How a Centered Random Walk on the Affine Group Goes to Infinity

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How a centered random walk on the affine group goes to infinity (Comment une marche aléatoire centrée sur le groupe affine tend vers l’infini)

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Abstract

We consider the processes obtained by (left and right) products of random i.i.d. affine transformations of the Euclidean space $\mathbb{R}^d$. Our main goal is to describe the geometrical behavior at infinity of the trajectories of these processes in the most critical case when the dilatation of the random affinities is centered. Then we derive a proof of the uniqueness of the invariant Radon measure for the Markov chain induced on $\mathbb{R}^d$ by the left random walk and prove a stronger property of divergence for the process on induced by the right random walk. \footnote{Laboratoire de Probabilités et Modèles Aléatoires, CNRS-UMR 7599, Université Paris VI et Université Paris VII, 4 place Jussieu, F-75252 Paris Cedex 05. E-mail: brofferi@ccr.jussieu.fr.}

We consider the group $\text{Aff}(\mathbb{R}^d)$ of affine transformations of the space $\mathbb{R}^d$:

$$(a, b) : x \mapsto ax + b$$

where $a$ is a positive real number and $b$ a vector of $\mathbb{R}^d$. Let $\{(A_n, B_n)\}_{n \in \mathbb{N}}$ be a sequence of random independent and identically distributed affine transformations. We are interested in the behavior of their composition products, that is in the behavior of the right and left random walks on $\text{Aff}(\mathbb{R}^d)$:

$$R_n = (A_1, B_1) \cdots (A_n, B_n) \text{ and } L_n = (A_n, B_n) \cdots (A_1, B_1).$$

We will identify the group $\text{Aff}(\mathbb{R}^d)$ with the half-space $\mathcal{H} = \mathbb{R}^+ \times \mathbb{R}^d$. General results on random walks ensure that random walks on the affine group

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are transient, so that accumulation points of the trajectories are on the boundary \( \partial \mathbb{H} = \mathbb{R}^d \cup \{ \infty \} \) of the geometrical compactification of \( \mathbb{H} \). In particular, it is known and easy to show that if the mean of the logarithm of the component on \( \mathbb{R}^*_+ \) is positive, then the random walks converge to \( \infty \), while, if this mean is negative, the right random walk converges to a random element of the boundary different from \( \infty \). Although random walks on this group are well studied (e.g. Kesten [Kes73], Grincevicius [Gri74], Élie [Elè82], Le Page and Peigné [LPP97] Goldie and Maller [GM00]), it was not yet known what happens in the so-called “critical case” when the projection on \( \mathbb{R}^*_+ \) is recurrent. We will prove, under a weak moment hypothesis, but without supposing that the step distribution has a density or is spread out, that a centered right random walk converges to the point \( \infty \). The argument is inspired by an analogous result on the affine group of a tree obtained by D.I.Cartwright, V.A.Kaimanovich and W.Woess [CKW94].

We will then apply this result to the study of the Markov chain induced by the left random walk on \( \mathbb{R}^d \), that is the process that defined recursively by the sequence of i.i.d. random variables \( \{(A_n, B_n)\}_n \) as:

\[
Y_{n+1}^y = A_{n+1}Y_n^y + B_{n+1} \quad Y_0^y = y
\]

Besides its intrinsic interest, this process, also known as first order random coefficient autoregressive model, has various applications (especially in economy and biology, see for instance Engle and Bollerslev[EB86], Nicholls and Quinn[NQ82], Goldie[Go91]). M.Babillot, Ph.Bougerol and L.Elie have already shown in [BBE97] that even when the coefficients \( A_n \) are centered, that is \( E[\log(A_n)] = 0 \), the trajectories of this process satisfy a property that may be seen as a global stability at finite distance or as a local contraction, that is:

\[
|Y_n^x - Y_n^y| 1_K(Y_n^x) \to 0 \text{ as } n \to +\infty
\]

almost surely for all compact set \( K \) of \( \mathbb{R}^d \). As they had noticed, this property is related to the uniqueness of an invariant Radon measure. We will show that the local contraction property is a straightforward geometrical consequence of the convergence of the right random walk to \( \infty \), and we will give a proof of the uniqueness of the Radon invariant measure for the chain \( \{Y_n\}_n \) via the Chacon-Ornstein theorem, thus correcting an error in [BBE97].

In the last section we will look at the projection of the right random walk on \( \mathbb{R}^d \), that is the series:

\[
Z_n^d = b(g) + a(g) \sum_{k=1}^n A_1 \cdots A_{k-1} B_k
\]  \hspace{1cm} (1)

Using a stronger moment hypothesis and a density condition for the marginal on \( \mathbb{R}^d \), we will prove that \( Z_n^d \) is transient, in the sense that almost surely
\[ \lim_{n} |Z_n^\beta| = +\infty. \] Although \( Z_n^\beta \) is not Markov, it is of some interest for various problems. For instance if we consider the continuous time process \( \bar{Z}_t = \int_0^t e^{W_s} dB_s \), where \( W_t \) and \( B_t \) are two independent Brownian motions, and we look at it at integer times we obtain a series of the type \( 1 \). \( \bar{Z} \) is a well known economic model (cf. for instance [Ver79] and [Yor97]) and it is easy to show that \( \lim_{t \to \infty} \bar{Z}_t = -\infty \) and \( \lim_{t \to \infty} \bar{Z}_t = +\infty \) so that, being continuous, \( \bar{Z} \) has to visit infinitely often every open set of \( \mathbb{R} \). The result of the last section implies that every discretization of time leads to a transient process and shows therefore that the recurrence of this model is not very robust.

We can remark that the results of section 2 and section 3 are still valid, and can be proved exactly with the same techniques, for a random walk on the group of affine conformal transformations, that is when the variables \( A_n \) are not real positive numbers but, more generally, matrices that live in a group that is direct product of \( \mathbb{R}^d_+ \) and a compact subgroup of \( GL(\mathbb{R}^d) \).

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1 Notations and hypotheses

We will denote by \( \text{Aff}(\mathbb{R}^d) \) the group of affine transformations of the Euclidean space \( \mathbb{R}^d \), that is transformations of the form \( x \mapsto ax + b \) with \( a \) a positive number and \( b \) a vector in \( \mathbb{R}^d \); thus \( \text{Aff}(\mathbb{R}^d) \) may be identified with the hyperbolic half-space \( \mathbb{H} = \mathbb{R}^d_+ \times \mathbb{R}^d \). We will denote by \( a \) and \( b \) the projections of \( \text{Aff}(\mathbb{R}^d) \) on \( \mathbb{R}^d_+ \) and \( \mathbb{R}^d \) respectively, so that \( g = (a(g), b(g)) \) for each \( g \in \text{Aff}(\mathbb{R}^d) \).

Adding a sphere at infinity leads to the geometrical compactification of \( \mathbb{H} \), where the boundary of \( \mathbb{H} \) is \( \partial \mathbb{H} = \mathbb{R}^d_+ \cup \{ \infty \} \). The group of hyperbolic isometries of \( \mathbb{H} \) that fix the point \( \infty \) is nothing but the group of affine conformal transformations which contains \( \text{Aff}(\mathbb{R}^d) \) as a subgroup. The action of \( \text{Aff}(\mathbb{R}^d) \) on \( \partial \mathbb{H} - \{ \infty \} \) is then the canonical action on \( \mathbb{R}^d_+ \) and will be denoted by:

\[ g \cdot x = a(g)x + b(g) \]

We will sometimes use the fact that the action and the projection on \( \mathbb{R}^d_+ \) coincide, in the sense that \( g \cdot x = b(g(a, x)) \) for every \( a \in \mathbb{R}^d_+ \).

For the composition of two affinities we have the identity:

\[ (a_1, b_1)(a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1), \]

so that, from an algebraic point of view, \( \text{Aff}(\mathbb{R}^d) \) is a semi-direct product of \( \mathbb{R}^d_+ \) and \( \mathbb{R}^d \).
We consider a sequence \( X_n = (A_n, B_n) \) of random variables from a probability space \((\Omega, \mathcal{F}, P)\) to \(\text{Aff}(\mathbb{R}^d)\), that we suppose independent and identically distributed with distribution \(\mu\). The right and left random walks of law \(\mu\) are the Markov chains on \(\text{Aff}(\mathbb{R}^d)\) defined respectively by:

\[
R_{n+1} = R_n X_{n+1} \quad R_0 = 1
\]

and

\[
L_{n+1} = X_{n+1} L_n \quad L_0 = 1
\]

For a fixed \( n \), \( R_n \) and \( L_n \) are both distributed as \(\mu^{(n)}\), the \( n \)th convolution power of \(\mu\). The expected number of visits of these random walks in a Borel set \( B \subset \mathbb{H} \) is given by

\[
U(B) = \sum_{n=0}^{+\infty} P[R_n \in B] = \sum_{n=0}^{+\infty} P[L_n \in B]
\]

that is by the potential measure \( U = \sum_{n=0}^{+\infty} \mu^{(n)} \). The potential kernels of the right and left random walk give the expected numbers of visits when the random walks start from a generic point \( g \in \text{Aff}(\mathbb{R}^d) \), and are respectively given by \( U^r 1_B(g) = \delta_g \ast U(B) \) and \( U^l 1_B(g) = U \ast \delta_g (B) \).

We observe that

\[
a(R_n) = a(L_n) = A_1 \cdots A_n
\]

is a classical multiplicative random walk on \( \mathbb{R}_+^* \).

We will suppose that the random walk is non degenerate in the sense that there exists no \( y \in \mathbb{R}^d \) fixed by \( X_1 \) and that we are really dealing with affinities and not just with translations, i.e.

\[
(H1) \quad \forall y \in \mathbb{R}^d : P[X_1 \cdot y = y] < 1 \text{ and } P[a(X_1) = 1] < 1
\]

Under this hypothesis the closed group generated by the support of \(\mu\) is non-unimodular, thus the random walks are transient and the expected number of visits, \( U(K) \), in every compact set \( K \) is finite (cf.[GKR77]).

We will not need, at least in the first part, any density hypothesis but only a weak moment condition that is

\[
(H2) \quad \mathbb{E}[[\log(a(X_1))]] < +\infty \text{ and } \mathbb{E}[\log^+ |b(X_1)|] < +\infty
\]

As announced, we will only consider the most critical case for which the random walk projected on \( \mathbb{R}_+^* \) is recurrent, that is

\[
(H3) \quad \mathbb{E}[\log(a(X_1))] = 0
\]
2 Convergence to infinity

In this section we shall prove:

**Theorem 1** Under the hypothesis (H1), (H2) and (H3), almost surely for all $g \in \text{Aff}(\mathbb{R}^d)$:

$$\lim_{n \to +\infty} gR_n = \infty$$

in the topology of the compactification $\mathbb{H} \cup \partial \mathbb{H}$.

To prove that $gR_n$ converges to $\infty$ is equivalent to show that the sequence $gR_n$ is definitely in every neighborhood of $\infty$, or equivalently that every set of the form $C_{s,t} = \{g \in \text{Aff}(\mathbb{R}^d) : a(g) < s$ and $|b(g)| < t\}$, with $s$ and $t$ real positive number, is transient. On the other hand we will see that the left random walk visits this set infinitely often, so that the potential measure of $C_{s,t}$ is infinite. The transience of the set $C_{s,t}$ for the right random walk is thus a quite subtle phenomenon, and instead to prove it directly we will show, following the ideas of [CKW94], that the right random walk can cross the border between $C_{s,t}$ and its complement only a finite numbers of times.

**Proposition 1** Suppose that the hypotheses (H1) and (H2) are verified. Then if $C = \{g \in \text{Aff}(\mathbb{R}^d) : a(g) < 1$ and $|b(g)| < 1\}$

$$\mathbb{P}[gR_{n+1} \in C, gR_n \notin C \text{ infinitely often}] = 0$$    \hspace{1cm} (2)

for almost all $g \in \text{Aff}(\mathbb{R}^d)$ with respect to the Haar measure.

This proposition cannot be proven as in [CKW94] because in our case the group does not acts on a discrete space, such as the tree. So we have to find a different way and we will need the following lemma that estimates the potential of integrable functions on a locally compact group.

**Lemma 1** Let $U^r$ be the potential kernel of a transient right random walk on a locally compact second countable group $G$ and $dg$ the right Haar measure of $G$. Then for every positive function $f \in L^1(dg)$, $U^r f(g)$ is $dg$-almost surely finite.

**Proof**: We will show that $U^r f$ is locally $dg$-integrable and thus necessarily $dg$-almost surely finite. If $\mu$ is the distribution of the random walk, we recall that potential measure is $U = \sum_{n=0}^{+\infty} \mu^{(n)}$. Let $K$ be a compact set of $G$. 


Then by the right invariance of $dg$:

$$\int_G U^r f(g_1) \mathbf{1}_K(g_1) dg_1 = \int_G \int_G f(g_1 g_2) \mathbf{1}_K(g_1) U(dg_2) dg_1$$

$$= \int_G \int_G f(g_1) \mathbf{1}_K(g_1 g_2^{-1}) dg_1 U(dg_2)$$

$$= \int_G f(g_1) \tilde{U}^r \mathbf{1}_K(g_1) dg_1$$

$$\leq \sup_{g \in G} \tilde{U}^r \mathbf{1}_K(g) \int_G f(g_1) dg_1$$

where $\tilde{U}^r$ is the potential kernel of the right random walk whose law $\tilde{\mu}$ is the image of $\mu$ under the map $g \mapsto g^{-1}$. The transience of the random walk of law $\tilde{\mu}$ follows from the transience of random walk of law $\mu$ by duality. Therefore $\tilde{U}^r \mathbf{1}_K(g)$ is finite and thus uniformly bounded by the maximum principle (cf.[Rev75]). Since $f$ is integrable $\int_G U^r f(g_1) \mathbf{1}_K(g_1) dg_1$ has to be finite.

\[\square\]

**Proof of Proposition 1:** By the Borel-Cantelli lemma, in order to prove (2), it is sufficient to show that

$$\sum_{n=0}^{\infty} P[g R_{n+1} \in C, g R_n \notin C] < +\infty.$$ 

On the other hand we can write

$$P[g R_{n+1} \in C, g R_n \notin C] = E[P[g R_n X_{n+1} \in C | R_n] 1_{C^c}(g R_n)]$$

$$= E[\phi(g R_n)]$$

where

$$\phi(g) = P[g X_1 \in C] 1_{C^c}(g)$$

and therefore

$$\sum_{n=0}^{\infty} P[g R_{n+1} \in C, g R_n \notin C] = U^r \phi(g).$$

So, if $U^r \phi$ is almost everywhere finite, (2) will hold for almost all $g$. Using the previous lemma, we just need to show that $\phi$ is integrable with respect to the right Haar measure $dg = \frac{da db}{a}$ of $\text{Aff}(\mathbb{R}^d)$. As

$$C^c = \{g \in \text{Aff}(\mathbb{R}^d) : a(g) \geq 1\} \cup \{g \in \text{Aff}(\mathbb{R}^d) : a(g) < 1, \ |b(g)| \geq 1\}$$

we can split the integral of $\int_{\text{Aff}(\mathbb{R}^d)} \phi(g) dg$ into two parts. On the first set we have:

$$\int \int \phi(a, b) 1_{a \geq 1} \frac{db da}{a} = E \left[ \int_1^{+\infty} \int_{\mathbb{R}^d} 1_C((a, b)(A_1, B_1)) \frac{db da}{a} \right]$$
where $v_d$ is the volume of the disc of radius 1 in $\mathbb{R}^d$. On the second set we have that:

$$
\int \int \phi(a, b) 1_{|a| < 1, |b| \geq 1} \frac{db \, da}{a} \leq \mathbb{E} \left[ \int_0^1 \left( \int_{\mathbb{R}^d} 1_{|\|b\|_2| \geq 1} 1_{|a| B_1 + i < 1} db \right) \frac{da}{a} \right]
$$

As the Lebesgue measure of the intersection of the complement of the unitary disc of center the origin and of the unitary disc of center $aB_1$ is bounded by the volume of the unitary disc and goes to zero as $|aB_1|$, it is possible to find a constant $c_d$ depending only on the dimension $d$ such that

$$
\int \int \phi(a, b) 1_{|a| < 1, |b| \geq 1} \frac{db \, da}{a} \leq \mathbb{E} \left[ \int_0^1 (v_d \wedge c_d a B_1) \frac{da}{a} \right]
\leq c \mathbb{E} \left[ \log^+ (|B_1|) \right]
$$

where $c$ is a suitable constant.

We are now able to prove theorem 1

**Proof of theorem 1**: A direct consequence of proposition 1 is that, for almost all $g \in \text{Aff}(\mathbb{R}^d)$, $gR_n$ is either definitely in $C$ or definitely in $C^c$. As we assumed that $\mathbb{E}[\log(a(X_1))] = 0$, $\log(a(gR_n))$ is a recurrent random walk on the real line; hence $gR_n$ visits the set $[1, +\infty] \times \mathbb{R} \subset C^c$ infinitely often. Therefore we can conclude that for almost all $g$, almost surely, $gR_n$ is definitely in $C^c$, or equivalently that for almost all $g$, $R_n$ is definitely in $gC^c$.

Combining the fact that $\{gC^c\}_{g \in \text{Aff}(\mathbb{R}^d)}$ is a family of open neighborhoods of $\infty$ and the fact that for every fixed $g_0$ the set of $g$ such that $gC^c \subset g_0C^c$ has positive Haar measure, it is possible to choose a sequence $g_k$ such that almost surely $R_n$ is definitely in every $g_kC^c$ and such that $\{g_kC^c\}_n$ remain a family of open neighborhoods of $\infty$. Hence almost surely $\lim_{n \to \infty} R_n = \infty$.

To conclude that almost surely for every $g$ in $\text{Aff}(\mathbb{R}^d)$, $gR_n$ converges to $\infty$ we only need to notice that the action of $g$ on $\mathbb{H} \cup \partial \mathbb{H}$ is continuous, so that $\lim_{n \to \infty} gR_n = g \cdot \infty = \infty$.
3 Local contraction

Let $y_0$ be a random vector independent of the $\{X_i\}$. The left random walk induces a Markov chain on $\mathbb{R}^d$:

$$Y_{n_0}^y = L_n \cdot y_0$$

for every $n \geq 0$. Since $Y_n = X_n \cdot Y_{n-1}$, this process satisfies the random difference equation

$$Y_n = a(X_n)Y_{n-1} + b(X_n).$$

From the geometrical viewpoint that we have adopted, this process may be seen as the projection on $\mathbb{R}^d$ of the left random walk starting from a point whose $\mathbb{R}^d$ component is $y_0$, that is

$$Y_{n_0}^y = L_n \cdot y_0 = b(L_n(a, y_0))$$

for every $a$ in $\mathbb{R}^d$.

A consequence of the previous theorem concerns the dependence on the starting point of the process $Y_{n}^y$. Let us consider the distance $Y_{n}^x - Y_{n}^y = a(L_n) |x - y|$ between two trajectories starting from two different points $x$ and $y$ of $\mathbb{R}^d$; since we assumed that $\mathbb{E}[\log(A_1)] = 0$, we have that

$$\lim_{n \to \infty} |Y_{n}^x - Y_{n}^y| = +\infty \text{ while } \lim_{n \to \infty} |Y_{n}^x - Y_{n}^y| = 0.$$

Thus it is not possible to globally control this distance. However M.Babillot, Ph.Bougerol and L.Elie have noticed that if we look at the Markov chain only at the times it visits a compact set $K$, $Y_n$ becomes a contractive process, in the sense that almost surely for every $x$ and $y$:

$$\lim_{n \to \infty} |Y_{n}^x - Y_{n}^y| \mathbf{1}_K(Y_{n}^y) = 0.$$

This property was obtained in [BBE97] using an asymptotic estimate of the potential. The proof we propose here is more geometrical and relies on the convergence of the right random walk to $\infty$.

**Theorem 2** Under the hypotheses H1-3, almost surely for every compact set $K \subset \mathbb{R}$ and every $x, y \in \mathbb{R}$

$$\lim_{n \to +\infty} |L_n \cdot x - L_n \cdot y| \mathbf{1}_K(L_n \cdot y) = \lim_{n \to +\infty} a(L_n) \mathbf{1}_K(L_n \cdot y) = 0 \quad (3)$$

**Proof:** We first observe that

$$L_n = (X_1^{-1} \cdots X_n^{-1})^{-1} = \tilde{L}_n^{-1}$$
where $\tilde{R}_n$ is the right random walk of law $\tilde{\mu}$, obtained from $\mu$ by composing with the inversion on the group; therefore, as $(a, b)^{-1} = \left(\frac{1}{b}, -\frac{a}{b^2}\right)$, we have

$$b(L_n) = -\frac{b(\tilde{R}_n)}{a(\tilde{R}_n)} = -a(L_n)b(\tilde{R}_n)$$

Let $k$ be a real positive number such that $K$ is contained in the disc centered at the origin and of radius $k$; then $L_n \cdot y = a(L_n)y + b(L_n) \in K$ implies that $|b(L_n)| \leq k + a(L_n) |y|$. Using the equality (4) we have $|b(\tilde{R}_n)| \leq \frac{k}{a(L_n)} + |y|$, so that:

$$L_n \cdot y \in K \Rightarrow \max\{a(\tilde{R}_n), b(\tilde{R}_n)\} \leq (k + 1) \frac{1}{a(L_n)} + |y|$$

As the right random walk $\tilde{R}_n$ satisfy the hypotheses of theorem 1, we have that $\max\{a(\tilde{R}_n), b(\tilde{R}_n)\}$ converges to $+\infty$ and we conclude.

\[\square\]

Let us consider the attractor set $A(\omega, y) \subseteq \mathbb{R}^d$ of each trajectory, that is the set of accumulation point of $\{L_n(\omega) \cdot y\}_n$. It is known that if we add to the hypothesis H1-3 a little stronger moment condition, that is:

\[(H4) \quad \mathbb{E}\left[\log(a(X_1))\right]^2 < +\infty \text{ and } \mathbb{E}\left[\left(\log^+|b(X_1)|\right)^{2+\eta}\right] < +\infty\]

for some $\eta > 0$ the Markov chain $L_n \cdot y$ is recurrent in the sense that the attractor sets $A(\omega, y)$ are almost surely not empty. A direct consequence of the local contraction property is that the set $A(\omega, y)$ does not depend on $y$, and then on $\omega$ by the 0-1 law; thus there exists a set $A \subseteq \mathbb{R}^d$ such that $A = A(\omega, y) \mathbb{P}(d\omega)$-almost surely for all $y \in \mathbb{R}^d$.

Although in the centered case, $L_n \cdot y$ is not positive recurrent and has not an invariant probability measure, M. Babillot, Ph. Bougerol and L. Elie have constructed in [BBE97] an invariant Radon measure for this process. We will see in the next theorem that the invariant Radon measure is unique and therefore that its support is $A$.

**Theorem 3** Under the hypothesis $H1-4$, the Markov chain $L_n \cdot y$ has a unique Radon invariant measure on $\mathbb{R}^d$ up to a multiplicative constant.

Furthermore for every couple $f$ and $g$ of continuous functions with compact support such that $g$ is positive and not identically zero on $A$, the limit of

$$\sum_{k=0}^{n} f(L_k \cdot y)$$

is almost surely constant and does not depend on $y$.

**Proof:** We will see that it follows from the local contraction property (3) that if $f$ and $g$ are continuous functions with compact support and $g$ is...
positive and not identically zero on A, there exists a constant $c_{f,g}$ such that almost surely for every \( y \):

$$
\lim_{n \to +\infty} \frac{\sum_{k=0}^n f(L_k \cdot y)}{\sum_{k=0}^n g(L_k \cdot y)} = c_{f,g}
$$

(5)

On the other hand if we consider an invariant Radon measure \( m \) on \( \mathbb{R}^d \), the measure \( \mathbb{P}_m \) on the space \((\mathbb{R}^d)^\mathbb{N}\) of trajectories of the Markov chain \( Y_n^\theta \), obtained as the image of the measure \( m \times \mathbb{P} \) on \( \mathbb{R}^d \times (\text{Aff}(\mathbb{R}^d))^\mathbb{N} \) by the mapping \((y_0, (x_1, x_2, \ldots)) \mapsto (y_0, x_1 \cdot y_0, x_2 x_1 \cdot y_0, \ldots)\), is invariant by the shift \( \theta \) on \((\mathbb{R}^d)^\mathbb{N}\). So that the linear transformation induced by the shift on \( L^1(\mathbb{P}_m) \) is a positive contraction. Moreover the recurrence and the local contraction imply that this linear transformation is also conservative. Indeed, if we consider the positive integrable function \( 1_D \), where \( D \) is an open and relatively compact set such that \( A \cap D \neq \emptyset \), we have that \( \mathbb{P}_m \)-almost surely:

$$
\left\{ y \in (\mathbb{R}^d)^\mathbb{N} : \sum_{k=0}^{\infty} 1_D(\theta^k y) = \sum_{k=0}^{\infty} 1_D(y_k) = +\infty \right\} = (\mathbb{R}^d)^\mathbb{N}.
$$

We can then apply the Chacon-Ornstein theorem and we have that for every positive function \( f \) and \( g \) in \( L^1(m) \) on the set \( \{(y, \omega) | \sum_{k=0}^n g(L_k(\omega) \cdot y) > 0 \} \):

$$
\lim_{n \to +\infty} \frac{\sum_{k=0}^n f(L_k \cdot y)}{\sum_{k=0}^n g(L_k \cdot y)} = \frac{\mathbb{E}_m[f(Y_0) | I]}{\mathbb{E}_m[g(Y_0) | I]} d m(y) \times \mathbb{P}-\text{almost surely}
$$

(6)

where \( I \) is the \( \sigma \)-algebra of invariant set for the shift on \((\mathbb{R}^d)^\mathbb{N}\).

Because of (5) we have then:

$$
\frac{\mathbb{E}_m[f(Y_0) | I]}{\mathbb{E}_m[g(Y_0) | I]} = c_{f,g} d m(y) \times \mathbb{P}-\text{almost surely}
$$

and this implies that for every invariant measure \( m \) we have:

$$
m(f) = \mathbb{E}_m[\mathbb{E}_m[f(Y_0) | I]] = \mathbb{E}[c_{f,g} \mathbb{E}[g(Y_0) | I]] = c_{f,g} m(g)
$$

So \( m \) is unique up to a constant.

We will start to prove that the ratio limit in (5) does not depend on the starting point. We will denote \( \sum_{k=0}^n f(L_k \cdot y) \) by \( S_n f(y) \).

Let \( f \) and \( g \) two continuous functions with compact support, such that \( g \) is positive and not identically zero on \( A \), and let \( K \) be a compact set which contains their support. For every \( \delta > 0 \), let \( K_\delta = \{ z \in \mathbb{R}^d | \text{dist}(z, K) \leq \delta \} \).

Since \( f \) is uniformly continuous, using (3), almost surely for every \( y \) and
$x$ in $\mathbb{R}^d$ and for every positive number $\epsilon$ there exists a $N \in \mathbb{N}$ such that if $k \geq N$:

$$|f(I_k \cdot y) - f(I_k \cdot x)| \leq \epsilon \max\{1_K(I_k \cdot y), 1_K(I_k \cdot x)\} \leq \epsilon 1_{K_k}(I_k \cdot y)$$

As $g$ is a positive function and not identically zero on $A$, then is possible, using (6), to choose $y$ such that $\mathbb{P}$-almost surely $\frac{S_n 1_{K_k}(y)}{S_n g(y)}$, converges. So that:

$$\lim_{n \to +\infty} \left| \frac{S_n f(y) - S_n f(x)}{S_n g(y)} \right| \leq \epsilon \lim_{n \to +\infty} \frac{S_n 1_{K_k}(y)}{S_n g(y)}$$

because $\lim S_n g(y) = +\infty$. As $\epsilon$ was arbitrary chosen, we have that almost surely, for all $x$, $\lim \left| \frac{S_n f(y) - S_n f(x)}{S_n g(y)} \right| = 0$.

If the support of $g$ contains the support of $f$ (which is not a restrictive hypothesis), we have:

$$\lim_{n \to +\infty} \left| \frac{S_n f(y) - S_n f(x)}{S_n g(y)} \right| \leq \lim_{n \to +\infty} \left| \frac{S_n f(y) - S_n f(x)}{S_n g(y)} \right| + \lim_{n \to +\infty} \left| \frac{S_n f(x)}{S_n g(y)} \right| \frac{S_n g(y) - S_n g(x)}{S_n g(y)} = 0$$

because $\frac{S_n f(x)}{S_n g(x)}$ is bounded. So if we had chosen $y$ such that $\frac{S_n f(y)}{S_n g(y)}$ converges to a random variable $c_{f,g}$ then almost surely, for every $x$,

$$\lim_{n \to +\infty} \frac{S_n f(x)}{S_n g(x)} = \lim_{n \to +\infty} \frac{S_n f(y)}{S_n g(y)} = c_{f,g} \quad (7)$$

The fact that the limit $c_{f,g}$ is constant is due to the 0-1 law. In fact for every $h \in \mathbb{N}$:

$$\lim_{n \to +\infty} \frac{\sum_{k=h}^{n} f(X_k \cdots X_{h+1} \cdot y)}{\sum_{k=h}^{n} g(X_k \cdots X_{h+1} \cdot y)} = \lim_{n \to +\infty} \frac{\sum_{k=h}^{n} f(I_k L^{-1}_h \cdot y)}{\sum_{k=h}^{n} g(I_k L^{-1}_h \cdot y)}$$

$$= \lim_{n \to +\infty} \frac{S_n f(L^{-1}_h \cdot y)}{S_n g(L^{-1}_h \cdot y)} = \lim_{n \to +\infty} \frac{S_n f(y)}{S_n g(y)} = c_{f,g}$$

because $S_n g(L^{-1}_h \cdot y)$ converges to $+\infty$ and because of (7), so that $c_{f,g}$ is measurable with respect to the $\sigma$-algebra of the $\{X_k\}_{k>h}$, for each $h \in \mathbb{N}$.

\[ \square \]

4 Divergence of the right projection

In this section we will reinforce the result of theorem 1 by showing that, under some density hypothesis, not only the right random walk goes to
infinity but its projection on $\mathbb{R}^d$ do the same, in other words the process

$$Z_n^g = b(gR_n) = b(g) + a(g) \sum_{k=1}^{n} A_1 \cdots A_{k-1} B_k$$

is transient. We can remark that it was known (cf. [Ver79]) that in the centered case both the process $|Z_n|$ and $|Y_n|$, that have the same law for a fixed $n$, converge in probability to $+\infty$; but while $Y_n$ is a recurrent process, we will prove that $|Z_n|$ converges almost surely to $+\infty$.

**Theorem 4** Suppose that the hypotheses $H1-3$ are satisfied, that the marginal of $\mu$ on $\mathbb{R}^d$, that is the law of $B_1$, has a bounded density and that $E[|B_1|^\rho] < \infty$ for some $\rho > 1$, then almost surely for every $g$ in $\text{Aff}(\mathbb{R}^d)$,

$$\lim_{n \to \infty} |b(gR_n)| = +\infty$$

**Proof**: We first observe that the $d$-dimensional affine group $\text{Aff}(\mathbb{R}^d)$ may be projected on a one-dimensional affine group just taking the first coordinate $b_1(g)$ of the vector $b(g)$. As obviously when the first coordinate diverges also the vector diverge, we can restrict to the case $d = 1$ without loss of generality.

We will proceed as in the proof of theorem 1 and we will start to show that if $S = \{g \in \text{Aff}(\mathbb{R})| |b(g)| \leq 1\}$, $gR_n$ does not cross the border of $S$ but a finite number of times. As we have seen in proposition 1, we only need to show that the potential of the function

$$\psi(g) = P[gX_1 \in S] 1_{S^c}(g)$$

is finite. We will need to split the function $\psi$ in two parts and study their potential with two different techniques. Let $S_1 = \{g \in \text{Aff}(\mathbb{R})| |b(g)| > 1, a(g) \leq 1\}$ and $S_2 = \{g \in \text{Aff}(\mathbb{R})| |b(g)| > 1, a(g) > 1\}$ so that:

$$\psi = \psi 1_{S_1} + \psi 1_{S_2} = \psi_1 + \psi_2$$

The integral of $\psi_1$ with respect with the right Haar measure was already calculated in the proof of the proposition 1 where we proved that

$$\int \int \psi_1(a,b) \frac{db \, da}{a} = E \left[ \int_0^1 \int_{\mathbb{R}} 1_{\{|b| \geq 1\}} 1_{|aB_1 + b| < 1} \frac{db \, da}{a} \right] \leq c \, E \left[ \log^+ \left( |B_1| \right) \right]$$

so that $U^\psi_1(g)$ is finite for almost all $g$.

It is easily checked that $\psi_2$ is not integrable for the right Haar measure so, to prove that its potential is finite, we will need to use a more adapted method. Let

$$F_1 = \{g \in \text{Aff}(\mathbb{R}) : 1 \leq a(g) < 2, 0 \leq b(g) < 1\}$$
and for every $k \in \text{Aff}({\mathbb{R}})$

$$F_k = kF_1 = \{ g \in \text{Aff}({\mathbb{R}}) : a(k) \leq a(g) < 2a(k), b(k) \leq b(g) < a(k) + b(k) \}$$

As the random walk is transient and the sets $F_k$ are relatively compact, their potential is bounded and we have for every $k$:

$$\|U^r \mathbf{1}_{F_k}\|_\infty = \sup_{g \in \text{Aff}({\mathbb{R}})} |\delta_g \ast U(kF_1)| = \|U^r \mathbf{1}_{F_1}\|_\infty$$

We denote $-F_{(2^m, 2^{m+1})}$ the image of the set $F_{(2^m, 2^{m+1})}$ under the mapping $(a, b) \mapsto (a, -b)$ and we observe that $-F_{(2^m, 2^{m+1})} = F_{(2^m, -2^{m+1} - 1)}$; thus the family $\{F_{(2^m, 2^{m+1})}\}_{n \in \mathbb{N}, m \in \mathbb{N}} \cup \{-F_{(2^m, 2^{m+1})}\}_{n \in \mathbb{N}, m \in \mathbb{N}}$ is a partition of $S_2$ into “equipotential” squares. We observe that:

$$U^r \psi_2 = \sum_{k = (2^m, 2^{m+1}) \cap n, m \in \mathbb{N}} U^r(\psi_2 \mathbf{1}_{F_k}) + U^r(\psi_2 \mathbf{1}_{-F_k})$$

$$\leq \sum_{k = (2^m, 2^{m+1}) \cap n, m \in \mathbb{N}} \|U^r \mathbf{1}_{F_k}\|_\infty \|\psi_2 \mathbf{1}_{F_k}\|_\infty + \|U^r \mathbf{1}_{-F_k}\|_\infty \|\psi_2 \mathbf{1}_{-F_k}\|_\infty \quad (8)$$

so that to prove that the potential is bounded we need to estimate the function $\psi_2$ on the sets $F_k$.

It may be worth observing that this approach allows us to compare the right potential kernel with something that roughly looks like a left invariant measure, in the sense that we sum the maximum of the function over a collection of sets obtained by left translation (and $\psi_2$ is integrable for the
left Haar measure). The problem is that the size of the square on which we sum is fixed and we need the function \( \psi_2 \) to be smooth enough on them. Now if \( f \) is the bounded density of the law of \( B_1 \), we have:

\[
\mathbb{P}[gX_1 \in S] = \mathbb{P}
\left[
\frac{-1 - b}{a} < B_1 < \frac{1 - b}{a}
\right] 
\leq \|f\|_\infty \frac{2}{a}
\]

To control \( \psi_2 \) when \( b \) is big we observe that for every \( p > 1 \):

\[
\mathbb{P}[gX_1 \in S] \leq \left( \frac{|b| - 1}{a} \right)^{-\frac{2}{p}} \mathbb{E}[|B_1|^{p-1} \mathbf{1}_{\left[ -\frac{|b| - 1}{a}, \frac{1 - b}{a} \right]}(B_1)]
\leq \left( \frac{|b| - 1}{a} \right)^{-\frac{2}{p}} \mathbb{E}[|B_1|^{p}] \left( \int_{-1/a}^{1/a} f(x) dx \right)^{\frac{1}{p}}
\leq 2^{\frac{1}{p}} \|f\|_\infty \mathbb{E}[|B_1|^{p}] ^{\frac{1}{p}} \frac{a^\frac{1}{p}}{(|b| - 1)^{\frac{1}{p}}}
\]

so that there exist \( 0 < \alpha < \beta \) and \( \beta > 1 \) such that \( \psi(a, b) \leq C(a^\alpha |b|^{-\beta} \wedge a^{-1}) \) for a suitable constant \( C \). Then for every \( g \in F(2^m, 2^m n + 1) \cup F(2^m, 2^m n + 1) \)

\[
\psi(g) \leq C(2^{\alpha(m + 1)}(2^{m n} + 1)^{-\beta} \wedge 2^{-m}) \leq C(2^{(\alpha - \beta)(m + 1)} n^{-\beta} \wedge 2^{-m})
\]

and using the decomposition (8) we have

\[
U^r \psi_2 \leq \|U^r \mathbf{1}_{F_1} \|_\infty 2C \sum_{m, n \in \mathbb{N}} 2^{(\alpha - \beta)(m + 1)} n^{-\beta} \wedge 2^{-m} < +\infty
\]

So we proved that, for almost all \( g \in \text{Aff}(\mathbb{R}) \), \( g R_n \) can cross the border of \( S \) only a finite number of times.

Combining the recurrence of the real random walk \( a(g R_n) \) and the transience of \( g R_n \), we see that \( g R_n \) visits an infinite number of times the set \((1, 2] \times \mathbb{R}) \cap S^c \) so that, for almost all \( g \), \( g R_n \) is definitely in \( S^c \). As for every \( s > 0 \) the set of \( g \) such that

\[
g S^c \subset \{(a, b) \in \text{Aff}(\mathbb{R}) : |b| > s\}
\]

has positive Haar measure, we may conclude that for every \( s > 0 \) almost surely \( b(R_n) \) is definitely greater than \( s \), so that, letting \( s \) go to \( +\infty \) on a countable sequence, we obtain that, almost surely, for every \( g \in \text{Aff}(\mathbb{R}) \)

\[
\lim_{n \to +\infty} |b(g R_n)| = \lim_{n \to +\infty} a(g) b(R_n) + b(g) = +\infty
\]

\( \square \)
References


