Conformal Correlation Functions
Frobenius Algebras and Triangulations

Jürgen Fuchs
Ingo Runkel
Christoph Schweigert

Vienna, Preprint ESI 1103 (2001)  
December 3, 2001

Supported by Federal Ministry of Science and Transport, Austria
Available via http://www.esi.ac.at
CONFORMAL CORRELATION FUNCTIONS,
FROBENIUS ALGEBRAS AND TRIANGULATIONS

Jürgen Fuchs 1  Ingo Runkel 2  Christoph Schweigert 2

1 Institutionen för fysik
Universitetsgatan 1
S–651 88 Karlstad

2 LPTHE, Université Paris VI
4 place Jussieu
F–75252 Paris Cedex 05

Abstract
We formulate two-dimensional rational conformal field theory as a natural generalization of two-dimensional lattice topological field theory. To this end we lift various structures from complex vector spaces to modular tensor categories. The central ingredient is a special Frobenius algebra object \( A \) in the modular category that encodes the Moore-Seiberg data of the underlying chiral CFT. Just like for lattice TFTs, this algebra is itself not an observable quantity. Rather, Morita equivalent algebras give rise to equivalent theories. Morita equivalence also allows for a simple understanding of T-duality.

We present a construction of correlators, based on a triangulation of the world sheet, that generalizes the one in lattice TFTs. These correlators are modular invariant and satisfy factorization rules. The construction works for arbitrary orientable world sheets, in particular for surfaces with boundary. Boundary conditions correspond to representations of the algebra \( A \). The partition functions on the torus and on the annulus provide modular invariants and NIM-reps of the fusion rules, respectively.
1 Introduction

Attempts to understand the spectrum of bulk fields in two-dimensional conformal field theories gave rise to the formulation of a mathematical problem: Classify modular invariant torus partition functions. Considerable effort has been spent on this problem over the past 15 years. In theories of closed strings, modular invariance of the partition function guarantees the absence of anomalies in the low-energy effective field theory. Modular invariance is a much stronger property than anomaly freedom, i.e. string theory imposes strictly stronger conditions on the field content than field theory. It comes therefore as a disappointment that there exist modular invariants that obey all the usual constraints – positivity, integrality, and uniqueness of the vacuum – but are nevertheless unphysical (see e.g. [1, 2]). Thus, albeit a mathematically well-posed problem, classifying modular invariants is not exactly what is desired from a physical point of view.

The study of the open string field content of conformal field theories resulted in the formulation of a similar problem [3, 4]: Classify NIM-reps, that is, representations of the fusion rules by matrices with non-negative integral entries. Again, this classification yields (plenty of) spurious solutions that cannot appear in a consistent conformal field theory [5].

Motivated by these observations we pose the following questions: First, what is the correct structure that allows to classify full rational conformal field theories with given Moore-Seiberg data? And second, how does this structure determine the correlation functions of the full CFT, in particular how does one obtain a modular invariant partition function (as the 6-point correlator on the torus) and NIM-reps (as the 6-point correlators on the annulus)? In this note we present a natural structure that allows to construct correlation functions and that can be expected to arise in every RCFT. The structure in question is the one of a symmetric special Frobenius algebra in a modular tensor category. The resulting correlation functions are modular invariant and satisfy factorization rules.

Neither the presence of this structure nor its relevance to the questions above is a priori obvious. Moreover, as the choice of words indicates, to formulate our prescription we need to employ some tools which are not absolutely standard. In particular we work in the context of modular tensor categories. Their relevance to our goal emerges from the fact that they provide a powerful graphical calculus and a convenient basis-free formalization of the Moore-Seiberg data of a chiral conformal field theory, like braiding and fusing matrices and the modular S-matrix. A well-known example of a modular tensor category, which can serve as a guide to the general theory, is the category of finite-dimensional vector spaces over the complex numbers. This category is also relevant to the analysis of two-dimensional lattice topological theory [6], so that one may suspect a relation between those theories and our prescription. Indeed, one of our central observations is that much of the structure of (rational) conformal field theories can be understood in terms of constructions familiar from lattice TFT, provided that one adapts them so as to be valid in general modular tensor categories as well.

The rest of the paper is organized as follows. In Section 2 we provide the necessary background on modular tensor categories and motivate the appearance of Frobenius algebras. In Section 3 we develop the representation theory of these algebras. In Section 4 it is shown how to obtain consistent torus and annulus partition functions from a symmetric special Frobenius algebra. In Section 5 we give a prescription for general correlators that generalizes the one for two-dimensional lattice TFTs in the category of vector spaces. It turns out that the
Frobenius algebra itself is not observable; Morita equivalent algebras give rise to equivalent theories. When combined with orbifold techniques, this fact allows for a general understanding of T-duality in rational CFT. This is discussed in Section 6. Section 7 contains our conclusions.

A much more detailed account of our results, including proofs, will appear elsewhere.

2 Frobenius algebras

As already pointed out, our considerations are formulated in the language of modular tensor categories [7], a formalization of Moore–Seiberg [8] data. A modular tensor category $\mathcal{C}$ may be thought of as the category of representations of some chiral algebra $\mathfrak{A}$, which in turn correspond to the primary fields of a chiral conformal field theory. Accordingly the data of $\mathcal{C}$ can be summarized as follows. The (simple) objects of $\mathcal{C}$ are the (irreducible) representations of $\mathfrak{A}$, and the morphisms of $\mathcal{C}$ are $\mathfrak{A}$-intertwiners. For a rational conformal field theory, the category is semisimple, so that every object is a finite direct sum of simple objects. There is a tensor product $\otimes$ which corresponds to the (fusion) tensor product of $\mathfrak{A}$-representations, and the vacuum representation (identity primary field) provides a unit element $1$ for this tensor product.

The existence of conjugate $\mathfrak{A}$-representations gives rise to a duality on $\mathcal{C}$, in particular each object $V$ of $\mathcal{C}$ has a dual object $V^\vee$. The (exponentiated) conformal weight provides a twist on $\mathcal{C}$: To every object there is associated a twist endomorphism $\theta_V$; for a simple object $V$ corresponding to a primary field of conformal weight $\Delta_V$, $\theta_V$ is a multiple of the identity morphism, $\exp(-2\pi i \Delta_V) \text{id}_V$. Finally, the presence of braid group statistics in two dimensions is accounted for by a braiding, i.e. for every pair $V, W$ of objects there is an isomorphism $c_{V, W} \in \text{Hom}(V \otimes W, W \otimes V)$ that 'exchanges' the two objects. The modular $S$-matrix of the CFT is obtained from the trace of the endomorphisms $c_{V, W} c_{W, V}$ with simple $V, W$.

These data are subject to a number of axioms that can be understood as formalizations of various properties of primary fields in rational CFT (see e.g. appendix A of [9]). Essentially, the axioms guarantee that these morphisms can be visualized via ribbons and that the graphs obtained by their composition share the properties of the corresponding ribbon graphs, and they also include the requirement that the $S$-matrix is invertible. (That one really must use ribbons rather than lines is closely related to the fact that Wilson lines in Chern–Simons gauge theories require a framing.)

A particularly simple example of a modular tensor category is the category of finite-dimensional complex vector spaces. It has a single isomorphism class of simple objects – the class of the one-dimensional vector space $\mathbb{C}$ – and has trivial twist and braiding. In conformal field theory, this category arises for meromorphic models, i.e. models with a single primary field, such as the $E_8$ WZW theory at level 1. It is also the category in which one discusses two-dimensional lattice topological theory, and as we will discuss below, this is not a coincidence.

Central to our construction are certain objects in modular tensor categories that can be endowed with much further structure. To get an idea on the role of these objects, let us collect a few more results from CFT. First, recall that modular invariants are either of 'extension type' or of 'automorphism type', or a combination thereof [10]. A simple example of extension type arises for the $sl(2)$ WZW theory at level 4:

$$Z(\tau) = |\chi_0(\tau) + \chi_4(\tau)|^2 + 2|\chi_2(\tau)|^2. \quad (1)$$
This invariant corresponds to the conformal embedding of $sl(2)$ level 4 into $sl(3)$ at level 1, and can therefore be interpreted as follows: The vacuum sector of the extended theory $sl(3)_1$ gives rise to a reducible sector $A = (0) \oplus (4)$ of the $sl(2)_4$ theory. In category theoretic terms, $A$ is a reducible object of the modular tensor category $\mathcal{C} \equiv \mathcal{C}(sl(2)_4)$. Now $A$ is not merely an object, but comes with plenty of additional structure. First, the operator product provides an associative multiplication on $A$. Second, since in a chiral algebra the vacuum is unique, the $sl(2)_4$-descendants of the vacuum provide a distinguished subobject $(0)$ of $A$. Finally, crossing symmetry of the 4-point conformal blocks of the vacuum on the sphere, which relates the s-channel and the t-channel, results in additional properties of the product on $A$.

The formalization of these properties of $A$ as an object of $\mathcal{C}$ reads as follows [11]. There is a multiplication morphism $m \in \text{Hom}(A \otimes A, A)$ that is associative and for which there exists a unit $\eta \in \text{Hom}(1, A)$. There exists a coassociative coproduct $\Delta \in \text{Hom}(A, A \otimes A)$ as well, along with a co-unit $\varepsilon \in \text{Hom}(A, 1)$. It is convenient to depict these morphisms as follows:

$$m = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (0,1) -- (1,1);
\draw (1,1) -- (1,2);
\draw (1,2) -- (1,3);
\draw (1,3) -- (0,3);
\draw (0,3) -- (0,4);
\draw (0,4) -- (1,4);
\end{tikzpicture}
\end{array} \qquad \Delta = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (0,1) -- (1,1);
\draw (1,1) -- (1,2);
\draw (1,2) -- (1,3);
\draw (1,3) -- (0,3);
\draw (0,3) -- (0,4);
\draw (0,4) -- (1,4);
\draw (1,4) -- (1,5);
\draw (1,5) -- (0,5);
\end{tikzpicture}
\end{array} \qquad \eta = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (0,1) -- (1,1);
\draw (1,1) -- (1,2);
\draw (1,2) -- (1,3);
\draw (1,3) -- (0,3);
\draw (0,3) -- (0,4);
\draw (0,4) -- (1,4);
\draw (1,4) -- (1,5);
\end{tikzpicture}
\end{array} \qquad \varepsilon = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (0,1) -- (1,1);
\draw (1,1) -- (1,2);
\draw (1,2) -- (1,3);
\draw (1,3) -- (0,3);
\draw (0,3) -- (0,4);
\draw (0,4) -- (1,4);
\draw (1,4) -- (1,5);
\end{tikzpicture}
\end{array}
\] (2)

Such pictures are to be read from bottom to top, and composition of morphisms amounts to concatenation of lines; a single line labelled by an object $V$ stands for the identity morphism $id_V \in \text{End}(V) \equiv \text{Hom}(V, V)$. Lines labelled by the tensor unit $1$ can be omitted since $\text{End}(1) = \mathbb{C}$ so that $id_1 = 1$. In (2) we have suppressed the label $A$ which decorates each of the lines of the picture. (Also, throughout the paper the fattening of the lines to ribbons is implicitly understood.)

The (co-)associativity and (co-)unit properties, i.e.

$$m \circ (m \otimes id_A) = m \circ (id_A \otimes m), \quad m \circ (\eta \otimes id_A) = id_A = m \circ (id_A \otimes \eta),$$

$$\Delta \otimes id_A \circ \Delta = (id_A \otimes \Delta) \circ \Delta, \quad (\varepsilon \otimes id_A) \circ \Delta = id_A = (id_A \otimes \varepsilon) \circ \Delta,$$

are then drawn as

$$= \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (0,1) -- (1,1);
\draw (1,1) -- (1,2);
\draw (1,2) -- (1,3);
\draw (1,3) -- (0,3);
\draw (0,3) -- (0,4);
\draw (0,4) -- (1,4);
\draw (1,4) -- (1,5);
\end{tikzpicture}
\end{array} \quad = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (0,1) -- (1,1);
\draw (1,1) -- (1,2);
\draw (1,2) -- (1,3);
\draw (1,3) -- (0,3);
\draw (0,3) -- (0,4);
\draw (0,4) -- (1,4);
\draw (1,4) -- (1,5);
\end{tikzpicture}
\end{array} \quad = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (0,1) -- (1,1);
\draw (1,1) -- (1,2);
\draw (1,2) -- (1,3);
\draw (1,3) -- (0,3);
\draw (0,3) -- (0,4);
\draw (0,4) -- (1,4);
\draw (1,4) -- (1,5);
\end{tikzpicture}
\end{array} \quad = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (0,1) -- (1,1);
\draw (1,1) -- (1,2);
\draw (1,2) -- (1,3);
\draw (1,3) -- (0,3);
\draw (0,3) -- (0,4);
\draw (0,4) -- (1,4);
\draw (1,4) -- (1,5);
\end{tikzpicture}
\end{array}
\] (4)

Further, the crossing symmetry is taken into account by demanding $A$ to be a Frobenius algebra, i.e. to satisfy

$$= \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (0,1) -- (1,1);
\draw (1,1) -- (1,2);
\draw (1,2) -- (1,3);
\draw (1,3) -- (0,3);
\draw (0,3) -- (0,4);
\draw (0,4) -- (1,4);
\draw (1,4) -- (1,5);
\end{tikzpicture}
\end{array} \quad = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (0,1) -- (1,1);
\draw (1,1) -- (1,2);
\draw (1,2) -- (1,3);
\draw (1,3) -- (0,3);
\draw (0,3) -- (0,4);
\draw (0,4) -- (1,4);
\draw (1,4) -- (1,5);
\end{tikzpicture}
\end{array} \quad = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (0,1) -- (1,1);
\draw (1,1) -- (1,2);
\draw (1,2) -- (1,3);
\draw (1,3) -- (0,3);
\draw (0,3) -- (0,4);
\draw (0,4) -- (1,4);
\draw (1,4) -- (1,5);
\end{tikzpicture}
\end{array} \quad = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (0,1) -- (1,1);
\draw (1,1) -- (1,2);
\draw (1,2) -- (1,3);
\draw (1,3) -- (0,3);
\draw (0,3) -- (0,4);
\draw (0,4) -- (1,4);
\draw (1,4) -- (1,5);
\end{tikzpicture}
\end{array}
\] (5)
We also require $A$ to be a *special* Frobenius algebra, i.e. impose that (after suitably fixing the normalization of the co-unit)

$$
\varepsilon \circ \eta = \dim(A) \, id_1 \quad \text{and} \quad m \circ \Delta = id_A,
$$

where $\dim(V)$ stands for the quantum dimension of an object $V$, which is defined as the trace of the identity morphism $id_V$. To account for the uniqueness of the vacuum, one would impose that $\dim \text{Hom}(1,A) = 1$, a property that has been termed *haploidity* of $A$ in [11]. But with an eye on the relation with two-dimensional lattice TFT we prefer to relax this requirement and only require that $A$ is *symmetric* in the sense that the morphisms

$$
\Phi_1 := \quad \Phi_2 :=
$$

from $A$ to its dual $A^\vee$ are isomorphisms and coincide. It can be shown that haploidity implies symmetry, hence this is indeed a weaker requirement. But as it turns out, it still leads to a meaningful theory.

The presence of a braiding allows us to introduce a notion of commutativity: An algebra $A$ is called commutative if $m \circ c_{A,A} = m$. But it is well-known that certain off-diagonal modular invariants like the $D_{\text{odd}}$ series of $sl(2)$ correspond to a hidden chiral superalgebra, which is not commutative. Correspondingly we do *not* impose commutativity of $A$.

Let us now present more examples. The category of vector spaces contains just a single haploid algebra, the ground field $\mathbb{C}$ itself. $\mathbb{C}$ is obviously a special Frobenius algebra. Thus in a sense haploid Frobenius algebras generalize the ground field. Now as already pointed out, instead of haploidity we impose the weaker condition of symmetry, i.e. equality of the two morphisms (7). This proves to be most reasonable, as symmetric special Frobenius algebras in the category of complex vector spaces are known to describe two-dimensional lattice TFTs [6,12, 13]. Since a topological field theory is in particular a (rather degenerate) conformal field theory, it is gratifying that our formalism covers this case. More generally, an algebra that is symmetric but not haploid should be thought of as containing several vacua that constitute a topological subsector of the theory. In fact, this topological sector gives itself rise to the structure of a symmetric special Frobenius algebra $A_{\text{top}}$ over the complex numbers. $A$ is symmetric in the sense introduced above if and only if $A_{\text{top}}$ is a symmetric Frobenius algebra in the usual [14] sense.

For general modular tensor categories the situation is much more involved. But one haploid special Frobenius algebra is always present, namely the tensor unit $1$. The associated torus partition function is the charge conjugation invariant, and the boundary conditions are precisely those which preserve the full chiral symmetry (‘Cardy case’). The construction of correlation functions presented below then reduces to the one already given in [9].

Another large class of examples is supplied by simple currents, i.e. simple objects $J$ of $\mathcal{C}$ of quantum dimension $\dim(J) = 1$. Simple currents (more precisely, their isomorphism classes) form a subgroup of the fusion ring of the CFT that organizes the set of primary fields into
orbits. Haploid special Frobenius algebras all of whose simple subobjects are simple currents can be classified. As objects, they are direct sums of the form

$$A = \bigoplus_{J \in \mathcal{H}} J,$$

with $\mathcal{H}$ a subgroup of the group of simple currents. But not all objects of the form (8) can be endowed with an associative product; such a product exists if and only if the corresponding 6j-symbols yield the trivial class in the cohomology $H^3(\mathcal{H}, \mathbb{C}^\times)$. It follows from proposition 7.5.4 in [15] that this requirement restricts $\mathcal{H}$ to be a subgroup of the so-called ‘effective center’, i.e. to consist of simple currents for which the product of the conformal weight $\Delta_J$ and the order $N_J$ (the smallest natural number such that $J^{\otimes N} \cong 1$), is an integer.

Allowed objects of the form (8) can admit several different products $m \in \text{Hom}(A \otimes A, A)$; the possible products are classified by the second cohomology $H^2(\mathcal{H}, \mathbb{C}^\times)$. It is a fortunate fact about abelian groups that classes in $H^2(\mathcal{H}, \mathbb{C}^\times)$ are in one-to-one correspondence with alternating bihomomorphisms on $\mathcal{H}$, i.e. with maps $\phi$ from $\mathcal{H} \times \mathcal{H}$ to $\mathbb{C}^\times$ that are homomorphisms (i.e. compatible with the product on $\mathcal{H}$) in each argument and that obey $\phi(J, J) = 1$ for all $J \in \mathcal{H}$. Employing this result, the isomorphism class of the algebra structure $m$ can be encoded in the Kreuzer-Schellekens bihomomorphism (KSB) $\Xi$ [16], which graphically is represented as

$$\Xi(J, K) =$$

(Here the twist morphism $\theta_K$ appears; were we drawing ribbons instead of lines, this would amount to a full $2\pi$ rotation of the $K$-ribbon.) The possible KSBs are in one-to-one correspondence with the associative products $m$ on $\mathcal{H}$; different KSBs for a simple current group $\mathcal{H}$ are related by alternating bihomomorphisms. The choice of a KSB has been called a choice of ‘discrete torsion’ in [16]; there are indeed models where the KSB precisely describes discrete torsion. In the case of a free boson compactified on a lattice, the choice of a multiplication on the algebra object that characterizes the lattice corresponds to the choice of a background value of the $B$-field.

We finally quote examples of algebra objects for the exceptional modular invariants of the $sl(2)$ WZW theory. For the $E_6$-type modular invariant at level 10, one has $A = (0) \oplus (6)$, for $E_7$ at level 16, one finds $A = (0) \oplus (8) \oplus (16)$, and for the $E_8$-type invariant at level 28, $A = (0) \oplus (10) \oplus (18) \oplus (28)$ [17].

### 3 Representations

Just like for ordinary algebras, the next step to be taken in the analysis of the algebra $A$ is the study of its representation theory. Precisely as in the case of vector spaces, an $A$-representation
$M$ consists of two data: An object $\hat{M}$ of $\mathcal{C}$, corresponding to the vector space that underlies the module on which $A$ acts, and a representation morphism $\rho_M \in \text{Hom}(A \otimes \hat{M}, \hat{M})$ that specifies the action of $A$ on the module. And further, these data are subject to exactly the same constraint as in the vector space case; pictorially:

$$
\begin{array}{c}
\begin{array}{c}
\overset{\rho_M}{\bullet} \\
A
\end{array}
\begin{array}{c}
\overset{\rho_M}{\bullet} \\
M
\end{array}
\begin{array}{c}
\overset{\rho_M}{\bullet} \\
A
\end{array}
\begin{array}{c}
\overset{\rho_M}{\bullet} \\
M
\end{array}
\end{array}
\begin{array}{c}
= \\
\begin{array}{c}
\overset{\rho_M}{\bullet} \\
A
\end{array}
\begin{array}{c}
\overset{\rho_M}{\bullet} \\
A
\end{array}
\end{array}
$$

(In more detailed terminology, what we get this way is a left $A$-module. Right modules, as well as comodules, can be introduced analogously. One can develop the whole theory equivalently also in terms of those.)

The representations of $A$ are in one-to-one correspondence with conformally invariant boundary conditions. For any algebra $A$ in a modular tensor category, all $A$-modules are fully reducible. Irreducible modules correspond to elementary boundary conditions, while direct sums of irreducible modules describe boundary conditions with non-trivial Chan-Paton multiplicities. Commutative algebras $A$ with trivial twist, i.e. $\theta_A = \text{id}_A$, give rise to modular invariants of extension type. Irreducible modules then fall into two different classes, local and solitonic modules. Local modules can be characterized [17] by the fact that their twist is a multiple of the identity, i.e. that their irreducible subobjects all have the same conformal weight modulo integers. They correspond to boundary conditions that preserve all symmetries of the extension; solitonic representations describe symmetry breaking boundary conditions.

The standard representation theoretic tools, like induced modules and reciprocity theorems, generalize to the category theoretic setting (see e.g. [17, 18, 11]) and allow to work out the representation theory in concrete examples. The case of the $E_6$ modular invariant of the $\mathfrak{sl}(2)$ WZW theory has been presented in [17, 11]; here we briefly comment on the representations of simple current algebras (8). One starts from the observation that every irreducible module is a submodule of some induced module $A \otimes V$. $A$ acts on the induced module $A \otimes V$ from the left by multiplication, i.e. $\rho_{A \otimes V} = m \otimes \text{id}_V$. We only need to consider irreducible $V$, and abbreviate the irreducible $V_i$, with $i$ labelling the primary fields, by $i$. As an object in $\mathcal{C}$, the induced module $A \otimes i$ is just the $\mathcal{H}$-orbit $[i]$ of $i$ under fusion, with the order of the stabilizer $\mathcal{S}_i = \{J \in \mathcal{H} \mid J \otimes i \cong i\}$ of $i$ as the multiplicity:

$$
A \otimes i = |\mathcal{S}_i| \bigoplus_{J \in \mathcal{H}/\mathcal{S}_i} J \otimes i.
$$

To study the decomposition of (11) into irreducible modules, one needs to classify projectors in the endomorphism space $\text{End}_A(A \otimes i)$. This vector space possesses the structure of a twisted group algebra over the abelian group $\mathcal{S}_i$. The twist can be computed explicitly in terms of the KSB (9) and of certain gauge invariant $\mathcal{G}$-symbols. This way one recovers the list of boundary conditions proposed in [19], which was used in [20] to compute the correct B-type boundary states in Gepner models.
4 Partition functions

We now demonstrate how to extract partition functions, i.e. torus and annulus amplitudes, from a given symmetric special Frobenius algebra. To this end we make use the fact that to every modular tensor category one can associate a three-dimensional topological field theory \([8, 7]\). Such a 3-d TFT can be thought of as a machine that associates vector spaces – the spaces of conformal blocks – to two-manifolds, and linear maps between these vector spaces to three-manifolds. When a path integral formulation is available, such as the Chern–Simons theory in the case of WZW models \([21]\), the vector spaces can be thought of as the spaces of possible initial conditions, while the linear maps describe transition amplitudes between given initial and final conditions.

More precisely, to each oriented two-manifold \(\hat{X}\) with a finite number of embedded arcs, which are labelled by primary fields (and certain additional data that will not concern us here, see e.g. \([9]\) for details), the TFT associates a vector space \(\mathcal{H}(\hat{X})\) of conformal blocks. This assignment behaves multiplicatively under disjoint union, i.e. \(\mathcal{H}(X_1 \sqcup X_2) = \mathcal{H}(X_1) \otimes \mathcal{H}(X_2)\), and assigns \(\mathbb{C}\) to the empty set, \(\mathcal{H}(\emptyset) = \mathbb{C}\). To an oriented three-manifold \(M\) containing a Wilson graph, with a decomposition \(\partial M = \partial_+ M \sqcup \partial_- M\) of its boundary into two disjoint components, the TFT associates a linear map

\[
Z(M, \partial_- M, \partial_+ M) : \quad \mathcal{H}(\partial_- M) \to \mathcal{H}(\partial_+ M). 
\]

(12)

These linear maps are multiplicative, compatible with the gluing of surfaces, and obey a few further functoriality and naturality axioms (see e.g. chapter 4 of \([22]\)).

As explained in \([23]\), correlation functions of a full conformal field theory on an arbitrary world sheet \(X\) are nothing but special elements in the space \(\mathcal{H}(\hat{X})\) of conformal blocks on the complex cover \(\hat{X}\) of \(X\). This is an oriented two-manifold doubly covering \(X\), endowed with an anticonformal involution \(\sigma\) such that \(\hat{X}/\sigma = X\). Following the strategy of \([24, 9]\), we determine the correlators using the ‘connecting three-manifold’ \(M_X\), which has the complex cover as its boundary,

\[
\partial M_X = \hat{X}.
\]

(13)

The connecting manifold is obtained as the quotient of \(\hat{X} \times [-1, 1]\) by the involution that acts as \(\sigma\) on \(\hat{X}\) and as \(t \mapsto -t\) on the interval \([-1, 1]\). We can and will identify the image of \(\hat{X} \times \{0\}\) in \(M_X\) with \(X\). The world sheet \(X\) is a retract of the connecting manifold \(M_X\), and hence we may think of \(M_X\) as a fattening of \(X\). Upon choosing an appropriate Wilson graph in \(M_X\), the three-dimensional TFT then provides us with a map

\[
C(X) := Z(M_X, \emptyset, \hat{X}) : \quad \mathbb{C} \to \mathcal{H}(\hat{X}),
\]

(14)

and it is this map that yields the correlators on \(X\).

Let us now have a look at a specific situation, and explain how to obtain a modular invariant torus partition function from a symmetric special Frobenius algebra. The connecting manifold for a torus \(T\) is just a cylinder over \(T\), i.e. \(M_T = T \times [-1, 1]\). In this three-manifold, we consider
the following Wilson graph, labelled by $A$:

$$Z = \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{wilson_graph.png}}
\end{array}
\end{array}
\quad s^1 \times [-1,1] \times s^2 \quad (15)$$

Here (as well as in the figures (16), (18) and (19) below) the top and bottom are to be identified, and the lower trivalent vertex denotes the coproduct and the upper vertex the product of $A$. To determine the coefficients $Z_{ij}$ of the torus partition function, we glue to both sides of (15) two solid tori, containing a Wilson line labelled by $i$ and $j$, respectively. This yields the following Wilson graph in the three-manifold $S^2 \times S^1$:

$$Z_{ij} = \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{wilson_graph_glued.png}}
\end{array}
\end{array}
\quad s^2 \times s^1 \quad (16)$$

It is useful to interpret this result as $Z_{ij} = \text{tr} \ P_{ij}$, where $P_{ij}$ is the linear map given by the same graph in $S^2 \times [0,1]$. One can show that $P_{ij}$ is a projector, which implies that the numbers $Z_{ij}$ are non-negative integers, as befits a partition function. Furthermore, $Z_{\infty}$ equals the dimension of the center of the $\mathbb{C}$-algebra $\text{Hom}(1, A)$; in particular, when $A$ is haploid, then the vacuum sector appears precisely once, $Z_{\infty} = 1$. Finally, one can also prove modular invariance. We illustrate invariance under the U-transformation $\tau \mapsto \frac{\tau}{\tau + 1}$ (the pictures correspond to figure (15) with the interval $[-1,1]$ suppressed):

$$\begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{invariance.png}}
\end{array}
\end{array} \Rightarrow \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{invariance2.png}}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{invariance3.png}}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{invariance4.png}}
\end{array}
\end{array} \quad (17)$$

Here in the first equality the Frobenius property (5) of $A$ is used, while in the second step the periodicity of the graph is taken into account; equality with the original picture then follows by applying the Frobenius property once again. In the case of simple current algebras (8), our result (16) reproduces the formula of [16] that expresses the most general simple current
modular invariant in terms of the KSB and monodromy charges. In agreement with [17], for commutative algebras $A$ with $\theta_A = id_A$ the partition function can be shown to be of pure extension type in which only local modules appear.

For the annulus, a similar reasoning applies. The connecting manifold looks as follows:

\[ A_M^N = \]

It is a solid torus with two Wilson lines along its non-contractible cycle. These Wilson lines have opposite orientation; each of them is is labelled by a (left) $A$-module that characterizes the boundary condition. In addition there are Wilson lines labelled by $A$, which are attached to the modules by trivalent vertices that are given by the respective representation morphisms. Again, the annulus coefficients $A_{iM}^N$, i.e. the coefficients in an expansion of (18) in the characters $\chi_i$, are obtained by gluing, in this case with a single torus containing a Wilson line labelled by $i$. This leads to

\[ A_{iM}^N = \]

As in the torus case, one then shows that the annulus coefficients are non-negative integers. Moreover, we can show that the matrices $A_i$ with entries $(A_i)_{iM}^N := A_{iM}^N$ indeed furnish a NIM-rep of the fusion rules of $C$. It also follows that if the algebra $A$ is haploid and $M = N$ is an irreducible $A$-module, i.e. an elementary boundary condition, then the vacuum $\mathbf{1}$ appears just once in the annulus, i.e. $A_{iM}^M = 1$. From the annuli, one can read off the boundary states, and show that their coefficients provide the ‘classifying algebra’ [26]. Performing a modular transformation, one obtains the annuli in the closed string channel; one can check that only fields $i$ appear that are compatible with the torus partition function (15), and that they appear with the correct multiplicity $Z_{i,iv}$. 

10
5 General amplitudes

The results presented above can be summarized by the statement that every symmetric special
Frobenius algebra gives rise to a modular invariant torus partition function and to a \text{NIM-rep} of
the fusion rules that are compatible with each other. To show that we even obtain a complete
consistent conformal field theory, we should construct all correlators on world sheets $X$ of arbi-
trary genus, including an arbitrary number of boundary components, and show that they are
invariant under the relevant mapping class group and possess the correct factorization prop-
ties. (In the present note, we restrict ourselves to orientable world sheets. The unorientable
case requires additional structure, so as to account for the possibility of having several different
Klein bottle amplitudes.)

The construction of correlators consists in a beautiful combination of the construction of
\cite{24,9} with structures familiar from lattice TFTs in two dimensions. As is already apparent
from our prescription for the torus and annulus amplitudes, what is needed is a Wilson graph
in the connecting three-manifold $M_X$, which besides the Wilson lines that correspond to field
insertions and to the presence of a boundary contains in addition a suitable network of $A$-lines.
The general prescription is the following:

(0) To start, we fix once and for all an orientation of the world sheet $X$. The complex double
$\hat{X}$ of $X$ can then be decomposed as $\hat{X} = X^{+} \sqcup \partial X \sqcup X^{-}$, with the involution acting as $\sigma(X^{+}) = X^{-}$
and $\sigma(\partial X) = \partial X$.

(1) Each bulk insertion point $p_{\ell}$ on $X$ has two preimages on $\hat{X}$, $\hat{p}_{\ell}^{+}$ on $X^{+}$ and $\hat{p}_{\ell}^{-}$ on $X^{-}$, which
are joined by an interval in the connecting manifold $M_X$. We put a Wilson line along each such
interval. The image of $X$ in $M_X$ intersects the interval in a unique point, which we identify
with $p_{\ell} \in X$.

(2) Each component of the boundary of $X$ gives rise to a line of fixed points of $M_X$ under the
$\mathbb{Z}_2$ action. We place a circular Wilson line along each such line of fixed points.

(3) Boundary insertion points $q_{\ell}$ on $X$ have a unique preimage $\hat{q}_{\ell}$ on $\hat{X}$. We join $\hat{q}_{\ell}$ by a short
Wilson line to the image of $q_{\ell}$ in $M_X$. This results in a trivalent vertex on the circular Wilson
line for the relevant boundary component.

(4) We pick a triangulation of the image of $X$ in $M_X$. Without loss of generality, we do this
in the following manner. First, only trivalent vertices may appear. (But we allow for arbitrary
polygonal faces rather than just triangles, which is completely equivalent \cite{25} to the case with
only triangular faces; for brevity we still use the term ‘triangulation’). Moreover, each segment
of the boundary must contain the end point of an edge of the triangulation, and each of the bulk
points $p_{\ell}$ must lie in the interior of a separate face of the triangulation, while each boundary
point $q_{\ell}$ must lie in the interior of a separate edge. Finally, for each bulk insertion we join the
perpendicular bulk Wilson line in $p_{\ell}$ by an additional line to an interior point of an arbitrary
edge of the face to which $p_{\ell}$ belongs. (More precisely, in terms of ribbons, we first bend the
‘horizontal’ ribbon straight up, at a right angle to $X$, so that it points towards $\hat{p}_{\ell}^{-}$ and is parallel
to the vertical bulk ribbon, and then join it to the bulk ribbon.)

This completes the geometric part of our prescription. It is illustrated, in the case of a disk
with three bulk and three boundary insertions, in the following figure.

\begin{center}
\includegraphics[width=0.5\textwidth]{figure.png}
\end{center}

(20)

The remaining task is now to decorate the Wilson lines with labels specifying the corresponding object of the category and choose couplings (morphisms) for the trivalent vertices. Our prescription is inspired by the situation in two-dimensional lattice TFTs and contains that case as a special example.

(5) The short Wilson lines that connect the bulk insertion points \( p_\ell \) to the triangulation are labelled by the algebra \( A \), with the \( A \)-line pointing towards \( p_\ell \).

(6) The three lines that join at any vertex of the triangulation are interpreted as outgoing and are labelled with the algebra object \( A \). The morphisms at the vertices are constructed from the coproduct in \( A \): At every trivalent vertex at which these \( A \)-lines join we put the morphism \( \Delta \circ \Phi_{\text{triv}}^{-1} \), with \( \Phi_{\text{triv}} \) given by (7). (The condition that \( A \) is symmetric ensures that the resulting correlators do not depend on the choice of which of the three lines carries the \( \Phi_{\text{triv}}^{-1} \).) In the lattice case, this prescription corresponds to the assignment of structure constants with only lower indices.

(7) In the case of lattice TFT one also needs a metric. In the general case this corresponds to reversing the direction of the \( A \)-line. Accordingly we place the morphism (7) in \( \text{Hom}(A, A^\vee) \) in the middle of each edge of the triangulation.

(8) To characterize the bulk field inserted at \( p_\ell \), we need two irreducible objects \( j_\ell^\pm \); they label the Wilson lines that start from the points \( p_\ell^\pm \) on \( \hat{X} \). To account for the coupling to the short \( A \)-lines, we need as a third datum for a bulk field a morphism in \( \text{Hom}(A \otimes j_\ell^+, j_\ell^{-\vee}) \).

It turns out that some of the latter morphisms completely decouple, i.e. that every correlator containing a bulk field labelled by such a coupling vanishes. Physical bulk fields therefore correspond to a subspace of the couplings \( \text{Hom}(A \otimes j_\ell^+, j_\ell^{-\vee}) \); its dimension matches the value \( Z_{j_\ell^+ j_\ell^{-\vee}} \) of the torus partition function.

(9) The segments of the circular Wilson line correspond to boundary conditions; they are to be labelled by a module of \( A \). Wilson lines of the triangulation that end on such a boundary result in a trivalent vertex, which requires a morphism in \( \text{Hom}(A \otimes \hat{M}, \hat{M}) \). For this morphism we choose the representation morphism \( \rho_M \).

(10) The last ingredient in our construction are the boundary fields. They have a single chiral label \( k_\ell \), which we use to label the short Wilson line from \( q_\ell \) to \( q_\ell \). The trivalent vertex that is formed by this Wilson line and the two adjacent boundary conditions \( M, N \) requires the choice of a coupling in \( \text{Hom}(M \otimes k_\ell, N) \).
Again, only a subspace of these couplings is relevant. In this case the subspace can easily be characterized: $M \otimes k_\ell$ is a left $A$-module, and only the subspace $\text{Hom}_A(M \otimes k_\ell, N)$, of dimension $A_{k_\ell M}$, of $A$-morphisms gives rise to physical boundary fields that change the boundary condition from $M$ to $N$.

In the example of the disk, the picture then looks as follows (after rearranging, in one of several possible ways, the vertices that result from the presence of $\Phi_1$ and its inverse, so as to eliminate all occurrences of unit and co-unit):

![Diagram](image)

\begin{equation}
(21)
\end{equation}

Topological invariance of a two-dimensional lattice model means that all correlation functions are independent of the choice of triangulation. The same type of arguments as in the topological case can be used to deduce independence from the triangulation also in the CFT case. Concretely, triangulation independence can be reduced (in a dual formulation) to invariance under two local moves, the so-called fusion and bubble moves [6, 25]. These moves look like

\begin{equation}
\longleftrightarrow \quad \text{and} \quad \bigcirc \leftrightarrow
\end{equation}

Our construction amounts to regard the lines in this picture as morphisms and decorate all of them with the label $A$, and to interpret the trivalent vertices as products and coproducts, respectively. The fusion move is, in CFT terms, just crossing symmetry between the s- and t-channel, and hence invariance under this move is guaranteed by the associativity and the Frobenius property (5) of $A$. Similarly, the second the equality (22) is nothing but the second part of the formula (6), i.e. is guaranteed by the fact that $A$ is special. Together with the naturality properties of 3-d topological field theory, the independence from the triangulation implies that the correlators are invariant under the relative modular group of [27]. Furthermore, the correct factorization rule for boundary fields follows directly from dominance properties of the category $\mathcal{C}$. Bulk factorisation requires in addition a surgery move on the connecting three-manifold.
6 Morita equivalence and T-duality

We have shown how to obtain a full conformal field theory from a symmetric special Frobenius algebra $A$ in the modular tensor category of the underlying chiral CFT. The relation between such algebras and full conformal field theories is, however, not one-to-one. Rather, different algebras can describe one and the same conformal field theory. In other words, the algebra $A$ itself should not be thought of as an observable quantity.

This is already familiar from the degenerate realization of our construction that is provided by topological lattice theories. In that case, symmetric Frobenius algebras over the complex numbers with the same number of simple ideals (and thus isomorphic centers) give identical theories [6]. In particular, the algebra can therefore be chosen to be commutative.

The general case is treated as follows. We call two symmetric special Frobenius algebras $A$ and $B$ Morita equivalent iff there exist two bimodules $A M_B$ and $B M'_A$ (the first a left module of $A$ and a right module of $B$, the second a left module of $B$ and a right module of $A$) such that

$$(A M_B) \otimes_B (B M'_A) = A \quad \text{and} \quad (B M'_A) \otimes_A (A M_B) = B.$$  \hfill (23)

Using these relations, one can replace a triangulation of the world sheet $X$ that is labelled by $A$-lines with the dual triangulation labelled by $B$-lines. Moreover, by standard arguments [28] it follows that Morita equivalent algebras have the same representation theory, so that $A$-modules and $B$-modules, and hence boundary conditions, are in one-to-one correspondence. Since the correlation functions do not depend on the triangulation, this implies that our prescription yields, upon a relabelling of fields and boundary conditions, the same correlators when performed with either of the two algebras $A$ and $B$.

These observations also allow for a general derivation of T-duality in rational CFT. Suppose we are given a ‘symmetry’ $\omega$ of the chiral data. Technically, this is a group $G$ together with a functor from $\mathcal{C}[G]$ -- a category whose objects are pairs of objects of $\mathcal{C}$ and an action of $G$ by automorphisms of the objects – to $\mathcal{C}$; it includes in particular a permutation $i \mapsto \omega i$ of the primary fields such that $\omega 1 = 1$, $T_{\omega i} = T_i$ and $S_{\omega i \omega j} = S_{ij}$. It is straightforward to check that along with $Z_{ij}$ also $Z_{i \omega j}$ is a modular invariant partition function. (A familiar realization is the standard T-duality in the CFT of a free boson. The relevant symmetry is the one that reverses the sign of the U(1) charges.) The idea is to associate to the symmetric special Frobenius algebra $A$ that gives the partition function $Z_{ij}$ another algebra $A^\omega$ that yields the partition function $Z_{i \omega j}$. The construction is based on Morita equivalence in the orbifold theory $\mathcal{C}^\omega$ that is obtained by modding out the symmetry $\omega$ on $\mathcal{C}$. One can show that with respect to this orbifold theory, the algebras $A$ and $A^\omega$ give, upon appropriate identification of the fields and of the boundary conditions, isomorphic correlation functions on all surfaces.

7 Conclusions

We have shown how to obtain full rational conformal field theories based on the chiral data that are encoded in a modular tensor category $\mathcal{C}$ from (Morita equivalence classes of) symmetric special Frobenius algebras in $\mathcal{C}$. We gave a general prescription for the construction of correlation functions on orientable surfaces, including surfaces with boundary. We will demonstrate elsewhere that the so obtained correlators give rise to a fully consistent CFT.
Our results open up several lines of research. First, the generalization of these results to unorientable surfaces, a necessary step to a deeper understanding of type I string compactifications, will require more structure that has to account, in particular, for the different possible choices of orientifold projections.

Another important goal is classification. In the framework we propose, the problem of classifying rational conformal field theories can be divided in three independent tasks, each of which deals with a clearly posed (albeit very difficult) problem. The first is to classify modular tensor categories. Second, given a modular tensor category $C$, two distinct problems must be attacked. On one hand we must determine the Morita equivalence classes of symmetric special Frobenius algebras in $C$. Note that this issue can be studied entirely at a `topological’ level, i.e. no analytic properties of chiral blocks are involved. On the other hand, one also needs to get a handle on the control data that allow to completely reconstruct a chiral CFT from its modular tensor category. (The correspondence between chiral CFT and modular tensor categories is not one-to-one. For example, all WZW theories based on $so(2n+1)$ at level 1 with values of $n$ that differ by a multiple of 24 have equivalent tensor categories.) This part of the problem is a purely chiral issue. It is closely related to the classification of chiral algebras, which at present remains the least accessible part of the classification programme.

In order that this programme is complete, we also still must establish that every full rational CFT can be obtained from some symmetric special Frobenius algebra $A$. In all examples known to us -- in particular for all pure extensions as well as for all simple current modular invariants -- this is indeed the case, but at present we cannot give a general proof. What is desirable is a general prescription for reconstructing the algebra object $A$ from some collection of correlation functions of the full CFT. It would also be interesting to see whether this reconstruction is related to other algebraic structures, such as the double triangle algebras [29] which in the work of [30, 31] are regarded as a structure underlying every rational CFT. This would also allow for a comparison between the expressions for correlators obtained there and those which follow from our prescription.

We are confident that the framework we propose is flexible enough to allow for an extension to non-rational conformal field theories. In fact, rationality implies that the category $C$ is semi-simple. Generalizations of TFT for non semi-simple tensor categories have been studied in the literature, compare the recent book [32]. It can be expected that the concepts which underlie these generalizations will play a role in the analysis of non-rational theories. Finally, we point out that our results suggest an intimate relation between the problem of deforming algebras and the problem of deforming conformal field theories.

Acknowledgments:
We are grateful to Giovanni Felder and Albert Schwarz for helpful discussions and comments, and to the Erwin Schrödinger Institute and to Karlstads universitetet for hospitality. IR is supported by the EU grant HPMF-CT-2000-00747.
References


[17] A.A. Kirillov and V. Ostrik, On q-analog of McKay correspondence and ADE classification of \( sl(2) \) conformal field theories, preprint math.QA/0101219

[18] A.A. Kirillov, Modular categories and orbifold models, preprint math.QA/0104242


