Logarithmic fluctuations for the Internal Diffusion Limited Aggregation

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Abstract

The Internal Diffusion Limited Aggregation (Internal DLA) is a growth model on an infinite set $\mathcal{G}$, associated to a random walk on $\mathcal{G}$. It was introduced by Diaconis and Fulton in 1991 [2]. Lawler, Bramson, and Griffeath [5] studied this model for the simple random walk on $\mathbb{Z}^d$. They proved that the limiting shape of the cluster generated by this model is the trace of the Euclidean balls. Later, Lawler [4] gave bounds of order $n^{1/3}$ (with logarithmic corrections) on the fluctuations around this limiting shape. Here, we prove that the fluctuations are at most logarithmic, using an induction based on Lawler’s proof.

1 Introduction

Let $\mathcal{G}$ be an infinite discrete set. Let $S(k)$ be an irreducible Markov chain on the elements of $\mathcal{G}$ where $S(0) = O$ a fixed element. The Internal DLA is a Markov chain $A(n) \ (n \in \mathbb{N}^*)$ of increasing random set of points of $\mathcal{G}$ defined as follow:

- $A(1) = \{O\},$
- $P\{A(n + 1) = A(n) \cup \{y\} | A(n)\} = P\{S(\tau(A'(n))) = y\},$

where $\tau(A)$ is the hitting time of the set $A$.

Thus, the set $A(n)$ grows as follows. At each discrete time $n$, we start a particle at $O$, wait until the particle leaves the previous set $A(n - 1)$, and add the first element visited outside $A(n - 1)$, to obtain $A(n)$. Note that this Markov chain is well defined because, for all $n$, $\tau(A'(n)) < \infty$ almost surely, as $A(n)$ is finite and $S(k)$ is irreducible.

The name Internal DLA comes from the similitude with the (external) DLA. In the latter model, the particles start “at infinity”, conditioned to hit the previous cluster, and are stuck just before hitting it. See for example [3] for precise definition and properties.

Diaconis and Fulton studied the case where $\mathcal{G}$ is the graph of $\mathbb{Z}$ and $S(k)$ the simple random walk (nearest neighbor random walk with a uniform distribution). They showed that the cluster 'tends' to the interval $[-n/2, n/2]$ and the fluctuations are Gaussian of parameter $\sqrt{n}$. 

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Theorem 1.1 [2, Prop. 3.2] We denote $A^+(n)$ the number of positive site visited at time $n$. As $n$ goes to infinity:

$$
P\left\{ \frac{|A^+(n)| - n/2}{\sqrt{n}/2} \leq t \right\} \to \frac{1}{\sqrt{2\pi}} \int_0^t \exp(-x^2/2)dx.
$$

Lawler, Bramson, and Griffeath [5] and later Lawler [4] studied the case of $\mathbb{Z}^d$ ($d \geq 2$) with the simple random walk. They proved that the limiting shape of the cluster is the trace of the Euclidean balls on the graph of $\mathbb{Z}^d$ (from now we will just say $\mathbb{Z}^d$). In [4], Lawler studied the fluctuations of the cluster around the balls.

More precisely, let $|\cdot|$ be the Euclidean norm of $\mathbb{R}^d$ and $\omega_d$ the volume of the unit ball in $\mathbb{R}^d$. Let $B(n)$ be the trace of the Euclidean balls on $\mathbb{Z}^d$, so that $B(n)$ contains approximately $[\omega_d n^d]$ points. Let $\delta_I(n)$ and $\delta_O(n)$ be the inner and outer error defined by

$$
n - \delta_I(n) = \inf\{|z|: z \notin A([\omega_d n^d])\},
$$

$$
n + \delta_O(n) = \sup\{|z|: z \in A([\omega_d n^d])\}.
$$

Note that $\delta_I(n)$ and $\delta_O(n)$ are functions of the random set $A([\omega_d n^d])$.

Theorem 1.2 [4, Theorem 1] If $d \geq 2$, then with probability 1,

$$
\delta_I(n) = o(n^{1/3}(\ln n)^2) \quad \text{and} \quad \delta_O(n) = o(n^{1/3}(\ln n)^4).
$$

2 Logarithmic fluctuations

Our result is a refinement of Theorem 1.2:

Theorem 2.1 If $d \geq 2$, then with probability 1,

$$
\begin{align*}
\delta_I(n) &= O((\ln n)^{1/2}) & \text{for } d \geq 3, \\
\delta_I(n) &= O((\ln n \ln(\ln n))^{1/2}) & \text{for } d = 2
\end{align*}
\quad \text{and} \quad
\begin{align*}
\delta_O(n) &= O((\ln n)^{3/2}) & \text{for } d \geq 4, \\
\delta_O(n) &= O((\ln n)^{5/3}) & \text{for } d = 3, \\
\delta_O(n) &= O((\ln n)^2) & \text{for } d = 2.
\end{align*}
$$

The proof is adapted from Lawler’s one in [4] and is two steps. The first step deals with the inner error, the second step with the outer error. The new idea is to use an induction involving the size of the fluctuations.

For the inner error between the cluster at time $[\omega_d n^d]$ and the ball $B(n)$, Lawler’s proof uses the comparison of two random variables which depend on all the $[\omega_d n^d]$ particles. This comparison is based on sharp estimates of the stopped Green function and on the expected exit time from $B(n)$ starting at some point $y \in B(n)$. The proof of Theorem 2.1 relies on the comparison of similar random variables but, thanks to an induction argument, these variables depend only on a part of the $[\omega_d n^d]$ particles. The new ingredient needed for the comparison of these two random variables is an asymptotic expansion of the expected time spent inside an annulus of logarithmic width within $B(n)$. This leads to the logarithmic inner error estimates. Similar variations on Lawler’s proof give the logarithmic estimates for the outer error.

Like in [5] and [4], Theorem 2.1 concerns the study of the internal DLA on $\mathbb{Z}^d$ for the simple random walk. For other random walks, [1] gives polynomial error estimates depending on moment conditions for the random walk. The present improvement up to logarithmic errors
can probably be extended to random walks with exponential moments. It seems unlikely it holds under weaker conditions since one must avoid having too many jumps of more than logarithmic size.

Recently Moore and Machta [6] simulated the two dimensional internal DLA for the simple random walk. They obtain a logarithmic behavior for the square of the deviation between the average distance, from the origin, on the boundary and the expected one \((\sqrt{k/n})\) at time \(k\). These simulations tend to prove that the order \((\ln n)^{1/2}\) should be the bottom line for the inner and outer error, at least in dimension two.

2.1 Inner part

We give an alternative construction of the internal DLA. We let the particles run until they either add to the cluster or hit \(\partial B(n) = B(n + 1) \setminus B(n)\). At this point, the cluster is denoted \(A_0^d(\omega_d n^d)\). Then we let the particles that have not added the cluster run until they add to the cluster.

This process may appear slightly different from the internal DLA because it can happen that a particle \(S_i\) stopped on \(\partial B(n)\) would have add some point \(z \in B(n)\) to the cluster, if not stopped. Then this “empty space” can be taken by a later particle \(S_k\) \((k > i)\). In this case, when \(S_i\) restarts and finally hits \(z\), the initial construction of the internal DLA would require to stop \(S_i\) and restart \(S_k\). Then, as all the \(S_i\)’s are independent, we obtain the same cluster. See [5, Section 6] for details.

Remark that \(A_0^d(\omega_d n^d) \subset A(\omega_d n^d)\). That is why, estimating the inner error with respect to \(A_0^d(\omega_d n^d)\) is enough to get the inner part of Theorem 2.1.

We denote \(f_i(n) = (\ln n)^{1/2}\) for \(d \geq 3\) and \(f_i(n) = (\ln n \ln(\ln n))^{1/2}\) for \(d = 2\). Let define

\[
\mathcal{H}_i(n, C) = \{B(n - Cf_i(n)) \subset A_0^d(\omega_d n^d)\},
\]

for \(n \in \mathbb{N}^*\) and \(C > 1\). By [5], with probability 1, for all \(k\) large enough,

\[
B(k/2) \subset A_0^d(\omega_d k^d).
\]

Then, with \(C_k = k/(2f_i(k))\), we get \(\mathcal{H}_i(k, C_k)\) for \(k\) large enough, with probability 1. We need an upper bound for the probability that \(\mathcal{H}_i(n, C)\) does not occur, knowing that \(\mathcal{H}_i(n - Cf_i(n), C)\) occurs. Namely, it suffices to prove

\[
P\{\overline{\mathcal{H}_i(n, C)} | \mathcal{H}_i(n - Cf_i(n), C)\} \leq C n^{-1 - \frac{C}{\ln n}} ,
\]

for \(n\) and \(C\) large enough. Indeed, if we denote

\[
\overline{\mathcal{H}_i'(k)} = \mathcal{H}_i(k, C_k) \cap \{\exists n > k + C_k f_i(k), \text{ s.t. } \mathcal{H}_i(n, C_k)\},
\]

the inequality (1) yields

\[
P\{\overline{\mathcal{H}_i'(k)}\} \leq P\{\exists n > k, \text{ s.t. } \mathcal{H}(n, C_k) \cap \mathcal{H}_i(n - C_k f_i(n), C_k)\}
\leq \sum_{n = k + C_k f_i(n)}^{\infty} P\{\overline{\mathcal{H}_i(n, C_k)} | \mathcal{H}_i(n - C_k f_i(n), C_k)\}
\leq 12k^{-C_k/12} \leq \exp(-k).
\]
Hence, by the Borel-Cantelli lemma, \( \mathbf{P}\left( \bar{\mathcal{H}}'(k) \text{ i.o.} \right) = 0 \), so with probability 1, for \( k \) large enough,

\[
\mathcal{H}_I(k, C_k) \cup \left\{ \forall n > k + C_k f_I(k), \mathcal{H}_I(n, C_k) \right\}.
\]

But \( \mathcal{H}_I(k, C_k) \) has probability 1 for \( k \) large enough. Then, there exists a constant \( C < \infty \) such that for all \( n \) large enough,

\[
B(n - C f_I(n)) \subset A^n_0([\omega_d n^d]) \subset A([\omega_d n^d]),
\]

which proves the inner part of Theorem 2.1.

Remark that (1) follows from

\[
\mathbf{P}\left\{ B(n - C f_I(n)) \not\subset A^n_0([\omega_d n^d]) \mid B(n - 2 C f_I(n)) \subset A^n_0([\omega_d(n - C f_I(n))^d]) \right\} \leq Cn^{-1 - \frac{\nu}{d}}. \tag{2}
\]

In the sequel, we suppose \( \{B(n - 2 C f_I(n)) \subset A^n_0([\omega_d(n - C f_I(n))^d])\} \) without mentioning this conditioning. We study the model at time \([\omega_d n^d]\). Let \( m = m(n) = [\omega_d n^d] - [\omega_d(n - C f_I(n))^d] \)
and \( m'(n) = [\omega_d n^d] - [\omega_d(n - 2 C f_I(n))^d] \).

Now, we let the particles live even after they reach the point where they add to the cluster. Among the \([\omega_d n^d]\) particles, the \( j \)'th one is associated to the random walk \( S^j(k) \) starting at 0, and we define

\[
\xi_n^j = \inf \{ k > 0 : S^j(k) \in \partial B(n) \} \quad \text{exit time from } B(n),
\]
\[
\rho^j = \inf \{ k \geq 0 : S^j(k) \not\in A^n_0(j - 1) \} \quad \text{adding time to the cluster},
\]
\[
\tau_n^z = \inf \{ k > 0 : S^j(k) = z \} \quad \text{hitting time of } z.
\]

Without the \( j \), these stopping times refer to a general simple random walk starting at 0. And \( \xi_n^y, \rho^y, \tau_n^y \) refer to a general simple random walk starting at \( y \). We denote

\[
M = M(n, z) = \sum_{j = [\omega_d(n - C f_I(n))^d] + 1}^{[\omega_d n^d]} \mathbb{I}\{\tau_n^j < \xi_n^j\}
\]

\[
= \# \text{ of } j > [\omega_d(n - C f_I(n))^d] \text{ such that the walk } S^j \text{ visits } z \text{ before leaving } B(n)
\]

\[
L = L(n, z) = \sum_{j = [\omega_d(n - C f_I(n))^d] + 1}^{[\omega_d n^d]} \mathbb{I}\{\rho^j \leq \tau_n^j < \xi_n^j\}
\]

\[
= \# \text{ of } j > [\omega_d(n - C f_I(n))^d] \text{ such that the walk } S^j \text{ visits } z \text{ after adding the cluster and before leaving } B(n).
\]

Note that

\[
\{ z \not\in A^n_0([\omega_d n^d]) \mid z \not\in A^n_0([\omega_d(n - C f_I(n))^d]) \} \subset \{ M = L \mid z \not\in A^n_0([\omega_d(n - C f_I(n))^d]) \},
\]

so for any \( a > 0 \) (denoting \( A' = A^n_0([\omega_d(n - C f_I(n))^d]) \)),

\[
\mathbf{P}\left\{ z \not\in A^n_0([\omega_d n^d]) \right\} \leq \mathbf{P}\left\{ M = L \mid z \not\in A' \right\} \leq \mathbf{P}\{ M \leq a \mid z \not\in A' \} + \mathbf{P}\{ L \geq a \mid z \not\in A' \}.
\]

To estimate the right hand side we will need to compute the expectation of \( M \) and \( L \). The summands of \( M \) are i.i.d., then we can remove the index \( j \), and

\[
\mathbb{E}\{ M \mid z \not\in A' \} = \mathbb{E}\{ M \} = m \mathbf{P}\{ \tau_n < \xi_n \}.
\]
The summands of $L$ are not i.i.d. but only the particles such that $p_j \leq \xi_n$ and $S(p_j) \not\in A'$ contribute to the sum. Moreover, as $B(n - 2C f_1(n)) \subset A'$, any such particle must add to the cluster within $B(n) \setminus B(n - 2C f_1(n))$. Note also that, for each $y \in B(n) \setminus B(n - 2C f_1(n))$, there is at most one $j$ such that $S^j(p_j) = y$. So,

$$L \leq \sum_{y \in B(n) \setminus B(n - 2C f_1(n))} \mathbb{I}\{\tau^n_y < \xi_n, y \not\in A'\} = \tilde{L}.$$  

Therefore,

$$\mathbb{P}\{z \not\in A_n^0([\omega_0 n^d])\} \leq \mathbb{P}\{M \leq a | z \not\in A'\} + \mathbb{P}\{\tilde{L} \geq a | z \not\in A'\}. \quad (3)$$

Let $G_n(x, y)$ be the stopped Green function, that is the expected number of visits to $y$, starting at $x$, before leaving $B(n)$ ($x, y \in B(n)$),

$$G_n(x, y) = \mathbb{E}^x\left\{\sum_{k=0}^{\xi_n-1} \mathbb{I}\{S(k) = y\}\right\}.$$ 

For any $z$ in $B(n - C f_1(n)) \setminus B(n - 2C f_1(n))$,

$$G_n(z, z) \mathbb{E}\{M | z \not\in A'\} = mG_n(z, z) \mathbb{P}\{\tau_z < \xi_n\} = mG_n(0, z).$$

We denote $\delta = n - ||z||$. By [4, Lemma 2],

$$G_n(z, z) \mathbb{E}\{M | z \not\in A'\} = 2dC \delta f_1(n) + 2dC b(z) f_1(n) + O(1), \quad (4)$$

where $b(z) = \mathbb{E}^z\{||S(\xi_n^z)|| - n\}$. On the other hand, to compute $\mathbb{E}\{\tilde{L} | z \not\in A'\}$, we write

$$\mathbb{E}\{\tilde{L} | z \not\in A'\} = \sum_{y \in B(n) \setminus B(n - 2C f_1(n))} \mathbb{P}\{\tau^n_y < \xi_n, y \not\in A'\} - \mathbb{I}\{\sum_{y \in B(n) \setminus B(n - 2C f_1(n))} \mathbb{I}\{\tau^n_y < \xi_n, y \not\in A'\} \not\in E(n, z)\}.$$ 

But

$$E(n, z) = \mathbb{E}\left\{\sum_{j=1}^{[\omega_0(n-Cf_1(n))^d]} \sum_{y \in B(n)} \mathbb{I}\{\tau^n_y < \xi_n\} \mathbb{I}\{S(p_j) = y\}\right\} \times \mathbb{I}\{S(p_j) \not\in B(n - 2C f_1(n)) | z \not\in A'\}$$

$$= \mathbb{E}\left\{\sum_{j=1}^{[\omega_0(n-Cf_1(n))^d]} \sum_{y \in A(j-1)} \mathbb{I}\{\tau^n_y < \xi_n\} \mathbb{I}\{S(\tau_{A(j-1)}) = y\}\right\} \times \mathbb{I}\{S(p_j) \not\in B(n - 2C f_1(n)) | z \not\in A'\}$$

$$= \mathbb{E}\left\{\sum_{j=1}^{[\omega_0(n-Cf_1(n))^d]} \mathbb{I}\{S(p_j) \not\in B(n - 2C f_1(n))\} \mathbb{I}\{\tau_z < \xi_n\} | z \not\in A'\right\}$$

$$= \mathbb{P}\{\tau_z < \xi_n, z \not\in A'\} \mathbb{E}\left\{\sum_{j=1}^{[\omega_0(n-Cf_1(n))^d]} \mathbb{I}\{S(p_j) \not\in B(n - 2C f_1(n))\} | \tau_z < \xi_n, z \not\in A'\right\}$$

$$= (m' - m) \mathbb{P}\{\tau_z < \xi_n\}. \quad (5)$$
The third equality uses the Markov property and the fact that, under \( z \not\in A' \), for any \( j \leq \lfloor \omega_2(n - C_1(n)) \rfloor \), we have \( z \not\in A(j - 1) \). The last equality comes from the conditioning by \( \{B(n - 2C_1(n)) \subset A_0^n(\lfloor \omega_2(n - C_1(n)) \rfloor)\} \), which is always supposed in those computations. By the symmetry of the random walk, \( G_n(y, z) = G_n(z, y) \). So

\[
G_n(z, z) \mathbb{E}\{\tilde{L} | z \not\in A'\} = \sum_{y \in B(n) \setminus B(n - 2C_1(n))} G_n(z, y) - (m' - m)G_n(0, z),
\]

so we are left with the estimation of this last term. We give this estimate in a more general form, which will be useful in the proof of the outer error.

**Proposition 2.2** Let \( d \geq 2 \), \( f(n) = o(n^{1/3}) \), \( z \in B(n) \setminus B(n - f(n)) \) and \( \delta = n - \|z\| \), then

\[
\sum_{y \in B(n) \setminus B(n - f(n))} G_n(z, y) = 2d\delta f(n) + 2db_0 f(n) - d\delta^2 - 2db_0\delta + O(1),
\]

with \( b_0 \) defined below.

**Proof**

The left-hand side is the time spent in the annulus \( B(n) \setminus B(n - f(n)) \), starting at \( z \), before the exit time from \( B(n) \).

Let \( \tau_{n-f(n)}^z = \inf\{k \geq 0 : \|S^z(k)\| < n - f(n)\} \). We define the following stopping times \( (D_i) \) and \( (U_i) \), for \( i \geq 0 \), corresponding to the \( i^{th} \) downward and upward crossing of the sphere of \( \mathbb{R}^d \) of radius \( n - f(n) \).

- \( D_0 = 0 = U_0 \);
- If \( \tau_{n-f(n)}^z < \xi_n^z \), then \( D_1 = \tau_{n-f(n)}^{S(D_1)} \) and \( U_1 = \xi_n^{S(U_1)} \);
- If \( \tau_{n-f(n)}^z < \xi_n^z \) and \( \tau_{n-f(n)}^{S(U_1)} < \xi_n^{S(U_1)} \), then \( D_2 = \tau_{n-f(n)}^{S(U_1)} \) and \( U_2 = \xi_n^{S(U_2)} \);
- For all \( i \geq 2 \), if \( \forall j \leq i - 1, \tau_{n-f(n)}^{S(U_j)} < \xi_n^{S(U_j)} \), then \( D_i = \tau_{n-f(n)}^{S(U_{i-1})} \) and \( U_i = \xi_n^{S(D_i)} \).

With these notations, we can write

\[
\sum_{y \in B(n) \setminus B(n - f(n))} G_n(z, y) = \mathbb{E}^z \{\tau_{n-f(n)}^z \wedge \xi_n^z\} + \mathbb{P}^z \{\tau_{n-f(n)}^z < \xi_n^z\}
\]

\[
\times \sum_{i=1}^{\infty} \mathbb{E}^z \{\tau_{n-f(n)}^{S(U_i)} \wedge \xi_n^{S(U_i)}\} \mathbb{P}^{S(U_i)} \{\tau_{n-f(n)}^{S(U_j)} < \xi_n^{S(U_j)}\} | \forall j \leq i - 1, \tau_{n-f(n)}^{S(U_j)} < \xi_n^{S(U_j)}\}
\]

Now, we compute each term of this expression. For every \( i \) such that \( U_i \) exists, we denote:

- \( a_i = a_i(z) = \mathbb{E}^{S(U_i)} \{n - f(n) - \|S(D_{i+1})\| \mid \tau_{n-f(n)}^{S(U_i)} < \xi_n^{S(U_i)}\} \);
- \( b_i = b_i(z) = \mathbb{E}^{S(U_i)} \{\|S(\xi_n^{S(U_i)})\| - n \mid \tau_{n-f(n)}^{S(U_i)} > \xi_n^{S(U_i)}\} \);
- \( c_i = c_i(z) = \|S(U_i)\| - (n - f(n)) \).
By a martingale argument (see [3, Prop. 1.5.10]), we get, for $d \geq 3$,

$$
P^z \{ \tau^z_{n-f(n)} < \xi^z_n \} = \frac{G(z) - \mathbb{E}^z \{ G(S(\xi^z_n)) \mid \tau^z_{n-f(n)} > \xi^z_n \}}{\mathbb{E}^z \{ G(S(\tau^z_{n-f(n)})) \mid \tau^z_{n-f(n)} < \xi^z_n \} - \mathbb{E}^z \{ G(S(\xi^z_n)) \mid \tau^z_{n-f(n)} > \xi^z_n \}}.
$$

By [3, Th. 1.5.4], for $d \geq 3$,

$$
\omega_d G(z) = \frac{2}{d-2} \| z \|^{2-d} + O(n^{-d}) = \frac{2}{d-2} n^{2-d} + 2\delta n^{1-d} + (d-1)\delta^2 n^{-d} + O(n^{-d}),
$$

$$
\omega_d \mathbb{E}^z \{ G(S(\xi^z_n)) \mid \tau^z_{n-f(n)} > \xi^z_n \} = \frac{2}{d-2} \mathbb{E}^z \{ \| S(\xi^z_n) \|^{2-d} \mid \tau^z_{n-f(n)} > \xi^z_n \} + O(n^{-d})
$$

$$
= \frac{2}{d-2} \mathbb{E}^z \{ n^{2-d}(1 + (2-d)n^{-1})(\| S(\xi^z_n) \| - n) + O(n^{-2}) \mid \tau^z_{n-f(n)} > \xi^z_n \} + O(n^{-d})
$$

$$
= \frac{2}{d-2} n^{2-d} - 2\delta n^{1-d} + O(n^{-d}),
$$

and,

$$
\omega_d \mathbb{E}^z \{ G(S(\tau^z_{n-f(n)})) \mid \tau^z_{n-f(n)} < \xi^z_n \} = \frac{2}{d-2} \mathbb{E}^z \{ \| S(\tau^z_{n-f(n)}) \|^{2-d} \mid \tau^z_{n-f(n)} < \xi^z_n \} + O(n^{-d})
$$

$$
= \frac{2}{d-2} \mathbb{E}^z \{ (n - f(n))^{2-d}(1 - (2-d)(n - f(n))^{-1}
$$

$$
\times (n - f(n) - \| S(\tau^z_{n-f(n)}) \|) + O(n^{-2})) \mid \tau^z_{n-f(n)} < \xi^z_n \} + O(n^{-d})
$$

$$
= \frac{2}{d-2} (n - f(n))^{2-d} + 2(n - f(n))^{1-d} a_0 + O(n^{-d})
$$

$$
= \frac{2}{d-2} n^{2-d} - 2 n^{1-d} f(n) + 2 a_0 n^{-d} + (d-1) n^{-d} f(n)^2
$$

$$
+ 2(d-1) a_0 n^{d} f(n) + O(n^{-d}).
$$

Likewise, for $d = 2$, with the potential kernel $a(z)$ instead of the Green function,

$$
P^z \{ \tau^z_{n-f(n)} < \xi^z_n \} = \frac{\mathbb{E}^z \{ a(S(\xi^z_n)) \mid \tau^z_{n-f(n)} > \xi^z_n \} - a(z)}{\mathbb{E}^z \{ a(S(\tau^z_{n-f(n)})) \mid \tau^z_{n-f(n)} > \xi^z_n \} - \mathbb{E}^z \{ a(S(\tau^z_{n-f(n)})) \mid \tau^z_{n-f(n)} < \xi^z_n \}}.
$$

Moreover, by [3, Th. 1.6.2], there is a constant $C_0$ such that

$$
\omega_2 a(z) = 2 \ln n + C_0 - 2\delta n^{-1} - \delta^2 n^{-2} + O(n^{-2}),
$$

$$
\omega_2 \mathbb{E}^z \{ a(S(\xi^z_n)) \mid \tau^z_{n-f(n)} > \xi^z_n \} = 2 \ln n + C_0 + 2b_0 n^{-1} + O(n^{-2}),
$$

and,

$$
\omega_2 \mathbb{E}^z \{ a(S(\tau^z_{n-f(n)})) \mid \tau^z_{n-f(n)} < \xi^z_n \} = 2 \ln n + C_0 - 2n^{-1} f(n) - 2a_0 n^{-1} - n^{-2} f(n)^2
$$

$$
- 2a_0 n^{-2} f(n) + O(n^{-2}).
$$

So, we get, for $d \geq 2$

$$
P^z \{ \tau^z_{n-f(n)} < \xi^z_n \} = \frac{2\delta + 2b_0 + (d-1)\delta^2 n^{-1} + O(n^{-1})}{2f(n) + 2(a_0 + b_0) + (d-1)n^{-1} f(n)^2 + 2(d-1)a_0 n^{-1} f(n) + O(n^{-1})}
$$

$$
= \left( \frac{2f(n) + 2(a_0 + b_0) + (d-1)n^{-1} f(n)^2 + 2(d-1)a_0 n^{-1} f(n) + O(n^{-1})}{(2f(n) + 2(a_0 + b_0))^{-1}} \right) \frac{2\delta + 2b_0 + (d-1)\delta^2 n^{-1} + O(n^{-1})}{(2f(n) + 2(a_0 + b_0))^{-1} [2\delta + 2b_0 + (d-1)n^{-1} (\delta^2 - \delta f(n))^2 - (d-1)b_0 n^{-1} f(n) + (d-1)(b_0 - a_0)\delta n^{-1} + O(n^{-1})]},
$$

(8)
which can also be written as
\[
P^z \{ \tau^z_{n-f(n)} < \xi^z_n \} = \delta f(n)^{-1} + b_0 f(n)^{-1} - (a_0 + b_0)\delta f(n)^{-2} + O(f(n)^{-2}). \tag{9}
\]
Now, by another martingale argument (see [3, (1.21)]),
\[
\mathbb{E}^z \{ \tau^z_{n-f(n)} \land \xi^z_n \} = \mathbb{E}^z \{ \| S(\tau^z_{n-f(n)} \land \xi^z_n) \|^2 \} - \| z \|^2,
\]
and we easily get
\[
\mathbb{E}^z \{ \| S(\tau^z_{n-f(n)} \land \xi^z_n) \|^2 \} = \mathbb{E}^z \{ \| S(\xi^z_n) \|^2 \mid \tau^z_{n-f(n)} > \xi^z_n \} P^z \{ \tau^z_{n-f(n)} > \xi^z_n \} \\
+ \mathbb{E}^z \{ \| S(\tau^z_{n-f(n)}) \|^2 \mid \tau^z_{n-f(n)} < \xi^z_n \} P^z \{ \tau^z_{n-f(n)} < \xi^z_n \} \\
= (n^2 + 2b_0n)P^z \{ \tau^z_{n-f(n)} > \xi^z_n \} + (n^2 - 2f(n)n - 2a_0n \\
\quad + f(n)^2 + 2a_0 f(n))P^z \{ \tau^z_{n-f(n)} < \xi^z_n \} + O(1) \\
= n^2 + 2b_0n - n(2f(n) + 2(a_0 + b_0))P^z \{ \tau^z_{n-f(n)} < \xi^z_n \} \\
\quad + (f(n)^2 + 2a_0 f(n))P^z \{ \tau^z_{n-f(n)} < \xi^z_n \} + O(1) \\
= n^2 - 2\delta n + d\delta f(n) - (d-1)\delta^2 + db_0 f(n) + d(a_0 - b_0)\delta + O(1)
\]
by (8) and (9). Then, as \( \| z \|^2 = n^2 - 2\delta n + \delta^2 \),
\[
\mathbb{E}^z \{ \tau^z_{n-f(n)} \land \xi^z_n \} = d\delta f(n) - d\delta^2 + db_0 f(n) + d(a_0 - b_0)\delta + O(1). \tag{10}
\]
The same computation with \( n - \| S(U_i)\| = f(n) - c_i \) instead of \( \delta \) gives
\[
\mathbb{E} \{ \tau_{n-f(n)}^S \land \xi_n^S \mid \forall j \leq i - 1, \tau_{n-f(n)}^S < \xi_n^S \} \\
= d(f(n) - c_i)c_i + db_i f(n) + d(a_i - b_i)(f(n) - c_i) + O(1) \\
= df(n)(a_i + c_i) + O(1), \tag{11}
\]
and,
\[
\mathbb{E} \{ P^S \{ \tau_{n-f(n)}^S < \xi_n^S \} \mid \forall l \leq j-1, \tau_{n-f(n)}^S < \xi_n^S \} = 1 - (a_i + c_i)f(n)^{-1} + O(f(n)^{-2}). \tag{12}
\]
Remark that all the \( O(\cdot) \) functions used above do not depend on \( i \) nor \( j \). Remark also that
\[
a_i + c_i = \mathbb{E}^S \{ \| S(U_i) \| - \| S(D_{i+1}) \| \mid \tau_{n-f(n)}^S < \xi_n^S \} ,
\]
and, we easily get that \( S(U_i) \) has a neighbor \( x \) within \( B(n - f(n)) \) such that \( \| S(U_i) \| - \| x \| \geq 1/(2\sqrt{d}) \), so
\[
2 \geq a_i + c_i \geq 1/(4d\sqrt{d}) > 0. \tag{13}
\]
For \( p \geq 1 \), let
\[
\sigma_p = \sum_{i=1}^p \mathbb{E} \{ \tau_{n-f(n)}^S \land \xi_n^S \mid \forall j \leq i - 1, \tau_{n-f(n)}^S < \xi_n^S \} \\
\times \prod_{j=1}^{i-1} \mathbb{E} \{ P^S \{ \tau_{n-f(n)}^S < \xi_n^S \} \mid \forall l \leq j-1, \tau_{n-f(n)}^S < \xi_n^S \}.
\]
By (11),
\[ \sigma_1 = df(n)^2 \left[ 1 - (1 - (a_1 + c_1)f(n)^{-1} + O(f(n)^{-2})) \right] + O(1). \]

Now, suppose
\[ \sigma_p = df(n)^2 \left[ 1 - \prod_{j=1}^{p} (1 - (a_j + c_j)f(n)^{-1} + O(f(n)^{-2})) \right] + O(1), \]
then by (11) and (12),
\[ \sigma_{p+1} = \sigma_p + (df(n)(a_{p+1} + c_{p+1}) + O(1)) \prod_{j=1}^{p} (1 - (a_j + c_j)f(n)^{-1} + O(f(n)^{-2})) \]
\[ = df(n)^2 \left[ 1 - \prod_{j=1}^{p+1} (1 - (a_j + c_j)f(n)^{-1} + O(f(n)^{-2})) \right] + O(1). \]

Then by (13),
\[ \sigma_\infty = df(n)^2 \left[ 1 - \prod_{j=1}^{\infty} (1 - (a_j + c_j)f(n)^{-1} + O(f(n)^{-2})) \right] + O(1) = df(n)^2 + O(1). \quad (14) \]

Finally, putting (9), (10) and (14) into (7), we get
\[ \sum_{y \in B(n) \setminus B(n-f(n))} G_n(z, y) = 2d\delta f(n) + 2db_0 f(n) - d\delta^2 - 2db_0 \delta + O(1). \]

\[ \square \]

From (6), [4, Lemma 2] and Proposition 2.2 with \( f(n) = 2C f_1(n), \)
\[ G_n(z, z)\mathbb{E} \{ \tilde{L} \mid z \not\in A' \} = 2dC\delta f_1(n) + 2dC f_1(n)(2b_0 - b(z)) - d\delta^2 - 2db_0 \delta + O(1). \]

Together with (4), we obtain
\[ G_n(z, z)[\mathbb{E} \{ M \mid z \not\in A' \} - \mathbb{E} \{ \tilde{L} \mid z \not\in A' \}] = d\delta^2 + O(f_1(n)), \]

As \( \delta \geq C f_1(n), \) we get
\[ \mathbb{E} \{ M \mid z \not\in A' \} - \mathbb{E} \{ \tilde{L} \mid z \not\in A' \} \geq \frac{1}{10} [\mathbb{E} \{ M \mid z \not\in A' \} + \mathbb{E} \{ \tilde{L} \mid z \not\in A' \}]. \]

Now, we need a large deviation result [5, Lemma 4]:

**Lemma 2.3** Let \( S \) be a sum of \( k \) independent indicator functions and \( \mu = \mathbb{E} \{ S \}. \) Then for all sufficiently large \( k \) and all \( \alpha > 0, \)
\[ \mathbb{P} \{|S - \mu| \geq \alpha \mu \} \leq 2 \exp(-\alpha \mu/2). \]
Note that this result holds under a conditioning by an event \( \mathcal{E} \) with \( \mu = \mathbb{E}\{S|\mathcal{E}\} \).

Hence, taking \( a = \mathbb{E}\{M|z \not\in A'\} - \frac{1}{20}[\mathbb{E}\{M|z \not\in A'\} + \mathbb{E}\{\tilde{L}|z \not\in A'\}] \) in (3) and \( C \) large enough, yields for \( d \geq 3 \),

\[
\mathbb{P}\{M \leq a | z \not\in A'\} \leq \mathbb{P}\{|M - \mathbb{E}\{M|z \not\in A'\}| \geq \mathbb{E}\{M|z \not\in A'\}/20 | z \not\in A'\} \\
\leq 2 \exp(-\mathbb{E}\{M|z \not\in A'\}/40) \leq 2 \exp(-C^2G(0)^{-1} \ln n/10)
\]

and

\[
\mathbb{P}\{\tilde{L} \geq a | z \not\in A'\} \\
\leq \mathbb{P}\{|\tilde{L} - \mathbb{E}\{\tilde{L}|z \not\in A'\}| \geq \mathbb{E}\{\tilde{L}|z \not\in A'\}/20 | z \not\in A'\} \\
\leq 2 \exp(-\mathbb{E}\{\tilde{L}|z \not\in A'\}/40) \leq 2 \exp(-C^2G(0)^{-1} \ln n/10).
\]

As \( d \geq 3 \), we used \( G_n(z,z) \leq G(0) \), where

\[
G(x) = G(0, x) = \mathbb{E}\left\{ \sum_{k=0}^{\infty} \mathbb{I}\{S(k) = x\} \right\},
\]

is the Green function. Then, denoting \( \mathcal{A} = B(n - C \sqrt{\ln n}) \setminus B(n - 2C \sqrt{n}) \),

\[
\mathbb{P}\{A \not\subset A_0^n(\lfloor \omega_d n^d \rfloor) | B(n - 2C \sqrt{\ln n}) \subset A'\} \\
\leq \sum_{z \in \mathcal{A}} \mathbb{P}\{z \not\in A_0^n(\lfloor \omega_d n^d \rfloor) | z \not\in A', B(n - 2C \sqrt{\ln n}) \subset A'\} \\
\leq c_d C n^{d-1} n^{-C/10} \sqrt{\ln n}.
\]

It yields (2) for \( C > 60d \), and then the inner part of Theorem 2.1, for \( d \geq 3 \).

For \( d = 2 \), we need the following lemma.

**Lemma 2.4** Let \( d = 2 \) and \( \|z\| = n - \delta \), then

\[
G_n(z, z) \approx \ln \delta,
\]

where \( \approx \) means 'up to a multiplicative constant on both side'.

**Proof**

On \( \mathbb{R}^2 \), let \( H(z) \) be the half plane whose frontier is tangent to \( B(n) \) and such that \( \text{dist}(z, H(z)) = \delta \). We identify \( H(z) \) with its trace on \( \mathbb{Z}^d \) and denote \( \tau = \tau^z(H(z)) \) its hitting time from \( z \). By the central limit theorem, there exists a constant \( \mu < 1 \) such that

\[
\mathbb{P}^z\{S^z(k) = z \text{ for some } k \text{ such that } \delta^2 \leq k < \tau\} \leq \mathbb{P}^z\{\delta^2 < \tau\} \leq \mu < 1.
\]

Moreover, if we denote \( \varepsilon(z) \) the event \( \{S^z(k) = z \text{ for some } k \text{ such that } \delta^2 \leq k \leq \tau\}, \)

\[
\mathbb{E}^z\{\# \text{ of visits to } z \text{ before } \tau\} = \mathbb{E}^z\{\# \text{ of visits to } z \text{ before } \tau \wedge \delta^2\} \\
+ \mathbb{E}^z\{\# \text{ of visits to } z \text{ before } \tau \text{ and after } \delta^2|\varepsilon(z)\}\mathbb{P}\{\varepsilon(z)\}.
\]

But, by the Markov property,

\[
\mathbb{E}^z\{\# \text{ of visits to } z \text{ before } \tau\} = \mathbb{E}^z\{\# \text{ of visits to } z \text{ before } \tau \text{ and after } \delta^2|\varepsilon(z)\}.
\]
So, by (15),

\[ \mathbb{E}^z \{ \# \text{ of visits to } z \text{ before } \tau \} \leq \frac{1}{1 - \mu} \mathbb{E}^z \{ \# \text{ of visits to } z \text{ before } \tau \wedge \delta^2 \} \leq \frac{1}{1 - \mu} \mathbb{E}^z \{ \# \text{ of visits to } z \text{ before } \delta^2 \} \leq \frac{1}{1 - \mu} e \ln \delta^2. \]

The last inequality comes from classical diagonal estimates of the iterated transition probability. Finally, \( G_n(z, z) \leq \mathbb{E}^z \{ \# \text{ of visits to } z \text{ before } \tau \} \) gives the upper bound of the lemma, and the lower bound is given by the central limit theorem since \( G_n(z, z) \geq G_\delta(0, 0) \).

Using this lemma, we get

\[ \mathbf{P} \{ M \leq a \mid z \notin A' \} \leq 2 \exp(-C^2 \ln n), \]

and

\[ \mathbf{P} \{ \tilde{L} \geq a \mid z \notin A' \} \leq 2 \exp(-C^2 \ln n), \]

for \( n \) large enough. The inner part of Theorem 2.1 follows then as in the case \( d \geq 3 \).

2.2 Outer part

Denote \( \alpha = 3/2 \) for \( d \geq 4 \), \( \alpha = 5/3 \) for \( d = 3 \) and \( \alpha = 2 \) for \( d = 2 \). We give another alternative construction of the internal DLA, in the spirit of the one used for the inner error. Let \( \beta = (3 - \alpha)/(d - 1) \) and \( n_\varepsilon = n + s(\ln n)^\beta \). The choice of \( \alpha \) is the smallest possible with the techniques we use, according to a suitable choice of \( \beta \). Now the index \( s \) always refers to \( \partial B(n_\varepsilon) \) (for simplicity, we use \( s \) instead of \( n_\varepsilon \) for indices), except when \( s = 0 \) for which we keep \( n \) not to be confusing. We let the first \( j \) particles run until they either add to the cluster or hit \( \partial B(n) \). Then we let the particles that have not added the cluster run until they either add to the cluster or reach \( \partial B(n + (\ln n)^\beta) \). And we continue this process with the spheres \( \partial B(n_\varepsilon) \) until all the particles add to the cluster.

As before, the final cluster has the same law than the one obtained by the initial definition. We denote \( A^n_s(j) \) \((s \geq 0)\) the cluster obtained by the above process with \( j \) particles, when those that have not added the cluster are stopped on \( \partial B(n_\varepsilon) \). We easily see that

\[ A^n_s(j) \subset A^n_s(j + 1), \quad A^n_s(j) \subset A^n_{s+1}(j), \text{ and } A([\omega_d n^\delta]) = \lim_{s \to \infty} A^n_s([\omega_d n^\delta]). \]

Let

\[ \mathcal{H}_O(n, C) = \{ A([\omega_d n^\delta]) \subset B(n + C(\ln n)^\alpha) \} \]

and \( \mathcal{H}(n, C) = \mathcal{H}_I(n, C^{1/2}) \cap \mathcal{H}_O(n, C) \) for \( n \in \mathbb{N}^+ \) and \( C > 1 \). By [4], with probability 1, for all \( k \) large enough,

\[ B(k - k^{1/2} f_1(k)(\ln k)^{-\alpha/2}) \subset A_0^k([\omega_d k^\delta]) \subset A([\omega_d k^\delta]) \subset B(2k). \]

Then, with \( C_k = k(\ln k)^{-\alpha} \), we get \( \mathcal{H}_O(k, C_k) \) and \( \mathcal{H}_I(k, C_k^{1/2}) \), and so \( \mathcal{H}(k, C_k) \), for \( k \) large enough, with probability 1. It suffices to prove

\[ \mathbf{P} \{ \mathcal{H}(n, C) \cap \mathcal{H}_O(n - 2C(\ln n)^\alpha, C) \cap \mathcal{H}_I(n - 2C(\ln n)^\alpha, C^{1/2}) \} \leq 3C^{1/2}n^{-1-C^{1/4}}, \quad (16) \]
for \( n \) and \( C \) large enough. Indeed, if we denote
\[
\mathcal{H}'(k) = \mathcal{H}(k, C_k) \cap \{ \exists n > k + 2C_k(\ln k)^\alpha, \text{ s.t. } \mathcal{H}(n, C_k) \},
\]
the inequality (16) yields
\[
\mathbf{P}\{\mathcal{H}'(k)\} \leq \mathbf{P}\{\exists n > k + 2C_k(\ln k)^\alpha, \text{ s.t. } \mathcal{H}(n, C_k) \cap \mathcal{H}(n - 2C_k(\ln n)^\alpha, C_k)\}
\leq \sum_{n=k}^{\infty} \mathbf{P}\{\mathcal{H}(n, C_k) \cap \mathcal{H}_0(n - 2C_k(\ln n)^\alpha, C_k) \mid \mathcal{H}_1(n - 2C_k(\ln n)^\alpha, C_k)\}
\leq 3C_k^{1/2} - C_k^{1/4} \leq \exp(-k^{1/4}).
\]
Hence, by the Borel-Cantelli lemma, \( \mathbf{P}\{\mathcal{H}'(k) \text{ i.o.}\} = 0 \), so with probability 1, for \( k \) large enough,
\[
\mathcal{H}(k, C_k) \cup \{ \forall n > k + 2C_k(\ln k)^\alpha, \mathcal{H}(n, C_k) \}.
\]
But \( \mathcal{H}(k, C_k) \) has probability 1 for \( k \) large enough. Then, there exists a constant \( C < \infty \) such that for all \( n \) large enough,
\[
A(\lfloor \omega_d n^d \rfloor) \subset B(n + C(\ln n)^\alpha),
\]
which proves the outer part of Theorem 2.1. In the sequel, we condition by the event
\[
\{B(n - 2C(\ln n)^\alpha - C_1^{1/2} f_1(n)) \subset A_0'(\lfloor \omega_d(n - 2C(\ln n)^\alpha)^d \rfloor)\}.
\]
We study the model at time \( \lfloor \omega_d n^d \rfloor \). Let \( m = m(n) = \lfloor \omega_d n^d \rfloor - \lfloor \omega_d(n - 2C(\ln n)^\alpha)^d \rfloor \) and \( m' = m'(n) = \lfloor \omega_d n^d \rfloor - \lfloor \omega_d(n - 2C(\ln n)^\alpha - C_1^{1/2} f_1(n))^d \rfloor \), where \( f_1(n) \), defined in the proof of the inner part, is \( \lfloor (\ln n)^{1/2} \rfloor \) for \( d \geq 3 \) and \( \lfloor (\ln n \ln(\ln n))^{1/2} \rfloor \) for \( d = 2 \). We denote \( \rho^j \) the adding time of the \( j^{th} \) particle in the new construction of the internal DLA at time \( \lfloor \omega_d n^d \rfloor \).

Now, we let the particles live even after they reach the point where they add the cluster. We denote \( \mathcal{A}' = B(n - C_1^{1/2} f_1(n)) \setminus B(n - 2C(\ln n)^\alpha - C_1^{1/2} f_1(n)) \), and we define, for any \( s \geq 0 \) and \( z \in \partial B(n_s) \),
\[
M(s, n, z) = \sum_{j = \lfloor \omega_d(n - 2C(\ln n)^\alpha)^d \rfloor + 1}^{\lfloor \omega_d n^d \rfloor} \mathbb{I}\{S^j(x^j) = z\},
\]
\[
L(s, n, z) = \sum_{j = \lfloor \omega_d(n - 2C(\ln n)^\alpha)^d \rfloor + 1}^{\lfloor \omega_d n^d \rfloor} \mathbb{I}\{S^j(x^j) = z, x^j > \rho^j\},
\]
\[
\bar{L}(s, n, z) = \sum_{y \in \mathcal{A}'} \mathbb{I}\{S^n(y^n) = z, y \notin A(\lfloor \omega_d(n - 2C(\ln n)^\alpha)^d \rfloor)\},
\]
\[
W(s, n, z) = M(s, n, z) - L(s, n, z) = \# \text{ of } j > \lfloor \omega_d(n - 2C(\ln n)^\alpha)^d \rfloor \text{ such that } \text{ the walk } S^j \text{ hits } \partial B(n_s) \text{ at } z, \text{ before adding to the cluster}.
\]
For \( U \subset \mathbb{Z}^d \), we wrote
\[
W(s, n, U) = \sum_{z \in U \cap \partial B(n_s)} W(s, n, z), \text{ and } W(s, n) = W(s, n, \partial B(n_s)),
\]
and likewise for \( M, L \) and \( \bar{L} \).
The event \( \{B(n - C^{1/2} f_1(n)) \subset A^\circ_n(\omega_d n^d)\} \) is included into
\[
\{\forall y \in \mathcal{A}' \setminus \mathcal{A}(\omega_d (n - 2C \ln n)^\alpha) \exists j \leq \omega_d n^d \text{ s.t. } S^j(\rho^i) = y \text{ and } \rho^i < \xi^i\}
\subset \{\forall s \geq 0, \forall z \in \partial B(n), (\# \text{ of r.w. after } \omega_d (n - 2C \ln n)^\alpha \text{ s.t. } S^j(\xi^i) = z, \xi^i > \rho^i) \geq (\# \text{ of r.w. starting in } \mathcal{A}' \setminus \mathcal{A}(\omega_d (n - 2C \ln n)^\alpha)) \text{ s.t. } S(\xi) = z\}
= \{\forall s \geq 0, \forall z \in \partial B(n), L(s, n, z) = \tilde{L}(s, n, z)\}.
\]

We split the left-hand side of (16) into three parts:
\[
P\{\mathcal{H}(n, C) \cap \mathcal{H}_0(n - 2C \ln n)^\alpha, C) \mid \mathcal{H}_1(n - 2C \ln n)^\alpha, C^{1/2}\}\}
\leq P\{\mathcal{H}_1(n, C^{1/2}) \mid \mathcal{H}_1(n - 2C \ln n)^\alpha, C^{1/2}\}\}
+ P\{\exists s \geq 0 \text{ and } z \in \partial B(n) \text{ s.t. } \tilde{L}(s, n, z) > L(s, n, z) \mathcal{H}_1(n - 2C \ln n)^\alpha, C^{1/2}\}
+ P\{\mathcal{H}_0(n, C) \mathcal{H}_1(n - 2C \ln n)^\alpha, C^{1/2} \cap \mathcal{H}_0(n - 2C \ln n)^\alpha, C\}
\cap \{\forall s \geq 0, \forall z \in \partial B(n), L(s, n, z) = \tilde{L}(s, n, z)\}\}.
\] (18)

The first term of the right hand side is bounded by \(C^{1/2}n^{1-\frac{1}{2}C^{1/2}/16}\) by iterating (1). The second one is also bounded by \(C^{1/2}n^{1-\frac{1}{2}C^{1/2}/16}\) because \(\{\exists s \geq 0 \text{ and } z \in \partial B(n) \text{ s.t. } \tilde{L}(s, n, z) > L(s, n, z)\}\) is included into \(B(n - C^{1/2} f_1(n)) \subset A^\circ_n(\omega_d n^d)\). So, we are left with the third term. Thus, in the sequel, we suppose, in addition to (17), \(\forall s \geq 0, \forall z \in \partial B(n), L(s, n, z) \geq \tilde{L}(s, n, z)\) and \(A(\omega_d (n - 2C \ln n)^\alpha) \subset B(n - C \ln n)^\alpha)\).

Note that \(W(s, n)\) is the number of particles that hit \(\partial B(n)\) before adding to the cluster, then
\[
\{\mathcal{H}_0(n, C)\} \subset \{A(\omega_d n^d) \cap B^\circ(n_s) \geq 1\} \subset \{W(s, n) \geq 1\}. \tag{19}
\]

We will show that, for all \(s > 0,\)
\[
\mathbb{E}\{W(s, n)\} \leq (1 - C^{1/2} \ln n)^{1-\alpha} \mathbb{E}\{W(s - 1, n)\} + n^{-C^{1/2}}. \tag{20}
\]

If we iterate this inequality from \(s = 1\) to \(C \ln n)^{-\frac{\alpha}{\beta} + \alpha}\), we get for \(n\) large enough, plugging the values of \(\alpha\) and \(\beta,\)
\[
\mathbb{E}\{W(C \ln n)^{-\beta}, n\} \leq (1 - C^{1/2} \ln n)^{1-\alpha} \mathbb{E}\{W(0, n)\} + n^{-C^{1/2}}\]
\[
\leq \omega_d \exp(-C^{1/2} \ln n)n^d + n^{-C^{1/2}}\]
\[
\leq n^{-1-C^{1/4}},
\]
with \(C\) large enough. The second inequality comes from \(\mathbb{E}\{W(0, n)\} \leq |B(n)|\) and \(\alpha + \beta(d - 1) = 3\). Then, by (19), for \(C\) large enough, the 3 parts of (18) are bounded by \(C^{1/2}n^{1-\frac{1}{2}C^{1/4}}\). So, as for the inner error, the Borel-Cantelli Lemma gives the result.

Now, we will prove (20). As we follow closely [4], we omit the proofs of some intermediate results. But, for the sake of completeness, we will give the main step of the proof of (20) at the end. Using (4) with the new value of \(m\) and \(m'\), and Proposition 2.2 for \(f(n) = C^{1/2} f_1(n)\) and \(f(n) = 2C(\ln n)^\alpha + C^{1/2} f_1(n)\), yields for \(\|x\| = n - C^{1/2} f_1(n),\)
\[
\sum_{y \in \mathcal{A}'} G_n(x, y) - (m' - m)G_n(x) = O(C^{3/2} f_1(n)(\ln n)^\alpha),
\]
\[
mG_n(x) = O(C^{3/2} f_1(n)(\ln n)^\alpha),
\]

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and
\[ \sum_{y \in A^t} G_n(x, y) - m'G_n(x) = 4dC(b_0 - b(x))(\ln n)^{\alpha} + O(C f_1(n)^2), \]
where \( b_0 \) comes from Proposition 2.2 for \( f(n) = 2C(\ln n)^{\alpha} + C^{1/2} f_1(n) \). So, with the definitions of \( b(x) \) and \( b_0 \)
\[
b_0(1 - P^x \{ r^{x}_{n-2C(\ln n)^{\alpha}-C^{1/2} f_1(n)} < \xi^\alpha_n \}) \leq b(x) \\
\leq b_0(1 - P^x \{ r^{x}_{n-2C(\ln n)^{\alpha}-C^{1/2} f_1(n)} < \xi^\alpha_n \}) + cP^x \{ r^{x}_{n-2C(\ln n)^{\alpha}-C^{1/2} f_1(n)} < \xi^\alpha_n \}.
\]
So, (9) gives
\[ \sum_{y \in A^t} G_n(x, y) - m'G_n(x) = O(C f_1(n)^2), \]
which implies
\[ \sum_{y \in A^t} G_n(x, y) - (m' - m)G_n(x) = mG_n(x)(1 + O(C^{-1/2} f_1(n)(\ln n)^{-\alpha})). \]
(21)
Note that all the \( O(.) \) functions depend on \( C \) only in the specified way. So, like in [4, Lemma 8], (21) implies

**Lemma 2.5** For any \( s \geq 0 \),
\[ \mathbb{E}\{M(s, n, z)\} = \mathbb{E}\{\tilde{L}(s, n, z)\}(1 + O(C^{-1/2} f_1(n)(\ln n)^{-\alpha})). \]

**Proof**
Let \( y \in B(n - C^{-1/2} f_1(n)) \) and \( x \in \partial B(n - C^{-1/2} f_1(n)) \). Using a last-exit decomposition argument, see [3, Lemma 2.1.1],
\[
P^y \{ S(\xi^y) = z \} = \sum_{x \in \partial B(n - C^{-1/2} f_1(n))} G_s(y, x)P^x \{ S(\xi^x \land \tau^{x}_{n-C^{-1/2} f_1(n)}) = z \}
\]
\[
= \sum_{x \in \partial B(n - C^{-1/2} f_1(n))} G_s(y, x)P^x \{ S(\xi^x \land \tau^{x}_{n-C^{-1/2} f_1(n)}) = x \}
\]
\[
= P^z \{ \tau^{x}_{n-C^{-1/2} f_1(n)} < \xi^x_n \} \mathbb{E}^z \{ G_s(S(\tau^{x}_{n-C^{-1/2} f_1(n)}), y) \mid \tau^{x}_{n-C^{-1/2} f_1(n)} < \xi^x_n \}.
\]
Then, as the summands of \( M(s, n, z) \) are i.i.d.,
\[
\mathbb{E}\{M(s, n, z)\} = P^z \{ \tau^{x}_{n-C^{-1/2} f_1(n)} < \xi^x_n \}m \mathbb{E}^z \{ G_s(S(\tau^{x}_{n-C^{-1/2} f_1(n)}), 0) \mid \tau^{x}_{n-C^{-1/2} f_1(n)} < \xi^x_n \}.
\]
Like we obtain (5), using
\[ A([\omega_2(n - 2C(\ln n)^{\alpha})^4]) \subset B(n - C(\ln n)^{\alpha}) \subset B(n - C^{-1/2} f_1(n)), \]
we get
\[
\mathbb{E}\{\tilde{L}(s, n, z)\} = P^z \{ \tau^{x}_{n-C^{-1/2} f_1(n)} < \xi^x_n \}
\times \mathbb{E}^z \{ \sum_{y \in A^t \setminus A([\omega_2(n - 2C(\ln n)^{\alpha})^4])} G_s(S(\tau^{x}_{n-C^{-1/2} f_1(n)}), y) \mid \tau^{x}_{n-C^{-1/2} f_1(n)} < \xi^x_n \}
\]
\[
= P^z \{ \tau^{x}_{n-C^{-1/2} f_1(n)} < \xi^x_n \} \mathbb{E}^z \{ \sum_{y \in A^t} G_s(S(\tau^{x}_{n-C^{-1/2} f_1(n)}), y) \mid \tau^{x}_{n-C^{-1/2} f_1(n)} < \xi^x_n \}
\]
\[
-(m' - m)G_s(S(\tau^{x}_{n-C^{-1/2} f_1(n)}), 0) \mid \tau^{x}_{n-C^{-1/2} f_1(n)} < \xi^x_n \}.
\]
Then, the result follows from (21).

By [3, Lemma 1.7.4], $\mathbb{P}\{S(\xi) = z\} \approx n^{1-d}$ (we remind that $\approx$ means ‘up to a multiplicative constant on both sides’), so

$$\mathbb{E}\{M(s, n, z)\} \approx C(\ln n)^{\alpha},$$

(22)

and with Lemma 2.5, we get

$$\mathbb{E}\{\tilde{L}(s, n, z)\} \approx C(\ln n)^{\alpha},$$

(23)

and

$$\mathbb{E}\{M(s, n, z) - \tilde{L}(s, n, z)\} \approx C^{1/2}f_1(n).$$

(24)

Now, we get the following result like in [4, Lemma 9].

**Lemma 2.6** There exist constants $c_1, c_2$ such that, if $s \geq 0$ and $V \subset \partial B(n_s)$ with $|V| \approx (\ln n)^{\beta(d-1)}$,

$$\mathbb{P}\{W(s, n, V) \geq c_1 C^2(\ln n)^2\} \leq c_2 \exp(-c_2 C^{1/3}(\ln n)).$$

**Proof**

We write $W, M, L$ and $\tilde{L}$ for $W(s, n, V), M(s, n, V), L(s, n, V)$, and $\tilde{L}(s, n, V)$. By assumption $L \geq \tilde{L}$, so for any $a, b > 0$,

$$\mathbb{P}\{W \geq a + b + \mathbb{E}\{M - \tilde{L}\}\} \leq \mathbb{P}\{M \geq \mathbb{E}\{M\} + a\} + \mathbb{P}\{\tilde{L} \leq \mathbb{E}\{\tilde{L}\} + b\}.$$

By (22), (23), (24),

$$\mathbb{E}\{M\} \approx C(\ln n)^{\alpha + \beta(d-1)},$$

$$\mathbb{E}\{\tilde{L}\} \approx C(\ln n)^{\alpha + \beta(d-1)},$$

$$\mathbb{E}\{M - \tilde{L}\} \approx C^{1/2}f_1(n)(\ln n)^{\beta(d-1)} \leq C^{1/2}(\ln n)^{\alpha - 1 + \beta(d-1)}.$$

But $\alpha + \beta(d-1) = 3$, so by [5, Lemma 4] for $a = \mathbb{E}\{M\}^{\frac{1}{2 + \frac{\beta}{d}}}$ and $b = \mathbb{E}\{\tilde{L}\}^{\frac{1}{2 + \frac{\beta}{d}}}$, we get the result since

$$a + b + \mathbb{E}\{M - \tilde{L}\} \leq c_1 C^2(\ln n)^2,$$

for some constant $c_1$. We also have an immediate corollary.

**Corollary 2.7** There exist constants $c_1, c_2 > 1$ such that

$$\mathbb{P}\{W(s, n, B(x, (\ln n)^{\beta}) \leq c_1 C^2(\ln n)^2 \text{ for all } s \geq 0, x \in \partial B(n_s)\} \geq 1 - c_2 n^{-c_1 C^{1/3}}.$$

We recall two results from [4].

**Lemma 2.8** [4, Lemma 11] For every $\varepsilon > 0$, there exists $\nu = \nu(\varepsilon) > 0$ such that, if $A \subset B(n)$ with $|A| \geq \varepsilon n^d$, then

$$\mathbb{P}^{\nu}\{\tau_A < \xi_n\} \geq \nu.$$

**Lemma 2.9** [4, Lemma 12] There is a constant $K$ such that for every $D > 0$ and $V \subset \mathbb{Z}^d$, there exists a subset $\tilde{V} \subset V$ satisfying

1. $V \subset \bigcup_{x \in \tilde{V}} B(x, D),$

2. For every $y \in \mathbb{Z}^d$, $|B(y, 2D) \cap \tilde{V}| \leq K.$
By this last lemma, for each \( s \geq 0 \), we can find \( x_1, \ldots, x_u \in \partial B(n_s) \) such that
\[
\partial B(n_s) \subset \bigcup_{k=1}^u B(x_k, (\ln n)^{\beta}),
\]
and each \( y \in \mathbb{Z}^d \) is contained in at most \( K \) balls \( B(x_k, 2(\ln n)^{\beta}) \).

Fix \( x \in \partial B(n_s) \). Write \( U = B(x, (\ln n)^{\beta}) \) and \( 2U = B(x, 2(\ln n)^{\beta}) \). For all \( y \in \partial B(n_s) \cap U \), there exists \( z \in \partial B(n_s) \) such that \( B(z, (\ln n)^{\beta}/4) \subset B(y, (\ln n)^{\beta}) \cap U \). So, there exists a constant \( c_d \) such that
\[
|B(y, (\ln n)^{\beta}) \cap U \cap B^c(n_s)| \geq |B(z, (\ln n)^{\beta}/4) \cap B^c(n_s)| \geq 2c_d(\ln n)^{d\beta}.
\]

Remark that, for any set \( A \),
\[
|U \cap B^c(n_s) \cap A| \leq c_d(\ln n)^{d\beta} \Rightarrow |B(y, (\ln n)^{\beta}) \cap B^c(n_s) \cap A| \geq c_d(\ln n)^{d\beta} \text{ for every } y \in \partial B(n_s) \cap U.
\]

Let \( Y(s, j, U) \) be the indicator function of the event \( \{S^j(\xi) \in U, \rho^j > \xi\} \) and \( H \) be the set \( A_{s+1}(j-1)^c \cap B(S^j(\xi), (\ln n)^{\beta}) \cap B^c(n_s) \). Then,
\[
P\{\xi < \rho^j \leq \xi_{s+1}^j, S^j(\rho^j) \in 2U\}
\geq \mathbb{P}\{\xi < \rho^j \leq \xi_{s+1}^j, S^j(\rho^j) \in 2U, S^j(\xi) \in U, |H| \geq c_d(\ln n)^{d\beta}\}
= \mathbb{P}\{\rho^j \leq \xi_{s+1}^j, S^j(\rho^j) \in 2U, S^j(\xi) \in U, \xi < \rho^j, |H| \geq c_d(\ln n)^{d\beta}\}
\times \mathbb{P}\{S^j(\xi) \in U, \xi < \rho^j, |H| \geq c_d(\ln n)^{d\beta}\}
\geq \mathbb{E}\{\mathbb{P}^{S^j(\xi)}(\tau_{\beta^2 U}^j < \tau_{\beta^2 Z}) | S^j(\xi) \in U, \xi < \rho^j, |H| \geq c_d(\ln n)^{d\beta}\}
\times \mathbb{E}\{Y \text{ if } |U \cap B(S^j(\xi), (\ln n)^{\beta}) \cap B^c(n_s)| \geq c_d(\ln n)^{d\beta}\}\}
\geq \nu \mathbb{E}\{Y \text{ if } |A_{s+1}(\omega_n^d)^c \cap U \cap B^c(n_s)| \leq c_d(\ln n)^{d\beta}\}\}.
\]
The next to last inequality comes from Lemma 2.8 for the ball \( B(S^j(\xi), (\ln n)^{\beta}) \) and the fact that
\[
S^j(\xi) \in U \Rightarrow B(S^j(\xi), (\ln n)^{\beta}) \subset 2U.
\]
The last inequality comes from (26) and the fact that, on the support of \( Y \), \( S^j(\xi) \in U \). By summing this inequality over all \( j \) from \( \lfloor \omega_{\beta^2} \rfloor \) to \( \lceil \omega_{\beta^2} \rceil \), we get
\[
\mathbb{E}\{|A_{s+1}(\omega_n^d) \cap A_s(\omega_n^d) \cap 2U|\} \geq \nu \mathbb{E}\{W(s, n, U) \text{ if } Z\} ,
\]
where \( Z \) denotes the event \( \{A_{s+1}(\omega_n^d) \cap U \cap B^c(n_s)| \leq c_d(\ln n)^{d\beta}\} \). On \( Z \),
\[
|A_{s+1}(\omega_n^d) \cap A_s(\omega_n^d) \cap 2U| \geq c_d(\ln n)^{d\beta}.
\]
Hence,
\[
\mathbb{E}\{|A_{s+1}(\omega_n^d) \cap A_s(\omega_n^d) \cap 2U|\} \geq c_d(\ln n)^{d\beta} \mathbb{E}\{\text{ if } Z\} .
\]
Therefore, as \( d\beta \leq 2 \),
\[
\mathbb{E}\{|A_{s+1}(\omega_n^d) \cap A_s(\omega_n^d) \cap 2U| \} \geq \frac{1}{2} \mathbb{E}\{\nu W(s, n, U) \text{ if } Z \} + c_d(\ln n)^{d\beta} \mathbb{E}\{\text{ if } Z\} \}
\geq \frac{1}{2} \mathbb{E}\{\min\{\nu W(s, n, U), c_d(\ln n)^{d\beta}\}\}
\geq \frac{\nu c_d}{2c_1} C^{-\frac{2}{\beta^2}} (\ln n)^{d\beta-2} \mathbb{E}\{W(s, n, U) \}
\times \mathbb{E}\{\text{ if } W(s, n, U) \leq c_1 C^{\frac{1}{\beta^2}} (\ln n)^2\}
\geq \frac{\nu c_d}{2c_1} C^{-\frac{2}{\beta^2}} (\ln n)^{d\beta-2} \mathbb{E}\{W(s, n, U) \} - n^{-C^2/4} ,
\]
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for $C$ large enough. The last inequality comes from Corollary 2.7. If we cover $\partial B(n_s)$ by the balls $U_k = B(x_k, (\ln n)^{\beta})$ as in (25), we get

$$
\mathbb{E}\{ |A_{s+1}(\lceil \omega_d n^d \rceil) \setminus A_s(\lceil \omega_d n^d \rceil) \cap B(n_s + 1) | \} \geq K^{-1} \sum_{k=1}^n \mathbb{E}\{ |A_{s+1}(\lceil \omega_d n^d \rceil) \setminus A_s(\lceil \omega_d n^d \rceil) \cap 2U_k | \}
$$

$$
\geq \frac{\nu d \mu}{2Kc_1} C^{-\frac{2}{\beta}} (\ln n)^{\frac{d-2}{\beta}} \mathbb{E}\{ W(s, n) \} - \frac{u}{K} n^{-C_{1/\beta}}.
$$

Finally (20) follows from

$$
W(s, n) - W(s + 1, n) \geq |A_{s+1}(\lceil \omega_d n^d \rceil) \setminus A_s(\lceil \omega_d n^d \rceil) \cap B(n_s + 1) |.
$$

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References


