mu)^N$ and $\{y_j\}$ from $(Y, \nu)^N$. Let the family $\{C_{n,k}\}_{k=1}^\infty$ as before is the basis of sigma-fields of the borel sets in $\mathbb{R}^n$ (say cubes with rational vertices), $n \in \mathbb{N}$. Define two systems of the subsets of naturals $\mathbb{N}$:

$$N_{n,k}^2 = \{m \in \mathbb{N} : \{r_{i,m} = f(x_i, y_m)\}_{i=1}^n \in C_{n,k}\} = \{m \in \mathbb{N} : y_m \in B_{n,k}^2\}$$

and

$$N_{n,p}^1 = \{m \in \mathbb{N} : \{r_{m,j} = f(x_m, y_j)\}_{j=1}^n \in C_{n,p}\} = \{m \in \mathbb{N} : x_m \in B_{n,p}^1\}$$

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where, $B_{n,k}^2(B_{n,k}^1)$ are preimages of the set $C_{n,k}$ under the maps $Y \ni y \mapsto (f(x_1, y) \ldots f(x_n, y)) \in \mathbb{R}^n$
(resp. $X \ni x \mapsto (f(x_1, y), \ldots f(x_n, y)) \in \mathbb{R}^n$). By lemma the system of sets $B_{n,k}^2(B_{n,k}^1)$
generated the basis in the spaces $(Y, \nu)$ $[(X, \mu)]$ and consequently the system of rectangles
$
\{B_{n,p}^1 \times B_{m,k}^2\}_{n,m,p,k=1}^{\infty}$ is a basis in the space $(X \times Y, \mu \times \nu)$.

By ergodic theorem for almost all sequences $\{(x_i, y_j)\}$ from the space $(X \times Y)^N$, with
respect to measure $(\mu \times \nu)^N$ the following limits exist:

$$\lim_{T \to \infty} T^{-1} |N_{n,k}^1 \cap [0, T]| = \mu(B_{n,k}^1), \quad \lim_{T \to \infty} T^{-1} |N_{n,k}^2 \cap [0, T]| = \nu(B_{n,k}^2),$$

$$\lim_{T \to \infty} T^{-2} \sum_{(i,j) \in W_T} f(x_i, y_j) = \int_{B_{n,k}^1 \times B_{n,p}^2} f(x, y) d\mu(x) d\nu(y)$$

where summation (in the LHS) runs over all pairs from the set $W_T = \{(i, j) : i, j = 1 \ldots T, i \in N_{n,k}^1, j \in N_{n,p}^2\}$.

Because for almost all choice of the sequences $\{x_i, y_j\}$ all countable many of equations are valid we can find the values of the integrals of the function $f$ over all sets from the basis in $(X \times Y, \mathcal{B}_X \times \mathcal{B}_Y, \mu \times \nu)$ using the values of that function $f(x, y_j)$ at the countable set of points or simply using the matrix $\{r_{i,j}\} \in M_N(\mathbb{R})$ from the set of $D_f$-measure one. Because each measurable function is defined $\text{mod}0$ uniquely by thier integrals over the measurable sets of some basis we have restored a function $f$.

Suppose now that we have two measurable pure functions $f_1, f_2$ on the spaces $(X \times Y, \mu \times \nu)$ and $(\bar{X} \times \bar{Y}, \bar{\mu} \times \bar{\nu})$ with the same invariant measure $D_{f_1} = D_{f_2} = D$ as a measure in the space of matrices. We choose one matrix $\{r_{i,j}\}$ from the set of full $D$-measure. The familly of borel sets $C_{n,k}$ in $\mathbb{R}^n$ defined as above generates the basis of rectangles $B_{n,k}^1 \times B_{n,p}^2$ in $(X \times Y)$ and correspondingly the basis of rectangles $B_{n,k}^1 \times B_{n,p}^2$ in $(\bar{X} \times \bar{Y})$. The measures of corresponding rectangles are coincided in both spaces because of our equations.

So the correspondence between rectangles $B_{n,k}^1 \times B_{n,k}^2$ and $\bar{B}_{n,k}^1 \times \bar{B}_{n,k}^2$ defines unique isomorphism between $(X \times Y, \mu \times \nu)$ and $(\bar{X} \times \bar{Y}, \bar{\mu} \times \bar{\nu})$. That isomorphism must be of type $S_1 \times S_2$ because it preserves the rectangles, here $S_1 : (X, \mu) \to (\bar{X}, \bar{\mu})$ and $S_2 : (Y, \nu) \to (\bar{Y}, \bar{\nu})$. The second group of equations gives the equality of the integrals of the functions $f_1$ and $f_2$ over corresponding rectangles because we used the same matrix $\{r_{i,j}\}$ from the set of full $D$-measure in the cases of both functions. So, the isomorphism above brings the function $f_1$ to function $f_2$ because their integrals over corresponding rectangles from our basis are coincided and we have $f_1(x, y) = f_2(S_1 x, S_2 y)$.
3.3 Symmetries of the pure functions

Speaking on classification of the functions of several arguments it is naturally to say about the group of symmetries of function. This question is tightly connected with the properties of the map $F_j$. Pure functions by definition have no symmetries of type $f(x, y) = f(Sx, y)$, where $S$ is a measure preserving transformation of the space $(X, \mu)$. But it is possible to have the following property: $f(S_1 x, S_2 y) = f(x, y)$, where $(S_1, S_2) \in \mathcal{A}(X, \mu) \times \mathcal{A}(Y, \nu)$. In this case $S_2$ is uniquely defined by $S_1$ (or vice versa) because if $(S_1, S_2)$ and $(S_1, S_2')$ both are symmetries of the pure function $f$ then $(Id, S_2^{-1}S_2')$ also is the symmetry which contradict to the purity of $f$. So we have two groups $G_1 \subset \mathcal{A}(X, \mu), G_2 \subset \mathcal{A}(Y, \nu)$ and isomorphism $\tau : G_1 \rightarrow G_2$, all the symmetries are of the type

$$f(Sx, \tau(S)y) = f(x, y), S \in G_1.$$ 

Consequently, the symmetries has the map $F_j; F_j^\infty$.

**Lemma 4** The partitions on the levels of constancy of the map $F_j - F_j^{-1}(e)$ in the space $(X^N \times Y^N, \mu^N \times \nu^N)$ coinside with the partition on the ergodic components of the group \{S, $\tau(S); S \in G_1\}^N$ Or in the naive form: if $F_j(\{x_i\}, \{y_j\}) = F_j(\{x'_i\}, \{y'_j\})$ then $x'_i = Sx_i$ for all $i$ and $y'_j = \tau(S)y_j$ for all $j$.

We will not prove this lemma because the proof is routine.

**Corollary 2** The map $F_j$ is univalent iff $f$ has no symmetries.
4 Properties of matrix distributions as invariant measures on the space of matrices.

4.1 Distributions of the columns and rows

In this section we will obtain the characteristic properties of matrix distributions as the measures in $M_N(R)$. We will see that those measures are very degenerated with comparison with other $S_N \times S_N$-invariant measures. Even the metric type of the action of that group is special. From technical point of view sometimes it is more convinient to use also actions of one-sided shifts instead action of the group $S_N$.

**Definition 3** The measure on $M_N(R)$ called stationary if it is invariant with respect to the horizontal and vertical shifts: the horizontal shift $H$ is erasing of the first column: $\{H(r)\}_{i,j} = r_{i,j+1}$ and vertical shift is erasing of the first row: $\{V(r)\}_{i,j} = r_{i+1,j}$.

Makes sense to mention that in all our setting about invariant measures in the space of matrices one could consider the matrices with four infinities - $\{\{r_{i,j}\}_{(i,j)\in \mathbb{Z}^2}\} = M_{2}(R)$ instead of $M_N(R)$. In terms of functions it means that the numeration of the sequences of arguments is also two-sided: $\{x_i\}_{i\in \mathbb{Z}}, \{y_j\}_{j\in \mathbb{Z}}$. That approach is more natural form dynamical point of view becuse in that case we have a group of shifts - $\mathbb{Z}^2$ on $M_{2}(R)$ instead of semigroup $\mathbb{Z}_+^2$ in $M_N(R)$, but there are no serious differences for us in between two cases and we will follow to our previous denotations.

**Lemma 5** Each $S_N \times S_N$-invariant measure on the space $M_N(R)$ is stationary measure. Action of $S_N$ on the set of row (columns) is ergodic iff vertical (horizontal) shift is ergodic.

**Proof.** Consider a matrix $\{r_{i,j}\}_{i,j=1}^{\infty}$ as sequence of $\{\bar{r}_i\}_{i=1}^{\infty}$ where $\bar{r}_i$ is the $i$-th column of matrix $\{r_{i,j}\}$. So we are looking on the matrix as a sequence of columns and we can apply well-known de Finetti theorem which claimed that $S_N$-invariant measure on the space of sequences with arbitrary values (in our case $R^N$) is a mixture of product-measures which is invariant under horysontal shift. The same is true for the rows. So we can apply ergodic theorem for both shifts simultaneously.

**Remark 2** Both vertical and horizontal shifts are Bernoulli shifts because they are factors of true Bernoulli shifts in $(X \times Y, \mu \times \nu)^N$ (vertical is a factor-shift of the shift in $(X, \mu)^N$ and horizontal - in $(Y, \nu)^N$). But simultaneous action of both factorshifts as well as action of group $S_N \times S_N$ in $(M_N(R), D_f)$ are of very intriguing type.
The lemma of section 2 allows us to define important isomorphisms.
Define the following maps are the isomorphisms between the corresponding measure spaces:

\[ L_{\{x_i\}} : (Y, \nu) \rightarrow (\mathbb{R}^N, \nu_{\{x_i\}}) \quad L_{\{x_i\}}(y) = \{f(x_i, y)\}_{i \in \mathbb{N}}; \]

resp.

\[ L_{\{y_j\}} : (X, \mu) \rightarrow (\mathbb{R}^N, \mu_{\{y_j\}}) \quad L_{\{y_j\}}(x) = \{f(x, y_j)\}_{j \in \mathbb{N}}, \]

and

\[ L_{\{y_j\}} \times L_{\{x_i\}} : (X \times Y, \mu \times \nu) \rightarrow (\mathbb{R}^N \times \mathbb{R}^N, \mu_{\{y_j\}} \times \nu_{\{x_i\}}) \quad L_{\{y_j\}} \times L_{\{x_i\}}(x, y) = (\{f(x, y_j)\}, \{f(x_i, y)\}). \]

The interpretation of the measures \( \nu_{\{x_i\}} \) \( \mu_{\{y_j\}} \) as a measure on \( \mathbb{R}^N \) is very simple - this is nothing more than a joint distribution of the infinite sequence of functions \( \{f(x_i, y)\} \) \( \{f(x, y_j)\} \).

**Corollary 3** 1. For almost all with respect to measure \( \mu_N \) (resp. \( \nu_N \) and \( \mu_N \times \nu_N \)) sequences \( \{x_i\}_{i \in \mathbb{N}} \) \( \{y_j\}_{j \in \mathbb{N}} \) and \( \{x_i, y_j\} \) the maps \( L_{\{x_i\}}, L_{\{y_j\}}, L_{\{y_j\}} \times L_{\{x_i\}} \) are the isomorphisms between the corresponding measure spaces.

2. The function of two variables \( \tilde{f} \) on the space \( (\mathbb{R}^N \times \mathbb{R}^N, \mu_{\{y_j\}} \times \nu_{\{x_i\}}) \) which is defined by formula

\[ \tilde{f}(\{f(x, y_j)\}_{j \in \mathbb{N}}, \{f(x_i, y)\}_{i \in \mathbb{N}}) = f(x, y) \]

is isomorphic (mod0) to the function \( f \) (defined on the space \( (X \times Y, \mu \times \nu) \)) as the functions of two variables because \( D_{\tilde{f}} = D_f \).

All assertions followed from the lemma above. The function \( \tilde{f} \) is a lifting of the function \( f \) to the space of matrices equipped with the product of conditional measures. We will define it in direct terms below (Theorem 4).

Using this corollary and the theorem we can analyze more deeply the properties of matrix distributions \( D_f \).

### 4.2 Matrix distribution is a simple measure in the space \( \mathcal{M}_N(\mathbb{R}) \)

We will consider now the actions on the measure space \( (\mathcal{M}_N(\mathbb{R}), D_f) \) of the first and second summands - \( S_N \times 1 \) and \( 1 \times S_N \) of the group \( S_N \times S_N \). The first subgroup acts “vertically” - as the permutations of the rows and the second - group acts “horizontally” - as permutations of columns. Next theorem claims that both actions are far to be ergodic but the action of its product does ergodic as we mentioned. As any actions of countable group we can correctly decompose this action, or more exactly can decompose the space \( (\mathcal{M}_N(\mathbb{R}), D_f) \), on the ergodic components: this means that there is an invariant with respect to action of
the group a measurable partition such that on almost all of its elements the action of the
group is ergodic. Denote the partition on the ergodic components of the action of subgroup
$S_N \times 1$ as $\xi_V$ and subgroup $1 \times S_N$ as $\xi_H$. As it follows form lemma 4 the same ergodic
decompositions have correspondingly the vertical and horizontal shifts.

The definition of those partitions make sense for arbitratry $S_N \times S_N$-invariant measure
and we preserve the denotation for the general case of the measures in $M_N(R)$. For matrix
distributions we have very special property of those partitons which shows in what sense the
matrix distribution is degenerated measure:

**Theorem 3** Let $f$ is a measurable function of two variables with values in borel space $R$ and
measure $D_j$ on $M_N(R)$ is its matrix distribution. The product (supremum) of measurable
partitions $\xi_V$ - on the ergodic components of the actions of $S_N \times 1$, and $\xi_H$ - on the ergodic
components of the actions $1 \times S_N$ is (mod0) the partition of the space $(M_N(R), D_j)$ on the
separate points (mod0) (or $e$ in traditional denotation of ergodic theory):

$$\xi_V \vee \xi_H = e,$$

moreover, these partitions are independent

**Definition 4** Let us call an arbitrary $S_N \times S_N$-invariant ergodic measures $\alpha$ in $M_N(R)$ a
simple measure if the condition $\xi_V \vee \xi_H = e$ satisfies and those partitions are independent.

**Proof.** The assertion of the theorem is trivial if the map $F_j : (X \times Y, \mu \times \nu)^N \rightarrow
(M_N(R), D_j)$ is univalent (has no multiplicties). In this case $\xi_V$ and $\xi_H$ are independent
and $\xi_V \vee \xi_H = e$ because the same is trivially true in the space: $(X \times Y, \mu \times \nu)^N$ where
preimages of $\xi_V, \xi_V$ are the partitions on the first and second coordinates. So we must
consider the case when $F_j$ is not univalent and consequently $f$ has non trivial symmetries.
(see corollary in 3.3.) The proof below uses a special structure of the elements of the partition
$\xi_V$ and $\xi_H$ in $(M_N(R), D_j)$, this analysis is useful for the studying of the general picture. It
is possible to avoid this and to use more abstract arguments.

We can reformulate the claim of the theorem means that the sigma-field of the measure
space $(M_N(R), D_j)$ is a sigma-field generated by the union of basis of each partitions, or in
another words: intersection of $D_j$-almost all element of the partition $\xi_V$ with $D_j$-almost all
element of partition $\xi_H$ is at most one point. We will prove the last formulation.

Let us examine the matrix $r = \{f(x_i, y_j)\} \in M_N(R)$ from the set of full measure and
consider the element of the partitions $\xi_V$ which contains that matrix -denote it as $C_V^r$, let a
conditional measure of this element is some measure $D_j^V(r)$. (analogously for $\xi_H$: the element
of partition $\xi_H$ which contained point $r$ is $C_H^r$ and conditional measure on that element is
$D_j^H(r)$). The measure $D_j^V(r)$ could be considered of course also on the space $M_N(R)$ and
by definition it is ergodic with respect to the action of \( S_\mathbb{N} \times 1 \) on space of the columns - we again consider matrices as sequence of columns. Then by theorem of de Finetti this conditional measure is the infinite product of some measure on the space of columns e.g. on the space \( \mathbb{R}^N \). Applying ergodic theorem (or martingale convergence theorem) to the group \( S_\mathbb{N} \) as for inductive limit of the finite group acting on the spaces of columns we can say that that measure \( D^V_f (r) \) is a weak limit of empirical measures for the sequence of columns of the matrix \( r \). But this is noting more than the distribution of the one column (all columns have the same distributions) or joint distribution of vector-function \( \{ f(x_i, \cdot) \}_{i=1}^\infty \) as function of the second argument with fixed sequence of the first argument: \( \{ x_i \} \) or -just the measure \( \nu_{\{ x_i \}} \) on \( \mathbb{R}^N \) which we had defined in the corollary 1. Consequently we have \( D^V_f (r) = \nu_{\{ x_i \}}^N \).

Analogously, conditional measure \( D^H_f (r) \) on the element \( C_H^\prime \) of partition \( \xi_H \) is ergodic with respect to the action of horizontal action of the group \( 1 \times S_\mathbb{N} \) (on rows) and we obtain that \( D^H_f (r) = \mu_{\{ y_j \}}^N \).

Now we use the previous corollary. Because for given (but for \( \mu^N \)-almost all) sequence \( \{ x_i \}_{i \in \mathbb{N}} \) we have isomorphism \( L_{\{ x_i \}} \) between \( (Y, \nu) \) and \( (\mathbb{R}^N, \nu_{\{ x_i \}}) \) then infinite product \( L_{\{ x_i \}}^N \) defined the isomorphism between \( (Y, \nu)^N \) and element \( (C_V^\prime, D_f^V (r)) \) of the partition \( \xi_V \). The same is true for the elements of \( C_H^\prime \). Consequently, we have equality for conditional measures:

\[
D^V_f (r) = \nu_{\{ x_i \}}^N \text{ for the same reason almost all element } C_H^\prime \text{ of } \xi_H \text{ with conditional measure } D^H_f (r) \text{ is isomorphic the to space } (X, \mu)^N \text{ as } S_\mathbb{N} \text{-spaces and } D^H_f (r) = \mu_{\{ y_j \}}^N.
\]

So we have proved that each elements of the partitions \( \xi_V, \xi_H \) as the measures spaces \((C_V^\prime, D_f^V (r))\) and \((C_H^\prime, D_f^H (r))\) is a monomorphic \((mod\) image under projection \( F_f \), of the corresponding spaces \((\{ x_i \}_{i=1}^\infty \times Y^N, \nu^N)\) (the set of \( \{ x_i \}_{i=1}^\infty \) is fixed, and \((X^N \times \{ y_j \}_{j=1}^\infty, \mu^N)\) (where sequence \( \{ y_j \}_{j=1}^\infty \) is fixed) So, preimage under the map \( F_f : (X \times Y) \to M_\mathbb{N}(R) \) (see section 2) of the element \( C_V^\prime \) is the set \((\{ x_i \}_{i=1}^\infty \times Y^N) \subset (X^N \times Y^N)\) and preimage of the \( C_H^\prime \) is the set \((X^N \times \{ y_j \}_{j=1}^\infty) \subset (X^N \times Y^N)\) Consequently they can intersect only by one point as it happened in the space \((X \times Y)^N\) - which the point \((\{ x_i \}_{i=1}^\infty \times \{ y_j \}_{j=1}^\infty)\). Finally the intersection of the element of \( \xi_V \) generated by sequence \( \{ x_i \} \) and element of \( \xi_H \) generated by sequence \( \{ y_j \} \) is just one point - the matrix \( \{ f(x_i, y_j) \} \).

**Corollary 4** The following formula gives two integral decompositions of the measure \( D_f \) by conditional measures:

\[
D_f = \int \nu_{\{ x_i \}} d\mu^N(\{ x_i \}) = \int \mu_{\{ y_j \}} d\nu^N(\{ y_j \})
\]

Consequently, the matrix distribution is defined uniquely by system of measures \( \nu_{\{ x_i \}} \) and measure \( \mu \) or with system \( \mu_{\{ y_j \}} \) and measure \( \nu \). Moreover applying ergodic theorem we can restore a matrix distribution using only one typical measure \( \nu_{\{ x_i \}} \) or \( \mu_{\{ y_j \}} \).
Remark 3 1. That fact that restriction of the map $F_I$ on the elements of those subsets is monomorphism (mod0) of course does not mean that $F_I$ is monomorphisms (mod0) of whole space $(X \times Y, \mu \times \nu)^N$ onto $(M_N(R), D_I)$ as we had mentioned. 2. The partitions $\xi_V$ and $\xi_H$ are mutually complementary as it was proved but it does not mean in generally that they are independent.

The definition of the simple measures could be reformulated as follows. Suppose we have two groups $G_1, G_2$ (it is possible that $G_1 = G_2$) and the actions of its on the spaces $(X_1, \mu_1$ and $X_2, \nu_2$ correspondingly. So called space-product of these actions is the natural action of $G_1 \times G_2$ in $(X_1 \times X_2, \mu \times \nu)$. So in previous theorem we have proved that the action of the $S_N \times S_N$ in $(M_N(R), D_I)$ is the space-product of two natural actions of $S_N$. (A natural action of $S_N$ is the action in a product-space, say $Z^N$ with product-measure $\gamma^N$). This is not evident from initial definition on matrix distributions.

Let us consider now quotient spaces of $M_N(R)$ over $\xi_V$, and $\xi_H$. It is clear from the previous consideration that the points of $M_N(R)/\xi_V$ could be identified with measures $\nu_{\{x_i\}}$ and of $M_N(R)/\xi_H$ - with $\mu_{\{y_j\}}$ -distributions of the columns (rows) when the sequence $\{x_i\},\{y_j\}$ is fixed. The quotient measures $\alpha_{x_iV}$ (resp $\alpha_{x_iH}$) is a “measure on measures” or distribution on the set of measures on $R^N$. Of course all information about $\alpha$ could be restored from $\alpha_{x_iV}$ or $\alpha_{x_iH}$, or even from $\nu_{\{x_i\}}(\mu_{\{y_j\}})$ for typical $\{x_i\},\{y_j\}$). We will see that these measures far form arbitrary measures $R^N$. In order to describe all (not necessary simple) $S_N \times S_N$-invariant measures on $M_N(R)$ it is enough to describe all measures on the measures on $R^N$ which are invariant with respect to natural action of one copy of $S_N$ on the space of all measures on $R^N$. In a sense this is de Finetti problem of the second level. It is not clear if these reformulations gives a simplification of the problem.

The following theorem gives a charaterization of the matrix distributions as an invarinat measure in $M_N(R)$.

Theorem 4 Each $S_N \times S_N$-invariant ergodic simple measure $\alpha$ on the space of matrices $M_N(R)$ is a matrix distribution for some measurable function $f$ of two variables with values in $R$ in another words - $D_I = \alpha$.

Proof. Starting from the measure $\alpha$ we must construct the space $(X \times Y, \mu \times \nu)$ and a function $f$ on it with the given matrix distribution $D_I = \alpha$. We will start with some matrix $r = \{r_{i,j}\} \in M_N(R)$, and our construction depends on the choice of $r$. But for $\alpha$-almost all various choices of $r$ we obtain the isomorphic functions. The conditions on $r$ in the existence of some ergodic limits do exist. Namely, let the weak limits in $R^N$ (now $R = R$) of the empirical distributions of the columns and rows which do exist because of invarinace $\alpha$ under the shifts denotes as measure $\alpha_{\xi_V}$, and $\alpha_{\xi_H}$. Also we suppose that for $r$
the limit of empirical distribution of whole matrix does exists (and equal to \( \alpha \)) - all this true for the set of full measure \( \alpha \) (shortly “\( r \) is generic”)

So the space which will plays role of the space \((X \times Y, \mu \times \nu)\) will be direct product \((R^N \times R^N, \alpha_V \times \alpha_H)\) The measures depend on matrix \( r \) but of course for almost all points which belong to the same ergodic component are the same. In order to distinguish the columns and rows we will denote the first space \( R^N_V \equiv X_V \) and the second as \( R^N_H \equiv X_H \). Now we will define a needed function. In order to determinie it we will be defined firstly some new measure on \( R^N \times R^N = X_V \times X_H \) which will be absolutely continuous with respect to \( \alpha_V \times \alpha_H \) and our function will nothing more than Radon-Nikodim derivative of that measure. The measure will be defined on the basis of space \( X_V \times X_H = R^N \times R^N \) which is the product of basis of cylindric sets in the space \( R^N \). Choose two cylindric sets \( \{C_{n,k}\}_{k=1}^\infty \) is a countable basis of the borel sets in the space \( R^n \), \( n = 1 \ldots \); define the sets of naturals

\[
N^2_{n,k} = \{ j \in \mathbb{N} : \{r_{i,j}\}_{i=1}^n \in C_{n,k} \}
\]

and

\[
N^1_{n,p} = \{ i \in \mathbb{N} : \{r_{i,j}\}_{j=1}^n \in C_{n,p} \}
\]

where \( n, k, p \) run over \( \mathbb{N} \) and \( \{C_{n,k}\}_{k=1}^\infty \) is a basis of borel sets in \( R^n \), \( n \in \mathbb{N} \). Define the

\[
\lim_{T \to \infty} T^{-2} \sum_{W_T} r(i,j) \equiv \alpha_r(\bar{C}_{n,k} \times \bar{C}_{n,p})
\]

where \( W_T = \{(i,j) : i,j = 1 \ldots T, i \in N^1_{k,n}, j \in N^2_{p,n}\} \) and \( \bar{C}_{n,k} \) is the cylinder in \( R_N \) which is defined by the borel set \( C_{n,k} \).

Consider \( \alpha_r \) as a (nonpositive generally speaking) measure on the the basis of the cylindric sets. It is clear from definition of it and sets \( N^1_{k,n}, N^2_{p,n} \) that the condition of absolutely continuity with respect to product measure \( \alpha_V \times \alpha_H \) is fulfilled. Define now a function on \( X_V \times X_H \):

\[
f_r = d\alpha_r / d\alpha
\]

as Radon-Nikodim derivative of the measure \( \alpha_r \). This is a measurable function of two variables on \( R^N \times R^N \) (product of the space of columns and space of rows) and we claim that the matrix distribution of this function - \( D_f \) - is equal to \( \alpha \). In order to prove this we will use that fact that \( \alpha \) is a simple measure and that the space \( X_V \), and \( X_H \) are nothing more than the elements of the partitions \( \xi_V \), and \( \xi_H \) correspondingly. The element of the partition \( \xi_V \) which contains the matrix \( r \) is \( C_V \) and it is the set of random matrices with independent columns which have the distribution \( \alpha_V \), the same is true for rows and elements \( C_H \) and distribution \( \alpha_H \). So we can consider its as a set of independent functions of one arguments \( f_r(x_{i,j}) \) on \( X_V \) and \( f_r(., y_{j}) \) on \( X_H \), the values of function \( f_r \) on \( i-th \) column and \( j-th \) row is \( r_{i,j} \). So we define the map (see section 3) \( F_{f_r} : ((X_V)^N \times (X_H)^N) \to M_N(R) \) for special set
of \( \{x_i\} \) and \( \{y_j\} \) the image will be conditional measures on the elements \( C^r_V \) and \( C^r_H \) namely \( \alpha^r_V \) and \( \alpha^r_H \). But in order to restore a whole map \( F_J \) we can apply the action of \( S_N \times S_N \) in the domain and target. The map \( F_J \) commutes with that action, and because \( r \) is generic the \( S_N \times S_N \)-invariant envelope of the measure \( \alpha^r \times \alpha^r \) is the measure \( \alpha \). So \( D_{f_r} = \alpha \). 

If we have started from the beginning from the measure \( \alpha = D_f \) where \( f \) is some measurable function and \( r_{ij} = f(x_i, y_j) \) with the sequences \( \{x_i\}, \{y_j\} \) of independent variables, then our construction coincides with the definition from the corollary 1 (item 2):

\[
f_r(\{z_i\}, \{u_j\}) = \frac{d\alpha_r}{d\alpha}(\{z_i\}, \{u_j\}) = \tilde{f}(\{f(x_i, y_j)\}_{j \in N}, \{f(x_i, y_j) \in N\}) = f(x, y)
\]

where \( z_i = f(x_i, y), u_j = f(x, y_j) \). In another words, the construction with Radon-Nikodim derivative allows to avoid assumption that \( \alpha = D_f \) -we found the need \( f \) using only simplicity of the measure \( \alpha \).

This construction is far generalization of the following representation of the function of one variable: each such univalent function is isomorphic to the function \( f(t) = t \) on the real line \( \mathbb{R} \) with some measure borel measure.

The conclusion of last two theorems is the following. We have define sequence of the operations:

[Starting from \( r \) which is \( \alpha \)-generic point of \( M_N(\mathbb{R}) \) \( \rightarrow \) [function \( f_r \) on the space \( \mathbb{R}^N \times \mathbb{R}^N \) (product of the space of columns and space of rows)] \( \rightarrow \) [\( D_{f_r} \)].

We had obtained a characteristic property of matrix distributions as the measures on \( M_N(\mathbb{R}) \). In another words: the operations above are defined for an arbitrary stationary measures \( \alpha \); what we had proved is that fixed point of this sequence of operations is the simple measure only: we can obtain a measure \( \alpha \) as \( D_{f_r} \) iff we had started with simple measure \( \alpha \). In particular product-measure is not simple: if \( \alpha \) is of product-type then \( f_r \equiv const \) and \( D_{f_r} \) will be delta measure on some constant matrix.

### 4.3 Additional properties of the distributions of rows and columns

For any measurable functions \( f \) of two arguments the joint distribution of the sequence of functions \( \{f(x_i, \cdot)\} \) of the second arguments for infinite sequences of independent values of \( x_i \) - which is measure \( \nu_{\{x_i\}} \) is not arbitrary measure on \( \mathbb{R}^N \). These measures have some special properties. For example these measures can not be infinite product measures. Roughly speaking measurability of the function \( f \) imposed some properties like compactness on the measures \( \nu_{\{x_i\}} \). It is more convenient to consider \( \{f(x_i, \cdot)\} \) as a sequence of random variables with values in the some Polish space \( R \) with metric \( \rho \). In this case we can define a metric in
the space of random variables with values in $R$:

$$d_\rho(\eta, \zeta) = \mathbb{E} \frac{\rho(\eta, \zeta)}{1 + \rho(\eta, \zeta)}.$$  

(Expectation $\mathbb{E}$ with respect to join distribution of random variable $\eta, \zeta$).

**Lemma 6** For $\mu^N$-almost all sequences $\{x_i\}$ random the sequence $\eta_i = f(x_i, \ldots)$ has the following property: for any $\delta$ there exists $n$ and set of naturals $N' \subset \mathbb{N}$ of the density more than $\delta$ such that for each $m \in N'$

$$\min_{i=1, \ldots, n} d_\rho(\eta_m, \eta_i) \leq \delta$$

We omit the proof of this lemma which actually uses only ergodic theorem and an approximation of measurable function by sequence of the stepfunctions.

Easy to check that the following property takes place.

**Corollary 5** The entropy of action of the group $\mathbb{Z}^2$ as horizontal and vertical shifts on $M_N(R)$ with simple measure equal to zero.

A very interesting problem is to study the special properties of joint distributions of $\{f(x_i, \ldots)\}$ or those measures $\nu_{\{x_i\}}$ for distinguish classes of functions. For measurable function the condition of the lemma is closed to sufficient. Perhaps in the theory of functions that question had been considered - Under what conditions we can construct a measurable (smooth, integrable and so on) function of two starting with joint distribution of the countable sequence of function of one variable which used to be a sections of unknown function?

We will add a few words about arbitrary invariant measures in $M_N(R)$ of general type - we will return to this in the next paper.

**Definition 5** A $S_N \times S_N$-invariant measure $\alpha$ called product-type if the following equivalent conditions do satisfy: 1. Action of both components of the group $S_N \times S_N$ is ergodic;

2. Vertical and horizontal shift are ergodic;

3. Measure $\alpha$ is Bernoulli measure which means that this is a product measure on $M_N \equiv R^{Z^2}$.

All claims easily follows from simple variant of de Finetti’s theorem.

So product-type measure is defined by one distribution on $R$ which is distribution of all entries of random matrix which are all independent. As we had proved product-type measures could not be a matrix distribution for measurable functions.
A general case could be reduced to the two extremal cases -simple and product type. as follow. 1. Let \( \xi = \xi_V \vee \xi_H \) then a quotient measure \( \alpha_\xi \) on \( M_N(R)/\xi \) is simple. The partition \( \xi \) is invariant under the action of \( S_N \times S_N \) on \( M_N(R) \).

2. The quotient (under \( \xi \)) action of \( S_N \times S_N \) is isomorphic to the action with the simple measure, and the actions on the fiber is of product type action, so the general action of \( S_N \times S_N \) on \( M_N(R) \) with ergodic invariant measure is a skew product of two kind of actions - with simple and product-type measures.

This is refining of the result due to Aldous ([1]) which claims that any ergodic \( S_N \times S_N \)-invariant measures is a distribution of the function \( f(\xi_i, \eta_j, \zeta_{i,j}) \) where \( f \) is some measurable function of real arguments and \( \{\xi_i\}, \{\eta_j\}, \{\zeta_{i,j}\} \) are the sequences of independent real random variables.

The details will be done elsewhere.

References


