The spectrum of Completely Positive Entropy Actions of Countable Amenable Groups

A.H. Dooley
V. Ya. Golodets

Vienna, Preprint ESI 1078 (2001)  
September 27, 2001
The spectrum of completely positive entropy actions of countable amenable groups

A. H. Dooley
School of Mathematics, University of N.S.W.
Sydney NSW 2052, Australia
(e-mail: a.dooley@unsw.edu.au)

V. Ya. Golodets*
Institute for Low Temperatures Physics & Engineering,
Ukrainian National Academy of Sciences
47 Lenin Ave., 61103 Kharkov, Ukraine
(e-mail: golodets@ilt.kharkov.ua)

Abstract

We prove that an ergodic free action of a countable discrete amenable group with completely positive entropy has a countable Lebesgue spectrum. Our approach is based on the Rudolph-Weiss result on the equality of conditional entropies for actions of countable amenable groups with the same orbits. Relative completely positive entropy actions are also considered. An application to the entropic properties of Gaussian actions of countable discrete abelian groups is given.

1 Introduction

Rokhlin and Sinai [15] introduced the notion of actions of the group \( \mathbb{Z} \) with completely positive entropy (cpe for short) on a Lebesgue space. This property is also known as a \( K \)-action of \( \mathbb{Z} \). They proved that such actions of \( \mathbb{Z} \) have very simple spectral properties: they are isomorphic to a sum of countably many copies of the regular representation of \( \mathbb{Z} \). Subsequently Kamiński [8] extended these results to actions of \( \mathbb{Z}^d \), \( d < \infty \), and recently Kamiński and Liardet [9] found a nice proof of these results for \( \mathbb{Z}^d \), \( d = 1, 2, \ldots \). The same results were obtained for the groups of upper triangular matrices over \( \mathbb{Z} \) and their subgroups in [7].

On the other hand, Connes, Feldman and Weiss [1] proved that every free action \( T \) of a countable discrete amenable group \( G \) on a Lebesgue space is orbit equivalent to an action of the group \( \mathbb{Z} \). Recently Rudolph and Weiss [16] showed, under some additional assumptions,
that these actions of $G$ and $\mathbb{Z}$ have the same conditional mean entropies (see Theorem 2.6 below). Now conditional mean entropy for $\mathbb{Z}$ is a well studied concept. This leads to the idea of applying the result of Rudolph and Weiss to the spectrum problem for cpe actions of amenable groups (see Section 5 [16]).

In this paper we consider the above approach to the spectrum. Using it, we are able to prove that every free cpe ergodic action of a countable amenable group has a countable Lebesgue spectrum (see Definitions in Section 2 and Theorem 3.1 below). To prove this we use also results from [1], Dye’s approach to actions with the same orbits [5] and the relative version of the Rokhlin-Sinai results [9]. For the case of groups $G$ of type I we apply the ‘duals’ of Mackey [12] (see Proposition 3.4) and for ‘wild’ groups we have to use von Neumann algebras (see Lemma 4.1). In section 5 we consider relative cpe actions. In particular, we show that an action of $G$ has relative cpe with respect to its Pinsker algebra (Theorem 5.4) and apply this theorem to Gaussian actions of countable discrete abelian groups (Theorem 5.7).

In this version of our paper we have taken into account some useful comments of Daniel Rudolph and Benjamin Weiss. In particular, we have used relative perfect partitions, as suggested to us by B. Weiss (see Remark 2.5). We also have taken into account some remarks suggested by Jacob Feldman, his nice idea concerning the spectra of Bernoulli shifts was used in Lemma 4.2. We removed some misprints pointed out by Alexandre Danilenko and presented his refined proof of Proposition 5.5 (below).

An interesting but different approach to this problem was suggested by Jean-Paul Thouvenot, and we believe that he was the first to apply Rudolph-Weiss theory to the spectral properties of dynamical systems. In a collaboration with Sergey Sinel’shchikov begun in November 2000, he used cohomological transformations of orbital cocycles to get a version of Lemma 3.2 below, for the case of abelian groups, and solved the problem for this case. We understand that Jean-Paul Thouvenot reported these results at a Conference in Santiago in December 2000 and at several seminars, though a detailed text has not been published as yet. Unfortunately we did not know about their approach when we studied this problem in Sydney (November 2000 – January 2001).

2 Preliminaries

Let $G$ be a countable discrete amenable group, and $g \mapsto U_g$ a unitary representation of $G$ in a Hilbert space $\mathcal{H}$. If this representation is unitarily equivalent to the regular representation of $G$ (for example to the left regular representation) we say that this representation has a Lebesgue spectrum in $\mathcal{H}$. If $g \mapsto U_g$ has a Lebesgue spectrum of countable multiplicity, we say that $g \mapsto U_g$ has a countable Lebesgue spectrum in $\mathcal{H}$.

Let $(X, \mathcal{B}(X), \mu_X)$ be a Lebesgue space, where $\mathcal{B}(X)$ is the set of all measurable subsets of $X$. Let $T$ be an action of $G$ on the space $(X, \mathcal{B}(X), \mu_X)$, preserving the measure $\mu_X$. Then we may define a unitary representation of $G$ on the space $L^2(X, \mu_X)$ by

\[(U_g f)(x) = f(T_g^{-1} x), \quad g \in G, \quad f \in L^2(X, \mu_X)\]  \hspace{1cm} (1)

We say that the action $T$ has a countable Lebesgue spectrum if the representation (1) has a
countable Lebesgue spectrum on the subspace

\[ \mathcal{L}^2_0(X, \mu_X) = \{ f \in \mathcal{L}^2(X, \mu_X) : \int f(x) d\mu_X(x) = 0 \} . \]

**Lemma 2.1** Suppose that a countable discrete amenable group \( G \) has a measure preserving free action \( T \) on \((X, \mathcal{B}(X), \mu_X)\). Then there is a freely acting automorphism \( S \) of \((X, \mathcal{B}(X), \mu_X)\) and a family of subsets \( \mathcal{D} \subset \mathcal{B}(X) \) such that \( Sx = T_{g} x \) for \( x \in \mathcal{D}_g \), \( g \neq e \), and \( \mathcal{D}_{g_1} \cap \mathcal{D}_{g_2} = \emptyset \) for \( g_1 \neq g_2 \), \( \cup_g \mathcal{D}_g = X \), \( g \mathcal{D}_g = \mathcal{D}_g \), \( \emptyset = \emptyset \), \( g_1 \neq g_2 \).

Moreover, for every \( g \in G \) there is a family of subsets \( \{ \mathcal{D}_n \} \), \( n \in \mathbb{Z} \), such that \( T_{g} x = S^n x \) for \( x \in \mathcal{D}_n \), \( n \neq 0 \), and \( \mathcal{D}_{n+1} \cap \mathcal{D}_{n+2} = \emptyset \) for \( n_1 \neq n_2 \), and \( \cup_n \mathcal{D}_n = X \).

Lemma 2.1 is a consequence of the theory developed in [1]. The partitions considered in this lemma were introduced by Dye [5].

**Remark 2.2** Let \((X, \mathcal{B}(X), \mu_X)\) and \( G \) be as above. Then we can define a unitary representation of the group \( G \) in the space \( \mathcal{L}^2(X, \mu_X) \) by (1). If \( (P_g f) = \chi_{\mathcal{D}_g}(x) f(x) \) where \( f \in \mathcal{L}^2(X, \mu_X) \) and \( \chi_D \) is the indicator function for \( D \subset X \) then \( U_S = \sum_g U_g P_g \) is also a unitary operator corresponding to the automorphism \( S \).

Let us recall some results and definitions concerning the entropy of an action of \( T \) on a countable discrete amenable group \( G \) on a Lebesgue space \((X, \mathcal{B}(X), \mu)\) (see [13], [6]). If \( P \) is a finite partition of \( X \) and \( F \) a subset of \( G \), we let \( P^F = \bigvee_{g \in F} T_g^{-1} P \). The entropy of the process \((P, G)\) is given by

\[ h(P, T) = \lim_{n \to \infty} \frac{1}{|F_n|} H(P^F_n) \]

where \( \{F_n\} \) is a Følner sequence in the group \( G \), and \( H \) is the usual partition entropy. This limit exists and is independent of the choice of Følner sequence [13].

**Definition 2.3** We say a \( G \)-action \( T \) has completely positive entropy (or cpe) if for any nontrivial finite partition \( P \)

\[ h(P, T) > 0. \]

Let \( \mathcal{A} \) be a \( G \)-invariant \( \sigma \)-subalgebra of \( \mathcal{B}(X) \). Then one defines the conditional entropy of a partition \( P \) by

\[ h(P, T | \mathcal{A}) = \lim_{n \to \infty} \frac{1}{|F_n|} H(P^F_n | \mathcal{A}) \]

One can again deduce the existence of this limit and its independence of the sequence \( \{F_n\} \) [13]. For the definitions and basic properties of conditional entropy the reader is referred to [14].

Similarly to the above, we say that a \( G \)-action \( T \) has relative completely positive entropy (or relative cpe) if for any finite partition \( P \) such that \( H(P | \mathcal{A}) > 0 \) we have

\[ h(P, T | \mathcal{A}) > 0. \]

Recall the definition of the Pinsker algebra \( \Pi(T) \) of the action \( T \) of \( G \) on \((X, \mathcal{B}(X), \mu_X)\). \( \Pi(T) \) is the maximal \((T\text{-invariant}) \sigma\text{-subalgebra of} \mathcal{B}(X) \text{ such that for any finite partition} \)
\( P \subset \Pi(T) \) one has \( h(P, T) = 0 \). The relative Pinsker algebra \( \Pi(T \mid \mathcal{A}) \) with respect to a \( T \)-invariant \( \sigma \)-subalgebra \( \mathcal{A} \subset \mathcal{B}(X) \) can be introduced in a similar way [2, 6, 9, 13, 15].

Just as in the standard case, an action \( T \) of \( G \) has a relative cpe with respect to \( \mathcal{A} \subset \mathcal{B}(X) \) if and only if \( \Pi(T \mid \mathcal{A}) = \mathcal{A} \).

The theory of relative cpe (or relative \( K \)) actions for \( \mathbb{Z} \) is well-developed and completely parallel to the standard case. The following theorem is a relative version of well-known Rokhlin-Sinai results [15].

**Theorem 2.4** Let \( S \in Aut(X, \mathcal{B}(X), \mu_X) \), \( \mu_X \circ S = \mu_X \), and \( \mathcal{A} \) an \( S \)-invariant \( \sigma \)-subalgebra of \( \mathcal{B}(X) \).

(i) For any finite measurable partition \( P \)

\[
h(P, S \mid \mathcal{A}) = H(P \mid \mathcal{P}^{-S}_n \vee \mathcal{A})
\]

where

\[
\mathcal{P}^{-S}_n = \bigvee_{n<0} S^n P.
\]

(ii) If \( S \) has a relative cpe action with respect to \( \mathcal{A} \) then there exists a measurable partition \( \zeta \) of \( (X, \mu_X) \) such that

(a) \( S\zeta \geq \zeta \)

(b) \( \zeta = \bigvee_{n \in \mathbb{Z}} S^n \zeta = \mathcal{B}(X) \)

(c) \( \bigwedge_n S^n \zeta = \mathcal{A} \)

(d) \( H(\zeta \mid \mathcal{P}^{-S}_n \vee \mathcal{A}) = h(T \mid \mathcal{A}) \)

where \( h(T \mid \mathcal{A}) = \text{supp} h(P, T \mid \mathcal{A}) \) and \( P \) is a finite measurable partition of \( X \).

**Proof.** Part (i) of this theorem was proved in [9, section 2]. To prove (ii), we need a relative version of the Pinsker formula and its consequence (see [9, equations (9), (10)]):

\[
\lim_{n \to \infty} H(P \mid \mathcal{P}^{-S}_n \vee \mathcal{S}^{-n} \mathcal{P}^{-S}_n \vee \mathcal{D}) = h(P, S \mid \mathcal{D}),
\]

where \( P_S = \bigvee_{n \in \mathbb{Z}} S^n P \), \( \mathcal{P}^{-S}_n = \bigvee_{n<0} S^{-n} P \), and \( \mathcal{D} \) is an \( S \)-invariant subalgebra of \( \mathcal{B}(X) \).

Now we remark that (ii) is a relative version of [15, Theorem 1], which corresponds to the case \( \Pi(T) = \mathcal{N}(X) \), the trivial subalgebra of \( \mathcal{B}(X) \). One can thus prove (ii) by the argument of that theorem using (2), (3). The necessary modifications are almost obvious.

**Remark 2.5** It is natural to call a partition \( \zeta \) satisfying the conditions of 2.4 (ii) a relative perfect partition for \( T \) with respect to \( \mathcal{A} \). The perfect partitions were introduced in [15] (see also [8, 7]) and applied, in particular, to studying the spectral properties. Relative perfect partitions are very useful in our considerations, in particular, in the proof of Proposition 3.5 below.

Now we cite the following result from [16]. This is a crucial result for our approach.
Theorem 2.6 Suppose $T$ is a free and ergodic action of a countable discrete amenable group $G$ and $\mathcal{A}$ is a $T$-invariant $\sigma$-subalgebra of $\mathcal{B}(X)$. Suppose also that $S$ is a free (and necessarily ergodic) action of $Z$ with the same orbits as $T$. Suppose the orbit change from $T$ to $S$ is $\mathcal{A}$-measurable. Then for any finite partition $P$ of $X$ we have

$$h(P,T|\mathcal{A}) = h(P,S|\mathcal{A}).$$

3 CPE actions and spectrum

In this section we shall begin the proof of our main theorem, whose precise statement is as follows:

Theorem 3.1 Suppose that a countable discrete amenable group $G$ has a free ergodic measure preserving action $T$ on a Lebesgue space $(X,\mathcal{B}(X),\mu_X)$. If $T$ has cpe (Definition 2.3) then $T$ has a countable Lebesgue spectrum (see Section 2).

We shall proceed in several steps. Let $T_1 = T$ as above and let $T_2$ be a standard Bernoulli action of $G$ on $(Y,\mathcal{B}(Y),\mu_Y)$. Consider the action $T' = T_1 \otimes T_2$ of $G$ on the Lebesgue space $(Z,\mathcal{B}(Z),\mu_Z)$ where

$$Z = X \times Y, \quad \mu_Z = \mu_X \times \mu_Y, \mathcal{B}(Z) = \mathcal{B}(X) \times \mathcal{B}(Y).$$

Let $\mathcal{A} = \mathcal{N}(X) \otimes \mathcal{B}(Y)$ and $\mathcal{B} = \mathcal{B}(X) \otimes \mathcal{N}(Y)$ be the $\sigma$-subalgebras of $\mathcal{B}(Z)$, with $\mathcal{N}(X)$ (resp. $\mathcal{N}(Y)$) be the trivial $\sigma$-subalgebra of $\mathcal{B}(X)$ (resp. $\mathcal{B}(Y)$) generated by $X$ (resp. $Y$) and the empty set.

Consider the subsets $\mathcal{D}_g, g \in G$, for the action $T_2$ on $(Y,\mathcal{B}(Y),\mu_Y)$ which was introduced in Lemma 2.1. Then $\mathcal{D}_g^i = X \times \mathcal{D}_g^i \subset \mathcal{A}$, and we can define an automorphism $S$ of $(Z,\mathcal{B}(Z),\mu_Z)$ by

$$S_z = T_g^i z \quad \text{for} \quad z \in \mathcal{D}_g^i, \quad g \in G.$$  \hfill (4)

It follows from Lemma 2.1 that

$$T_g^i z = S^n z \quad \text{for} \quad z \in \mathcal{D}_g^i, \quad n \in \mathbb{Z}.  \hfill (5)$$

The automorphism $S$ of $(Z,\mathcal{B}(Z),\mu_Z)$ has the same orbits as $T'$ and acts freely. Now we can apply Theorem 2.6 to $S$ and $T'$. We obtain

$$h(P,S|\mathcal{A}) = h(P,T'|\mathcal{A})$$  \hfill (6)

where $P$ is any finite measurable partition of $Z$. By [6] theorem 3 the action $T'$ of the group $G$ on the space $(Z,\mathcal{B}(Z),\mu_Z)$ has cpe. Thus, in view of (6), the relative Pinsker algebra $\Pi(S|\mathcal{A})$ is just $\mathcal{A}$, and $S$ has relative $K$-action on $(Z,\mathcal{B}(Z),\mu_Z)$. Hence we may apply Theorem 2.4 to $S$.

Let $\zeta$ be the measurable partition of $Z$ defined as in Theorem 2.4 (ii). Denote by $F_0$ the conditional expectation from $\mathcal{B}(Z)$ on the $\sigma$-subalgebra $\zeta$, and by $F_i$ the conditional expectation on the $\sigma$-subalgebra $S^i \zeta$. Then $F_i \leq F_{i+1}, \quad i \in \mathbb{Z}$. It is evident that $\lim_{i \to -\infty} F_i = \mathcal{A}$. 

$F_{-\infty}$ is the conditional expectation on the $\sigma$-subalgebra $A$. Let $A_i = S^i \zeta$. Since $A \subset A_i$, $i \in \mathbb{Z}$, we have

$$a F_i = F_i a,$$  

$a \in A$,  

$i \in \mathbb{Z}$.

It is well-known that $F_i$ can be considered as a projection from $L^2(Z, \mu_Z)$ to $L^2(A_i)$ where $L^2(A_i)$ is the subspace of $L^2(Z, \mu_Z)$ consisting of all $A_i$-measurable functions. Then $\{F_i - F_{i-1}, i \in \mathbb{Z}\}$ is a family of pairwise orthogonal projections such that

$$I = \sum_{i \in \mathbb{Z}} (F_i - F_{i-1}) + F_{-\infty}$$  

(7)

is a decomposition of unity, and

$$a(F_i - F_{i-1}) = (F_i - F_{i-1})a, \quad a \in A.$$  

If $H_n = (F_n - F_{n-1})H$, where $H = L^2(Z, \mu_Z)$ then

$$H \oplus L^2(A) = \sum_{n \in \mathbb{Z}} \oplus H_n$$  

(8)

$$L^\infty(A)H_n \subseteq H_n, \quad n \in \mathbb{Z},$$  

(9)

where $L^\infty(A)$ is the subspace of $L^\infty(Z, \mu_Z)$ consisting of $A$-measurable functions. For $S \in \text{Aut}(Z, \mu_Z)$ define the unitary operator $V_S$ in the Hilbert space $H = L^2(Z, \mu_Z)$ by

$$(V_S f)(z) = f(S^{-1}z), \quad f \in H.$$  

Then it is clear that

$$V_S H_i = H_{i+1}, \quad i \in \mathbb{Z}.$$  

Lemma 3.2 Let $g \rightarrow V_g$ be a representation of $G$ in $H = L^2(Z, \mu_Z)$ which we define by (1). If $H_g = V_g(F_0 - F_{-1})H$ for $g \in G$ (in particular, $H_0 = H_\epsilon$) then

(i) $V_{g_1} H_g = H_{g_1 g}$,  

$g_1, g \in G$;

(ii) $L^\infty(A)H_g = H_g$;

(iii) $H_{g_1} \perp H_{g_2}$ for $g_1 \neq g_2$;

(iv) $\sum_{g \in G} \oplus H_g = H \oplus L^2(A)$.

Proof. It follows from (5) (see also Remark 2.2) that $V^{-1}_g = \sum_{n \in \mathbb{Z}\setminus\{0\}} \chi_{D^\ast_n} V^{-n}_S$ where $D^\ast_n, n \in \mathbb{Z}, g \in G$, are defined as above. (Since $T^1$ acts freely, $\chi_{\bar{D}^\ast_0} = 0$ for $g \neq \epsilon$.) Because $\chi_{\bar{D}^\ast_g}$ is $A$-measurable we have in view of (9), (10)

$$V^{-1}_g (F_0 - F_{-1})H = \sum_{n \in \mathbb{Z}\setminus\{0\}} \chi_{\bar{D}^\ast_n} H_{-n}.$$  

(11)

It follows from (9) that statement (ii) is true. It is also clear that $H_g \perp H_\epsilon$ for $g \in G$ (see (11)). Let $g_1, g_2 \in G$. Then

$$H_{g_2^{-1} g_1} \perp H_\epsilon \quad \text{or} \quad V_{g_2^{-1} g_1} H_{g_1} \perp H_\epsilon,$$

and so $H_{g_1} \perp H_{g_2}$.
Hence (iii) is also true.

To prove (iv) let us use Lemma 2.1 and (8). We obtain

$$
\sum_{g \in G} \oplus \mathcal{H}_g = \sum_{g} \sum_{n} V_S^n \chi_{\mathcal{D}_g} (F_0 - F_{-1}) \mathcal{H}
$$

$$
= \sum_{n} V_S^n \left( \sum_{g \in G \backslash \{e\}} \chi_{\mathcal{D}_g} \right) (F_0 - F_{-1}) \mathcal{H} + (F_0 - F_{-1}) \mathcal{H}
$$

$$
= \sum_{n} V_S^n (F_0 - F_{-1}) \mathcal{H} = \mathcal{H} \oplus \mathcal{L}^2(\mathcal{A})
$$

\[\square\]

**Corollary 3.3** The restriction of the representation $g \mapsto V_g$ to the subspace $\mathcal{H} \oplus \mathcal{L}^2(\mathcal{A})$ has a countable Lebesgue spectrum.

**Proof.** Indeed, this follows from (i) – (iv) of Lemma 3.2. \[\square\]

Now we are ready to begin proving Theorem 3.1. Consider first the case when $G$ is a group of type I, that is all factor-representations of $G$ are of type I in the von Neumann classification [3].

For such a group, Mackey [12] defined the dual object $\hat{G}$. Elements of $\hat{G}$ are equivalence classes of unitary irreducible representations of the group $G$. Mackey introduced a Borel structure in $\hat{G}$. He showed that if $g \mapsto U_g$ is a unitary representation of $G$ in a Hilbert space $\mathcal{H}$, then there exists a decomposition

$$
\mathcal{H} = \int_{\hat{G}} \mathcal{H}_x d\mu(x) \quad \text{and} \quad U_g = \int_{\hat{G}} U_g(x) d\mu(x)
$$

where $x \mapsto \mathcal{H}_x$ is a Borel field of Hilbert spaces, $x \mapsto U_g(x)$ is a Borel field of factor-representations of type I for $G$, and $\mu$ is a Borel measure on $\hat{G}$. Moreover, $\mathcal{H}_x = \mathcal{H}_x^1 \otimes \mathcal{H}_x^2$ and $U_g(x) = U_g^1(x) \otimes I_{n(x)}$ where $x \mapsto \mathcal{H}_x^i, \ i = 1, 2$, is a Borel field of Hilbert spaces, $x \mapsto U_g^1(x)$ is a Borel field of irreducible representations of $G$, and $I_{n(x)}$ is the identity operator in $\mathcal{H}_x^2$ such that $\dim \mathcal{H}_x^2 = n(x)$ for almost all $x \in \hat{G}$. Thus one can define for every unitary representation of the group $G$, a Borel measure $\mu$ on $\hat{G}$ and a multiplicity function $x \mapsto n(x)$ on $\hat{G}$. Mackey proved that two unitary representations $g \mapsto U_g^1$ and $g \mapsto U_g^2$ of $G$ are unitary equivalent if and only if $\mu_1 \sim \mu_2$ and $n_1(x) = n_2(x)$ for a.a. $x \in \hat{G}$.

Dixmier in [4] considered the Plancherel measure on $\hat{G}$ for the decomposition of the regular (left and right) representations of $G$. We denote this measure as $m_G$. This measure $m_G$ has many interesting properties (see 18.8 [4]), in particular, it is used in the non-commutative version of the Plancherel formula for $G$.

It is important to note that, for an abelian group $G$, the dual $\hat{G}$ coincides with the Pontryagin dual group $\hat{G}$, and $m_G$ coincides with the Haar measure of $\hat{G}$.

**Proposition 3.4** Let $G, T, (X, \mathcal{B}(X), \mu_X)$ and be as in the formulation of Theorem 3.1. If $G$ is a group of type I then the representation $g \mapsto U_g$ in $\mathcal{L}^2(X, \mu_X)$ defined by (1) has a countable Lebesgue spectrum.
Proof. Let $\nu$ be the Borel measure on $\dot{G}$ corresponding to the decomposition of the representation $g \mapsto U_g$ in $L_0^2(X, \mu_X)$. Then $\nu$ has the decomposition $\nu = \nu_a + \nu_s$ where the measure $\nu_a$ is absolutely continuous with respect to the Plancherel measure $m_G$, and $\nu_s$ is singular with respect to $m_G$. Now we can consider $L_0^2(X, \mu_X)$ as a $G$-invariant subspace of $\mathcal{H} \subset L^2(A)$ (see Lemma 3.2). Moreover

$$V_g|_{L_0^2(X, \mu_X)} = U_g, \quad g \in G. \quad (12)$$

But $g \mapsto V_g$ has a countable Lebesgue spectrum in $\mathcal{H} \subset L^2(A)$ by Corollary 3.3. This means that the measure of the decomposition of $g \mapsto V_g$ is equivalent to $m_G$. Since by (12) $g \mapsto U_g$ is a subrepresentation of $g \mapsto V_g$ then $\nu_s = 0$. Now by the generalized Sinai theorem \cite{13} there exists a finite partition $P$ of $X$ such that $P_G = \bigvee_{g \in G} T_g P$ is a Bernoulli factor ($\sigma$-subalgebra) for the action $T$ of $G$. It follows from Lemma 4.2 that $\nu_a \sim m_G$. So $\nu = \nu_a \sim m_G$. Thus $G$ has a Lebesgue spectrum in $L_0^2(X, \mu_X)$. But the restriction of the representation $g \mapsto U_g$ onto the subspace $\mathcal{H}_F$ (see Lemma 4.2 below) has a countable Lebesgue spectrum. Hence $g \mapsto U_g$ also has a countable Lebesgue spectrum in $L_0^2(X, \mu_X)$ (see also \cite{10}).

4 Spectrum for actions of “wild” groups

Now we consider groups whose regular representations contain factor-representations of type $II_1$.

Let us recall some results about von Neumann algebras. For more detailed information about this subject, the reader is referred to Dixmier’s book \cite{3}. Let $M$ be a von Neumann algebra in a separable Hilbert space $\mathcal{H}$, $M'$ its commutant, and $Z(M)$ its center. Then $Z(M) = M \cap M'$. If $Z(M) = \mathbb{C}I$, where $I$ is the identity operator in $\mathcal{H}$, then $M$ is a factor. If $Z(M) \neq \mathbb{C}I$ then $Z(M)$ is a commutative von Neumann algebra, and $Z(M)$ can be realized as $L^\infty(\Omega, \nu)$ where $(\Omega, \nu)$ is a Lebesgue space. In this case there are Borel fields of Hilbert spaces $\omega \mapsto H_\omega, \omega \in \Omega$, and factors $\omega \mapsto M_\omega$ such that $\mathcal{H}$ is the direct sum of Hilbert spaces

$$H = \int_\Omega H_\omega d\nu(\omega)$$

and $M$ is the direct sum of factors

$$M = \int_\Omega M_\omega d\nu(\omega),$$

that is, for each $m \in M$ there is a Borel field of operators $\{m_\omega\}, \ \omega \in \Omega$, such that $m = \int_\Omega m_\omega d\nu(\omega)$. Factors of type $I_\infty, II_\infty$ and $III$ in the von Neumann classification are called infinite factors. If the von Neumann algebra $M$ can be disintegrated into factors of infinite type, then $M$ is called an infinite algebra.

Lemma 4.1 Suppose that the von Neumann algebra $M$ acts in a separable Hilbert space $\mathcal{H}$.

Consider the Hilbert space $l^2(\mathcal{H}) = \{\tilde{f} = (f_i)_{i=1}^\infty : f_i \in \mathcal{H}, \|\tilde{f}\|^2 = \sum_{i=1}^\infty \|f_i\|^2 < \infty\}$, where $\|f_i\|$ is the norm of $f_i$ in $\mathcal{H}$ and define the representation $\pi$ of $M$ in $l^2(\mathcal{H})$ by

$$\pi(m)\tilde{f} = (mf_i)_{i=1}^\infty, \quad m \in M.$$
Then $\pi(M)'$ is an infinite algebra in $l^2(\mathcal{H})$.

**Proof.** We give a sketch of the (standard) proof. Note that $l^2(\mathcal{H}) = \mathcal{H} \otimes l^2(\mathbb{N})$, and $\pi(m) = m \otimes I_{\mathbb{N}}$, $m \in M$, where $I_{\mathbb{N}}$ is the identity operator in $l^2(\mathbb{N})$. Then $\pi(M) = M \otimes I$ in $\mathcal{H} \otimes l^2(\mathbb{N})$, and $M' \otimes \mathcal{B}(l^2(\mathbb{N})) \subset \pi(M)'$ where $\mathcal{B}(l^2(\mathbb{N}))$ is the algebra of all bounded operators in $l^2(\mathbb{N})$. A more detailed calculation shows that $\pi(M)' = M' \otimes \mathcal{B}(l^2(\mathbb{N}))$. Hence $\pi(M)'$ is an infinite algebra. \qed

Now we recall some definitions concerning infinite projections in von Neumann algebras [3]. Two projections $P_i \in M$, $i = 1, 2$, are equivalent in $M$ if there is a partial isometry $u \in M$ such that $P_1 = uu^*$. If $M$ is a factor then a projection $P \in M$ is called infinite if $P$ is equivalent to a proper subprojection. Two infinite projections belonging to the same factor are equivalent. If $M$ is an algebra then a projection $P = (P_\omega)$ is called infinite if $P_\omega$ is an infinite projection in $M_\omega$ for almost all $\omega \in \Omega$. For each projection $P$ in the von Neumann algebra $M$, there exists a smallest projection $Z(P)$ in the centre of $M$ such that $P \leq Z(P)$. Recall that two infinite projections $P_i \in M$, $i = 1, 2$, with $Z(P_1) = Z(P_2)$ are equivalent in $M$. (see [3, III 2.5, Exercise 15]).

The following result was suggested by Jack Feldman. In an earlier version we used the fact that a Bernoulli action has a countable Lebesgue spectrum, but have no proof. However, J. Feldman pointed out that we actually need only the weaker property that a Bernoulli action has a countable Lebesgue spectrum when it is restricted to an appropriate subspace. We give a short proof of this result.

**Lemma 4.2** Let $G, T$, $(X, \mathcal{B}(X), \mu_X)$ be as in the formulation of Theorem 3.1, and assume in addition that $T$ is a Bernoulli action of $G$. Then there exists a projection $R$ which is $U_g$-invariant for all $g \in G$, such that the restriction of the representation $g \mapsto U_g$ into the subspace $H_F = RL^2_0(X, \mu_X)$ has a countable Lebesgue spectrum. Moreover, if $M$ is the von Neumann algebra of operators in $L^2_0(X, \mu_X)$ generated by $\{U_g; g \in G\}$, then $R$ is an infinite projection in $M'$.

**Proof.** Since two Bernoulli shifts of the same entropy are isomorphic [13], there exists a partition $P = (P_i)_{i=1}^\infty$ of $X$ such that $\mu_X(P_i) > 0$ for all $i \in \mathbb{N}$, $h(T) = -\sum \mu_X(P_i) \log \mu_X(P_i)$, such that the partitions $T_{g} P$, $g \in G$, are pairwise independent. Let us consider the Hilbert subspace $L^2(P, \nu)$ of $L^2_0(X, \mu_X)$, with $\nu$ being the restriction of $\mu_X$ to the class of $P$-measurable sets. Then for each $g \in G$, $L^2_0(T_{g} P, T_{g}^* \nu)$ is orthogonal to $L^2_0(P, \nu)$ due to the independence of $P$ and $T\_g P$. Denote by $\mathcal{H}_F$ the Hilbert subspace spanned by $L^2_0(P, \nu)$, for all $g \in G$. Then

$$\mathcal{H}_F = \bigoplus_{g \in G} L^2_0(T_{g} P, T_{g}^* \nu).$$

It is clear that $\mathcal{H}_F$ is $U_g$-invariant for $g \in G$, and the restriction of $U$ to $\mathcal{H}_F$ has a countable Lebesgue spectrum.

If $R$ is an orthogonal projection onto $\mathcal{H}_F$ then $R \in M'$. Consider the von Neumann algebra $MR$ in the space $\mathcal{H}_F = RL^2_0(X, \mu_X)$, then $(MR)' = RM' R$. As the representation $g \mapsto RU_g$ has a spectrum of countable multiplicity in $\mathcal{H}_F$, the subalgebra $(MR)' = RM'R$ is infinite by Lemma 4.1. On the other hand, $R$ is the identity in $(MR)'$, so $R$ is an infinite projection in $(MR)'$, and hence $R$ is an infinite projection in $M'$. \qed
Proposition 4.3 Let $G$, $T$, $(X, \mathcal{B}(X), \mu_X)$ and $g \mapsto U_g$ be as in the statement of Theorem 3.1. If $G$ is a group not of type I then the representation $g \mapsto U_g$ in $\mathcal{L}^2_0(X, \mu_X)$ has a countable Lebesgue spectrum.

Proof. We preserve the notation of the proof of Proposition 3.4. Let us consider the representation $g \mapsto V_g$ of the group $G$ in the space $\mathcal{K} = \mathcal{H} \oplus \mathcal{L}^2(\mathcal{A})$, and denote by $M$ the von Neumann algebra generated by $\{V_g; g \in G\}$. Then the commutant $M'$ is an algebra of infinite type by Lemma 4.1 and Corollary 3.3. Hence the identity operator $I_{\mathcal{K}}$ in $\mathcal{K}$ is an infinite projection with respect to $M'$.

Since $\mathcal{L}^2_0(X, \mu_X)$ may be considered as a $G$-invariant subspace of $\mathcal{K}$, the orthogonal projection $Q$ on $\mathcal{L}^2_0(X, \mu_X)$ belongs to $M'$. Thus we obtain $U_g = V_g Q$.

Let us show that $Q$ is an infinite projection in $M'$. Let $P$ be a finite partition of $X$ such that $P_G$ is a Bernoulli factor $\sigma$-subalgebra of $\mathcal{B}(X)$ (see the proof of Proposition 3.4) and $\mathcal{L}^2_0(P_G)$ the Hilbert subspace of $\mathcal{L}^2_0(X, \mu_X)$, generated by all $P_G$-measurable functions in $\mathcal{L}^2_0(X, \mu_X)$. Denote by $R$ the orthogonal projection of Lemma 4.2 such that the representation $g \mapsto U_g R$, $g \in G$, has a countable Lebesgue spectrum in the subspace $R\mathcal{K}$ and $\mathcal{K}$ respectively, we have $Z(R) = I_{\mathcal{K}}$.

Now we know that $R \leq Q \leq I_{\mathcal{K}}$, and therefore $Z(Q) = I_{\mathcal{K}}$ and $Q$ is an infinite projection in $M'$. Thus $Q$ and $I_{\mathcal{K}}$ are infinite projections in $M'$, and there is a partial isometry $u \in M'$ such that $Q = u^* u$, $I_{\mathcal{K}} = uu^*$. Now we can conclude that, for all $g \in G$

$$U_g = V_g Q = u^* V_g u,$$

and hence the representations $g \mapsto U_g$ and $g \mapsto V_g$ are unitary equivalent. Since $g \mapsto V_g$ has a countable Lebesgue spectrum (see Corollary 3.3), it follows that $g \mapsto U_g$ also has a countable Lebesgue spectrum. 

Proposition 4.3 proves Theorem 3.1 for the case of a wild group. To complete the proof of Theorem 3.1 we need the following remark.

Remark 4.4 Let $g \mapsto V_g$ be a unitary representation of the group $G$ with a countable Lebesgue spectrum in a Hilbert space $\mathcal{H}$, and $\mathcal{M}$ a von Neumann algebra generated by $(V_g, g \in G)$, $\mathcal{M} = (V_g, g \in G)'$. Then $M'$ is a von Neumann algebra of infinite type, that is, $M'$ disintegrates into factors of infinite types $(I_\infty$ and $I_\infty$).

To prove this one can repeat the proof of Lemma 4.1. Now we can prove Theorem 3.1 in the general case by using the same argument as in the proof of Proposition 4.3.

This remark gives a general proof of the result without recourse to the Mackey dual. However, we feel that the approach of Section 3 gives a better understanding of the problem and its resolution in the type I case.

5 Relative cpe actions and spectrum

It is natural to investigate the spectral properties of relative cpe actions for countable amenable group actions.
Theorem 5.1 Suppose that the countable discrete amenable group $G$ has a free ergodic measure-preserving action $T$ on a Lebesgue space $(X, \mathcal{B}(X), \mu_X)$. Suppose that $\mathcal{D}$ is a $T$-invariant $\sigma$-subalgebra of $\mathcal{B}(X)$. If $T$ has a relative cpe action with respect to $\mathcal{D}$ then $T$ has a countable Lebesgue spectrum in the subspace $\mathcal{L}_0^2(X, \mu_X) \subseteq \mathcal{L}_0^2(\mathcal{D})$, where $\mathcal{L}_0^2(\mathcal{D})$ is the subspace of $\mathcal{L}_0^2(X, \mu_X)$ consisting of all $\mathcal{D}$-measurable functions.

Proof. To prove this theorem, we follow the approach of Section 3. We again consider the Bernoulli action $T_2$ of $G$ on $(Y, \mathcal{B}(Y), \mu_Y)$ and introduce the action $T' = T_1 \otimes T_2$ of $G$ on the space $(Z, \mathcal{B}(Z), \mu_Z)$. We define $S \in \text{Aut}(Z, \mathcal{B}(Z), \mu_Z)$ which acts freely and has the same orbits as $T'_g$, $g \in G$. It follows from Theorem 2.6 that

$$h(P, S|\mathcal{A} \vee \mathcal{T}) = h(P, T'|\mathcal{A} \vee \mathcal{T}),$$

where $P$ is a finite partition of $Z$, $\mathcal{A} = \mathcal{N}(X) \times \mathcal{B}(Y)$ and $\mathcal{T} = \mathcal{D} \times \mathcal{B}(Y)$.

Lemma 5.2 Let $(X, \mathcal{B}(X), \mu_X)$, $T$ and $\mathcal{D}$ be as in the formulation of Theorem 5.1. Let $(Y, \mathcal{B}(Y), \mu_Y)$ and $T_2$ be the Bernoulli action of $G$ as above. If $\mathcal{E}$ is the relative Pinsker algebra of $T' = T_1 \otimes T_2$, where $T_1 = T$, with respect to the $\sigma$-subalgebra $\mathcal{D} \times \mathcal{B}(Y)$ then

$$\mathcal{E} = \mathcal{D} \times \mathcal{B}(Y).$$

Proof. It is clear that $\mathcal{E} \supseteq \mathcal{D} \times \mathcal{B}(Y)$. If we repeat the argument of the proof of [16, Theorem 4.10], taking into account the proof of [16, Corollary 4.11] we obtain that $\mathcal{E} = \mathcal{E}_0 \times \mathcal{B}(Y)$, where $\mathcal{E}_0$ is a $\sigma$-subalgebra of $\mathcal{B}(X)$. But $T_1$ has a relative cpe action with respect to $\mathcal{D}$ by our assumption. Hence $\mathcal{D} = \mathcal{E}_0$.

By this Lemma we may conclude, in view of (13), that the relative Pinsker algebra $\Pi(S|\mathcal{A} \vee \mathcal{T})$ is just $\mathcal{A} \vee \mathcal{T} = \mathcal{D}$. So $S$ has a relative $K$-action on $(Z, \mathcal{B}(Z), \mu_Z)$. Applying Theorem 2.4 to $S$ we obtain an analogue of Lemma 3.2 with

$$\sum_{g \in G} \mathcal{H}_g = \mathcal{H} \oplus \mathcal{L}_0^2(\mathcal{D} \times \mathcal{B}(Y))$$

instead of (iv) of that Lemma. As before, the representation $g \to V_g$ has a countable Lebesgue spectrum in $\mathcal{H} \oplus \mathcal{L}_0^2(\mathcal{D} \times \mathcal{B}(Y))$, and we must prove that $g \to U_g$ has a countable Lebesgue spectrum in $\mathcal{H}_0 = \mathcal{L}_0^2(X, \mu_X) \oplus \mathcal{L}_0^2(\mathcal{D})$. Again we can consider $\mathcal{H}_0$ as the $G$-invariant subspace of $\mathcal{H} \oplus \mathcal{L}_0^2(\mathcal{D} \times \mathcal{B}(Y))$, and we have the analogue of formula (12):

$$V_g |\mathcal{H}_0 = U_g, \quad g \in G.$$

Now to prove analogues of Proposition 3.4 and 4.2 we use the following version of the generalized Sinai theorem (see also [6], lemma 3.3).

Theorem 5.3 If $T$ is an ergodic free action of the countable discrete amenable group $G$ with $h(T|\mathcal{D}) > 0$ then there is a finite partition $P$ of $X$ such that $P_G$ is a Bernoulli factor $\sigma$-subalgebra of $\mathcal{B}(X)$ which is independent of $\mathcal{D}$.
Note that this theorem was proved in [17] for the case $G = \mathbb{Z}$ and in [13] for the general case. The approach developed in this paper allows one to give an alternative proof of this result using the techniques of [17].

Thus, Theorem 5.1 can be proved via the approach of Theorem 3.1, by using Lemma 5.2 and Theorem 5.3. For this approach, we need to describe the $T$-invariant $\sigma$-subalgebras $\mathcal{D} \subset \mathcal{B}(X)$ such that $T$ has a relative cpe action with respect to $\mathcal{D}$. Now we demonstrate that the Pinsker algebra $\Pi(T)$ actually possesses the above property (see also Corollary 5.6).

**Theorem 5.4** Suppose that a countable discrete amenable group $G$ has a free ergodic measure preserving action $T$ on a Lebesgue space $(X, \mathcal{B}(X), \mu_X)$ and let $\Pi(T)$ be the Pinsker subalgebra of $T$. Then the action of $T$ is relative cpe with respect to $\Pi(T)$, and has a countable Lebesgue spectrum in the subspace $\mathcal{L}^0_\alpha(X, \mu_X) \oplus \mathcal{L}^0_\alpha(\Pi(T))$.

Note that results of this type were obtained for $\mathbb{Z}$ by Rokhlin and Sinai [15], for $\mathbb{Z}^d$ ($d = 1, 2, \ldots, \infty$) by Kaminski and Liardet [9] (see also [8]), and by Golodets and Sinel’shchikov [7] for countable nilpotent groups.

The following result is a version of [15, Theorem 2], and is an example where we can transfer the approach of Rokhlin-Sinai to the relative case. We present two proofs of this result.

**Proposition 5.5** Let $\Pi(S|\mathcal{D})$ be the relative Pinsker algebra of an automorphism $S$ with respect to $\mathcal{D}$. Then $S$ has relative cpe action with respect to $\Pi(S|\mathcal{D})$.

**Proof.** We give an outline of the proof. Let $C = \Pi(S|\mathcal{D})$ and let $Q$ be a finite partition of $(X, \mathcal{B}(X), \mu_X)$ such that $H(Q|C) > 0$. We prove first that $h(Q, S|C) > 0$.

Let us assume the contrary: $h(Q, S|C) = H(Q|Q^+_S \vee C) = 0$. It follows from the relative version of [15, Theorem 1] that there is a measurable partition $\zeta$ of $X$ with the following properties:

(i) $S\zeta \geq \zeta$,

(ii) $\bigvee_k S^k \zeta = \mathcal{B}(X)$,

(iii) $\bigwedge_k S^k \zeta = C$,

(iv) $H(T \zeta|\zeta \vee C) = h(T)$.

Consider a finite partition $\zeta$ of $X$ such that $\zeta \leq S^m \zeta$ for some integer $m$. We claim that $H(\zeta|C \vee Q_S) = H(\zeta|C)$, where $Q_S = \bigvee_{n \in \mathbb{Z}} S^n Q$. In view of (ii) it will follow that $C \vee Q_S = C$ or $Q_S \leq C$ and so $Q \leq C$. This means that $H(Q|C) = 0$, which contradicts our assumption. We can then deduce that $h(Q, S|C) > 0$, which was to be proved.

Now let us prove that $H(\zeta|C \vee Q_S) = H(\zeta|C)$. For any integer $p$ one has

$$H(\zeta|C) \geq H(\zeta|C \vee Q_S) \geq H(\zeta|\zeta^*_S \vee C \vee Q_S),$$

where $\zeta^*_S = \bigvee_{k \geq 1} S^{-pk} \zeta$, $SC = C$. As $Q \leq Q^-_S \vee C$, one deduces that $Q_S \vee C = Q^-_S \vee C = S^{-p} Q^-_S \vee C$, $n \in \mathbb{Z}$, and an application of (3) gives

$$H(\zeta|\zeta^*_S \vee C \vee Q^-_S) = \lim_{n \to \infty} H(\zeta|\zeta^*_S \vee S^{-p} Q^-_S \vee C) = H(\zeta|\zeta^*_S \vee C).$$

12
Since the sequence of partitions $\xi_p \vee C$ decreases and converges to $C$ as $p \to \infty$, we obtain $H(\xi \mid \xi_p \vee C) \to H(\xi \mid C)$. Thus $H(\xi \mid C \vee Q_S) = H(\xi \mid C)$. \hfill $\Box$

**Proof of Proposition 5.5 (A. I. Danilenko).** Let $C = \Pi(S \mid D), E = \Pi(S \mid C)$, and $Q$ be a finite partition from $E$. It is evident that $C$ is an $S$-invariant partition of $(X, B(X), \mu)$. Denote by $P$ a countable generator for $S$ restricted to $C$. Such generator exists by a well known Rokhlin result. It follows from the relative version of the Pinsker formula (2)

$$h(S, Q \vee P \mid D) = h(P, S \mid D) + H(Q \mid Q_S \vee P_S \vee D).$$

But $h(P, S \mid D) = 0$ by the definition of $P$, and

$$H(Q \mid Q_S \vee P_S \vee D) = H(Q \mid Q_S \vee E) = 0$$

by the definition of $Q \subset E$. Hence

$$h(S, Q \mid D) \leq h(S, Q \vee P \mid D) = 0.$$

This implies $Q \subset C$ and hence $E = C$. \hfill $\Box$

Now we are in a position to finish the proof of Theorem 5.4.

**Proof of Theorem 5.4.** Let $T_1 = T$ and $T_2$ be a Bernoulli action of $G$ on $(Y, B(Y), \mu_Y)$. Consider the action $T' = T_1 \otimes T_2$ of $G$ on the space $(Z, B(Z), \mu_Z)$ defined as at the beginning of section 3. As in section 3, there exists an automorphism $S$ of $(Z, B(Z), \mu_Z)$ such that (6) is valid. By [16, Theorem 4.10], the relative Pinsker algebra $\mathcal{E}$ of $T'$ with respect to $\mathcal{A} = \mathcal{N}(X) \otimes B(Y)$ coincides with $\Pi(T) \otimes B(Y)$. Hence $\Pi(T) \otimes B(Y)$ is the relative Pinsker algebra for $S$ with respect to $\mathcal{A}$. By Proposition 5.5 the action of $S$ has relative cpe with respect to $\Pi(T) \otimes B(Y)$. So $T'$ has a relative cpe action with respect to $\Pi(T) \otimes B(Y)$. We conclude that $T = T_1$ also has relative cpe with respect to $\Pi(T)$. Hence by Theorem 5.1 $T$ has a countable Lebesgue spectrum in $L^2_0(X, \mu_X) \oplus L^2_0(\Pi(T))$. \hfill $\Box$

**Corollary 5.6** Let $G, T,$ and $(X, B(X), \mu_X)$ be as in the statement of Theorem 5.4, and $D$ a $T$-invariant $\sigma$-subalgebra of $B(X)$. If $\Pi(T \mid D)$ is the relative Pinsker subalgebra of $T$ with respect to $D$, then the action $T$ is relative c.p.e. with respect to $\Pi(T \mid D)$. Moreover, $T$ has a countable Lebesgue spectrum in the subspace $L^2_0(X, \mu_X) \oplus L^2_0(\Pi(T \mid D))$.

**Proof.** We keep the notation of the proof of Theorem 5.4. If $\mathcal{E} = \Pi(T' \mid \mathcal{A})$ where $\mathcal{A} = \mathcal{N}(X) \otimes B(Y)$ and $T' = T_1 \otimes T_2$, then the argument of [16, Theorem 4.10] establishes that $\mathcal{E}$ has the form $\mathcal{E} = \Pi(T \mid D) \otimes B(Y)$. The Corollary now follows by the argument used in the proof of Theorem 5.4. \hfill $\Box$

Let us present an application of Theorem 5.4. Let $G$ be a countable discrete abelian group, and $\hat{G}$ its Pontryagin dual group. Consider a weakly mixing Gaussian action $T$ of $G$ on $(X, B, \mu_X)$ (see [10], [11]). By the spectral measure of $T$ we mean the measure $\sigma$ on $\hat{G}$ determined by

$$\hat{\sigma}(g) = \int_X \chi(g)d\sigma(\chi) = \int_X fT_g f d\mu,$$
where \( f \in L^2_{0,\mu}(X,\mu) \), the space of real-valued square integrable functions with zero mean [11]. This measure is symmetric; it is also well known that, given a symmetric finite Borel measure \( \sigma \) on \( \hat{G} \), one can define in a unique way a Gaussian \( G \)-action whose spectral measure is \( \sigma \).

**Theorem 5.7** Let \( G \) be a countable discrete abelian group. The entropy of a Gaussian \( G \)-action \( T \) is either zero or infinity. The former case holds iff \( \sigma \perp \lambda \) where \( \lambda \) is the Haar measure of \( \hat{G} \).

**Proof.** This nice theorem was proved by Lemańczyk [11] for abelian groups \( G \) which satisfy the Thouvenot conjecture, namely, that the conclusion of Theorem 5.4 is valid for every action of \( G \). But we have seen that this property holds for any countable discrete abelian group. \( \square \)

**Acknowledgements.** We are grateful to Sergey Sinel’shchikov for pointing out some misprints in an earlier version of this paper and for his help in preparing the present version. We thank Dan Rudolph and Benji Weiss for their interest to this paper and their support. We are grateful to Jean-Paul Thouvenot for drawing our attention to spectral problems. We thank Jack Feldman for his helpful remarks concerning the spectrum of Bernoulli actions. We thank Alexander Danilenko for his help. We are grateful to the Australian Research Council for its support.

**References**


