Canonical Semigroups of States and Cocycles for the Group of Automorphisms of a Homogeneous Tree

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Supported by Federal Ministry of Science and Transport, Austria
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Introduction.
Canonical states on semisimple groups were defined in the papers [GGV1] [GGV2]
for the group $SL(2, \mathbb{R})$ first and then for $SO(n, 1)$, $SU(n, 1)$ or for all semisimple
groups which do not have Kazdan property T.

The reason was to construct an irreducible representation of the current groups
with values in Lie groups.

Roughly speaking canonical semigroups $\exp(\mu \psi(g)) = \phi^\mu(g)$, $(\mu \geq 0)$ on a group
$G$ is a semigroup of positive definite functions such that $\psi$ is non-positive definite
but conditionally positive definite.

The main (and characteristic) property of the semigroup of canonical states
$\phi^\mu(g)$ is its infinitesimal generator: $\frac{d \phi^\mu}{d \mu}|_{\mu=0} = \log(\phi) = \psi$ which is a conditionally
positive definite function.

In terms of this paper this generator is pure conditionally positive definite function
vanishing on a given maximal compact subgroup and given by the square of the
norm of an unbounded cocycle. At the same time $\psi$ is the derivative of spherical
function $\phi_z$ of complementary series at the point $z = 0$ which corresponds to
the identity.

The representation giving the cocycle cannot be separated from the identity in
the Fell topology. We call these representations infinitesimally small.

Remark that for sufficiently small parameter $\mu > 0$ the unitary representation
of $\phi_\mu$ contains complementary series but for large values of $\mu$ is
square integrable and decomposes by means of the principal series.

It happened that for completely different reasons some years later U. Haagerup
in [H] proved that the function $g \to \exp(-\mu d(g))$ is positive definite on the free
group; where $d(g)$ is word distance on a free group but can also be viewed as

1991 Mathematics Subject Classification. Primary 43A35, 20E08; Secondary 22D10.

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tree distance on a homogeneous tree. He mentioned that only for large $\mu$ the corresponding representation is square integrable. Later it was shown (see e.g. [K-S]) that for small $\mu$ the representation contains so called complementary series of the free group.

Actually (without mentioning the notion) Haagerup constructed a nontrivial cocycle of the free group with values in some unitary representation.

In this paper we enlarge the definition of canonical state by defining a large class of semigroups of canonical states. We can say that in both cases we have to deal with a general canonical semigroup of states on the group $\text{PSL}(2, \mathbb{R})$ in [V-G-G1] and free group in [H] having an unbounded infinitesimal generator.

Nontheless we shall show that there is an important difference between these two examples: in the first case the semigroup is pure in the sense that will be precise in Section 1.4 while in the second case this is not true.

This difference is due to the fact that in the first case the infinitesimal generator $\psi$ corresponds to the limit of complementary series representations of $\text{PSL}(2, \mathbb{R})$.

The main result of this paper is formula (3.3) which shows the relation between the word metric $d(g)$ on the free group and the canonical state $\psi_\mu$ in sense of [V-G-G1] (derivative of the spherical function of the comlementary series representation). In particular we shall prove that $-d(g)$ and $\psi_\mu$ come from cohomologous cocycles.

Both functions appeared in very natural way and both of them defined the same class of cohomology. We believe that the analog of such formula takes place for other classical groups and homogeneous spaces.

As a consequence of this we will see that the natural semigroup $g \to \exp(-\mu d(g))$ is not pure for $\text{Aut}(T)$, the group of automorphisms of a homogeneous tree. We shall produce a pure semigroup for $\text{Aut}(T)$ and consequently also for $\text{PGL}(2, \mathbb{Q}_p)$, which is unique up to choice of the compact subgroup and equivalent (see Theorem 6) to the semigroup $\exp(-\mu d(g))$.

We give some new information about canonical semigroups and a precise description of the canonical semigroups of states for the group of automorphisms of the tree.

We show the connection with the theory of "pure" cocycles of the group and the special representation which can be realized on the Poisson boundary of the tree.

Our Theorems in Section 1 establish the connection between

1. semigroups of canonical states
2. conditionally positive definite functions
3. cohomology of the group

in a general context. These results generalize the results in [V-G-G] and [V-Kar].

The main examples are for $\text{Aut}(T)$, the group of automorphisms of a homogeneous tree but can also be applied to $\text{PGL}(2, \mathbb{Q}_p)$. We shall show the connection between Haagerup's cocycle and one of the two special representations of $\text{Aut}(T)$. It is surprising that the natural semigroup $\exp(-\mu d(g))$ is not pure in the sense of complementary series of $\text{Aut}(T)$.

We shall use actively results from [F-P] [F-N] [K-V1].
Theory of canonical states is closely related to the so-called Beresin kernel and quantization problem, as well as to the geometry of the group and becomes very important for current groups.

In a forthcoming paper we shall describe the representations of current groups for $\text{Aut}(T)$ and $\text{PGL}(2, \mathbb{Q}_p)$, based on this theory.

1. Canonical semigroups of states

1.1 Conditionally positive definite functions.

Let $G$ be a locally compact topological group. Fix once for all a maximal compact subgroup $K$ (maximal it is not necessary for the general definition, but in our examples it will play an important role). We recall that a continuous function $\psi : G \to \mathbb{C}$ is said to be conditionally positive definite, briefly c.p.d., if

$$\sum_{i,j} c_i \overline{c_j} \psi(g_j^{-1} g_i) \geq 0 \quad \text{whenever} \quad \sum_i c_i = 0$$

A c.p.d. is said to be normalized if it is real and $\psi(1) = 0$.

A first example of normalized c.p.d. is the following: take any real positive definite function $\varphi$ normalized so that $\varphi(1) = 1$ and let

$$\psi(g) = C(\varphi(g) - 1) \quad \text{for some positive constant } C$$

It is obvious that such $\psi$ is c.p.d. and normalized: we shall call such a function trivial.

Conditionally positive definite normalized functions define a real cone $\mathcal{K}$:

$$\mathcal{K} = \{ \psi : \psi(1) = 0, \psi = \overline{\psi}, \psi \text{ is c.p.d.} \}$$

Functions belonging to extreme rays of this cone are called pure c.p.d.:

$\psi \in \mathcal{K}$ is pure if $\psi = \alpha_1 \psi_1 + \alpha_2 \psi_2$ for some positive constants $\alpha_i$ and $\psi_i \in \mathcal{K}$ then $\psi_1 = \psi_2 = \psi$.

We shall see below connections between pure c.p.d. and irreducible representations.

1.2 Cocycles.

Let $(H_\tau, \pi)$ be a continuous unitary representation of $G$. Throughout this paper we shall call cocycle a 1-cocycle $\beta$ defined on $G$ with values in $H_\tau$, that is, a continuous function $\beta : G \to H_\tau$ satisfying the following

$$\beta(g_1 g_2) = \beta(g_1) + \pi(g_1) \beta(g_2) \quad \beta(1) = 0 \text{ for all } k \in K$$

Here are some basic definitions that we shall need later:

1. A trivial cocycle is a cocycle cohomologous to zero, that is,

$$\beta(g) = \pi(g)v - v \quad \text{for some } v \in H_\tau$$
(2) A cocycle is **total** if the linear span of $\beta(g)_{g \in G}$ is dense in $H_z$.

(3) A **real** cocycle is a cocycle $\beta$ such that

$$\Im \langle \beta(h), \beta(g) \rangle = 0 \quad \text{for all } h, g \in G.$$ 

(4) A cocycle is **pure** if the corresponding representation is irreducible.

(5) Two cocycles $\beta_1$ and $\beta_2$ associated respectively to $(\pi_1, H_1)$ and $(\pi_2, H_2)$, are said equivalent if there exists a bijective $G$-isometry $U : H_1 \to H_2$ such that $U \beta_1 = \beta_2$.

(6) Two cocycles $\beta_1$ and $\beta_2$ are cohomologous iff $\beta_1 - \beta_2$ is trivial.

The cohomology group $H^1(G, \pi)$ is the quotient of the vector space of all cocycles and the subspace of trivial cocycles.

**Lemma 1.** If $\beta : G \to H_z$ is a real cocycle then the function

$$\psi(g) = -\frac{1}{2}||\beta(g)||^2$$

is normalized c.p.d.

**Proof.** Since $\beta(e) = 0 = \beta(g^{-1}g) = \beta(g^{-1}) + \pi(g^{-1})\beta(g)$ for every $g$, we have $\pi(g^{-1})\beta(g) = -\beta(g^{-1})$ and $\pi(g)\beta(g^{-1}) = -\beta(g)$.

Keeping this in mind assume that $\sum c_i = 0$ and compute

$$\sum_{i,j} c_i \overline{c_j} \psi(g_j^{-1}g_i) = \frac{1}{2} \sum_{i,j} c_i \overline{c_j} ||\beta(g_j^{-1}g_i)||^2 =$$

$$-\frac{1}{2} \sum_{i,j} c_i \overline{c_j} ||\beta(g_j^{-1})||^2 - \frac{1}{2} \sum_{i,j} c_i \overline{c_j} ||\beta(g_i)||^2 + \sum_{i,j} c_i \overline{c_j} \Re(\pi(g_j)\beta(g_j^{-1}), \beta(g_i)) =$$

$$||\sum c_i \beta(g_i)||^2 - \left(\sum c_i\right) \left(\sum_i c_i ||\beta(g_i)||^2\right) \geq 0$$

because $\sum c_i = 0$.

Conversely, using a construction which is very similar to the G-N-S construction in the case of positive definite functions, we can see that to every normalized conditionally positive definite function there corresponds a continuous unitary representation $(\pi, H_\psi)$.

**Lemma 2.** [V-Kar]. Let $\psi$ be normalized c.p.d. For every $g \in G$ let $\delta_g$ denote the Dirac delta at $g$. Define

$$H_\psi = \{m : m = \sum c_i \delta_{g_i} \text{ where } g_i \in G, c_i \in \mathbb{C} \text{ and } \sum c_i = 0\}$$
and set, for every \( m = \sum c_i \delta_{g_i} \) and \( m' = \sum c'_i \delta_{g'_i} \) in \( H_o \)

\[
\langle m, m' \rangle = \sum_{i,j} c_i c'_j \psi(g^{-1}_j g_i).
\]

Define a representation on \( H_o \) by letting

\[
(\pi(g)m)(g') = m(g^{-1}g')
\]

Let \( H_o' \) denote the completion of the quotient of \( H_o \) with respect to the null space of \( \langle , \rangle \). Then \( \pi \) extends to a continuous unitary representation of \( G \) on \( H_o' \). Moreover, if

\[
\beta(g) = \delta_{g} - \delta_e
\]

one has

\[
\psi(g) = -\frac{1}{2} \|\beta(g)\|^2
\]

It should be noticed that if we started with a right \( K \)-invariant \( \psi \), the corresponding cocycle will also be right \( K \)-invariant since

\[
\|\beta(gk) - \beta(g)\|^2 = \|\delta_{gk} - \delta_g\|^2 = \|\psi(k) - \psi(k^{-1})\| = 0
\]

Theorem 1 (see e.g. [D][G][V-Kar]). Assume that \( G \) is a locally compact topological group and \( K \) a maximal compact subgroup. There exists a bijection between equivalence classes of nontrivial real total cocycles vanishing on \( K \) and real conditionally positive definite normalized functions which are right \( K \)-invariant. Under this bijection irreducible representations correspond to pure c.p.d.

A real cocycle \( \beta : G \to H_e \) is said to be pure if the corresponding representation \( \pi \) is irreducible.

Let \( \beta \) be a real cocycle and denote by \( \psi_\beta \) the corresponding conditionally positive definite function. We shall now investigate in details the condition of nontriviality.

Theorem 2. The following conditions are equivalent

1. \( \beta \) is non trivial
2. \( \beta \) is unbounded, that is \( \text{Sup}_{g \in G} \| \beta(g) \| = +\infty \)
3. \( \psi_\beta = -\frac{1}{2} \|\beta(g)\|^2 \) is conditionally positive definite but not positive definite

Proof. We shall prove the equivalence between 1 and 2. For the equivalence between 3 and 2 we refer to [D][G]. It is obvious that 2 implies 1. Assume now that \( \beta \) is bounded. Let \( (\pi_\beta, H_\beta) \) denote a Hilbert space representation associated with \( \beta \). We may assume

\[
\text{Sup}_{g \in G} \|\beta(g)\| = 1
\]

Define \( \phi(g) = 1 - \frac{1}{2} \|\beta(g)\|^2 \). By Lemma 1 \( \phi \) is conditionally positive definite. Since \( \phi \) is also bounded, it follows that \( \phi \) is positive definite (see e.g. [G]). Let \( (\pi, H_e) \)
be the representation associated with \( \phi \) according to the G-N-S construction and let \( v \) be a unit vector so that
\[
\phi(g) = \langle \pi(g)v, v \rangle
\]
Form the Hilbert space \( H = H_\pi \oplus H_\beta \). Since \( \pi(g)v \) and \( \beta(g) \) are orthogonal for all \( g \), it is easy to see that
\[
\|\pi(g)v - v - \beta(g)\|^2 = 2 - 2\langle \pi(g)v, v \rangle - \|\beta(g)\|^2 = 0
\]
telling us that \( \beta \) is trivial.

1.3 Haagerup’s Cocycle.

Let \( \Gamma \) be a free group on a finite set \( A^+ = \{a_1, \ldots, a_n\} \) (\( r \geq 2 \)) of generators. Let \( A^- \) consist of the inverse of the generators. Set \( A = A^+ \cup A^- \). Each \( x \in \Gamma \), \( x \neq e \), can be uniquely represented as a reduced word, i.e., a product \( a_1a_2\ldots a_n \) of elements of \( A \) with \( a_ia_{i+1} \neq e \). The number of factors in this representation is called the length of \( x \) with respect to \( A \) and will be denoted by \( |x| \). The function \( d(x, y) = |y^{-1}x| \), \( x \) in \( \Gamma \) is a left invariant metric on \( \Gamma \).

In [H] Haagerup proved that the function \(-d(x) = -d(e, x) = -d(x, e) = -|x|\) is c.p.d. by constructing explicitly a nontrivial cocycle \( \beta_h : \Gamma \to H_\Lambda \) such that
\[
d(x) = \|\beta_h(x)\|_{H_\Lambda}^2
\]
We recall briefly the definition of \( \beta_h \). In the next section we shall extend \( \beta_h \) to a bigger group and we shall also see that \( \beta_h \) is not pure in our sense.

Let \( \Lambda \) consist of pairs \((x, y) \in \Gamma \times \Gamma \) for which \( x^{-1}y \) is an element of \( A^+ \). Construct a Hilbert space \( H_\Lambda \) having \( \{e_{(x, y)}(x, y) \in \Lambda \) as an orthonormal basis.

Define also
\[
\Lambda = \{(x, y) \in \Gamma \times \Gamma : x^{-1}y \in A^- \} = \{(x, y) : (y, x) \in \Lambda \}
\]
For \((x, y) \in \Lambda \) define
\[
e_{(x, y)} = -e_{(y, x)}
\]
Assume that \( x = a_1a_2\ldots a_n \) is a reduced word for \( x \). Then for each \( k \), with \( 1 \leq k \leq n-1 \), \((a_1\ldots a_k, a_1\ldots a_k a_{k+1}) \) belongs to \( \Lambda \) or to \( \Lambda \) according to the fact that \( a_{k+1} \) belongs to \( A^+ \) or to \( A^- \).

Haagerup’s cocycle is given by
\[
\beta_h(x) = e_{(x, a_1)} + e_{(a_1, a_2, a_3)} + \ldots e_{(a_1\ldots a_{n-1}, a_1\ldots a_{n-1} a_n)}
\]
Obviously we have
\[
\|\beta_h(x)\|^2 = |x|.
\]
The diagonal action of \( \Gamma \) on \( \Lambda \) gives a unitary representation \( \pi \) on \( H_\Lambda \):
\[
\pi(x)e_{(x, y)} = e_{(x, x)}
\]
It is straightforward to check that \( \beta_h(x_1x_2) = \beta_h(x_1) + \pi(x_1)\beta_h(x_2) \) and it is also clear that \( \beta_h \) is unbounded, hence not cohomologous to zero. Nonetheless there is a natural map taking \( H_\Lambda \) to \( \ell^2(\Gamma) \) and the image of \( \beta_h \) under this map is trivial.
**Remark 1.**

Define $P : \mathcal{H} \rightarrow \ell^2(\Gamma)$ by the rule

$$P(c(x,y)c(x,y)^*) = c(x,y)(\delta_y - \delta_x)$$

Then

$$P(\beta_k(x)) = \delta_x - \delta_e$$

is a trivial cocycle corresponding to the left regular representation of $\Gamma$ on $\ell^2(\Gamma)$.

1.4 Canonical Semigroups of States.

In this paper we generalize the notion of canonical state which was defined in [V-G-G1]. We shall see that, according to this generalization, other remarkable examples of canonical semigroups which are not pure can arise, namely Haagerup’s semigroup $g \rightarrow \exp(-\mu d(g))$.

A canonical semigroup of states associated to a subgroup $K$ is a semigroup of continuous functions $\phi_\mu(g) : \mathbb{R}^+ \rightarrow G$ satisfying the following conditions:

1. $\phi_\mu(g) = \exp(\mu \psi(g))$ where $\mu \geq 0$.
2. $\psi(g) : G \rightarrow \mathbb{R}$ is bi-$K$ invariant.
3. $\psi$ is conditionally positive definite and normalized.
4. $\psi$ is unbounded.

**Remark 2.** It is clear from the definition that $\phi$ is positive definite

Pure canonical semigroups.

A pure canonical state is a canonical state for which $\psi$ is pure. If $\psi$ is pure $c.p.d.$ we shall call the canonical semigroup $\exp(\mu \psi)$ pure.

1.5 Core.

According with [V-Kar], a continuous unitary representation $\pi$ is called a core if it is a limit, in the Fell topology, of irreducible representations which are also tending to one in the same topology. In other words $\pi$ cannot be separated from $I$ in the Fell topology: we shall also call these representations infinitesimally small.

The following Theorem is from [V-Kar]

**Theorem 3.** Let $G$ be a locally compact separable group and $(\pi, H_\pi)$ a continuous unitary irreducible representation of $G$. Assume that $H^1(G, \pi) \neq 0$, that is, there exists a nontrivial cocycle $\beta : G \rightarrow H_\pi$. Then $\pi$ is infinitesimally small.

The description of $H^1(G, \pi)$ for semisimple Lie groups $G$ and irreducible representations $\pi$ is known now. The first examples for $SL(2, \mathbb{R}), SO(n, 1), SU(n, 1)$ are in [V-G-G1][V-G-G2] (see also [V-Kar] and the papers [D] [Gu][S] by Delorm, Guichardet, Shalom).

Concerning conditionally positive definite functions, Gangolli [Ga] found all pure (in our terms) $c.p.d.$ for Gel’fand pairs $(G, K)$. 

2. The group of automorphisms of the tree

2.1 Notation.
Let $\mathcal{T}$ be a homogeneous tree of degree $q + 1 \geq 3$ and $\text{Aut}(\mathcal{T})$ the group of all automorphisms of $\mathcal{T}$.
Let $I$ denote the identity of $\text{Aut}(\mathcal{T})$. Define a open basis of neighborhoods of $I$ as follows: fix a finite subtree $\mathcal{F}$ and let

$$U_x(I) = \{g \in \text{Aut}(\mathcal{T}) : g(x) = x \text{ for all } x \in \mathcal{F}\}.$$ 

The topology generated by the sets $U_x$ as $\mathcal{F}$ varies among all finite subtrees of $\mathcal{T}$ is the restriction to $\text{Aut}(\mathcal{T})$ of the natural topology as a subspace of the power space $\text{Map}(\mathcal{T}, \mathcal{T})$ and turns $\text{Aut}(\mathcal{T})$ into a locally compact totally disconnected group.

Fix a vertex, say $e$ once for all. The subgroup $K_e$ of $\text{Aut}(\mathcal{T})$ consisting of all automorphisms of $\mathcal{T}$ which fix $e$ is a maximal compact subgroup and $(\text{Aut}(\mathcal{T}), K_e)$ is a Gelfand pair (see [F-N]).

$\text{Aut}(\mathcal{T})$ is a very huge group, having many interesting subgroups, at least for special values of $q$. When $q$ is a prime number $\text{PGL}(2, \mathbb{Q}_q)$ is a closed subgroup whose natural topology coincides with the restriction of the topology of $\text{Aut}(\mathcal{T})$.

Denote by $V$ the set of vertices of $\mathcal{T}$.

When $q + 1$ is even, say $q + 1 = 2r$, $\mathcal{T}$ can be thought as the Cayley graph of a free group $\Gamma$ on $r$ generators which is embedded as a lattice in $\text{Aut}(\mathcal{T})$. This allows us to identify $V$, as a set, with $\Gamma$, moreover tree distance will correspond to group length.

Since right $K_e$ invariant functions can be easily described as functions on $V$ and hence on $\Gamma$, we recall briefly the construction of the Cayley graph.

As in 1.3 we fix a set $A^+$ of generators for $\Gamma$. Let $A$ consist of the generators and the inverses of generators. Denote by $|\cdot|$ the length with respect to these generators.

The Cayley graph of $\Gamma$ with respect to $A$ is a homogeneous tree having $\Gamma$ as its vertex set. The oriented edges of $\text{Aut}(\mathcal{T})$ are identified with the ordered pairs $(x, xa)$ where $x \in \Gamma$ and $a \in A$. The action of $\Gamma$ is given simply by left multiplication. One has $q + 1 = |A|$ and the vertex $e$ corresponds to the identity of $\Gamma$.

If $x = a_1 a_2 \ldots a_n$ is the reduced word for $x$, the unique geodesic from $e$ to $x$ in the tree is $[e, x] = (e, a_1, a_1 a_2, \ldots, x)$. Tree distance between vertices $x$ and $y$ is simply the length $|y^{-1} x|$.

Every element of $\text{Aut}(\mathcal{T})$ can be uniquely represented as a product of an element of $\Gamma$ and an element of $K_e$ so that $\text{Aut}(\mathcal{T}) = \Gamma \cdot K_e$ (a disjoint product). The Haar measure on $\text{Aut}(\mathcal{T})$ is the product of counting measure on $\Gamma$ and Haar measure on $K_e$.

It is known that $\text{Aut}(\mathcal{T})$ is type I and its unitary representation theory is understood (see [O] and [F-N]). Continuous unitary irreducible representations of $\text{Aut}(\mathcal{T})$ are “indexed” by compact subgroups, in the sense that to every finite complete subtree $X$ corresponds a series of representations $\tau_X$ having nontrivial $K_X$ fixed
vectors where \( K_X \) is the compact subgroup fixing \( X \) pointwise. Letting \( Y = g \cdot X \) \((g \in Aut(T))\) we get equivalent representations. Up to equivalence no other series of irreducible representation of \( Aut(T) \) will admit such nonzero invariant vectors.

2.2 The spherical series realized on the boundary.

The boundary \( \Omega \) of \( T \) is the set of semi-infinite geodesics starting at \( e \). One thinks of each element \( \omega \in \Omega \) as an infinite reduced word \( \omega = a_1 a_2 a_3 \ldots \). There is a natural action of \( \Gamma \) on \( \Omega \) which extends the left action of \( \Gamma \) on itself. Indeed, for \( x \in \Gamma \) and \( \omega \in \Omega \), \( x \cdot \omega \) can be calculated as the product of the finite reduced word \( x \) with the infinite reduced word \( \omega \).

It should be mentioned that \( \Omega \) coincides with the Martin (in this case the same as the Poisson) boundary of any symmetric nearest neighborhood random walk on \( \Gamma \) whose support generates all \( \Gamma \) (see for example [Kai-V]).

Give \( \Omega \) the natural topology as a subspace of the power space \( Map(\mathbb{N}, \text{vertices of } T) \). This makes \( \Omega \) compact and totally disconnected, isomorphic to the Cantor set.

A simple set of generators for this topology may be obtained as follows: fix a vertex \( x = a_1 a_2 \ldots a_n \). Define \( \Omega(x) \) as the set of all half infinite reduced words "starting" with \( x \). It is easy to see that the sets \( \{ \Omega(x) \}_{x \in \Gamma} \) form an open basis for this topology.

Denote by \( \nu \) the unique \( K \)-invariant probability measure on \( \Omega \) which assigns the measure \( \frac{q^{|x|}}{q+1} q^{-|x|} \) to each of sets \( \Omega(x) \).

The spherical series of \( Aut(T) \) consists of all irreducible unitary representations of \( Aut(T) \) which contain \( K \)-fixed vectors.

Because of this property they can also be realized as acting on some suitable space of functions defined on the set of vertices \( V \) of \( T \) but we shall present here the boundary realization given in [F-T-N].

Let \( \omega = \omega_1 \omega_2 \ldots \omega_n \ldots \) and \( \omega' = \omega'_1 \omega'_2 \ldots \omega'_n \ldots \) be two distinct elements of \( \Omega \). Assume that they agree through the first \( k \) letters, but \( \omega_{k+1} \neq \omega'_{k+1} \). The common "starting" word \( \omega_1 \omega_2 \ldots \omega_k \) will be denoted by \( \omega \wedge \omega' \) (\( \omega \wedge \omega' = e \) when \( \omega_1 \neq \omega'_1 \)).

For any given \( g \in Aut(T) \), let \( y e = a_1 a_2 \ldots a_n \) denote the vertex \( y e \) and let \( \omega = \omega_1 \omega_2 \ldots \omega_n \ldots \).

Define analogously \( |x \wedge \omega| = N(y, \omega) \):

\[
N(y, \omega) = N(g, \omega) = k \quad \text{if } a_1 = \omega_1, \ldots, a_k = \omega_k, \text{ but } a_{k+1} \neq \omega_{k+1}
\]

Say also \( N(y, \omega) = 0 \) if \( a_1 \neq \omega_1 \)

So that \( N(y, \omega) \) denotes the length of the maximum common geodesic between \( [e, y e] \) and \( [e, \omega] \).

The Radon-Nikodym derivative of the group action on is called the Poisson kernel and is given by

\[
\frac{d\nu(g^{-1}\omega)}{d\nu(\omega)} = P(g, \omega) = q^{2N(y, \omega) - d(e, y e)}.
\]
Let $\mathcal{K}(\Omega)$ be the space of locally constant complex functions on $\Omega$. Denote by $1$ the constant function.

For a fixed $g$ in $\text{Aut}(T)$, $P(g, \omega)$ belongs to $\mathcal{K}(\Omega)$.

Define a representation of $\text{Aut}(T)$ on $\mathcal{K}(\Omega)$ by letting

$$(\pi_z(g)f)(\omega) = P_z^z(g, \omega)f(g^{-1} \cdot \omega).$$

When $z = \frac{1}{2} + it$ the representation $\pi_z$ is unitary with respect to the inner product

$$\langle f, g \rangle = \int_{\Omega} f(\omega)g(\omega) d\nu(\omega).$$

As described in [F-T-N] the principal spherical series is obtained by completing $\mathcal{K}(\Omega)$ with respect to the $L^2(\Omega, d\nu)$ norm.

When $z$ is real and $0 < z < 1$ the representation $\pi_z$ is unitarizable and the complementary spherical series is obtained by completing $\mathcal{K}(\Omega)$ with respect to a suitable inner product.

The spherical functions $\phi_z$ are obtained, as usual, as matrix coefficients with respect to the unique $K_z$-invariant vector in $\mathcal{K}(\Omega)$. So that

$$\phi_z(g) = \langle \pi_z(g)1, 1 \rangle = \int_{\Omega} P_z^z(g, \omega) d\nu.$$

A direct computation gives

$$\phi_z(g) = c(z)q^{-z d(e, g \cdot e)} + e(1 - z)q^{(1 - z)d(e, g \cdot e)} \quad \text{if } z \neq \frac{1}{2} + \frac{2k\pi i}{\log q},$$

where

$$c(z) = \frac{1}{q + 1} \frac{q^{1 - z} - q^{-1}}{q^z - q^{-1}}$$

and

$$\phi_z(g) = (1 + \frac{q - 1}{q + 1} \cdot d(e, g \cdot e))q^{-z d(e, g \cdot e)} \quad \text{if } z = \frac{1}{2} + \frac{2k\pi i}{\log q}.$$

Observe that the endpoints of the principal series are obtained when $z = \frac{1}{2} + k\pi i / \log q$ while the endpoints of the complementary series correspond to the cases $z = 1$ and $z = 0$.

In particular when

$$z = 1 \quad \text{or} \quad z = 0$$

the spherical function becomes identically one and the endpoint representation obtained by letting $z \to 0$ (or $z \to 1$) splits into the sum of $\pi_z$, one of the two so called "special" representations of $\text{Aut}(T)$ and the trivial representation (see [O]) as mentioned in the introduction.
The complementary series representations are not unitary as acting on $L^2(\Omega, dv)$, but need to be unitarized. Since we will be interested in computing limits of their matrix coefficients we shall describe this inner product in details.

Let $z$ be real between 0 and 1 and let $\pi_z$ be the corresponding complementary series representation.

$\langle \cdot, \cdot \rangle_z$ will denote the inner product in $L^2(\Omega, dv)$ while $\langle \cdot, \cdot \rangle_z$ will be the inner product which makes $\pi_z$ unitary. Define

$$K_z(\omega, \omega') = \frac{q + 1 (1 - q^{1-2z})}{q} q^{2(1-z)||\omega \omega'||}$$

It can be recovered from [F-T-P, page 173] that

$$\langle \xi, \xi \rangle_z = \int_{\Omega} \int_{\Omega} K_z(\omega, \omega') \xi(\omega) \overline{\xi'(\omega')} dv(\omega) dv(\omega')$$

for every $\xi \in K(\Omega)$. Also, since $K_z(\omega, \omega')$ is normalized so that

$$\int_{\Omega} K_z(\omega, \omega') dv(\omega') = 1$$

we can see that

$$\langle \xi, \xi \rangle_z = \langle \xi, \xi \rangle - \frac{1}{2} \int_{\Omega} \int_{\Omega} |\xi(\omega) - \xi'(\omega'')|^2 K_z(\omega, \omega') dv(\omega) dv(\omega')$$

This formula for the scalar product will be strongly used in the following section.

3. Special Representations

3.1 Special Representations. As shown in [O] there are two inequivalent irreducible special representations $sp_+$ and $sp_-$ of $Aut(T)$.

They are characterized by admitting nonzero $K_z$-invariant vectors where $K_z$ is the subgroup of $Aut(T)$ fixing an oriented edge $\varepsilon$.

Fix $\varepsilon^-, \varepsilon^+$ norm one $K_z$-invariant vectors in the representation space of $sp_-$, respectively $sp_+$. The matrix coefficients

$$\varphi_\pm(g) = \langle \varepsilon^\pm, sp_\pm(g) \varepsilon^\pm \rangle$$

can be thought as functions on the directed edges. They belong to $L^2(Aut(T))$ but not to $L^1(Aut(T))$. So that $sp_\pm$ can be thought as subrepresentations of the natural representation of $Aut(T)$ on $\ell^2$(directed edges). Here is the standard realization taken from [F-T-N].

Fix $\varepsilon_0 = (e, a)$ ($a \in A^+$). Recall from the construction of the Cayley graph of $\Gamma$ that unoriented edge $\{\varepsilon\}$ can be identified with a pair $\{x, xa^\pm\}$ where $x$ is reduced $a \in A^+$ and $xa^\pm$ is not necessary reduced. An oriented edge $\varepsilon$ is an ordered
pair \((x, xa^\pm)\), that is, an element on \(\Gamma \times \Gamma\). An orientation on the set of edges can be thought as a subset of \(\Gamma \times \Gamma\). We remark that the subset \(\Lambda\) defined in section 1.3 is a natural orientation on the set of edges.

Let \(\mathcal{E}\) denote the set of oriented edges of \(\mathcal{T}\) with respect to the orientation given by \(\Lambda\).

If \(\varepsilon = (x, y)\), denote by \(i(\varepsilon)\) the initial vertex \(x\) of \(\varepsilon\) and by \(t(\varepsilon)\) the terminal vertex \(y\) of \(\varepsilon\).

For every \(\varepsilon = (x, y) \in \mathcal{E}\) let \(\varepsilon'\) denote the edge \((y, x)\).

A function \(f: \mathcal{E} \to C\) is called even if \(f(\varepsilon) = f(\varepsilon')\) and odd if \(f(\varepsilon) = -f(\varepsilon')\). Observe that an element of \(\mathcal{H}_\Lambda\) can be uniquely extended to an odd or to an even function on \(\mathcal{E}\).

Denote by \(\ell^2(\mathcal{E}^+)\) and \(\ell^2(\mathcal{E}^-)\) the spaces of square integrable even, respectively, odd functions. One has

\[
\ell^2(\mathcal{E}) = \ell^2(\mathcal{E}^+) \oplus \ell^2(\mathcal{E}^-)
\]

an orthogonal sum.

It is obvious that \(\ell^2(\mathcal{E}^\pm)\) are both \(\text{Aut}(\mathcal{T})\)-invariant. Denote by \(\lambda_+\), respectively \(\lambda_-\) the restriction to \(\ell^2(\mathcal{E}^+)\), respectively to \(\ell^2(\mathcal{E}^-)\) of the left regular representation of \(\text{Aut}(\mathcal{T})\) coming from the natural action on \(\mathcal{E}\). Define

\[
V_0^+ = \{ f \in \ell^2(\mathcal{E}^+) : \sum_{i(\varepsilon) = x} f(\varepsilon) = 0 \} \text{ for every } x \in \mathcal{T}
\]

and

\[
V_0^- = \{ f \in \ell^2(\mathcal{E}^-) : \sum_{i(\varepsilon) = x} f(\varepsilon) = 0 \} \text{ for every } x \in \mathcal{T}
\]

The two special representations of \(\text{Aut}(\mathcal{T})\) \(sp_+\) and \(sp_-\) are the restriction of \(\lambda_+\) and \(\lambda_-\) to \(V_0^+\) and to \(V_0^-\).

We shall now identify \(\varphi_\pm\). Since we are interested only in \(sp_-\) we shall consider only odd functions being the other case analogous.

Let \(\varepsilon_1\) and \(\varepsilon_2\) be two distinct oriented edges. Denote by \([\varepsilon_1, \varepsilon_2]\) the shortest geodesic containing the vertices of \(\varepsilon_1\) and \(\varepsilon_2\) and by \(\ell(\varepsilon_1, \varepsilon_2)\) its length. Define

\[
D(\varepsilon_1, \varepsilon_2) = \ell(\varepsilon_1, \varepsilon_2) - 1.
\]

Let \(g \in \text{Aut}(\mathcal{T})\) be such that \(g(\varepsilon, a) = \varepsilon\). One has

\[
\varphi_- (g) = (-1)^{d(\varepsilon, g \varepsilon)} \left( \frac{-1}{q} \right)^{D(\varepsilon, \varepsilon)}
\]
3.2 Special Representations and Cocycles.

Let us turn now to Haagerup's cocycle. Identify \( \ell^2(\mathcal{V}) \) with \( \ell^2(\Gamma) \).

Recall the map defined in Remark 1 and extend \( P \) to a bounded map from \( \ell^2(\mathcal{E}) \) to \( \ell^2(\Gamma) \) by letting

\[
P(\varepsilon \varepsilon \chi_{\varepsilon}) = c_\varepsilon (\delta_{t(\varepsilon)} - \delta_{i(\varepsilon)})
\]

where \( \chi_{\varepsilon} \) is the characteristic function of the oriented edge \( \varepsilon \). Observe that \( \|P\| \leq 2(q + 1) \) and that even functions are killed by \( P \).

Hence \( P \) can be thought as a function from \( \ell^2(\mathcal{E}^-) \) to \( \ell^2(\Gamma) \). Identify \( \mathcal{H}_A \) with \( \ell^2(\mathcal{E}^-) \) and an element \( x \in \Gamma \) with the corresponding translation element of \( Aut(\mathcal{T}) \).

We have a natural action \( \pi \) of \( Aut(\mathcal{T}) \) on \( \mathcal{H}_A \). Moreover, when restricted to \( \Gamma \), \( \pi \) is the action considered in section 1.3. A moment's reflection shows that \( P \) commutes with the action of \( Aut(\mathcal{T}) \) and

\[
P(f)(x) = -2 \sum_{\varepsilon : i(\varepsilon) = x} f(\varepsilon)
\]

telling us that the kernel of \( P \) is \( V_0^- \).

Finally compute \( P^* \):

\[
P^*(\delta_y)(\varepsilon) = \begin{cases} 
1 & \text{if } t(\varepsilon) = y \\
-1 & \text{if } i(\varepsilon) = y
\end{cases}
\]

So that

\[
P^* f(\varepsilon) = f(y) - f(x) \quad \text{if } \varepsilon = (x, y)
\]

and \( P^* \) is injective, telling us that the range of \( P \) is \( \ell^2(\Gamma) \). So that

\[
\mathcal{H}_A \simeq \ell^2(\Gamma) \oplus V_0^-
\]

suggesting us a decomposition of \( \beta_b \) as sum of a trivial cocycle plus a cocycle taking values in \( V_0^- \).

Before doing this we shall extend \( \beta_b \) to a cocycle on \( Aut(\mathcal{T}) \).

Since every element of \( Aut(\mathcal{T}) \) can be uniquely represented as a product of an element of \( \Gamma \) and an element of \( K_\varepsilon \) we shall define

\[
\beta_b(xk) = \beta_b(x) \quad \text{for } x \in \Gamma \text{ and } k \in K_\varepsilon
\]

To see that \( \beta_b \) is still a cocycle we must check that, for every \( x, y \in \Gamma \) and \( k \in K_\varepsilon \) one has

\[
\beta_b(xk) = \beta_b(x) + \varepsilon(x) \beta_b(y) = \beta_b(x) + \pi(x) \beta_b(ky)
\]

Let \( y' \) be the unique element of \( \Gamma \) such that \( y' = ky \). We have

\[
\beta_b(xy') = \beta_b(x) + \pi(x) \beta_b(y')
\]

because \( \beta_b \) is a cocycle on \( \Gamma \). Now \( \pi(xk) \beta_b(y) = \pi(x) \pi(k) \beta_b(y) = \pi(x) \beta_b(ky) \) and (3.1) follows.
3.3 The Special Representation $sp_-$ realized on $\Omega$.

In this section we shall give a description of $sp_-$ which is a little different from the usual one and will lead to a precise formula for Haagerup's semigroup.

**Theorem 4.** Let $H^-$ denote the completion of the orthogonal complement of constant functions in $K(\Omega)$ with respect to the inner product

\[
\langle F, F^- \rangle = \frac{q^2 - 1}{q} \int_{\Omega} \int_{\Omega} |F(\omega) - F(\omega')|^2 q^{2|\omega \land \omega'|} d\nu(\omega) d\nu' \tag{3.2}
\]

and let $(\pi_-(g)F)(\omega) = F(g^{-1}\omega)$. Then $\pi_-$ is unitarily equivalent to $sp_-$.

**Proof.** It is well known [K Sect. 2.4.1] that the measure $q^{2|\omega \land \omega'|} d\nu(\omega) d\nu'$ is invariant under the diagonal action of $Aut(T)$ on $\Omega \times \Omega$ telling us that $\pi_-$ is unitary.

Fix, as before, a neighbour $a \in A^+$ of $\epsilon$. Let $K_{(\epsilon, a)}$ be the subgroup of $Aut(T)$ which fixes the edge $(\epsilon, a)$. Denote by $\mathbf{1}_a$ the characteristic function of the set $\Omega(a)$. Define $F_a = (q + 1)\mathbf{1}_a - 1$.

It is obvious that $F_a$ is $K_{(\epsilon, a)}$-invariant.

The matrix coefficient

\[
f_0(g) = \langle F_a, \pi_-(g)F_a \rangle
\]

can be thought as function $\tilde{f}_0$ on oriented edges by letting

\[
\tilde{f}_0(\epsilon) = f_0(g) \quad \text{if} \quad g(\epsilon, a) = \epsilon
\]

We prove first that $\tilde{f}_0$ is a scalar multiple of the function $\varphi_-$ described before.

Let $g^1$ be any element of $Aut(T)$ such that $g^1(\epsilon, a) = (a, \epsilon)$. Since $\pi_-(g^1)F_a = q\mathbf{1} - (q + 1)\mathbf{1}_a$, it is easy to see that

\[
f_0(g^1) = \langle F_a, \pi_-(g^1)F_a \rangle = \frac{q^2 - 1}{q} \int_{\Omega} \int_{\Omega} (q + 1)^2 [\mathbf{1}_a(\omega) - \mathbf{1}_a(\omega')] [-\mathbf{1}_a(\omega) + \mathbf{1}_a(\omega')] q^{2|\omega \land \omega'|} d\nu(\omega) d\nu(\omega') = -f_0(\epsilon)
\]

In general, if $g(\epsilon, a) = (v, v')$, then $gg^1(\epsilon, a) = (v', v)$, it is not hard to see that $\tilde{f}_0(v, v') = -\tilde{f}_0(v', v)$ so that $f_0$ is an odd function.

Let $K_a$ be the subgroup of $Aut(T)$ fixing the vertex $a$, so that $K_{(\epsilon, a)} = K_\epsilon \cap K_a$. Let $b$ denote any neighbour of $\epsilon$ different from $a$. Let $k_b$ be any element of $Aut(T)$ which fixes $\epsilon$ and interchanges $a$ and $b$. A direct computation gives

\[
\langle F_a, \pi_-(k_b)F_a \rangle = f_0(k_b) = \tilde{f}_0(\epsilon, b) = -\frac{1}{q} f_0(\epsilon) = \frac{1}{q} \tilde{f}_0(\epsilon, a)
\]

Since $K_\epsilon$ is the union of $q + 1$ distinct cosets of $K_{(\epsilon, a)}$, each corresponding to one of the $q + 1$ neighbours of $\epsilon$ using $K_{(\epsilon, a)}$ invariance we get

\[
\int_{K_\epsilon} f_0(k) dk = \sum_{\epsilon : i(\epsilon) = \epsilon} \tilde{f}_0(\epsilon) = 0
\]
Replacing now $K_\varepsilon$ with $K_\alpha$ we get also
\[ \sum_{i(\varepsilon) = \alpha} \hat{f}(\varepsilon) = 0. \]

To see that the above equality holds for every $\varepsilon$ consider $g \in \mathcal{A}ut(T)$ such that $g(\varepsilon, a) = (\varepsilon, \varepsilon')$. Write as before $K_\varepsilon = K_{(\varepsilon, \alpha)} \bigcup_{b \neq \alpha} k_b K_{(\varepsilon, \alpha)}$. We have
\[ \sum_{i(\varepsilon) = \alpha} \hat{f}_b(\varepsilon) = \sum_{b \neq \alpha} f_b(gk_b) + f_\varepsilon(g) \]

But
\[ f_b(gk_b) = \langle \pi(g^{-1}) F_\alpha, \pi(k_b) F_\alpha \rangle_\pi = -\frac{1}{q} \langle \pi(g^{-1}) F_\alpha, F_\alpha \rangle_\pi = -\frac{1}{q} f_\varepsilon(g) \]
telling us that $\hat{f}_b \in V_\varepsilon^-.$

To conclude that $\mathcal{S}p_\varepsilon^-$ is equivalent to the representation above described it is enought to show that $F_\alpha$ is a cyclic vector for $\pi_-$: we omit the computation which can be found in [M-Z].

3.4 Main Result.

For every $g \in \mathcal{A}ut(T)$ and $\omega \in \Omega$ let
\[ \beta_\varepsilon(g, \omega) = \log_q (P(g, \omega)) = 2N(g, \omega) - d(\varepsilon, g \cdot \varepsilon) \]

where the log$_q$ denotes the logarithm in base $q$.

For any fixed $g$, $\beta_\varepsilon(g)(\omega) = \beta_\varepsilon(g, \omega)$ belongs to $H^-$. Since $P(g, \omega)$ is a Radon-Nikodym derivative we can see that $\beta_\varepsilon$ is a cocyle, moreover since $d\nu$ is $K_\varepsilon$ invariant, $\beta_\varepsilon$ will be right $K_\varepsilon$ invariant.

Let
\[ \psi_\varepsilon(g) = -\frac{1}{2} \langle \beta_\varepsilon(g), \beta_\varepsilon(g) \rangle_\pi = -\frac{1}{2} \| \beta_\varepsilon(g) \|_\pi^2 \]

We have the following

**Theorem 5.** Let $d(g) = d(\varepsilon, g \cdot \varepsilon)$. Then $-d$, as a conditionally positive definite function, admits a representation

\[ -d(g) = \psi_\varepsilon(g) + \frac{2q}{q^2 - 1} (q^{d(\omega)} - 1) \]

where $\psi_\varepsilon$ is pure c.p.d., corresponding to the special representation $\mathcal{S}p_\varepsilon^-$, and the second summand is trivial, corresponding to $C(\phi - 1)$, where $\phi$ is square integrable.

**Proof.**
By the corollary of Theorem 4 in [V-Kar] \( \psi_a = -\frac{1}{2} \| \beta_a (g) \|^2 \) must be the derivative, with respect to \( z \) at the point \( \psi_a \), of a continuous family of irreducible representation which are tending to \( \psi_a \) as \( z \to \psi_a \).

The only possible case for our group is the complementary series representations and the possible values for \( \psi_a \) are 0 or 1, being the two cases equivalent.

So that we are interested in computing

\[
\lim_{z \to 0} \frac{\| \pi_z (g) \mathbf{1} - \mathbf{1} \|^2}{z} = \lim_{z \to 0} \frac{\langle \pi_z (g) \mathbf{1} - \mathbf{1}, \pi_z (g) \mathbf{1} - \mathbf{1} \rangle}{z}
\]

Remembering that \( \langle \pi_z (g) \mathbf{1}, \mathbf{1} \rangle = \langle \pi_z (g) \mathbf{1}, \mathbf{1} \rangle = \phi_z (g) \) and that \( \phi_z (g) \) is real, above limit gives us the derivative of \( \phi_z (g) \):

\[
\lim_{z \to 0} \frac{\langle \pi_z (g) \mathbf{1}, \mathbf{1} \rangle - 1}{z} = \lim_{z \to 0} \frac{\phi_z (g) - 1}{z} = -\frac{1}{2} \lim_{z \to 0} \frac{\| \pi_z (g) \mathbf{1} - \mathbf{1} \|^2}{z}
\]

Letting \( \xi = \pi_z (g) \mathbf{1} - \mathbf{1} \) in (2.2) we get

\[
\frac{\| \pi_z (g) \mathbf{1} - \mathbf{1} \|^2}{z} = \frac{\langle \pi_z (g) \mathbf{1} - \mathbf{1}, \pi_z (g) \mathbf{1} - \mathbf{1} \rangle - \frac{1}{2} I_z}{z}
\]

where

\[
I_z = \int_{\Omega} \int_{\Omega} P_z (g, \omega) - P_z (g, \omega') K_z (\omega, \omega') d \nu(\omega) d \nu(\omega').
\]

Remembering that \( z \) is real we have

\[
\langle \pi_z (g) \mathbf{1}, \pi_z (g) \mathbf{1} \rangle = \langle P_z (g, \omega), P_z (g, \omega) \rangle = \langle P_z (g, \omega), \mathbf{1} \rangle = \phi_z (g)
\]

Observing that

\[
\lim_{z \to 0} \frac{\phi_z (g) - 1}{z} = \frac{1}{4} \lim_{z \to 0} \frac{I_z}{z}
\]

we finally get

\[
\lim_{z \to 0} \frac{\phi_z (g) - 1}{z} = \frac{1}{4} \lim_{z \to 0} \frac{I_z}{z}
\]

A direct computation gives

\[
\frac{1}{4} \lim_{z \to 0} \frac{I_z}{z} = \frac{-(q^2 - 1) \log q}{8q} \int_{\Omega} \int_{\Omega} \left| 2N (x, \omega) - 2N (x, \omega') \right| |q^{2 |\omega \wedge \omega'|} d \nu(\omega) d \nu(\omega')
\]

so that

\[
(3.4) \quad \frac{d \phi_z}{dz}|_{z=0} = -\frac{(q^2 - 1) \log q}{2q} \int_{\Omega} \int_{\Omega} \left| N (x, \omega) - N (x, \omega') \right| q^{2 |\omega \wedge \omega'|} d \nu(\omega) d \nu(\omega')
\]
But
\[
\frac{d\hat{\rho}}{dz}|_{z=0}(g) = \left[-d(e, g \cdot e) + \frac{2q}{q^2 - 1}(1 - q^{-d(e, g \cdot e)})\right] \log q.
\]
as was shown in [K-V1].

Putting together (3.4) and (3.5) we have
\[
-d(e, g \cdot e) = -\frac{2q}{q^2 - 1} \left(1 - q^{-d(e, g \cdot e)}\right) - \frac{q^2 - 1}{2q} \int_\Omega \int_\Omega |N(x, \omega) - N(x, \omega')|^2 q^{2t} |\omega \lambda \omega'| \ d\nu(\omega) d\nu(\omega')
\]
Remembering the definition of \(\psi_\alpha\) and using (3.2)
\[
-d(e, g \cdot e) = -\frac{1}{2} \langle \beta_\alpha(g), \beta_\alpha(g) \rangle_\nu + \frac{2q}{q^2 - 1} (q^{-d(e, g \cdot e)} - 1)
\]
where \(\langle \cdot, \cdot \rangle_\nu\) is the scalar product defined in (3.2).

Let us compare now the two canonical semigroups having respectively \(-\frac{1}{2} \|\beta_\alpha(g)\|^2\) and \(-\frac{1}{2} \|\beta_\alpha(g)\|^2\) as infinitesimal generators. Set
\[
\psi^\mu_\alpha(g) = q^{-\mu d(e, g \cdot e)} \quad \psi^\beta_\alpha(g) = q^{-\beta \frac{1}{2} \|\beta_\alpha(g)\|^2}
\]
and denote by \(\Pi^\mu_\alpha\), respectively by \(\Pi^\beta_\alpha\) the representations of \(\text{Aut}(T)\) corresponding to \(\psi^\mu_\alpha\) and \(\psi^\beta_\alpha\).

We have the following

**Theorem 6.** The representations \(\Pi^\mu_\alpha\) and \(\Pi^\beta_\alpha\) behave in the same way in the following sense:

1. when \(\mu > \frac{1}{2}\) they are both square integrable.
2. when \(\mu = \frac{1}{2}\) they are both weakly contained in the regular representation.
3. when \(\mu < \frac{1}{2}\) they both contain the same complementary series.

**Proof.**

Since
\[
\psi^\beta_\alpha(g) = q^{-\frac{1}{2} \|\beta_\alpha(g)\|^2} q^{-\mu c} q^{-\mu c} q^{-d(e, g \cdot e)} \quad \text{beeing} \quad c = \frac{2q}{q^2 - 1}
\]
the first assertion follows from the fact that the Haar measure on \(\text{Aut}(T)\) is the product of counting measure on \(\Gamma\) and Haar measure on \(K_e\).

The second assertion was proved in [H] for \(\Pi^\beta_\alpha\) as a representation of \(\Gamma\). But \(\psi^\beta_\alpha(g)\) is also positive definite when thought as bi-\(K_e\) invariant function on \(\text{Aut}(T)\) telling us that
\[
\psi^\beta_\alpha(g) = \lim_{\epsilon \to 0} \psi^\beta_{\alpha + \epsilon}(g)
\]
is a limit, uniformly on compact sets, of positive definite square integrable functions and the same is true for $\psi^\mu$.

For sake of completeness we observe that, when $\mu \geq \frac{1}{2}$, none of the two representations is irreducible as a representation of $\text{Aut}(T)$. Since the functions $\psi^\mu_\nu$ and $\psi^\mu$ are both $\text{bi}-\mathcal{K}_\nu$-invariant the Gelfand transform can be easily computed and shows that the corresponding representations are direct integral of the principal series of $\text{Aut}(T)$ (see [Ku-V]).

Let us turn to (3). Assume that $\mu < \frac{1}{2}$. Remember the expression for $\phi_\mu$:

$$\phi_\mu (g) = c(\mu) q^{-\mu d(\nu, \xi, \epsilon)} + c(1 - \mu) q^{(1-\mu) d(\nu, \xi, \epsilon)}$$

where

$$c(\mu) = \frac{1}{q + 1} \frac{q^{1-\mu} - q^{\mu-1}}{(q^{-\mu} - q^{\mu-1})} \quad \text{and} \quad c(1 - \mu) = \frac{1}{q + 1} \frac{q^{\mu} - q^{-\mu}}{(q^{\mu-1} - q^{-\mu})}.$$  

Observe that

$$c(1 - \mu) < 0 \quad \text{and} \quad c(\mu) > 0.$$  

Write

$$q^{-\mu d(\nu, \xi, \epsilon)} = \frac{1}{c(\mu)} \phi_\mu (g) - \frac{c(1 - \mu)}{c(\mu)} q^{(1-\mu) d(\nu, \xi, \epsilon)}$$

as the sum of two positive definite functions on $\text{Aut}(T)$: the second is square integrable while the first is pure and

$$\frac{1}{c(\mu)} - \frac{c(1 - \mu)}{c(\mu)} = 1.$$  

To finish the proof recall from [Ku-V]) that

$$\psi^\mu (g) = \frac{k(\mu)}{c(\mu)} \phi_\mu (g) + \int_\mathbb{R} k(t) \phi_{\nu + it} (g) \, dm(t)$$

where $k(\mu) = q^{\frac{2\mu}{q^\mu - 1}}$, $c(\mu)$ is as before and $\phi_{\nu + it} (g)$ belong to the principal series of $\text{Aut}(T)$.

References


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