Meromorphic Continuation of the Spectral Shift Function

Vincent Bruneau
Vesselin Petkov

Vienna, Preprint ESI 1073 (2001)  
September 24, 2001  

Supported by Federal Ministry of Science and Transport, Austria
Available via http://www.esi.ac.at
MEROMORPHIC CONTINUATION OF THE SPECTRAL SHIFT FUNCTION

VINCENT BRUNEAU AND VESSELIN PETKOV

ABSTRACT. We obtain a representation of the derivative of the spectral shift function $\xi(\lambda, h)$ in the framework of semi-classical "black box" perturbations. Our representation implies a meromorphic continuation of $\xi(\lambda, h)$ involving the semi-classical resonances. Moreover, we obtain a Weyl type asymptotics of the spectral shift function as well as a Breit-Wigner approximation in an interval $(\lambda - \delta, \lambda + \delta), \ 0 < \delta < \epsilon h$.

AMS classification: 35B34, 35P25

1. Introduction

The purpose of this paper is to obtain a meromorphic continuation of the derivative of the spectral shift function $\xi(\lambda, h)$. This problem is closely related to the trace formulae (see [13], [34], [35] [21], [23], [30], [28], [29]) and to resonances expansions ([7], [32]). For compact perturbations the function $\xi(\lambda, h)$ coincides with the scattering phase

$$\sigma(\lambda, h) = \frac{1}{2\pi i} \log \det S(\lambda, h), \ \lambda \in \mathbb{R},$$

where $S(\lambda, h) = I + A(\lambda, h) : L^2(S^{n-1}) \to L^2(S^{n-1})$ is the scattering operator and for more information about the spectral shift function we refer to [33]. In the classical case ($h = 1$) the first result proving a representation of $\sigma(\lambda) = \sigma(\lambda, 1)$ containing the resonances $z_j \in \mathbb{C}_- = \{z \in \mathbb{C} : \text{Im} z < 0\}$ was established by Melrose [16] for obstacle scattering in odd dimensions $n \geq 3$. More precisely, given a function $\chi(t) \in C^\infty(\mathbb{R})$ such that $0 \leq \chi(t) \leq 1, \ \chi(t) = 1$ for $t \leq 2, \ \chi(t) = 0$ for $t \geq 3$, Melrose showed that

$$\sigma(\lambda) = \sigma_{\text{sing}}(\lambda) + \sigma_{\text{reg}}(\lambda),$$

with

$$\frac{d}{d\lambda} \sigma_{\text{sing}}(\lambda) = -\frac{1}{\pi} \sum_j \chi \left( \frac{|z_j|}{\lambda} \right) \frac{\text{Im} z_j}{\lambda - z_j}^2, \ \sigma_{\text{sing}}(0) = 0, \ \lambda \in \mathbb{R},$$

$$\sigma_{\text{reg}}(\lambda) \in S^n(\mathbb{R}).$$

Since $\sigma(\lambda, h)$ is the logarithmic derivative of the scattering determinant

$$s(\lambda, h) = \det(I + A(\lambda, h)),$$

it is natural to examine the behavior of $s(z, h)$ for $z$ in the "physical half plane", where we have no resonances. This idea was developed by Guillopé and Zworski [13] for the analysis of the scattering resonances for certain Riemann surfaces and in the classical case $h = 1$, Zworski [34], [35] gave an elegant proof of the trace formula for "black box" compact perturbations based on the meromorphic continuation of $s(z)$ (see [34] for other works on trace formulae).

In [21], [23] the Breit-Wigner approximation for the scattering phase has been justified for "black box" scattering with compact perturbations in the classical and the semi-classical cases.
Among the ideas introduced in [21], [23], one of the main point in [23] was the estimate of the holomorphic function $g(z, h)$,

$$|g(z, h)| \leq C(\Omega)h^{-n}, \quad n \geq n$$

in the local factorization

$$s(z, h) = e^{a(z, h)} \frac{P(z, h)}{P(z, h)}$$

where

$$P(z, h) = \prod_{\nu \in \text{Res} L(h) \cap \mathbb{R}} (z - w),$$

$$\Omega = (a, b) + i(-c, c), \quad 0 < a < b, \quad c > 0, \quad \Omega_\epsilon = \{ z \in \mathbb{C} : d(\Omega, z) < \epsilon \}, \quad \epsilon > 0.$$

Here $L(h)$ is a compactly supported perturbation of the operator $-h^2 \Delta$, $0 < h \leq h_0$, and $n$ depends on the estimates of the number of the eigenvalues of the reference operator. The local factorization implies immediately

$$\partial_z \sigma(z, h) = \frac{1}{2\pi i} \partial_z g(z, h) + \frac{1}{2\pi i} \sum_{\nu \in \text{Res} L(h) \cap \mathbb{R}} \left( \frac{1}{z - w} - \frac{1}{z - \bar{w}} \right), \quad z \in \Omega$$

and for $\lambda \in (a, b)$ we obtain an analogue of the formula of Melrose mentioned above. Combining (1.2) with the Birman-Krein formula one obtains easily the trace formula of [28] exploiting the meromorphic continuation of $\partial_z \sigma(z, h)$ in $\{ z \in \mathbb{C} : \text{Im} \leq 0 \}$ (see Theorem 1 in [23]). Moreover, a similar factorization has been established in [23] in domains $\lambda + h\Omega$ with an improved estimate for the holomorphic function $g(z, h)$.

In the case of ”black box” long-range perturbations the existence of the scattering operator and that of the scattering determinant are far from apparent. In this direction Sjöstrand [28], [29] proposed powerful techniques based on the complex scaling operators, introduced in [30], and complex analysis. The scattering determinant is replaced by $D(z, h) = \det(I + K(z))$, where $K(z)$ is trace class operator which is not uniquely determined and the resonances are the zeros of $D(z, h)$. Applying the approach of Sjöstrand, J.-F. Bony [1], [2], established upper and lower bounds on the number of the semi-classical resonances in small domains and the Breit-Wigner approximation has been extended to long-range perturbations in [3]. For a pair of self-adjoint operators $L_j(h)$, $j = 1, 2$, satisfying some assumptions (see Section 2) the spectral shift function $\xi(\lambda, h)$ is a distribution in $\mathcal{D}'(\mathbb{R})$ such that

$$\langle \xi'(\lambda, h), f(\lambda) \rangle_{\mathcal{D}'(\mathbb{R}), \mathcal{D}(\mathbb{R})} = \text{tr}_{L_h} \left( f(L_2(h)) - f(L_1(h)) \right), f(\lambda) \in C_0^\infty(\mathbb{R}),$$

where $\text{tr}_{L_h}$ is a generalized trace defined in Section 2. We denote by $\text{Res } L_j(h)$, $j = 1, 2$ the set of the resonances $w \in \overline{\mathbb{C}}$ of $L_j(h)$.

In this work we are strongly inspired by the approach in [23] and our main goal is to obtain an analogue of (1.2) in the cases when a scattering determinant is not available. We show that the representation (1.2) remains true in the general case of semi-classical ”black box” scattering, replacing $\sigma'(\lambda, h)$ by the ”regular part”

$$\xi'(\lambda, h) - \left[ \sum_{w \in \text{Res } L_j(h) \cap [a, b]} \delta(\lambda - w) \right]_{j=1}^2,$$
where here and throughout the paper we use the notation $[a_j]_{j=1}^2 = a_2 - a_1$. Our principal result is the following.

**Theorem 1.** Assume that $L_j(h)$, $j = 1, 2$, satisfy the assumptions of Section 2. Let $Ω ⊂ \mathbb{C} \setminus e^{[-2\theta, 2\theta]}', 0 < \theta < \pi/2$, be an open simply connected set and let $W ⊂ \subset Ω$ be an open simply connected and relatively compact set which is symmetric with respect to $\mathbb{R}$. Assume that $J = Ω \cap \mathbb{R}^+$, $I = W \cap \mathbb{R}^+$ are intervals. Then for $λ \in I$ we have the representation

$$
ξ'(λ, h) = \frac{1}{π} \Im r(λ, h) + \left[ \sum_{w \in \text{Res} L_j \cap \mathbb{R}, \Im w \neq 0} \frac{-\Im w}{π|λ - w|^2} + \sum_{w \in \text{Res} L_j \cap J} \delta(λ - w) \right]_{j=1},
$$

where $r(z, h) = g_+(z, h) - \overline{g_+(\overline{z}, h)}$, $g_+(z, h)$ is a function holomorphic in $Ω$ and $g_+(z, h)$ satisfies the estimate

$$
|g_+(z, h)| \leq C(W)h^{-m}, \quad z \in W
$$

with $C(W) > 0$ independent on $h \in [0, h_0]$.

**Remarks.**

- The terms related to the resonances are measures. In fact, the resonances $w$, $\Im w < 0$, are related to harmonic measures

$$
ω_{C_+}(w, E) = -\frac{1}{π} \int_E \frac{\Im w}{|w - t|^2} dt, \quad E \subset \mathbb{R} = \partial C_+,
$$

while the resonances $w \in \mathbb{R}^+$ coincide with the embedded eigenvalues of $L_j(h)$, $j = 1, 2$. Moreover, in a small neighborhood $U_λ(h)$ of every $λ \in I \setminus \cup_j \{λ \in \mathbb{R} : λ \in σ_{pp}(L_j(h))\}$ the derivative $ξ'(λ, h)$ coincides with a real analytic function on $U_λ(h)$. In particular, if we have no embedded positive eigenvalues of $L_j(h)$ in $I$, then $ξ'(λ, h)$ is real analytic in $I$.

- The representations of $ξ'(λ, h)$ obtained in [25], [6] involve the traces of the cut-off resolvents

$$
χ(L_j - λ \mp i0)^{-1} χ, \quad χ \in C_0^∞(\mathbb{R}^n),
$$

and some regular terms whose meromorphic continuation is far from apparent. The form of $ξ'(λ, h)$ in [25], [6] has been used for the investigation of the Weyl type asymptotics of $ξ(λ, h)$ (see also [17], [5] for semi-classical asymptotics in the trapping case).

The proof of (1.3) relies heavily on the work of Sjöstrand [29], while the arguments in [23] were self-contained and based on the semi-classical estimates of the scattering determinant. Having in mind (1.3), we obtain in the general case of ”black box” semi-classical scattering several results:

1) We establish a Weyl type asymptotics of the spectral shift function in the general framework of semi-classical “black box” perturbations improving our previous result [6] and working without any assumption on the behavior of the resonances close to the real axis. We generalize the results of Christiansen [8] for compact perturbations and those of Robert [25] for long-range perturbations. Theorem 1 allows to consider the sum of the harmonic measures related to the resonances $w$, $\Im w \neq 0$, as a monotonic function and to apply a Tauberian argument as in [16].

11) We present a new direct and short proof of the recent result of J.-F. Bony and Sjöstrand [3] on the Breit-Wigner approximation in the long-range case (see Theorem 3). For this purpose the Weyl asymptotics obtained in Theorem 2 plays an essential role. Moreover, Theorem 2 and
Theorem 3 are established under the “black box” assumptions in Section 2 and the condition (5.1). Thus we have an unified approach to these problems. Next, assuming the existence of free resonances domain, we obtain a Breit-Wigner approximation involving only the resonances $w$ lying in small “boxes”

$$\{ w \in \mathbb{C} : |\text{Re} w - \lambda| \leq R(h), |\text{Im} w| \leq R_1(h) \}$$

with $R(h) = \sqrt{h R_1(h)} = O(h^{-\infty}).$

III) In the same way as in [23], we obtain the local trace formula of Sjöstrand [28], [29] in a slightly stronger version (see Section 7). Moreover, we prove a trace formula involving the unitary groups $e^{-it \sqrt{\lambda} L_j(h)}$, $j = 1, 2$ (see Theorem 5) which is a semi-classical version of the classical trace formulae.

We expect that the approach of our work could be useful in other situations as in the analysis of periodic potentials [10] or the study of matrix Schrödinger operators [18] if a representation like (1.3) is established.

The plan of the paper is the following. In Section 2 we introduce the ”black box” scattering assumptions and in Section 3 we obtain a formula for $\xi'(\lambda, h)$ involving the limits of the functions $\sigma_{\pm}(z)$ as $\text{Im} z \to 0$. Theorem 1 is proved in Section 4 and in Section 5 we establish a Weyl type asymptotics for the spectral shift function $\xi(\lambda, h)$. The semi-classical Breit-Wigner approximation is established in Section 6 together with a stronger approximation based on some recent results of Stefanov [31]. In Section 7 we prove some trace formulae combining (1.3) with the arguments of [23]. In particular, we obtain a trace formula involving the unitary groups $e^{-it \sqrt{\lambda} L_j}$. Finally, in Section 8 the Breit-Wigner approximation is applied to establish the existence of clusters of resonances close to the real axis.

Acknowledgments. The authors are grateful to J. Sjöstrand and M. Zworski for many helpful discussions.

2. Preliminaries

We start by the abstract “black box” scattering assumptions introduced in [30], [28] and [29]. The operators $L_j(h) = L_j$, $j = 1, 2$, $0 < h \leq h_0$, are defined in domains $D_j \subset H_j$ of a complex Hilbert space $H_j$ with an orthogonal decomposition

$$\mathcal{H}_j = \mathcal{H}_{R_0-j} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)), \quad B(0, R_0) = \{ x \in \mathbb{R}^n : |x| \leq R_0 \}, \quad R_0 > 0, \quad n \geq 2.$$ 

Below $h > 0$ is a small parameter and we suppose the assumptions satisfied for $j = 1, 2$. We suppose that $D_j$ satisfies

$$\| L_{n \setminus B(0, R_0)} D_j \| = H^2(\mathbb{R}^n \setminus B(0, R_0)), \quad (2.1)$$

uniformly with respect to $h$ in the sense of [28]. More precisely, equip $H^2(\mathbb{R}^n \setminus B(0, R_0))$ with the norm $\| < h D >^2 u \|_2$, $< h D >^2 = 1 + (h D)^2$, and equip $D_j$ with the norm $\|(L_j + i)u\|_{K_j}$. Then we require that $\| L_{n \setminus B(0, R_0)} : D_j \to H^2(\mathbb{R}^n \setminus B(0, R_0))$ is uniformly bounded with respect to $h$ and this map has a uniformly bounded right inverse.

Assume that

$$\| B(0, R_0) (L_j + i)^{-1} \| \text{is compact} \quad (2.2)$$
and
\[ (L_j u)|_{B^c (0, R_0)} = Q_j \left( \frac{u}{|x|^\alpha} \right), \quad (2.3) \]
where \( Q_j \) is a formally self-adjoint differential operator
\[ Q_j u = \sum_{|\nu| \leq 2} a_{j,\nu} (x; h) (hD_x)^\nu u, \quad (2.4) \]
with \( a_{j,\nu} (x; h) = a_{j,\nu} (x) \) independent of \( h \) for \( |\nu| = 2 \) and \( a_{j,\nu} \in C^\infty_b (\mathbb{R}^n) \) uniformly bounded with respect to \( h \).

We assume also the following properties:
There exists \( C > 0 \) such that
\[ l_{j,0} (x, \xi) = \sum_{|\nu| = 2} a_{j,\nu} (x) \xi^\nu \geq C |\xi|^2, \quad (2.5) \]
\[ \sum_{|\nu| \leq 2} a_{j,\nu} (x; h) \xi^\nu \to |\xi|^2, \quad |x| \to \infty \quad (2.6) \]
uniformly with respect to \( h \).

There exists \( \overline{\tau} > n \) such that we have
\[ \left| a_{1,\nu} (x; h) - a_{2,\nu} (x; h) \right| \leq O(1) \langle x \rangle^{-\overline{\tau}} \quad (2.7) \]
uniformly with respect to \( h \). This assumption will guarantee that for every \( f \in C^\infty_0 (\mathbb{R}) \) the operator \( f(L_1) - f(L_2) \) is “trace class near infinity”.

There exist \( \theta_0 \in ]0, \frac{\pi}{2}[ \), \( \epsilon > 0 \) and \( R_1 > R_0 \) so that the coefficients \( a_{j,\nu} (x; h) \) of \( Q_j \) can be extended holomorphically in \( x \) to
\[ \Gamma = \{ r \omega; \omega \in \mathbb{C}^n, \text{dist} (\omega, S^{n-1}) < \epsilon, \ r \in \mathbb{C}, \ r \in \epsilon [0, \theta_0] \} \quad (2.8) \]
and (2.6), (2.7) extend to \( \Gamma \).

Let \( R > R_0 \), \( T = (\mathbb{R}/\mathbb{Z})^n, \hat{R} > 2R \). Set
\[ \mathcal{H}^\#_j = \mathcal{H}_R \oplus L^2 (T \setminus B(0, R_0)) \]
and consider a differential operator
\[ Q_j^\# = \sum_{|\nu| \leq 2} a_{j,\nu}^\# (x; h) (hD_x)^\nu \]
on \( T \) with \( a_{j,\nu}^\# (x; h) = a_{j,\nu} (x; h) \) for \( |x| \leq R \) satisfying (2.3), (2.4), (2.5) with \( \mathbb{R}^n \) replaced by \( T \).
Consider a self-adjoint operator \( L_j^\# : \mathcal{H}^\#_j \to \mathcal{H}^\#_j \) defined by
\[ L_j^\# u = L_j \phi u + Q_j^\# (1 - \phi) u, \quad u \in \mathcal{D}^\#_j, \]
with domain
\[ \mathcal{D}^\#_j = \{ u \in \mathcal{H}^\#_j : \phi u \in \mathcal{D}_j, \ (1 - \phi) u \in \mathcal{H}^2 \}, \]
where \( \phi \in C^\infty_0 (B (0, R); [0, 1]) \) is equal to 1 near \( B (0, R_0) \).
Denote by \( N (L_j^\#, [-\lambda, \lambda]) \) the number of eigenvalues of \( L_j^\# \) in the interval \( [-\lambda, \lambda] \). Then we assume that
\[ N (L_j^\#, [-\lambda, \lambda]) = O \left( \left( \frac{\lambda}{h^2} \right)^{n_j^\# /2} \right), \quad n_j^\# \geq n, \ \lambda \geq 1. \quad (2.9) \]
Finally, we suppose that with some constant $C \geq 0$ independent on $h$ we have

$$\text{sp } L_j(h) \subset [-C, \infty[, \ j = 1, 2,$$

where $\text{sp } (L)$ denotes the spectrum of $L$. This condition is a technical one and we expect that by a more fine version of Proposition 1 we could cover the general case.

Given $f \in C^\infty_0(\mathbb{R})$ independent on $h$ and $\chi \in C^\infty_{0\alpha}(\mathbb{R}^n)$ equal to 1 on $B(0,R_0)$ we can define $\text{tr}_{\text{rb}}[f(L_j)]_{j=1}^2$, as in [28], [29], by the equality

$$\text{tr}_{\text{rb}} \left( f(L_2) - f(L_1) \right) = \left[ \text{tr}(\chi f(L_j) \chi + \chi f(L_j)(1 - \chi) + (1 - \chi) f(L_j) \chi) \right]_{j=1}^2$$

$$+ \text{tr}[(1 - \chi) f(L_j)(1 - \chi)]_{j=1}^2.$$ 

Following [28], [29], we can define the resonances $w \in \mathbb{C}_-$ by the complex scaling method as the eigenvalues of the complex scaling operators $L_{j,\theta}$, $j = 1, 2$. We denote by $\text{Res } L_j(h)$, $j = 1, 2$, the set of resonances and set $n^\# = \max \{n_1^\#, n_2^\# \}$.

3. Representation of the derivative of the spectral shift function

Consider the resolvents

$$R_j(\lambda \pm i\epsilon) = i \int_0^{\pm \infty} e^{it\lambda} e^{-it(L_j \mp i\epsilon)} dt, \ \lambda \in \mathbb{R}, \ \epsilon > 0,$$

$$R_j(\lambda - i\epsilon) = -i \int_{-\infty}^0 e^{it\lambda} e^{-it(L_j + i\epsilon)} dt.$$

Given a function $f(\lambda) \in C^\infty_0(\mathbb{R})$, we have

$$\frac{1}{2\pi i} \int R_j(\lambda + i\epsilon) f(\lambda) d\lambda = \frac{1}{2\pi} \int_0^\infty \hat{f}(-t) e^{-itL_j - t\epsilon} dt,$$

$$- \frac{1}{2\pi i} \int R_j(\lambda - i\epsilon) f(\lambda) d\lambda = \frac{1}{2\pi} \int_{-\infty}^0 \hat{f}(-t) e^{-itL_j + t\epsilon} dt,$$

where $\hat{f}$ denotes the Fourier transform of $f$. Choose $z_0 \in \mathbb{R}^-$ which is away from $\text{sp } (L_j)$, $j = 1, 2$, and set $g(\lambda) = (\lambda - z_0)^m f(\lambda)$, where the integer $m > n/2$ will be taken sufficiently large and independent on $h$. Applying the above formula, we obtain

$$\frac{1}{2\pi i} \text{tr}_{\text{rb}} \int \left[ (L_j - z_0)^{-m} \left( (\lambda + i\epsilon - z_0)^m R_j(\lambda + i\epsilon) - (\lambda - i\epsilon - z_0)^m R_j(\lambda - i\epsilon) \right) \right]_{j=1}^2 f(\lambda) d\lambda$$

$$= \frac{1}{2\pi} \text{tr}_{\text{rb}} \left[ (L_j - z_0)^{-m} \left( \int_0^\infty e^{-atL_j} (\hat{g}(-t) + i\epsilon G_{+,\epsilon}(t)) dt \right. \right.$$

$$+ \left. \int_{-\infty}^0 e^{-atL_j} (\hat{g}(-t) + i\epsilon G_{-,\epsilon}(t)) dt \right]_{j=1}^2 \quad (3.1)$$

Here $G_{+,\epsilon}(t)$ are some functions in $\mathcal{S}(\mathbb{R})$ related to the Fourier transform of $\lambda^k f(\lambda)$, $0 \leq k \leq m - 1$, which are uniformly bounded with respect to $0 < \epsilon < 1$. To justify the limit $\epsilon \downarrow 0$ in (3.1), we need to establish the estimates of the trace uniformly with respect to $\epsilon > 0$. To do this we will prove the following.
Lemma 1. For any $t \in \mathbb{R}$, the trace $t_{nb} \left[ (L_j - z_0)^{-m} e^{-itL_j} \right]_{j=1}^{2}$ is well defined, and
\[
t_{nb} \left[ (L_j - z_0)^{-m} e^{-itL_j} \right]_{j=1}^{2} = O \left( h^{-n^#} (1 + |t|) \right) .
\]

Proof. Let $\chi \in C_0^\infty (\mathbb{R}^n)$ be equal to 1 near $B(0, R_1)$, $R_1 > R_0$. Since the operators $\chi (L_j - z_0)^{-m}$ and $(L_j - z_0)^{-m} \chi$ are trace class (see [28]) and $e^{-itL_j}$ is uniformly bounded with respect to $t$, it is clear that $\chi (L_j - z_0)^{-m} e^{-itL_j}$ and $(L_j - z_0)^{-m} e^{-itL_j} \chi$ are trace class ones with trace bounded by $O(h^{-n^#})$. To be more precise let us note that in [29] the condition (2.10) is not assumed and we can formally apply the results of [29] for $z_0 \in \mathbb{C} \setminus \mathbb{R}$. In our case $z_0 \in \mathbb{R}^-$ and according to the resolvent equation we have
\[
(L_j - z_0)^{-m} = (L_j - z_1)^{-m} \left( I + (z_0 - z_1)(L_j - z_0)^{-1} \right)^m .
\]
So taking $z_1 \in \mathbb{C} \setminus \mathbb{R}$, we obtain the trace class properties mentioned above.

Now consider the operator
\[
\left[ (1 - \chi)(L_j - z_0)^{-m} e^{-itL_j} (1 - \chi) \right]_{j=1}^{2} .
\]
By Duhamel formula we obtain
\[
(1 - \chi)(L_j - z_0)^{-m} e^{-itL_j} (1 - \chi) = e^{-itQ_j} (1 - \chi)(L_j - z_0)^{-m} (1 - \chi) + i \int_0^t e^{-i(t-s)Q_j} \chi \hat{L}_j (L_j - z_0)^{-m} e^{-isL_j} ds .
\]
The integrand is a trace class operator with trace bounded by $O(h^{-n^#})$ and it remains to study the operator
\[
\left[ e^{-itQ_j} (1 - \chi)(L_j - z_0)^{-m} (1 - \chi) \right]_{j=1}^{2} .
\]
For $R_1 > R_0$, $\chi_0 \in C_0^\infty (\mathbb{R}^n)$ equal to 1 near $B(0, R_1)$ and $\chi_0 \prec \chi$ we have
\[
(L_j - z_0)^{-1} (1 - \chi) = (1 - \chi_0)(Q_j - z_0)^{-1} (1 - \chi) + (L_j - z_0)^{-1} [Q_j, \chi_0] (Q_j - z_0)^{-1} (1 - \chi) .
\]
Here and below the notation $\varphi \prec \psi$ means that $\psi = 1$ on supp $\varphi$. Choose cut-off functions $\theta_N < \ldots < \theta_0 \prec \chi$ so that $\theta_N = 1$ on $\overline{B(0, R_0)}$ and apply the telescopic formula
\[
(L_j - z_0)^{-1} [Q_j, \chi_0] (Q_j - z_0)^{-1} (1 - \chi) = (L_j - z_0)^{-1} [Q_j, \chi_0] (Q_j - z_0)^{-1} [Q_j, \theta_{N-1}] \ldots [Q_j, \theta_1] (Q_j - z_0)^{-1} (1 - \chi) .
\]
For $N > n/2$ this operator is trace class. In fact, for $\hat{\chi} \in C_0^\infty$ equal to 1 on supp $\theta_N$ the operator
\[
\hat{\chi} (Q_j - i)^{-N/2} (Q_j - i)^{N/2} [Q_j, \theta_N] (Q_j - z_0)^{-1} \ldots [Q_j, \theta_1] (Q_j - z_0)^{-1} (1 - \chi)
\]
is trace class, while $(L_j - z_0)^{-1} [Q_j, \chi_0] (Q_j - z_0)^{-1}$ is bounded. Here we have used the fact that $Q_j$ are elliptic operators and
\[
(Q_j - z_0)^{-1} = O(1) : H^N (\mathbb{R}^n) \longrightarrow H^{N+2} (\mathbb{R}^n), \forall N \in \mathbb{N} .
\]
Repeating this procedure, we obtain modulo trace class operators
\[
e^{-itQ_j} (L_j - z_0)^{-m} (1 - \chi)
\]
\[
e^{-itQ_j} (1 - \theta_m) (Q_j - z_0)^{-1} \ldots (1 - \theta_1) (Q_j - z_0)^{-1} (1 - \chi) .
\]
In the same way, since $\theta_k < \theta_{k-1}$, each term $\theta_k(Q_j - z_0)^{-m} (1 - \theta_k^{-1})$ in the above product is trace class operator and modulo a trace class operator we are going to study

$$[e^{-iQ_j}(Q_j - z_0)^{-m} (1 - \chi)]_{j=1}^2.$$ 

Consider the difference

$$(Q_2 - z_0)^{-m} e^{-iQ_2} - (Q_1 - z_0)^{-m} e^{-iQ_1}$$

$$= e^{-iQ_2} \left( (Q_2 - z_0)^{-m} - (Q_1 - z_0)^{-m} \right) + \left( e^{-iQ_2} - e^{-iQ_1} \right) (Q_1 - z_0)^{-m}.$$ 

For the first term at the right hand side observe that the operator $(Q_2 - z_0)^{-m} - (Q_1 - z_0)^{-m}$ for $m > \frac{n}{2}$ is a trace class one (see [9], [24], [28]). To handle the second term, notice that

$$\left( e^{-iQ_2} - e^{-iQ_1} \right) (Q_1 - z_0)^{-m} = i \int_0^t e^{-i(t-s)Q_2} (Q_1 - Q_2)(Q_1 - z_0)^{-m} e^{-iQ_1} ds$$

and use the fact that $(Q_1 - Q_2)(Q_1 - z_0)^{-m}$ is trace class for $m > \frac{n}{2} + 1$. 

According to Lemma 1, in the equation (3.1) we can take the limit $\epsilon \downarrow 0$ with respect to the norm in the space of trace class operators and taking into account the definition of $\text{tr}_{\text{th}}(\cdot)$, we get

$$\lim_{\epsilon \downarrow 0} \frac{1}{2\pi} \text{tr}_{\text{th}} \left[ (L_j - z_0)^{-m} \left( \int_{0}^{\infty} e^{-i\epsilon - iL_j (\hat{g}(-t) + i\epsilon G_{+,\epsilon}(t) \right) dt 

$$

$$= \frac{1}{2\pi} \text{tr}_{\text{th}} \left[ (L_j - z_0)^{-m} \left( \int_{-\infty}^{\infty} e^{-iL_j \hat{g}(-t) dt \right) \right)^2$$

$$= \text{tr}_{\text{th}} \left[ (L_j - z_0)^{-m} \right]_{j=1}^2 = \text{tr}_{\text{th}} \left( f(I_1) - f(I_2) \right) = \langle \xi^2(h), f(\lambda) \rangle_{D'(\mathbb{R}), D(\mathbb{R})}.$$ 

Thus we have proved the following.

**Proposition 1.** We have

$$\xi^2(h) = \frac{1}{2\pi i} \lim_{\epsilon \downarrow 0} \text{tr}_{\text{th}} \left[ ((\lambda + i\epsilon - z_0)^m (L_j - \lambda - i\epsilon)^{-1} \right.$$ 

$$\left. - (\lambda - i\epsilon - z_0)^m (L_j - \lambda + i\epsilon)^{-1} (L_j - z_0)^{-m} \right]_{j=1}^2,$$ 

where the limit is taken in the sense of distributions $D'(\mathbb{R})$.

Introduce the functions

$$\sigma_{\pm}(z) = (z - z_0)^m \text{tr}_{\text{th}} \left[ (L_j - z)^{-1} (L_j - z_0)^{-m} \right]_{j=1}^2, \pm \text{Im} z > 0.$$ 

which are well defined (see [29] and Proposition 2 below). The relation

$$\text{tr}_{\text{th}} \left[ (L_j - (\lambda - i\epsilon))^{-1} (L_j - z_0)^{-m} \right]_{j=1}^2 = \text{tr}_{\text{th}} \left[ (L_j - (\lambda + i\epsilon))^{-1} (L_j - z_0)^{-m} \right]_{j=1}^2,$$ 

implies immediately

$$\sigma_{-}(z) = \sigma_{+}(z), \text{ Im } z < 0.$$ 

The equality (3.4) plays a crucial role in the proof of (1.3) and our choice of real $z_0$ is related to the above relation.
4. Meromorphic continuation of the spectral shift function

In this section we prove our principal result given in Theorem 1. Taking $0 < \theta < \theta_0 < \pi/2$, consider the complex scaling operators $L_{j,\theta}$ related to $L_j$, $j = 1, 2$, introduced by Sjöstrand and Zworski (see [30], [28] and Section 2 in [29]). More precisely, given $\epsilon_0 > 0$, $R_1 > R_0$, consider a function

$$f_\theta(t) : ]0, \frac{\pi}{2} [ \times ]0, \infty [ \ni (\theta, t) \mapsto \mathbb{C},$$

which is injection for every $\theta$ and has the properties:

$$f_\theta(t) = t \ \text{for} \ \theta \leq t \leq R_1,$$

$$0 \leq \arg f_\theta(t) \leq \theta, \ \partial_t f_\theta \neq 0,$$

$$\arg f_\theta(t) \leq \arg \partial_t f_\theta(t) \leq \arg f_\theta + \epsilon_0,$$

$$f_\theta(t) = e^{i\theta}t, \ \text{for} \ t \geq T_0,$$

where $T_0$ depends on $\epsilon_0$ and $R_1$. Next consider the map

$$\kappa_\theta : \mathbb{R}^n \ni x = t\omega \mapsto f_\theta(t)\omega \in \mathbb{C}^n, \ t = |x|$$

and introduce $\Gamma_\theta = \kappa_\theta(\mathbb{R}^n)$ which coincides with $\mathbb{R}^n$ along $B(0, R_1)$. We define

$$\mathcal{H}_{j,\theta} = \mathcal{H}_{R_0, j} \oplus L^2(\Gamma_\theta \setminus B(0, R_0))$$

and $L_{j,\theta} : \mathcal{H}_{j,\theta} \to \mathcal{H}_{j,\theta}$ with domain $\mathcal{D}_j$ as the operator

$$L_{j,\theta}u = L_j(\chi_1 u) + Q_j \text{Re} \left( \frac{1 - \chi_1}{\chi_1} \right) u,$$

$\chi_1 \in C_0^\infty(B(0, R_1))$ being a function equal to 1 near $B(0, R_0)$.

Let $\Omega \subset e^{[1-2\delta, 2\delta]0, +\infty} \mathbb{C}$ be a simply connected open relatively compact set such that $\Omega \cap \mathbb{R}^+ = J$ is an interval. The spectrum of $L_{j,\theta}$ outside of $e^{-2\delta}[0, +\infty]$ consists of the negative eigenvalues of $L_j$ and the eigenvalues in $e^{-[0,2\delta]}[0, +\infty]$ (see [28]). Since the spectrum of $L_j$ is bounded from below, we may choose $z_0 \in \mathbb{R}^+$, $z_0 \notin \Omega$, so that $z_0$ is away from $\text{sp}(L_j)$ and $\text{sp}(L_{j,\theta})$, $j = 1, 2$. Given a positive number $\delta > 0$, we can apply Proposition 4.1 of Sjöstrand [29], saying that for all $z \in \Omega \cap \{z : \text{Im } z \geq \delta\}$ we have

$$\text{tr}_{\mathcal{H}_j} [(L_j - z)^{-1}(L_j - z_0)^{-m}]_{i=1}^2 = \text{tr}_{\mathcal{H}_j} [(L_{j,\theta} - z)^{-1}(L_{j,\theta} - z_0)^{-m}]_{i=1}^2,$$

where in the definition of the complex scaling operators $L_{j,\theta}$ the parameter $\epsilon_0$ is chosen small enough. Notice that the choice of $z_0 \in e^{[3\delta_0, \min[\pi, 2\pi-2\delta-3\delta_0]}[0, +\infty]$ in [29] says that we may take $z_0 \in \mathbb{R}^+$, assuming $\theta < \frac{\pi}{2} - \frac{3}{2}\epsilon_0$.

Below we assume $\delta$ and $\theta$ fixed and we will drop in the notations $L_j$ the index $j$ writing $L$, when the properties are satisfied for both operators $L_j$, $j = 1, 2$. Following [29], Section 4, there exists an operator $\hat{L}_{\cdot,\theta} : \mathcal{D} \to \mathcal{H}$, so that

$$K_{\cdot,\theta} = \hat{L}_{\cdot,\theta} - L_{\cdot,\theta} \text{ has rank } \mathcal{O}(h^{-n\#})$$

and for all $N, M \in \mathbb{N}$ we have

$$K_{\cdot,\theta} = \mathcal{O}(1) : \mathcal{D}(L_N^\infty) \to \mathcal{D}(L_M^M).$$
Secondly, \( K_{\gamma, \theta} \) is compactly supported, that is if \( \chi \in C_0^\infty(\mathbb{R}^n) \) is equal to 1 on \( B(0, R) \) for \( R \geq R_0 \) large enough, we have \( K_{\gamma, \theta} = \chi K_{\gamma, \theta} \chi \) and, finally, for every \( N \in \mathbb{N} \) we have

\[
(\hat{L}_{\gamma, \theta} - z)^{-1} = \mathcal{O}(1) : \mathcal{D}(L_N) \rightarrow \mathcal{D}(L_{N+1}),
\]

uniformly for \( z \in \Omega \). These properties imply for \( z \in \Omega \cap \{ \text{Im} \ z > 0 \} \) the representation

\[
(\hat{L}_{\gamma, \theta} - z)^{-1} = (\hat{L}_{\gamma, \theta} - z)^{-1} + (\hat{L}_{\gamma, \theta} - z)^{-1} K_{\gamma, \theta} (\hat{L}_{\gamma, \theta} - z)^{-1}.
\]

(4.2)

The contributions related to the resolvent \((\hat{L}_{\gamma, \theta} - z)^{-1}\) are examined in the following.

**Proposition 2.** There exists a function \( a_+(z, h) \) holomorphic in \( \Omega \) such that for \( z \in \Omega \cap \{ \text{Im} \ z > 0 \} \) we have

\[
\sigma_+(z) = \text{tr} \left[ (L_{\gamma, \theta} - z)^{-1} K_{\gamma, \theta} (\hat{L}_{\gamma, \theta} - z)^{-1} \right]^2 + a_+(z, h).
\]

Moreover,

\[
|a_+(z, h)| \leq C(\Omega) h^{-n^*}, \quad z \in \Omega
\]

with a constant \( C(\Omega) \) independent on \( h \in [0, h_0] \).

**Remark.** The singularities of \( \sigma_+(z) \) for \( \text{Im} \ z \downarrow 0 \) are independent on \( z_0 \in \mathbb{R}^- \) and \( m \in \mathbb{N} \).

**Proof.** According to (4.2), for \( z \in \Omega \cap \{ \text{Im} \ z \geq \delta \} \) we have

\[
\sigma_+(z) = (z - z_0)^m \text{tr} \left[ (\hat{L}_{\gamma, \theta} - z)^{-1} (L_{\gamma, \theta} - z_0)^{-m} \right]^2 \]

+ \( (z - z_0)^m \text{tr} \left[ (L_{\gamma, \theta} - z)^{-1} K_{\gamma, \theta} (\hat{L}_{\gamma, \theta} - z)^{-1} (L_{\gamma, \theta} - z_0)^{-1} \right]^2 \).

(4.6)

From the resolvent equation we obtain

\[
(z - z_0)^m (L_{\gamma, \theta} - z_0)^{-m} (\hat{L}_{\gamma, \theta} - z)^{-1} = (L_{\gamma, \theta} - z)^{-1} - \sum_{k=1}^m (z - z_0)^{k-1} (L_{\gamma, \theta} - z_0)^{-k}.
\]

To treat (4.6) we use the cyclicity of the trace and the above equality and conclude that this term is equal to \( \text{tr} \left[ (L_{\gamma, \theta} - z)^{-1} K_{\gamma, \theta} (\hat{L}_{\gamma, \theta} - z)^{-1} \right]^2 \) modulo a function holomorphic in \( \Omega \) and bounded by \( \mathcal{O}(h^{-n^*}) \).

Now we pass to the analysis of (4.5). Our purpose is to show that (4.5) is holomorphic in \( \Omega \) and bounded by \( \mathcal{O}(h^{-n^*}) \). By construction, \((\hat{L}_{\gamma, \theta} - z)^{-1}\) is holomorphic on \( \Omega \) and for any cut-off function \( \chi \in C_0^\infty(\mathbb{R}^n) \), \( \chi = 1 \) on \( B(0, R_0) \) with \( \text{supp} \chi \subset B(0, R_1) \) the operators \( \chi(L_{\gamma, \theta} - z_0)^{-m}, (L_{\gamma, \theta} - z_0)^{-m} \chi \) are trace class ones. Hence the function \( \text{tr} \left[ (\hat{L}_{\gamma, \theta} - z)^{-1} (L_{\gamma, \theta} - z_0)^{-m} \chi \right] \) is holomorphic in \( \Omega \). On the other hand,

\[
(L_{\gamma, \theta} - z_0)^{-m} (\hat{L}_{\gamma, \theta} - z)^{-1} = (L_{\gamma, \theta} - z_0)^{-m} (L_{\gamma, \theta} - z_0)^{-1} K_{\gamma, \theta} (\hat{L}_{\gamma, \theta} - z)^{-1} (L_{\gamma, \theta} - z_0)^{-m}.
\]

(4.7)

Consequently, for \( \text{Im} \ z > 0 \) if \( \chi_1 \in C_0^\infty(\mathbb{R}^n) \) is a cut-off function and \( \chi_1 \propto \chi \), applying the cyclicity of the trace once more, we get

\[
\text{tr} \left( \chi_1 (\hat{L}_{\gamma, \theta} - z)^{-1} (L_{\gamma, \theta} - z_0)^{-m} (1 - \chi) \right) = 0.
\]
Thus it remains to examine
\[
\tau_+ (z) = \text{tr} \left[ (1 - \chi_1)(\hat{L}_{j,\theta} - z)^{-1}(1 - \chi)(L_{j,\theta} - z_0)^{-m}(1 - \chi) \right]_{j=1}^2.
\]

Consider the operator \( Q_{\cdot,\theta} = Q_{\cdot,\theta}I \), and note that for \( \psi \in C^\infty \) supported away from \( B(0, R_1) \) we have \( L_{j,\theta} \psi = Q_{\cdot,\theta} \psi \). Repeating the construction of \( \hat{L}_{j,\theta} \) in Section 4, [29], we can find an operator \( \hat{Q}_{\cdot,\theta} : H^2(\Gamma_\theta) \rightarrow L^2(\Gamma_\theta) \) so that
\[
\hat{Q}_{\cdot,\theta} - Q_{\cdot,\theta} \text{ has rank } O(h^{-n}),
\]
the operator \( \hat{Q}_{\cdot,\theta} - Q_{\cdot,\theta} \) is compactly supported and for \( z \in \overline{\Omega} \) we have
\[
(\hat{Q}_{\cdot,\theta} - z)^{-1} = O(1) : D(Q_{\cdot,\theta}^N) \rightarrow D(Q_{\cdot,\theta}^{N+1}), \quad \forall N \in \mathbb{N}.
\]
Moreover, for \( \psi \in C^\infty \) supported away from \( B(0, R_1) \) we have \( \hat{L}_{j,\theta} \psi = \hat{Q}_{\cdot,\theta} \psi \) and for \( \chi \in C_0^\infty(\Gamma_\theta) \) equal to 1 on a sufficiently large set, \( z \in \Omega \) and \( \chi_1 \ll \chi_0 \ll \chi \) we obtain
\[
(\hat{L}_{j,\theta} - z)^{-1}(1 - \chi) = (1 - \chi_0)(\hat{Q}_{\cdot,\theta} - z)^{-1}(1 - \chi)\]
\[
+ (\hat{L}_{j,\theta} - z)^{-1}[\hat{Q}_{\cdot,\theta} - z_0]_{j=1}^2.
\]

As above, we assume that \( z_0 \in \mathbb{R}^+ \) is chosen so that \( z_0 \notin \text{sp} (Q_j), \quad z_0 \notin \text{sp} (Q_{j,\theta}), \quad j = 1, 2 \). For simplicity of the notations below we omit the index \( \theta \) and we get
\[
\tau_+(z) = \text{tr} \left[ (1 - \chi_0)(\hat{Q}_j - z)^{-1}(1 - \chi)(L_j - z_0)^{-m}(1 - \chi) \right]_{j=1}^2
+ \text{tr} \left[ (1 - \chi_1)(\hat{L}_j - z)^{-1}[\hat{Q}_j, \chi_0](\hat{Q}_j - z)^{-1}(1 - \chi)(L_j - z_0)^{-m}(1 - \chi) \right]_{j=1}^2.
\]

Obviously, \( [\hat{Q}_j, \chi_0] = [Q_j, \chi_0] + M_j \) with a trace class operator \( M_j \). To show that the operator \( [Q_j, \chi_0](\hat{Q}_j - z)^{-1}(1 - \chi) \) is a trace class one, we apply the telescopic formula choosing cut-off functions \( \theta_N < \theta_{N-1} < \ldots < \theta_1 < \chi \) and write
\[
[Q_j, \chi_0](\hat{Q}_j - z)^{-1}(1 - \chi) = [Q_j, \chi_0][\hat{Q}_j, \chi_0](\hat{Q}_j - z)^{-1} \chi(\hat{Q}_j - i)^{-m}
\]
\[
\times \left[ [Q_j - i]^N[\hat{Q}_j, \theta_N](\hat{Q}_j - z)^{-1}[\hat{Q}_j, \theta_{N-1}]\ldots[\hat{Q}_j, \theta_1](\hat{Q}_j - z)^{-1}(1 - \chi) \right]
\]
with \( N \geq 2m > n \). The operator in the brackets \([...]\) and \( [Q_j, \chi_0](\hat{Q}_j - z)^{-1} \) are bounded, while \( \chi(\hat{Q}_j - i)^{-m} \) is trace class. Thus the term involving \( [\hat{Q}_j, \chi_0] \) is holomorphic in \( \Omega \) and bounded by \( O(h^{-n#}) \).

As in the proof of Proposition 1, we have
\[
\|(1 - \chi)(L_j - z_0)^{-m}(1 - \chi) - (1 - \chi)(Q_j - z_0)^{-m}(1 - \chi)\|_{L^2} = O(h^{-n#}).
\]
Moreover, \( (Q_j - z_0)^{-m} \chi \) is trace class and, consequently, there exists a function \( b(z, h) \) holomorphic in \( \Omega \) and bounded by \( O(h^{-n#}) \) so that
\[
\tau_+(z) = b(z, h) + \text{tr} \left[ (1 - \chi)(\hat{Q}_j - z)^{-1}(Q_j - z_0)^{-m}(1 - \chi) \right]_{j=1}^2.
\]

We write
\[
(\hat{Q}_2 - z)^{-1} (Q_2 - z_0)^{-m} - (\hat{Q}_1 - z)^{-1} (Q_1 - z_0)^{-m}
= (\hat{Q}_2 - z)^{-1} [(Q_2 - z_0)^{-m} - (Q_1 - z_0)^{-m}] + [(\hat{Q}_2 - z)^{-1} - (\hat{Q}_1 - z)^{-1}](Q_1 - z_0)^{-m} = I + II.
\]
According to [28], [29], the operator \((Q_2 - z_0)^{-m} - (Q_1 - z_0)^{-m}\) is trace class one and the contribution of \(I\) is holomorphic and bounded by \(O(h^{-n\#})\). For \(II\) we obtain the representation

\[ II = (\hat{Q}_2 - z)^{-1}(\hat{Q}_1 - \hat{Q}_2)(\hat{Q}_1 - z)^{-1}(Q_1 - z_0)^{-m}. \]

It is clear that \(\hat{Q}_1 - \hat{Q}_2 = Q_1 - Q_2 + K_{1,2}\) with a finite rank operator \(K_{1,2}\), and modulo a trace class operator we have

\[ II = (\hat{Q}_2 - z)^{-1}\left((Q_1 - Q_2)(Q_2 - z_0)^{-m}\right)(Q_2 - z_0)^{m}(\hat{Q}_1 - z)^{-1}(Q_1 - z_0)^{-m}. \]

The second factor is a trace class operator, while the first and the third ones are bounded operators. Consequently, \(II\) has the same property as \(I\). Combining the above results, we conclude that \(\tau_+(z)\) is holomorphic in \(\Omega\) and bounded by \(O(h^{-n\#})\).

To establish (4.3), notice that the right hand side of this equality is holomorphic for \(z \in \Omega \cap \{\text{Im} z > 0\}\). The left hand side is also holomorphic in this domain since we may apply (4.1) with different \(\delta > 0\), \(\epsilon_0 > 0\) and \(0 < \theta < \frac{\pi}{2} - \frac{\pi}{2} \epsilon_0\). By analytic continuation we deduce (4.3) and the proof of Proposition 2 is complete.

**Proof of Theorem 1.** To obtain a meromorphic continuation of \(\sigma_+(z)\) through the real axis, it suffices to do this for the trace involving \(K_{j,\theta}\). Next we will follow closely the argument of Sjöstrand [29] and since \(\theta\) is fixed, we will omit it in the notations. Setting \(\bar{K}(z) = K(z - \bar{L})^{-1}\), from (4.31) in [29] we get the representation

\[ -\text{tr}((L - z)^{-1}K_{j}(\bar{L} - z)^{-1}) = \text{tr}\left((1 + \bar{K}(z))^{-1}\frac{\partial}{\partial z} \bar{K}(z)\right) \]

and the resonances of \(L\) are precisely the zeros of the function

\[ D(z, h) = \text{det}(1 + \bar{K}(z)) = O(1) \exp(CH^{-n\#}). \quad (4.9) \]

Notice that the multiplicities of the resonances and the zeros coincide. Below in the notations we omit the subscript \(j\) since the argument does not depend on \(j = 1, 2\). Let \(\text{Res}(L)\) be the resonances of \(L\) and let

\[ D(z, h) = G(z, h) \prod_{w \in \text{Res}(L) \cap \Omega} (z - w), \]

where \(G(z, h)\) and \(\frac{1}{\sigma(z, h)}\) are holomorphic in \(\Omega\) and the resonances in the product are repeated following their multiplicity. Obviously,

\[ \partial_z \log D(z, h) = \partial_z \log G(z, h) + \sum_{w \in \text{Res}(L) \cap \Omega} \frac{1}{z - w} \]

and according to the estimate (4.5) in [29], we get

\[ \left| \frac{\partial}{\partial z} \log G(z, h) \right| \leq C(\Omega)|h^{-n\#}|, \quad z \in \tilde{\Omega}, \quad (4.10) \]

where \(\tilde{\Omega} \subset \subset \Omega\) is an arbitrary open simply connected domain and \(C(\Omega)\) is independent on \(h \in [0, h_0]\).
Going back to the representation (3.2) and taking into account (3.4), we observe that for \( \lambda \in I \subset \mathbb{R}^+, \, \Im w \neq 0 \), we have
\[
-\frac{1}{2\pi i} \lim_{\epsilon \to 0} \left( \frac{1}{\lambda + i \epsilon - w} - \frac{1}{\lambda - i \epsilon - w} \right) = -\frac{\Im w}{\pi |\lambda - w|^2},
\]
while for \( w \in \mathbb{R} \) we get
\[
-\frac{1}{2\pi i} \lim_{\epsilon \to 0} \left( \frac{1}{\lambda + i \epsilon - w} - \frac{1}{\lambda - i \epsilon - w} \right) = \delta(\lambda - w),
\]
where both limits are taken in the sense of distributions. Combining Propositions 1, 2 and the above arguments we complete the proof of Theorem 1.

The representation (1.3) shows that modulo a constant the spectral shift function \( \xi(\lambda, h) \) coincides with the distribution
\[
\xi(\lambda, h) = \frac{1}{\pi} \left[ \sum_{\mu \in \Sigma_\lambda} \int_{|\mu - w|^2}^{\lambda} \frac{\Im w}{|\mu - w|^2} d\mu \right]_{j=1}^\lambda
\]
\[
+ \left[ \#\{ \mu \in [\lambda_0, \lambda] : \mu \in \sigma_{pp}(L_j(h)) \} \right]_{j=1}^2 + \frac{1}{\pi} \int_{\lambda_0}^\lambda \Im r(\mu, h) d\mu, \, \lambda_0 > 0, \, \lambda_0 \notin I.
\]
In particular, for \( \lambda \in I \setminus \bigcup_{j=1}^2 \{ \lambda \in \mathbb{R} : \lambda \in \sigma_{pp}(L_j(h)) \} \) the distribution \( \xi(\lambda, h) \) is continuous and the function
\[
\eta(\lambda, h) = \xi(\lambda, h) - \left[ \#\{ \mu \in [\lambda_0, \lambda] : \mu \in \sigma_{pp}(L_j(h)) \} \right]_{j=1}^2
\]
is real analytic in \( I \).

5. Weyl asymptotics

In this section we obtain a Weyl type asymptotics for the spectral shift function. We generalize the results of Christiansen [8] and Robert [25] covering the "black box" long-range perturbations of the Laplacian and we improve our previous result (see Theorem 2 in [6]) working without any condition on the behavior of the resonances close to the real axis.

We will say that \( \lambda \in \mathbb{R} \) is a non-critical energy level for \( Q \) if for all \( (x, \xi) \in \Sigma_\lambda = \{ (x, \xi) \in \mathbb{R}^{2n} : l(x, \xi) = \lambda \} \) we have \( \nabla_x l(x, \xi) \neq 0 \), \( l(x, \xi) \) being the principal symbol of \( Q \). Given a Hamiltonian \( l(x, \xi) \), denote by
\[
\exp(tH_1)(x_0, \xi_0) = (x(t, x_0, \xi_0), \xi(t, x_0, \xi_0))
\]
the trajectory of the Hamilton flow \( \exp(tH_1) \) passing through \( (x_0, \xi_0) \in \Sigma_\lambda \). Recall that \( \lambda \in J \) is a non-trapping energy level for \( l(x, \xi) \) if for every \( R > 0 \) there exists \( T(R) > 0 \) such that for \( (x_0, \xi_0) \in \Sigma_\lambda, |x_0| < R \), the \( x \)-component of the trajectory of \( \exp(tH_1) \) passing through \( (x_0, \xi_0) \) satisfies
\[
|x(t, x_0, \xi_0)| > R, \, \forall |t| > T(R).
\]
Denote by \( N(L_j^#, I) \) the number of eigenvalues of \( L_j^# \) in the interval \( I \). From the assumptions (2.5) and (2.10) we deduce easily that there exists a constant \( C^# \) such that the spectrums of \( L_j^# \), \( j = 1, 2, \) do not intersect the interval \( ] - \infty, -C^# [ \) and consequently \( N(L_j^#, ] - \infty, -C^# [ ) = 0 \). In
fact, let $\chi_0, \chi_1 \in C_0^\infty(B(0,R) ; [0,1])$ be equal to 1 on $\overline{B(0,R_0)}$ and let $\chi_1 \gg \chi > \chi_0$. Using the resolvent equality we get

\[(L_j^\# - z)^{-1} = (L_j^\# - z)^{-1} + (L_j^\# - z)^{-1}(1 - \chi)
\]

\[= \chi_1(L_j - z)^{-1} \chi - (L_j^\# - z)^{-1}[Q_j^\#, \chi_1](L_j - z)^{-1} \chi
\]

\[+ (1 - \chi_0)(Q_j^\# - z)^{-1}(1 - \chi) + (L_j^\# - z)^{-1}[Q_j^\#, \chi_0](Q_j^\# - z)^{-1}(1 - \chi).
\]

Then

\[(L_j^\# - z)^{-1}(1 + [Q_j^\#, \chi_1](L_j - z)^{-1} \chi - [Q_j^\#, \chi_0](Q_j^\# - z)^{-1}(1 - \chi)
\]

\[= \chi_1(L_j - z)^{-1} \chi + (1 - \chi_0)(Q_j^\# - z)^{-1}(1 - \chi).
\]

According to the assumptions (2.5) and (2.10) there exists $C^\#$ such that spectrums of $L_j, Q_j^\#$, $j = 1, 2$, do not intersect the interval $]-\infty,-C^\#]$, hence for $z \in ]-\infty,-C^\#]$, the resolvents $(L_j - z)^{-1}, (Q_j^\# - z)^{-1}$ are bounded and we obtain immediately

\[|Q_j^\#, \chi_1](L_j - z)^{-1} \chi - [Q_j^\#, \chi_0](Q_j^\# - z)^{-1}(1 - \chi) = O(h).
\]

Consequently, for $h$ small enough and $z \in ]-\infty,-C^\#$, the resolvent $(L_j^\# - z)^{-1}$ is bounded and $z \not\in \text{sp}(L_j^\#)$. In the following we will use the notation

\[N(L_j^\#, \lambda) = N(L_j^\#, ]-\infty,-C^\#], \lambda), \ j = 1, 2.
\]

The spectral shift function $\xi(\lambda, h)$ is determined modulo a constant and from (2.10) we deduce that $\xi(\lambda, h)$ is constant on $]-\infty,-C_1]$ for $C_1$ sufficiently large. In the following, without loss of the generality, we may choose $\xi(\lambda, h)$ so that $\xi(\lambda, h) = 0$ on $]-\infty,-C^\#]$. Moreover, in this section we consider $\xi(\lambda, h) = \lim_{\delta \to 0} \xi(\lambda + \epsilon, h)$ as a function continuous from the right. The main result in this section is a Weyl type asymptotics for the spectral shift function.

**Theorem 2.** Assume that $L_j, j = 1, 2$ satisfy the assumptions of Section 2. Let $0 < E_0 < E_1$ and suppose that each $\lambda \in [E_0, E_1]$ is a non-critical energy level for $Q_j, Q_j^\#$, $j = 1, 2$. Assume that there exist positive constants $B, \epsilon_1, C_1, h_1$ such that for any $\lambda \in [E_0 - \epsilon_1, E_1 + \epsilon_1], h/B \leq \delta \leq B$ and $h \in [0, h_1]$ we have

\[N(L_j^\#, [\lambda - \delta, \lambda + \delta]) \leq C_1 \delta h^{-n^\#}, \ j = 1, 2.
\]

Then there exist $\omega(\lambda) \in C^{0}(\mathbb{R})$, $h_0 > 0$ such that

\[\xi(\lambda, h) = \left[N(L_j^\#, \lambda)\right]_{j=1}^2 + \omega(\lambda)h^{-n} + O(h^{1-n^\#})
\]

uniformly with respect to $\lambda \in [E_0, E_1]$ and $h \in [0, h_0]$.

**Remark.** Notice that if $\lambda$ is a non-critical energy level, then for $\epsilon > 0$ small enough each $\mu \in ]\lambda - \epsilon, \lambda + \epsilon]$ is also non-critical one. Consequently, (5.2) remains valid on some interval $[E_0 - \alpha, E_1 + \alpha], \alpha > 0$. Recall that the operators $L_j^\#, j = 1, 2$, have been defined in Section 2 by using the operators $Q_j^\#, j = 1, 2$, whose coefficients satisfy $a_{j,v}^\#(x; h) = a_{j,v}(x; h)$ for $|x| \leq R, R > R_0$. If the principal symbol $l_j(x, \xi)$ of $Q_j$ is non-critical for $\lambda \in [E_0, E_1]$, we can extend $a_{j,v}^\#(x; h)$ for $|x| > R$ in such way that $\lambda \in [E_0, E_1]$ become non-critical for $Q_j^\#$. This continuation changes the operator $L_j^\#$ but as it has been proved by J.-F. Bony [1], the assumption (5.1) does not depend on
the continuation of $a_{j,\nu}^\#(x; h)$.

To prove Theorem 2, we will introduce intermediate operators exploiting the following result of J.-F. Bony (see also [27]).

**Proposition 3.** ([2]) Assume that $L$ satisfy the assumptions of Section 2 and suppose that each $\lambda \in [E_0, E_1]$ is a non-critical energy level for $Q$. Given a fixed $\lambda \in [E_0, E_1]$, there exists a differential operator $\tilde{L}$, such that

(a) The pair $(\tilde{L}, \tilde{L})$ satisfies the assumptions of Section 2, with $\overline{\pi} = n + 1$,

(b) There exists an interval $I_0 \ni \lambda$, such that each $\mu \in I_0$ is non-trapping and non-critical energy level for $\tilde{L}$,

(c) The operator $\tilde{L}$ has no resonances in a complex neighborhood $\Omega_0$ of $I_0$ and $\Omega_0$ is independent on $h$.

Now denote by $\xi(\lambda; A, B)$ the spectral shift function related to the operators $A$ and $B$. Using the above proposition for each operator $L_j$, $j = 1, 2$, we can construct operators $\tilde{L}_j$, $j = 1, 2$, and decompose the spectral shift function $\xi(\lambda; L_1, L_2)$ as follows

$$\xi(\lambda; L_1, L_2) = \xi(\lambda; L_1, \tilde{L}_1) + \xi(\lambda; \tilde{L}_1, \tilde{L}_2) - \xi(\lambda; L_2, \tilde{L}_2).$$

Moreover, if every $\lambda \in [\alpha, \beta]$ is non-trapping and non-critical energy level for $\tilde{L}_j$, $j = 1, 2$, we have a complete asymptotic expansion of $\xi(\lambda; \tilde{L}_1, \tilde{L}_2)$ and its derivatives (see [25]) uniformly with respect to $\lambda \in [\alpha, \beta]$. Here we may estimate the difference $L_1 - \tilde{L}_2 = (\tilde{L}_1 - L_1) + (L_1 - L_2) + (L_2 - \tilde{L}_2)$ by applying our assumptions on $Q_1 - Q_2$. Thus it is sufficient to prove the theorem for $\lambda \in I_2 \subset I_0$ and the pair $(L_1, L_2)$ with $L_2 = \tilde{L}_2$ being a differential operator having no resonances in a complex neighborhood $\Omega_0$ of $I_0$ and such that every $\lambda \in I_0$ is non-trapping and non-critical energy level for $L_2$. Then the assertion follows by applying the local result and covering the compact interval $[E_0, E_1]$ by small intervals.

We denote $\xi(\lambda, h)$ the spectral shift function for the operators $(L_1, L_2)$. Applying Theorem 1 in the domain $\Omega_0$, we deduce that there exists a function $g_+(z, h)$ holomorphic in $\Omega_0$ such that for $\lambda \in I_0 = W_0 \cap \mathbb{R}$, $W_0 \subset \subset \Omega_0$ we have

$$\xi'(\lambda, h) = \frac{1}{\pi} \text{Im} g_+(\lambda, h) + \sum_{\omega \in \text{Res} L_1 \cap \Omega_0} \frac{-\text{Im} \omega}{\pi |\lambda - \omega|^2} + \sum_{\omega \in \text{Res} L_1 \cap I_0} \delta(\lambda - \omega), \quad (5.3)$$

where $g_+(z, h)$ satisfies the estimate

$$|g_+(z, h)| \leq C(\omega_0) h^{-n'}, \quad z \in W_0 \quad (5.4)$$

with $C(\omega_0) > 0$ independent on $h \in [0, \omega_0]$.

In the following, we fix an open interval $I_0 \subset \mathbb{R}^+$ so that each $\mu \in I_0$ is a non-critical energy level for $Q_j$, $j = 1, 2$, and we introduce open intervals $I_2 \subset \subset I_1 \subset \subset I_0$. It is convenient to decompose $\xi(\lambda, h)$ for $\lambda \in I_2$ into a sum of a term independent on $\lambda$ and a second one localized in $I_0$ where (5.3) holds.
Lemma 2. Let $C^\# > 0$ be such that the spectra of $I_j$ and $I_j^\#$, $j = 1, 2$, do not intersect the interval $[-\infty, -C^\#]$. Let $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}; \mathbb{R}^+)$ be such that $\text{supp } \varphi_1 \subset (-\infty, \gamma_1)$, $\text{supp } \varphi_2 \subset I_1$, $\varphi_2 = 1$ on $I_2 = (\gamma_1, \gamma_2)$ and $\varphi_1 + \varphi_2 = 1$ on $[-C^\# - \eta_0, \gamma_2]$, $\eta_0 > 0$. Then for $\lambda \in I_2$ we have

$$\xi(\lambda, h) = \text{tr}_{h_b} \left[ \varphi_1(L_j) \right]_j^2 + G_{\varphi_2}(\lambda) + M_{\varphi_2}(\lambda),$$

(5.5)

where

$$G_{\varphi_2}(\lambda) = \frac{1}{\pi} \int_{|\lambda|} \text{Im } g_\varphi(\mu, h) \varphi_2(\mu) d\mu,$$

$$M_{\varphi_2}(\lambda) = \sum_{\omega \in \text{Res } L_2 \cap \mathbb{R}} \int_{|\lambda|} \frac{|\text{Im } w|}{|\mu - w|} \varphi_2(\mu) d\mu + \sum_{\omega \in \text{Res } L_1 \cap [-C^\#, \lambda]} \varphi_2(\omega),$$

(5.6)

and we omit in $M_{\varphi_2}$ and $G_{\varphi_2}$ the dependence of $h$.

Proof. Roughly speaking, for $\lambda \in I_2$, if we express the action of the distributions as integrals, we must have

$$\xi(\lambda, h) = \int_{-\infty}^\lambda \varphi_1(\mu) \xi(\mu, h) d\mu + \int_{-\infty}^\lambda \varphi_2(\mu) \xi(\mu, h) d\mu.$$ 

Since $\varphi_1$ vanishes on $I_2$, the first term is independent on $\lambda \in I_2$ and equal to $\text{tr}_{h_b} \left[ \varphi_1(L_j) \right]_j^2$. For the second one we may apply (5.3) since $\varphi_2$ is supported in $I_1 \subset I_2$.

For a rigorous proof of the above representation, take $f \in C_0^\infty(I_2)$ and introduce

$$F(\lambda) = (\varphi_1 + \varphi_2(\lambda)) \int_{\lambda}^{+\infty} f(\mu) d\mu,$$

which is compactly supported. Since $\text{supp } f \subset I_2$ and $\varphi_1 + \varphi_2 = 1$ on $I_2$, we have

$$F'(\lambda) = -f(\lambda) + (\varphi_1' + \varphi_2'(\lambda)) \int_{\lambda}^{+\infty} f(\mu) d\mu,$$

where the second term vanishes on $[-C^\# - \eta_0, +\infty]$. Our choice of $\xi(\lambda, h) = 0$ on $]-\infty, -C^\#]$ makes possible to write

$$\langle \xi, f \rangle_{\mathcal{D}', \mathcal{D}} = -\langle \xi, F' \rangle_{\mathcal{D}', \mathcal{D}} = \langle \xi', F \rangle_{\mathcal{D}', \mathcal{D}}.$$

Next the equality $\varphi_1 \int_{\lambda}^{+\infty} f = \varphi_1 \int_{\mathbb{R}} f$ yields

$$\langle \xi', \varphi_1 \int_{\lambda}^{+\infty} f \rangle_{\mathcal{D}', \mathcal{D}} = \left( \int_{\mathbb{R}} f \right) \langle \xi', \varphi_1 \rangle_{\mathcal{D}', \mathcal{D}} = \left( \int_{\mathbb{R}} f \right) \text{tr}_{h_b} \left[ \varphi_1(L_j) \right]_j^2.$$

For the term involving $\varphi_2$, we apply (5.3) and we get

$$\langle \xi', \varphi_2 \int_{\lambda}^{+\infty} f \rangle_{\mathcal{D}', \mathcal{D}} = \langle G_{\varphi_2}, \psi \int_{\lambda}^{+\infty} f \rangle_{\mathcal{D}', \mathcal{D}} + \langle M_{\varphi_2}, \psi \int_{\lambda}^{+\infty} f \rangle_{\mathcal{D}', \mathcal{D}}$$

for $\psi \in C^\infty(\mathbb{R})$ equal to 1 on $\mathbb{R}^+$ and vanishing on $]-\infty, -1]$. The above relations imply (5.5) in the sense of distributions since $G_{\varphi_2} \psi = M_{\varphi_2} \psi = 0$ and $\psi f = f$. 

To prove Theorem 2, we will apply a Tauberian argument for the increasing function $M_{\varphi_2}(\lambda)$. Consider a function $\theta(t) \in C_0^\infty([-\delta_1, \delta_1])$, $\theta(0) = 1$, $\theta(-t) = \theta(t)$, such that the Fourier transform
\( \hat{\theta} \) of \( \theta \) satisfies \( \hat{\theta}(\lambda) \geq 0 \) on \( \mathbb{R} \) and assume that there exist \( 0 < \epsilon_0 < 1, \delta_0 > 0 \) so that \( \hat{\theta}(\lambda) \geq \delta_0 > 0 \) for \( |\lambda| \leq \epsilon_0 \). Next introduce

\[
(F_h^{-1} \hat{\theta})(\lambda) = (2\pi h)^{-1} \int e^{i\lambda/h} \hat{\theta}(t) dt = (2\pi h)^{-1} \hat{\theta}(-h^{-1} \lambda).
\]

**Remark.** It is obvious that the Lemma 2 holds if we take a partition of unity \( \varphi_1^2 + \varphi_2^2 \) over \( [-C^\# - \eta_0, \gamma_2] \) with cut-off functions \( \varphi_j, j = 1, 2 \).

The next lemma permits to establish a connection between the asymptotics of the functions \( M_{\varphi_2} \) and \( N_{\varphi_2}^# \).

**Lemma 3.** Let \( \varphi_2 \in C_0^\infty(I_1; \mathbb{R}^+) \) and let \( N_{\varphi_2}^#(\lambda) = \text{tr} \left( \varphi_2(L_1^#)1_{[-C^\#, \lambda]}(L_1^#) \right) \). Then there exists \( \omega_{\varphi_2}(\lambda) \in C_0^\infty(I_0) \) such that for any \( \lambda \in \mathbb{R} \) we have

\[
\frac{d}{d\lambda} (F_h^{-1} \theta \ast M_{\varphi_2})(\lambda) = \frac{d}{d\lambda} (F_h^{-1} \theta \ast N_{\varphi_2}^#)(\lambda) - G_{\varphi_2}(\lambda) h^{-n} + O(h^{1-n} #),
\]

where \( O(h^{1-n} #) \) is uniform with respect to \( \lambda \in \mathbb{R} \). Moreover, we have

\[
M_{\varphi_2}(\lambda) = (F_h^{-1} \theta \ast M_{\varphi_2})(\lambda) + O(h^{1-n} #)
\]

uniformly with respect to \( \lambda \in I_0 \).

**Proof.** For simplicity of the notations we omit the subscript \( \varphi_2 \) and denote by \( M, G, N, \omega \) the functions \( M_{\varphi_2}, G_{\varphi_2}, N_{\varphi_2}^#, \omega_{\varphi_2} \). According to (5.6) and (5.3), for any \( \lambda \in \mathbb{R} \) we have

\[
\frac{d}{d\lambda} (F_h^{-1} \theta \ast M)(\lambda) = (F_h^{-1} \theta \ast M')(\lambda) = (F_h^{-1} \theta \ast \varphi_2 \xi')(\lambda) - (F_h^{-1} \theta \ast G'(\lambda)).
\]

Using the Cauchy inequalities, it follows easily that \( G'(\lambda) = O(h^{-n} #) \) and \( G''(\lambda) = O(h^{-n} #) \) and we obtain immediately

\[
\frac{d}{d\lambda} (F_h^{-1} \theta \ast G)(\lambda) = G'(\lambda) + O(h^{1-n} #)
\]

uniformly with respect to \( \lambda \in \mathbb{R} \).

It remains to examine

\[
(F_h^{-1} \theta \ast \varphi_2 \xi')(\lambda) = \text{tr}_{1b} \left[ (F_h^{-1} \theta) \left( \lambda - L_j \right) \varphi_2(L_j) \right]^2_{j=1}
\]

\[
= \frac{1}{2\pi h} \int e^{i\lambda h^{-1} \theta}(t) \text{tr}_{1b} \left[ e^{-i\theta h^{-1} L_j} \varphi_2(L_j) \right]^2_{j=1} dt.
\]

We will prove that

\[
(F_h^{-1} \theta \ast \varphi_2 \xi')(\lambda) = \frac{d}{d\lambda} (F_h^{-1} \theta \ast N_{\varphi_2}^#)(\lambda) + \omega(\lambda) h^{-n} + O(h^{1-n}), \ \lambda \in \mathbb{R},
\]

where \( \omega(\lambda) \in C_0^\infty(I_0) \) has compact support and \( O(h^{1-n} #) \) is uniform with respect to \( \lambda \in \mathbb{R} \). As in Section 2, define the operator \( L_j^# \) on the torus \( T_{\bar{R}} = (\mathbb{R}/\mathbb{Z})^n \) with \( \bar{R} > 2R > 2R_0 \) and introduce \( \chi \in C_0^\infty(\{x : |x| \leq \bar{R}\}) \) equal to 1 for \( |x| \leq 2R > 2R_0 \). We have

\[
\text{tr}_{1b} \left[ e^{-i\theta h^{-1} L_j} \varphi_2(L_j) \right]^2_{j=1} = \left[ \text{tr} \left( \chi e^{-i\theta h^{-1} L_j} \varphi_2(L_j) \chi \right) \right]^2_{j=1} + \text{tr}_{1b} \left[ e^{-i\theta h^{-1} L_j} \varphi_2(L_j)(1 - \chi)^2 \right]^2_{j=1}.
\]
Applying the Duhamel formula and the semi-classical Egorov theorem (see Section 6 of [6] for more details), for $|t|$ sufficiently small we obtain
\[
\text{tr}_{L^2} \left[ e^{-i\hbar^{-1} L_j} \varphi_2(L_j)(1 - \chi^2) \right]_{j=1}^2 = \text{tr} \left[ e^{-i\hbar^{-1} Q_j} \varphi_2(Q_j)(1 - \chi^2) \right]_{j=1}^2 + O(h^\infty),
\]
\[
\text{tr} \left( e^{-i\hbar^{-1} L_1} \varphi_2(L_1) \chi \right) = \text{tr} \left( e^{-i\hbar^{-1} L_1#} \varphi_2(L_1#) \chi \right) + O(h^\infty)
\]
\[
= \text{tr} \left( e^{-i\hbar^{-1} L_1} \varphi_2(L_1) \right) - \text{tr} \left( e^{-i\hbar^{-1} Q_1#} \varphi_2(Q_1#)(1 - \chi^2) \right) + O(h^\infty),
\]
where $Q_1#$ is a differential operator
\[
Q_1# = \sum_{|\nu| \leq 2} a_{1,\nu}^#(x; h)(h D)^{\nu}
\]
on the torus $T_R$ introduced in Section 2 and $a_{1,\nu}^#(x; h) = a_{1,\nu}(x; h)$ for $|x| < r_0$, $r_0 > 2R_0$. Using the classical constructions of a parametrix for small $|t|$ for the unitary groups $e^{-i\hbar^{-1} Q_1#}$, $e^{-i\hbar^{-1} L_2}$, combined with the fact that $\lambda \in I_0$ is non-critical for $Q_1#$, $L_2$ we deduce for $\lambda \in I_0$
\[
\text{tr} \left( (F_h^{-1} \theta)(\lambda - Q_1#) \varphi_2(Q_1#)(1 - \chi^2) \right) = \omega_1(\lambda)h^{-n} + O(h^{1-n}),
\]
\[
\text{tr} \left( (F_h^{-1} \theta)(\lambda - L_2) \varphi_2(L_2) \chi \right) = \omega_2(\lambda)h^{-n} + O(h^{1-n}),
\]
with functions $\omega_1$, $\omega_2 \in C_0^0(I_1)$ and $O(h^{1-n})$ uniform with respect to $\lambda \in I_0$. The problem can be reduced to the application of the stationary phase method to some integrals where the integration is over a compact set. We refer to Chapter 10, [9], for more details. Since $\theta \in \mathcal{S}(\mathbb{R})$, we can extend the above relations to all $\lambda \in \mathbb{R}$ with $O(h^{1-n})$ uniform with respect to $\lambda \in \mathbb{R}$.

For the trace involving $Q_j$, $j = 1, 2$, we have for $\lambda \in I_0$
\[
\text{tr} \left[ (F_h^{-1} \theta)(\lambda - Q_j) \varphi_2(Q_j)(1 - \chi^2) \right]_{j=1}^2 = \omega_{ext}(\lambda)h^{-n} + O(h^{1-n})
\]
(5.10)
with $\omega_{ext} \in C_0^0(I_0)$ and $O(h^{1-n})$ uniform with respect to $\lambda \in I_0$. The proof of (5.10) is more technical since we must integrate over a non-compact domain. In fact, it is similar to the calculation of the traces in Section 4 in [2] and for the sake of completeness we present a proof in Appendix. Moreover, we show in the Appendix that we can extend (5.10) to all $\lambda \in \mathbb{R}$ with $O(h^{1-n})$ uniform with respect to $\lambda \in \mathbb{R}$. Taking together the asymptotics of the traces and the above relations, we obtain (5.9) and (5.7).

Now we will apply a Tauberian theorem (see for example, Theorem V-13 of [24]) for the increasing function $M_{\varphi_2}(\lambda)$. For this purpose we need the estimates
\[
M_{\varphi_2}(\lambda) = O(h^{-n#}), \quad \frac{d}{d\lambda} (F_h^{-1} \theta * M_{\varphi_2})(\lambda) = O(h^{-n#}), \quad \forall \lambda \in \mathbb{R}.
\]
(5.11)
The first one follows easily from (5.6). To establish the second one, we apply the equality (5.7). Thus it suffices to prove the estimate
\[
\frac{d}{d\lambda} (F_h^{-1} \theta * N_{\varphi_2}^#)(\lambda) = (2\pi h)^{-1} \text{tr} \left( \frac{1}{h} \langle L_1# - \lambda \rangle \varphi_2(L_1#) \right) = O(h^{-n#}), \quad \forall \lambda \in \mathbb{R}.
\]
(5.12)
To do this, assume first that $\lambda \in [E_0 - \epsilon_1, E_1 + \epsilon_1]$. Taking into account (5.1), we obtain

$$
\left| \text{tr} \left( \frac{\hat{L} - \lambda}{h} \right) \varphi_2(L_1^1) \right| = \left| \sum_{\mu \in \text{spec} \left( L_1^1 \right) \cap \text{supp} \varphi_2} \hat{\theta} \left( \frac{\mu - \lambda}{h} \right) \varphi_2(\mu) \right| \leq \frac{C}{h} \sum_{k=0}^{C/h} \sum_{\frac{k}{h} \leq |\mu - \lambda| \leq \frac{k+1}{h}} \left| \hat{\theta} \left( \frac{\mu - \lambda}{h} \right) \varphi_2(\mu) \right| \leq C \left( h^{1-n} + \sum_{k=1}^{C/h} \frac{k^{h-1-n}}{k^{3}} \right) \leq C h^{1-n},
$$

where we have used the inequality $|\hat{\theta}(\mu)| \leq C (1 + |\mu|)^{-3}$. On the other hand, for $\lambda \notin [E_0 - \epsilon_1, E_1 + \epsilon_1]$ and $\mu \in \text{supp} \varphi_2$, we have $|\mu - \lambda| \geq \delta_2 > 0$ and the term (5.11) is estimated by $O(h^\infty)$. Now a Tauberian argument implies the first assertion in (5.8). The second one is obtained by integration of (5.7) over $[\inf I_0, \lambda]$ combined with the equalities

$$
M_{\varphi_2}(\mu) = G_{\varphi_2}(\mu) = N_{\varphi_2}(\mu) = 0, \mu \leq \inf I_1
$$

and the fact that $\hat{\theta}(t) \in \mathcal{S}(\mathbb{R})$. \hfill \Box

**Proof of Theorem 2.** As mentioned above, it remains to show that

$$
\xi(\lambda, h) = \xi(\lambda; L_1, L_2) = N(L^1, \lambda) + \omega(\lambda) h^{-n} + O(h^{1-n})
$$

for a differential operator $L_2 = Q_2$ having no resonances in $\Omega_0$ and such that each $\lambda \in I_0$ is nontrapping and noncritical energy level for $L_2$. According to Lemma 2 and Lemma 3, for $\lambda \in I_2$ we have

$$
\xi(\lambda, h) = \text{tr}_{\mathbf{H}_d} \left[ \varphi_1(L_2^1) \right]_{j=1}^2 = \left[ \text{tr} \left( \chi \varphi_1(L_2^1) \right) \right]_{j=1}^2 + \text{tr} \left[ \chi \varphi_1(L_2^1) \right] - \text{tr} \left( \chi \varphi_1(L_2^1) \right) = \text{tr} \left( \chi \varphi_1(L_2^1) \right) - \text{tr} \left( \chi \varphi_1(L_2^1) \right) + \text{tr} \left( \varphi_1(Q_2) (1 - \chi^2) \right)_{j=1}^2 + O(h^\infty)
$$

where $C(\varphi_1)$ is a constant depending on $\varphi_1$.

On the other hand, applying a Tauberian theorem for $N_{\varphi_2}(\lambda) = O(h^{-n})$, we deduce

$$
N_{\varphi_2}(\lambda) = (F^{-1}_h \varphi \ast N_{\varphi_2})(\lambda) + O(h^{1-n}), \quad \forall \lambda \in \mathbb{R}.
$$

Consequently, for $\lambda \in I_2$ we get

$$
\xi(\lambda, h) = \text{tr} \left( \varphi_1(L_2^1) \right) + \text{tr} \left( \varphi_2(L_2^1) \right)_{1-C \# \lambda} \left( L_2^1 \right) + \left( C(\varphi_1) + \int_{-\infty}^{\lambda} \omega_{\varphi_2}(\mu) \right) h^{-n} + O(h^{1-n}).
$$

By construction we have

$$
\varphi_1(L_2^1) + \varphi_2(L_2^1) = 1_{1-C \# \lambda} \left( L_2^1 \right), \quad \forall \lambda \in I_2
$$

and this implies (5.13).
To obtain (5.2), we construct a covering of the interval $[E_0, E_1] \subset \bigcup_{\nu=1}^M J_\nu$ by small open intervals $J_\nu$ so that for every $J_\nu$ we can find an operator $Q_\nu$ with the properties of Proposition 3, where $I_0$ is replaced by $J_\nu$. Next we introduce a partition of unity

$$
\sum_{\nu=1}^M \varphi_\nu(x) = 1 \quad \text{on} \quad [E_0, E_1], \quad \varphi_\nu \in C_0^\infty(J_\nu; \mathbb{R}^+) 
$$

and we apply the above argument. This completes the proof of Theorem 2.

\[\square\]

6. BREIT-WIGNER APPROXIMATION

In this section we consider small domains of width $h$ and we prove a semi-classical analogue of the Breit-Wigner approximation for $\xi(\lambda, h)$ (see [21], [23], [3] for similar results, [12] for the case of a potential having the form of an "well in the island" and [11] for the one dimensional critical case). In the following $\eta(\lambda, h)$ denotes the real analytic function defined by

$$
\eta(\lambda, h) = \xi(\lambda, h) - \left[ \#\{\mu \in [E_0, \lambda] : \mu \in \text{sp}_{pp}(L_j(h)) \} \right]_{j=1}^2. 
$$

**Theorem 3.** Assume that $L_j(h)$, $j = 1, 2$ satisfy the assumptions of Theorem 2. Then for any $\lambda \in [E_0, E_1]$, any $0 < \delta < h/B$, $0 < B_1 < B$, and $h$ sufficiently small we have

$$
\eta(\lambda + \delta, h) - \eta(\lambda - \delta, h) = \left[ \sum_{\nu \in \text{Rez } L_j(h), \text{ Im } u \neq 0, |u-\lambda| < h/B_1} \omega_{\nu-}(w, [\lambda - \delta, \lambda + \delta]) \right]_{j=1}^2 + O(\delta) h^{-n#}, 
$$

where $B > 0$ is the constant introduced in Theorem 2.

**Remark.** Following the recent result of J.-F. Bony [1] the assumption (5.1) implies the existence of positive constants $D, \epsilon_2, C_2, h_2 > 0$ such that for any $\lambda \in [E_0 - \epsilon_2, E_1 + \epsilon_2]$, $h/D \leq \delta \leq D$ and $h \in [0, h_2]$ we have

$$
\#\{z \in \mathbb{C} : z \in \text{Res } L_j(h), |z - \lambda| \leq \delta \} \leq C_2 \delta h^{-n#}, \quad j = 1, 2. 
$$

**Proof.** We apply Theorem 1 in the interval $I_0 \supset (\lambda - \delta, \lambda + \delta)$, $0 < \delta \leq h/B_1$, and introduce the function

$$
F(z, h) = \left[ \sum_{\nu \in \text{Rez } L_j(h), \text{ Im } u \neq 0, h/B_1 \leq |u-\lambda| \leq C_4} \left( \frac{1}{z - w} - \frac{1}{\bar{z} - \bar{w}} \right) \right]_{j=1}^2, \quad z \in D(\lambda, h/B). 
$$

It is sufficient to show that

$$
|F(z, h)| \leq C h^{-n#}, \quad |z - \lambda| \leq h/B. 
$$

We have

$$
\partial_z F(z, h) = \left[ \sum_{\nu \in \text{Rez } L_j(h), \text{ Im } u \neq 0, h/B_1 \leq |u-\lambda| \leq C_4} \frac{1}{(z - w)^2} - \frac{1}{(\bar{z} - \bar{w})^2} \right]_{j=1}^2. 
$$
Let $l_0 \in \mathbb{N}$ be an integer such that $D \leq 2^{k_0 - 1} B$. Following the argument in [23] and applying (6.2), for any $z \in D(\lambda, h/B)$ we obtain

$$\sum_{u \in \text{Res} L_j, \text{Im } u \neq 0, \ \frac{h}{B} \leq |u - \lambda| \leq \frac{c_4}{h}} \frac{1}{|z - u|^2} \leq \sum_{u \in \text{Res} L_j, \text{Im } u \neq 0, \ \frac{h}{B} \leq |u - \lambda| \leq \frac{c_4}{h}} \frac{1}{|z - u|^2}$$

$$+ \sum_{k = k_0} C\log(1/h) + \sum_{k = k_0} \frac{1}{2^k h} \sum_{k = k_0} \frac{1}{2^{k+1} h} \left| \sum_{|w - \lambda| \leq 2^k h} \frac{1}{|z - w|^2} \right|^2 \leq C2^k D = 1 - h - n# + C \sum_{k = k_0} \frac{(2^{k+1} h)^{h-n#}}{(2^k h)^2} \leq C h^{1-n#}.$$ 

Here and below we denote by $C > 0$ different constants which may change from line to line and which are independent on $h$ and the choice of $\lambda$ in the interval $[E_0, E_1]$. Thus we get the estimate

$$|\partial_z F(z, h)| \leq C h^{-n#-1}, \ z \in D(\lambda, h/B).$$

It remains to find an estimate of $|F(\mu_0, h)| = |\text{Im } F(\mu_0, h)|$ at a suitable point $\mu_0 = \mu_0(h)$. \footnote{There is some similarity between the proof of the existence of $\mu_0(h)$ and that of the existence of a suitable point $z_0(h)$, $\text{Im } z_0(h) \geq \delta > 0$ in Section 4 in [23] so that $\log |\det S(z_0(h), h)| \geq -Ch^{-n#}$.

} Set $\nu = \frac{h}{B} \leq \frac{h}{B_1}$, and suppose that for all $\mu \in \mathbb{R}$, $|\mu - \lambda| \leq \nu$, we have $|\text{Im } F(\mu, h)| \geq M h^{-n#}$, $M > 0$. The continuity of the function $\text{Im } F(\mu, h)$ implies that $\text{Im } F(\mu, h)$ is either positive or negative in $[\lambda - \nu, \lambda + \nu]$. Assuming $\text{Im } F(\mu, h)$ positive, we get

$$\frac{M h^{-n#} + 1}{B \pi} \leq \frac{1}{2\pi} \int_{\lambda - \nu}^{\lambda + \nu} \text{Im } F(\mu, h) d\mu \leq \frac{1}{2\pi} \int_{\lambda - \nu}^{\lambda + \nu} \sum_{\text{Res } L_j, \text{Im } u \neq 0, |u - \lambda| \leq \frac{c_4}{h}} \frac{|\text{Im } w|}{|\mu - w|^2} d\mu$$

$$+ \frac{1}{2\pi} \sum_{j = 1}^2 \int_{\lambda - \nu}^{\lambda + \nu} \sum_{\text{Res } L_j, \text{Im } u \neq 0, |u - \lambda| \leq \frac{c_4}{h}} \frac{|\text{Im } w|}{|\mu - w|^2} d\mu$$

$$\leq |\eta(\lambda + \nu, h) - \eta(\lambda - \nu, h)| + C h^{1-n#}.$$

Here we have used the inequality

$$\int_{\lambda - \nu}^{\lambda + \nu} \frac{|\text{Im } w|}{|\mu - w|^2} d\mu \leq \int_{-\infty}^{\infty} \frac{|\text{Im } w|}{|\mu - w|^2} d\mu \leq \pi$$

and (6.2) to estimate the number of resonances in $\{w : |w - \lambda| < h/B_1\}$. Notice that if $D < B_1$, we have $\{w : |w - \lambda| < h/B_1\} \subseteq \{w : |w - \lambda| < h/D\}$. Next the assumption (5.1) combined with Theorem 2 yield the estimate

$$|\xi(\lambda + \nu, h) - \xi(\lambda - \nu, h)| \leq C h^{1-n#}.$$

Thus,

$$|\eta(\lambda + \nu, h) - \eta(\lambda - \nu, h)| \leq |\xi(\lambda + \nu, h) - \xi(\lambda - \nu, h)|$$

$$+ \sum_{j = 1}^2 2 \{\mu \in \text{sp}_p (L_j) : |\mu - \lambda| \leq \nu\} \leq C h^{1-n#},$$

where $p = p(h)$.
where for the second inequality we have used once more (6.2), observing that the positive eigenvalues of $L_j$ coincide with the resonances on $\mathbb{R}^+$. Consequently, we obtain a bound for $M$. Hence there exists a constant $C > 0$ and $\mu_0 \in [\lambda - \nu, \lambda + \nu]$ so that

$$|F(\mu_0, h)| \leq C h^{-n^\#}.$$  \hspace{1cm} (6.4)

Writing

$$F(z, h) = F(\mu_0, h) + \int_{\mu_0}^z \partial_z F(z, h)dz, \quad |z - \lambda| \leq h / B,$$

we obtain (6.3). The case $\text{Im} F(\mu, h) < 0$ can be treated by the same argument exploiting the inequality $-\text{Im} F(\mu, h) \geq M h^{-n^\#}, \quad |\mu - \lambda| \leq \nu$. By an integration over the interval $(\lambda - \delta, \lambda + \delta)$, we complete the proof of (6.1).

**Remark.** Our proof goes without a factorization in small domains $\{z \in \mathbb{C} : |z - \lambda| \leq Ch\}$ and a suitable trace formula (see Lemma 6.2 in [23] and Theorem 1.3 in [3]). The above argument can be applied to simplify the proof of Lemma 6.2 in [23].

Next, the estimate (6.3) of $F(z, h)$ yields immediately the following.

**Corollary 1.** Under the assumptions of Theorem 3 for $\mu \in \mathbb{R}, |\mu - \lambda| < h / B$ we have the representation

$$\xi(\mu, h) = \frac{1}{\sqrt{\pi}} \text{Im} q(\mu, h) + \left[ \sum_{\omega \in \text{Res} L_j, \text{Im} \omega \neq 0} \frac{-\text{Im} w}{|\mu - w|^2} + \sum_{\omega \in \text{Res} L_j, \text{Im} \omega \neq 0} \delta(\mu - w) \right]_{j=1},$$

where $q(z, h) = p(z, h) - \overline{p(z, h)}$, $p(z, h)$ is holomorphic in $D(\lambda, h / B)$ and $p(z, h)$ satisfies the estimate

$$|p(z, h)| \leq C h^{-n^\#}, \quad z \in D(\lambda, h / B)$$

with $C > 0$ independent on $h \in [0, h_0]$ and $\lambda \in [E_0, E_1]$.

We may slightly improve Theorem 3, noting that for every $0 < \epsilon < 1$ and $|\mu - \lambda| \leq \frac{h}{\epsilon^2}$ we have

$$\left[ \sum_{\omega \in \text{Res} L_j(h), \text{Im} \omega \neq \mu, |\omega - \lambda| \leq h / B} \frac{|\text{Im} w|}{|\mu - w|^2} \leq \frac{h}{\epsilon^2} C h^{1-n^\#} = O(\sqrt{h^{-n^\#}}).$$

Thus for $0 < \delta \leq \frac{h}{\epsilon^2}$ the right hand part in (6.1) can be replaced by

$$\left[ \sum_{\omega \in \text{Res} L_j(h), \text{Im} \omega \neq \mu, |\omega - \lambda| \leq \epsilon h / B} \omega \in (w, [\lambda - \delta, \lambda + \delta]) \right]_{j=1}^2 + O(\delta) h^{-n^\#}.$$

To obtain a stronger version involving the resonances in smaller "boxes", we need some additional information for the distribution of the resonances in $\{w \in \mathbb{C} : |w - \lambda| \leq ch\}$. In the case of the Schrödinger operator $L(h) = -h^2 \Delta + V(x)$ with $V(x) \in C_0^\infty(\mathbb{R}^n)$ real valued this is possible applying the recent result of Stefanov [31]. Set $a_0(x, \xi) = |\xi|^2 + V(x)$ and let $0 < E_0 < E_1$ be non-critical values of $a_0(x, \xi)$. Let

$$a^{-1}_0[E_0, E_1] = W_{\text{int}} \cup W_{\text{ext}},$$

where $W_{\text{ext}}$ is the unbounded connected component, while $W_{\text{int}}$ is the union of bounded ones if there are such connected components. Assume that all points in $W_{\text{ext}}$ are non-trapping (see [31] for a
precise definition). Then, according to Theorem 6.1 in [31], there exists a function $0 < R_1(h) = O(h^{\infty})$ such that for any $M \in \mathbb{N}$ the operator $L(h)$ has no resonances in the set

$$\Omega_M(\lambda, h) = [E_0, E_1] + i[-Mh, -R_1(h)], \ 0 < h \leq h(M).$$

(6.6)

Setting $0 < R(h) = \sqrt{hR_1(h)} = O(h^{\infty})$, an elementary argument shows that for $\lambda \in [E_0, E_1]$ and $\|\mu - \lambda\| \leq R(h)/2$ we have

$$\sum_{w \in \text{Res} \ L(h), \|w\| \leq R(h) / 2} \frac{|\text{Im} w|}{|\mu - w|^2} \leq C h^{-n\#}.$$

In the next result we treat a formally symmetric differential operator

$$L_1(h) = \sum_{|\nu| \leq 2} a_0(x, h)(hD_x)^\nu$$

on $L^2(\mathbb{R}^n)$ satisfying the assumptions of Section 2. Given a fixed $\lambda \in ]E_0, E_1[$, as in the previous section, we may construct an operator $L_2(h)$ having the properties (a) - (c) of Proposition 3. Applying Theorem 3 for $L_j(h)$, $j = 1, 2$, and $\{z \in \mathbb{C} : |z - \lambda| \leq h/B_1\} \subset W$, and assuming that we have a free resonances domain, we obtain the following improvement of Corollary 1.

**Corollary 2.** Let $E_0 < \lambda < E_1$ be fixed. Let $L_2(h)$ be chosen so that $L_j(h)$, $j = 1, 2$, satisfy the assumptions of Theorem 3 and $L_2(h)$ has no resonances in the disk $\{z \in \mathbb{C} : |z - \lambda| \leq h/B_1\}$. Suppose that there exists a function $0 < R_1(h) = O(h^{\infty})$ such that $L_1(h)$ has no resonances in the set

$$[E_0, E_1] + i[-\epsilon h, -R_1(h)], \ \epsilon > 0, \ 0 < h \leq h(\epsilon).$$

Then for $|\mu - \lambda| < R(h) / 2$ and $h$ sufficiently small we have

$$\xi^j(\mu, h) = \frac{1}{\pi} \text{Im} q(\mu, h) + \sum_{w \in \text{Res} L_j, \|w\| \leq R(h), \ |\mu - w| < R(h)} \frac{-\text{Im} w}{|\mu - w|^2} + \sum_{w \in \text{Res} L_j \cap \{w : |\mu - w| < R(h) / 2\}} \delta(\mu - w)$$

(6.7)

with $R(h) = \sqrt{hR_1(h)} = O(h^{\infty})$ and $q(\mu, h)$ as in Corollary 1.

**7. Local Trace Formula**

In this section we prove a local trace formula which is a slightly stronger version of that in [28], [29] (see [23] for compactly supported perturbations). Exploiting Theorem 1, we repeat with trivial modifications the argument of Section 5, [23], to get the following.

**Theorem 4.** Assume that $L_j(h)$ satisfy the assumptions of Section 2. Let $\Omega \subset e^{1/2g_0, g_0}|0, \infty[$ be an open, simply connected, relatively compact set such that $I = \Omega \cap \Re$ is an interval. Suppose that $f$ is holomorphic on a neighborhood of $\Omega$ and that $\psi \in C_0^\infty(\mathbb{R})$ satisfies

$$\psi(\lambda) = \begin{cases} 0, & d(I, \lambda) > 2 \epsilon, \\ 1, & d(I, \lambda) < \epsilon, \end{cases}$$

where $\epsilon > 0$ is sufficiently small. Then

$$\tau_{\Omega, h} \left[ \psi f(L_j(h)) \right]_{j=1}^2 = \left[ \sum_{z \in \text{Res} \ L_j(h) \cap \Omega} f(z) \right]_{j=1}^2 + E_{\Omega, f, \psi}(h)$$

(7.1)

with

$$|E_{\Omega, f, \psi}(h)| \leq M(\psi, \Omega) \sup \{|f(z)| : 0 \leq d(\Omega, z) \leq 2 \epsilon, \text{Im} z \leq 0\} h^{-n\#}.$$
Proof. Choose an almost analytic extension \( \tilde{\psi} \) of \( \psi \) so that \( \tilde{\psi} \in \mathcal{C}^\infty_c(\mathbb{C}) \), \( \tilde{\psi} = 1 \) on \( \Omega \) and

\[
\text{supp} \bar{\partial}_z \tilde{\psi} \subset \{ z \in \mathbb{C} : \epsilon \leq d(\Omega, z) \leq 2\epsilon \}.
\]

Setting \( \Omega_\epsilon = \{ z \in \mathbb{C} : d(\Omega, z) \leq \epsilon \} \), we have

\[
\text{tr}_{lh} \left[ \left( \psi f \right) \left( L_j(h) \right) \right]_{j=1}^2 = \xi' \left( \lambda, h \right), \left( \psi f \right)(\lambda) >
\]

\[
= \left[ \sum_\omega \left( \psi f \right)(w) \right]_{j=1}^2 + \frac{1}{2\pi i} \int \left( \psi f \right)(\lambda) r(\lambda, h) d\lambda
\]

\[
+ \frac{1}{2\pi i} \int \left( \psi f \right)(\lambda) \left[ \sum_{\omega \in \text{Res} \left( L_j(h) \right) \cap \Omega_2^1, \Im \omega 
eq 0} \left( \frac{1}{\lambda - w} - \frac{1}{\lambda - w} \right) \right]_{j=1}^2 d\lambda.
\]

The integral involving \( r(\lambda, h) \) can be estimated using (1.4) with \( W = \Omega_2^\epsilon \). For the integral containing the resonances we apply Green formula and we get the term

\[
\left[ \sum_\omega \left( \tilde{\psi} f \right)(w) \right]_{j=1}^2
\]

\[
+ \frac{1}{\pi} \int_{\partial \mathbb{C}} \left( \bar{\partial}_z \tilde{\psi} \right)(z) f(z) \left[ \sum_{\omega \in \text{Res} \left( L_j(h) \right) \cap \Omega_2^1, \Im \omega 
eq 0} \left( \frac{1}{z - w} - \frac{1}{\lambda - w} \right) \right]_{j=1}^2 d\lambda,
\]

where \( \mathcal{L}(dz) \) is the Lebesgue measure on \( \mathbb{C} \). As in the proof of Theorem 1 in [23], we apply the inequality

\[
\int_{\partial \Omega_1} \left| \frac{1}{z - w} \right| \mathcal{L}(dz) \leq 2\sqrt{2\pi |\Omega_1|}
\]

and an upper bound for the number of the resonances in \( \Omega_2^\epsilon \) to obtain the result. \( \square \)

Since we have no restrictions on the behavior of the holomorphic function \( f(z) \) on \( \Omega \cap \{ \Im z > 0 \} \), we may apply the above argument choosing \( f(z) = e^{-itz/h}, \ t \in \mathbb{R} \), to get the following.

**Theorem 5.** Let \( \Omega \) and \( \psi \) be as in Theorem 4 and let \( \tilde{\psi} \in \mathcal{C}^\infty_c(\mathbb{C}) \) be an almost analytic extension of \( \psi \) supported in \( \Omega_2^\epsilon \). Then for any \( 0 < \delta < 1 \) and \( t \geq h^{\delta} \) we have

\[
\text{tr}_{lh} \left[ \left( \psi f \right) \left( L_j(h) \right) e^{-it L_j(h)} \right]_{j=1}^2 = \left[ \sum_{\omega \in \text{Res} \left( L_j(h) \right) \cap \Omega_2^1} \tilde{\psi}(w) e^{-it w/h} \right]_{j=1}^2 + \mathcal{O}(h^\infty).
\]

Moreover, for \( t \geq \epsilon > 0 \) and \( N \in \mathbb{N} \) there exists \( h_N > 0 \) such that for \( 0 < h \leq h_N \) we have

\[
\text{tr}_{lh} \left[ \left( \psi \right)^2 \left( L_j(h) \right) e^{-it L_j(h)} \right]_{j=1}^2 = \left[ \sum_{\omega \in \text{Res} \left( L_j(h) \right) \cap \Omega_2^1, \Im \omega \leq -Nh \log h} \tilde{\psi}(w) e^{-it w/h} \right]_{j=1}^2 + \mathcal{O}(e^{Nh-N\log h}).
\]

**Proof.** Choose an almost analytic extension \( \tilde{\psi} \) of \( \psi \) as in Theorem 4. Applying Green formula, we must examine the integrals

\[
\int_{\partial \mathbb{C}} \bar{\partial}_z \tilde{\psi}(z) e^{-itz/h} r(z, h) \mathcal{L}(dz),
\]

\[
\int_{\partial \mathbb{C}} \bar{\partial}_z \tilde{\psi}(z) e^{-itz/h} \left[ \sum_{\omega \in \text{Res} \left( L_j(h) \right) \cap \Omega_2^1} \left( \frac{1}{z - w} - \frac{1}{\lambda - w} \right) \right]_{j=1}^2 \mathcal{L}(dz).
\]
Choose $\mu > 0$, $0 < \delta + \mu < 1$. For $-h^{\mu} \leq \text{Im} \, z \leq 0$ we have

$$|\bar{z} z^\nu| \leq C N |z|^N \leq C N h^{\mu N}, \forall N \in \mathbb{N}$$

and the integration over $-h^{\mu} \leq \text{Im} \, z \leq 0$ combined with the argument of the proof of Theorem 4 yield a term bounded by $O(h^{N^2})$. On the other hand, for $t \geq h^{\delta}$, $\text{Im} \, z \leq -h^{\mu}$ we get

$$|e^{-it z/h^N} - e^{-it z/h^N}| \leq e^{-\delta t} = O(h^{N^2})$$

and this implies (7.2). For the second assertion we have

$$|e^{-it w/h^N} - e^{i \log h} \leq h^N$$

for $|\text{Im} \, w| \geq -N h \log h$ and this completes the proof. \hfill \Box

**Remark.** For non-trapping compactly supported perturbations $L(h)$ (see [32], [4]) and for non-trapping long-range perturbations $L(h) = -h^2 \Delta + V(x)$ of the Laplacian (see [15]) there are no resonances of $L(h)$ in the domain

$$-N h \log \frac{1}{h} \leq \text{Im} \, z \leq 0, \quad 0 < h \leq h_N.$$ 

For such perturbations the left hand side of (7.3) is equal to $O(h^{N^2-n^2})$ and we obtain an analogue of the classical trace formula for non-trapping perturbations.

8. Existence of resonances close to the real axis

In this section we consider the operator $L(h) = -h^2 \Delta + V(x)$, where $\Delta$ is symmetric Laplace-Beltrami operator on $L^2(\mathbb{R}^n)$ associated to a metric $g(x) = \{g_{ij}(x)\}_{1 \leq i,j \leq n}$ and $V(x) \in C^\infty(\mathbb{R}^n)$ is a real valued function. We assume that there exists $\rho > n$ so that

$$|\partial_{\alpha}^\rho (g_{ij}(x) - \delta_{ij})| + |\partial_{\alpha}^\rho V(x)| \leq C_\rho < x \geq -\rho^\rho, \quad 1 \leq i,j \leq n, \forall \alpha.$$ 

Moreover, we assume that the coefficients $\{g_{ij}(x)\}$ and $V(x)$ can be extended holomorphically in $x$ to the domain given in (2.8) and the estimate (8.1) holds in this domain.

Consider the symbol

$$a_0(x, \xi) = \langle g(x)^{-1} \xi, \xi \rangle + V(x)$$

and denote by $H_{a_0}$ the Hamilton vector field associated to $a_0$ and by $\Phi^t = \exp(t H_{a_0})$ the Hamilton flow. Given $\lambda > 0$, let $\Sigma_\lambda = \{x, \xi \in \mathbb{R}^n : a_0(x, \xi) = \lambda\}$ be the energy surface and let $\nabla a_0(x, \xi) \neq 0$ on $\Sigma_\lambda$. A point $\nu \in \Sigma_\lambda$ is called **periodic** if there exists $T > 0$ such that $\Phi^T(\nu) = \nu$ and the smallest $T > 0$ with this property is called period $T(\nu)$ of $\nu$. Given a periodic point $\nu$, consider the trajectory

$$\gamma(\nu) = \{\Phi^t(\nu) : 0 \leq t \leq T(\nu)\} = \{(x(t), \xi(t)) : 0 \leq t \leq T(\nu)\}$$

and define the **action** $S(\nu)$ along $\gamma(\nu)$ by

$$S(\nu) = \int_{\gamma(\nu)} \xi dx = \int_0^{T(\nu)} \xi(t)x'(t) dt.$$ 

Next we denote by $m(\nu) \in \mathbb{Z}_4$ the Maslov index related to $\gamma(\nu)$ and set $q(\nu) = -\frac{\pi}{2} m(\nu)$. Let $\Pi$ be the set of all periodic points on $\Sigma_\lambda$ and let

$$Q(h, r) = (2\pi)^{-n} \int_{\Pi} \left[ \pi - h^{-1} S(\nu) + q(\nu) - r T(\nu) \right]_2 T(\nu)^{-1} d\nu,$$ 

(8.2)
where \( d\nu \) is the Liouville measure on \( \Sigma_\lambda \) and the residue \(-\pi < [z]_\pm \leq \pi\) is defined so that \( z = [z]_\pm + 2\pi k, k \in \mathbb{Z} \). The set \( \Pi \) is bounded, the integrand in (8.2) is a measurable function and \( T(\nu) \geq T_0 > 0, \forall \nu \in \Pi \). The oscillatory function \( Q(h, r) \) has been introduced in [19] for the analysis of the semi-classical behavior of the eigenvalues and it is a semi-classical analogue of the oscillating function defined by Guriev and Safarov [14] and Safarov [26]. Notice that the limits \( Q(h, r \pm 0) = \lim_{\epsilon \downarrow 0} Q(h, r \pm \epsilon) \) exist for each \( r \) and \( 0 < h \leq h_0 \) and, moreover,

\[
Q(h, r + 0) - Q(h, r - 0) = (2\pi)^{1-n} \int_{\Omega_{h,r}} \frac{d\nu}{T(\nu)},
\]

where \( \Omega_{h,r} = \{ \nu \in \Pi : h^{-1}S(\nu) - q(\nu) + rT(\nu) \equiv 0(2\pi) \} \). Following the arguments in Section 6, [21], we will prove the following.

**Theorem 6.** Let \( L(h) = -h^2 \Delta_g + V(x) \), where the metric \( g(x) \) and \( V(x) \) satisfy the estimates (8.1) and let \( \nabla a_0(x, \xi) \neq 0 \) on \( \Sigma_\lambda, \lambda > 0 \). Assume that there exist an integer \( p \in \mathbb{Z} \) and a subset \( \Pi_0 \subset \Pi \) with positive Liouville measure \( \mu(\Pi_0) > 0 \) so that

\[
\left( |g(\nu) - h^{-1}S(\nu)|^2 + 2\pi p \right) T(\nu)^{-1} = r(h), 0 < h \leq h_0
\]
does not depend on \( \nu \in \Pi_0 \). Then for \( 0 < h \leq h_1 \) we have

\[
\# \{ w \in \text{Res} L(h) : |w - \lambda - r(h)h| \leq h \} \geq \frac{(2\pi)^{1-n}}{2} h^{1-n} \int_{\Pi_0} \frac{d\nu}{T(\nu)}.
\]  

(8.3)

**Remark.** Clearly, \(|r(h)| \leq \max\{|2p - 1|, |2p + 1|\} \pi T_0^{-1} \) and applying the Remark after Theorem 3, we conclude that

\[
\# \{ w \in \text{Res} L(h) : |w - \lambda - r(h)h| \leq h \} \leq Ah^{1-n}.
\]

This shows that the order \( h^{1-n} \) in our estimate is optimal. On the other hand, it is clear that we may apply Theorem 3 with \( B > 1 \) and \( B_1 = 1 \).

**Proof.** Consider the scattering phase \( \sigma(\lambda, h) = \frac{1}{2\pi i} \text{det} S(\lambda, h) \), where the scattering operator \( S(\lambda, h) \) is related to \( L(h) \) and \( L_0(h) = -h^2 \Delta \). According to Birman-Krein theory (see for instance [33]), the scattering phase can be identified with the spectral shift function and, under our assumptions, we have not embedded positive eigenvalues. Following Theorem 2.1 in [5], and taking \(|r(h)| \leq r_0, 0 < \epsilon \leq \epsilon_0, 0 < h \leq h_0 \) and \( \lambda > 0 \) we have

\[
\sigma(\lambda + (r(h) + \epsilon)h, h) - \sigma(\lambda + (r(h) - \epsilon)h, h)
\]

\[
\geq h^{1-n} \left[ Q\left(h, (r(h) + \epsilon/2), h\right) - Q\left(h, (r(h) - \epsilon/2), h\right) \right] + 2\epsilon \gamma_0(\lambda) h^{1-n} - C_0 \epsilon h^{1-n} - o(h^{1-n}),
\]

where

\[
\gamma_0(\lambda) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left( \int_{a_0(x, \xi) \leq \lambda} d\xi - \int_{a_0(x, \xi) \leq \lambda} d\xi \right) dx,
\]

\( C_0 > 0 \) is independent on \( r(h), \epsilon \) and \( h \) and \( o(h^{1-n}) \) means that for any fixed \( \epsilon > 0 \) we have

\[
\lim_{h \downarrow 0} \frac{o(h^{1-n})}{h^{1-n}} = 0.
\]

On the other hand, for small \( \epsilon > 0 \) an application of Theorem 3 with \( \delta = \epsilon h \) yields the estimate

\[
\sigma(\lambda + (r(h) + \epsilon)h, h) - \sigma(\lambda + (r(h) - \epsilon)h, h)
\]
\[ \# \{ w \in \text{Res} L(h) : |w - \lambda - r(h)h| \leq h \} \leq C_1 \epsilon h^{1-n} + C_2 \epsilon^2, \]

with \( C_1 > 0 \) independent on \( \epsilon, r(h) \) and \( h \). We claim that

\[ Q(h, r(h) + \epsilon/2) - Q(h, r(h) - \epsilon/2) \geq -2(2\pi)^{1-n} \epsilon \mu(\Pi) + (2\pi)^{1-n} \int_{\Pi_0} \frac{d\nu}{T(\nu)}. \]  

(8.4)

In fact, according to the representation of the oscillatory function \( Q(h, r) \) (see for instance, Proposition 1, [26]), we have

\[ Q(h, r(h) + \epsilon/2) - Q(h, r(h) - \epsilon/2) = -\epsilon(2\pi)^{1-n} \mu(\Pi) \]

\[ + (2\pi)^{1-n} \int_{\Pi} T^{-1}(\nu) \sum_{k \in \mathbb{Z}} \chi_{h,k}(\nu) d\nu, \]

where \( \chi_{h,k}^\varepsilon \) is the characteristic function of the set

\[ \Omega_{h,k}^\varepsilon = \{ \nu \in \Pi : -\varepsilon T(\nu) \leq h^{-1} S(\nu) - q(\nu) + r(h) T(\nu) - 2k\pi < \varepsilon T(\nu) \}. \]

Obviously, for any \( \nu \in \Pi_0 \) we get

\[ h^{-1} S(\nu) - q(\nu) + r(h) T(\nu) + 2M(\nu, h) \pi - 2p\pi = 0 \]

with some \( M(\nu, h) \in \mathbb{Z} \). Consequently,

\[ \nu \in \Pi_0 \implies \sum_{k \in \mathbb{Z}} \chi_{h,k}(\nu) \geq 1 \]

and we obtain (8.4). Choosing \( \varepsilon > 0 \) small enough, we arrange the inequality

\[ -\epsilon(2\pi)^{1-n} \mu(\Pi) - \epsilon(C_0 + C_1) + 2\varepsilon \gamma_0(\lambda) \geq -\frac{\alpha_0}{4} \]

with \( \alpha_0 = (2\pi)^{1-n} \int_{\Pi_0} \frac{d\nu}{T(\nu)} \). Next we fix \( \varepsilon > 0 \) and choose \( 0 < h_1 \leq h_2 \) so that for \( 0 < h \leq h_2 \) we have

\[ |a_0(h^{-1} - 1)| \leq \frac{\alpha_0}{4} h^{1-n} \]

Combining the above estimates for the difference \( \sigma(\lambda + (r(h) + \epsilon)h, h) - \sigma(\lambda + (r(h) - \epsilon)h, h) \), we complete the proof.

\[ \square \]

**Example** (see Section 7 in [5]). Let \( L(h) = -h^2 \Delta + V(x) \) with

\[ V(x) = \Phi_\alpha(x - y_0) \left( |x - y_0|^2 + \hat{b} \right), \]

where \( \alpha > 0, \beta > 0 \) and \( y_0 \in \mathbb{R}^n \) are fixed and \( \Phi_\alpha(x) \in C_0^\infty(\mathbb{R}^n), \Phi_\alpha(x) = 1 \) for \( |x| \leq 2\alpha \). Let \( 0 < \epsilon < \alpha/2, |y_0| = \sqrt{\lambda - \beta} \) and let \( \lambda \in [\beta, \beta + \alpha^2] \) be a non-critical energy level for \( a_0(x, \xi) = |\xi|^2 + V(x) \). Therefore the set

\[ \Pi_0 = \{ (x, \xi) \in \Sigma_{\lambda} : |\xi - y_0|^2 + |x - y_0|^2 \leq \epsilon^2 \} \]

has a positive Liouville measure and \( \Pi_0 \subset \Pi \). Moreover, for every \( \nu \in \Pi_0 \) we have

\[ T(\nu) = \pi, \quad S(\nu) = (\lambda - b) \pi, \quad q(\nu) = \frac{\pi}{2} m \]

with \( m \in \mathbb{Z} \) independent on \( \nu \). We may apply Theorem 6 with \( r_p(h) = \frac{1}{2} \left[ \frac{\pi}{2} m - h^{-1}(\lambda - b)\pi \right] + 2p, \ p \in \mathbb{Z} \), to conclude that

\[ \# \{ w \in \text{Res} L(h) : |w - \lambda - r_p(h)h| < h \} \geq (2\pi)^{1-n} \mu(\Pi_0) h^{1-n}. \]
On the other hand, for \( p \neq j \) and \( 0 < h \leq h_0 \) we have
\[
\{ w : | \text{Re} \ w - \lambda - r_p(h)h | < h \} \cap \{ w : | \text{Re} \ w - \lambda - r_j(h)h | < h \} = \emptyset
\]
and the clusters related to \( p \neq j \) produce different resonances. Choosing \( \delta > 0 \) so that \( ] \lambda - \delta, \lambda + \delta[ \subset [b, b + a^2[ \), one obtains easily
\[
\# \{ w \in \text{Res} \ L(h) : | w - \lambda | \leq \delta \} \geq \alpha \delta (2\pi)^{-n} \mu(\Pi_0) h^{-n}
\]
with \( \alpha > 0 \) independent on \( \delta \). A stronger asymptotic for the number of the resonances in
\[
[b, b + a^2[ + i[-R(h), 0]
\]
has been obtained by Stefanov [31]. Notice that in the above result we count only the resonances lying in clusters.

9. Appendices

In this Appendix we present a proof of (5.10). Following the Remark after Lemma 2, we will assume that \( \varphi_2 = \psi^2, \psi \in C_0^\infty \left( I_1; \mathbb{R}^+ \right), I_1 \subset I_0 \). Recall that \( \lambda \in \Theta_0 \), supp \( \theta(t) \subset [-\delta_1, \delta_1] \) and \( \chi(x) = 1 \) for \( |x| \leq 2R \) and \( R > R_0 \). It is easy to see that
\[
\text{tr} \left[ \frac{1}{2\pi h} \int e^{i\lambda \theta(t)} e^{-iQ_2/h} \theta(t) \psi^2(Q_2) (1 - \chi^2) dt \right]_{j=1}^2
\]
\[
= \frac{1}{2\pi h} \int e^{i\lambda \theta(t)} \theta(t) \text{tr} \left[ \psi^2(Q_2) e^{-iQ_2/h} (1 - \chi^2) \right] dt
\]
\[
+ \frac{1}{2\pi h} \int e^{i\lambda \theta(t)} \theta(t) \text{tr} \left[ \psi^2(Q_1) e^{-iQ_1/h} (1 - \chi^2) \right] dt = A + B.
\]
This representation is justified by applying Lemma 4.1 in [2] saying that
\[
\| \psi^2(Q_2) \|_{j=1}^2 \| \theta \psi^2(Q_1) e^{-iQ_1/h} \|_{j=1}^2 \| \text{tr} = O(h^{-n}).
\]
We treat below \( A \) following closely the analysis of J.-F. Bony in Section 4.2, [2]. Put \( A = A_1 + A_2 \), where
\[
A_1 = \frac{1}{2\pi h} \int e^{i\lambda \theta(t)} \theta(t) \text{tr} \left[ \psi^2(Q_1) \psi^2(Q_2) (1 - \chi^2) \right] dt,
\]
\[
A_2 = \frac{1}{2\pi h} \int e^{i\lambda \theta(t)} \theta(t) \text{tr} \left[ \psi^2(Q_1) \psi^2(Q_2) (1 - \chi^2) \right] dt.
\]
We deal with the analysis of \( A_1 \) only, since that of \( A_2 \) is similar (see also Section 4.2, [2]). First, we find a pseudodifferential operator \( Q \) with symbol in \( S^0(1) \) so that
\[
A_1 = \frac{1}{2\pi h} \int e^{i\lambda \theta(t)} \theta(t) \text{tr} \left[ e^{-iQ_2/h} \psi^2(Q_2) Q(Q_1 - Q_2) \psi^2(Q_2) \right] dt,
\]
where \( \tilde{\psi} \in C_0^\infty (\mathbb{R}) \) is such that \( \tilde{\psi} = 1 \) on supp \( \psi \). We use the notations of [9] for \( h \)-pseudodifferential operators and set \( \langle x \rangle = (1 + |x|^2)^{1/2} \). Moreover, modulo a term in \( S^N(1) \), the symbol of \( Q \) is supported in \( \{ (x, \xi) : |x| > 2R \} \). Secondly, we obtain the existence of a pseudodifferential operator \( S \) with symbol
\[
s(x, \xi; h) \in S^0 \left( \langle x \rangle^{-N-1} \langle \xi \rangle^{-N} \right), \forall N \in \mathbb{N},
\]
having compact support in \( \xi \) and \( (x - y) \) and support in \( \{ (x, \xi) : |x| > 2R, (x, \xi) \in l^{-1}_2 (I_1) \} \) so that
\[
A_1 = \frac{1}{2\pi h} \text{tr} \left( \int e^{i\lambda \theta(t)} \theta(t) e^{-iQ_2/h} S dt \right) + O(h^\infty).
\]
Applying Theorem 2 in [2], we obtain the existence of a Fourier integral operator $\mathcal{U}_t$ such that for $|t| \leq \delta_1$ and $\delta_1$ sufficiently small we have

$$\|\mathcal{U}_t - e^{-i\lambda^2 t^2/h}S\|_{L^2} = \mathcal{O}(h^\infty).$$

Next, we write the kernel of the operator $\int e^{i\lambda \theta(t)} dt \mathcal{U}_t dt$ in the form

$$K(x, y; h) = \frac{1}{(2\pi h)^n} \int \int e^{i(\lambda + \Phi(t, x, z) - x, z)} \frac{\theta(t) A(t, x, y; h) dt dq}{\sqrt{\theta(t)}}$$

and deduce that

$$A_1 = \frac{1}{(2\pi h)^n} \int \int e^{i(\lambda + \Phi(t, x, z) - x, z)} \frac{\theta(t) A(t, x, y; h) dt dq}{\sqrt{\theta(t)}} + \mathcal{O}(h^\infty).$$

Here $\Phi(t, x, z)$ is the solution of the eikonal equation

$$\begin{cases} \frac{\partial \Phi}{\partial t} + l_2(x, \partial_x \Phi) = 0, \\ \Phi(0, x, z) = x, \end{cases}$$

$l_2(x, z)$ being the principal symbol of $Q_j, j = 1, 2$, and all derivatives $\partial_1^\alpha \partial_2^\beta \partial_3^\gamma \Phi(t, x, z) = x, \xi$ uniformly bounded for $(t, x, z) \in [-\delta_1, \delta_1] \times \mathbb{R}^n \times B(0, C_1)$ and $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. Moreover, the symbol $A(t, x, z)$ has support in $\{(x, z) : |x| > 2R, |z| \leq C_1, (x, z) \in \ell_2^1(I_1)\}$ so that for all $\alpha$ and $|t| \leq \delta_1$ we have

$$|\partial^\alpha A(t, x, z)| \leq C_\alpha (x^{-n-1}).$$

The last estimate enables us to calculate $A_1$ by using an infinite partition of unity

$$\sum_{\alpha \in \mathbb{N}^n} \Psi(x - \alpha) = 1, \forall x \in \mathbb{R}^n,$$

$\Psi \in C_\infty^0(K), \Psi(x) \geq 0, K$ being a neighborhood of the unit cube. Consequently, for every fixed $h \in [0, h_0]$ we have

$$A_1 = \frac{1}{(2\pi h)^n} \lim_{m \to \infty} \int \int e^{i(\lambda + \Phi(t, x, z) - x, z)} \frac{\theta(t) A(t, x, y, z; h) dt dq}{\sqrt{\theta(t)}}$$

$$\times \sum_{|\alpha| \leq m} \Psi(x - \alpha) A(t, x, y, z; h) dt dq + \mathcal{O}(h^\infty) = \lim_{m \to \infty} I_m + \mathcal{O}(h^\infty)$$

and we reduce the problem to the analysis of the integrals $I_m$ over a compact set in $(t, x, z)$. Concerning the phase function, we observe that

$$t\lambda + \Phi(t, x, z) = t\left(\lambda - l_2(x, z) + \mathcal{O}(t)\right),$$

where $\mathcal{O}(t)$ is uniformly bounded on the support of $\theta(t) A(t, x, z)$ since the derivatives of $\Phi(t, x, z) - x, \xi$ are bounded on this set. Finally, to have an uniform bound for the remainder with respect to $m \to \infty$, notice that

$$|\partial_\alpha l_2(x, z)| \geq \delta_2 > 0$$

(9.2)
for \(|\xi| \leq C_1\), \((x, \xi) \in l_{-1}^{-1}(\lambda), \lambda \in I_0\). The last condition follows easily from the form of the principal symbol
\[
l_2(x, \xi) = |\xi|^2 + \sum_{|\alpha| = 2} b_{\alpha, R}(x)\xi^\alpha + \sum_{|\alpha| \leq 1} b_{\alpha, R}(x)\xi^\alpha
\]
of the operator \(Q_2\), constructed in \([2]\), and the fact that
\[
|b_{\alpha, R}(x)| + |\partial_x b_{\alpha, R}(x)| \leq \epsilon_1(R)
\]
with \(\epsilon_1(R) \rightarrow 0\) as \(R \rightarrow +\infty\) (see Section 2.3 in \([2]\) for more details). Taking \(R \gg 1\) sufficiently large, we arrange \((9.2)\) uniformly with respect to \(|\xi| \leq C_1\) and \((x, \xi) \in l_{-1}^{-1}(\lambda)\). Now the critical points of the phase function \((t\lambda + \Phi(t, x, \xi) - x, \xi)\) become \(t = 0, l_2(x, \xi) = \lambda\) and by the stationary phase method we obtain
\[
I_m = \frac{1}{(2\pi h)^n} \psi(\lambda) \int_{l_2(x, \xi) = \lambda} \sum_{|\alpha| \leq m} \Psi(x - \alpha)A(0, x, \xi, \lambda)(1 - \lambda^2)(x)L_\lambda(d\omega) + O(h^{1-n}),
\]
where \(L_\lambda(d\omega)\) is the Liouville measure on \(l_2(x, \xi) = \lambda\) and the remainder \(O(h^{1-n})\) is uniform with respect to \(\lambda \in I_0\) and \(m \in \mathbb{N}\). Taking the limit \(m \to \infty\), we obtain an asymptotics of \(A_1\).

For the analysis of \(B\) we use the representation
\[
\left[ e^{-itQ_j/h} \right]_{j=1}^2 = \frac{t}{i h} \int_0^1 e^{-istQ_j/h} (Q_1 - Q_2) e^{-i(1-s)t} Q_2/h ds.
\]
Following the argument in Section 4.3, \([2]\), we find pseudodifferential operators
\[
Q \in Op_h \left(S^0(\langle \xi \rangle^{-n-1}(\xi)^{-N})\right), \tilde{Q} \in Op_h \left(S^0(\langle \xi \rangle^{-N})\right)
\]
with symbols \(q(x, y, \xi; h), \tilde{q}(x, y, \xi; h)\) having compact support in \(\xi\) and \((x - y)\) so that
\[
B = \frac{1}{2\pi h^2} \text{tr} \left( \int e^{it\lambda/h} t \theta(t) \int_0^1 e^{-istQ_1/h} Q e^{-i(1-s)t} Q_2/h Q_d s dt \right) + O(h^\infty).
\]
Moreover, modulo a term in \(S^N(1)\), the symbol of \(\tilde{Q}\) is supported in \(\{(x, \xi) : |x| > 2R\}\). Applying an approximation of the unitary groups \(e^{-itQ_j/h}, e^{-i(1-s)t} Q_2/h\) by Fourier integral operators, we are reduced to study the integral
\[
J = \frac{1}{(2\pi h)^{2n+2}} \int_0^1 \int_0^1 e^{it\lambda/h} t \theta(t) e^{i \left( \Phi_1(st, x, \xi) - z, \xi \right)/h} e^{i \left( \Phi_2((1-s)t, z, \eta) - x, \eta \right)/h} \\
\times B(t, s, X) dt ds dX,
\]
where \(X = (x, z, \xi, \eta)\) and the phase functions \(\Phi_1(t, x, \xi), \Phi_2(s, z, \eta)\) are related to the eikonal equations with symbols \(l_1(x, \xi)\) and \(l_2(z, \eta)\), respectively. The amplitude \(B(t, s, X)\) has a compact support with respect to \((\xi, \eta)\) and its support with respect to \(x\) is included in the set \(\{(x, \xi) : |x| \geq 2R\}\). Moreover, \(\partial^\alpha B(t, s, X)\) satisfy decreasing estimates with respect to \((x, z)\) like those in \((9.1)\).

In the same way, as in \([2]\), we check that the critical points of the phase in the integral \(J\) are related to the closed trajectories composed as union of a curve
\[
\{ \exp \left( \tau H_b \right)(\rho) : 0 \leq \tau \leq s \}
\]
of the Hamilton field $H_{ij}$ starting at same point $\rho \in \{(x, \xi) \in \mathbb{R}^n : |x| > 2R\}$ and a curve
\[
\{\exp(\tau H_{ij}) (\sigma) : 0 \leq \tau \leq (1 - s)t\}, \quad \sigma = \exp(st H_{ij})(\rho)
\]
of the Hamilton field $H_{ij}$. For $0 < t < \delta$, $\delta$, sufficiently small and $R > 0$ large enough, there are no such closed trajectories and the critical points are obtained for $t = 0$, only. We write the phase function in the form
\[
t \left[ \lambda - s l_1(x, \xi) - (1 - s)l_2(z, t) + O(t) \right] + (x - z)\xi = \eta
\]
and the critical points become
\[
t = 0, \quad s l_1(x, \xi) + (1 - s)l_2(x, \xi) = \lambda, \quad x = z, \xi = \eta.
\]
For $|x| \geq 2R$ and $0 \leq s < 1$, according to (2.6), we deduce
\[
m_s(x, \xi) = s l_1(x, \xi) + (1 - s)l_2(x, \xi) = |\xi|^2 + \eta_i(R)|\xi|^2
\]
with $\eta_i(R) \to 0$ as $R \to +\infty$, $i = 0, 1, 2$. Thus for $\lambda \in I_0$ and $R$ large enough the energy surface
\[
\Sigma_s(\lambda) = \{(x, \xi) : m_s(x, \xi) = \lambda, \quad |x| \geq 2R\}
\]
is non-degenerate. Repeating the argument used for $A_1$, and applying the stationary phase method, we get an asymptotics
\[
J = \frac{1}{(2\pi h)^n} b(\lambda) \int_0^1 \int_{m_s(x, \xi) = \lambda} B_1(s, x, \xi, \lambda)L_{s, \lambda}(d\omega)ds + O(h^{-n}),
\]
where $L_{s, \lambda}(d\omega)$ is the Liouville measure on $\Sigma_s(\lambda)$. Notice that the first term with power $h^{-1-n}$ vanishes because we have the factor $t \theta(t)$ and the term involving $h^{-n}$ yields the contribution to the leading term in (5.10). Moreover, $b(\lambda)$ has support in a small neighborhood of $I_1$ and taking $R > 0$ large, we may assume that $b(\lambda) \in C_0^0(I_0)$. This completes the proof of (5.10).

The above argument shows that for $\lambda \notin I_0$ the phase functions in $I_m$ and $J$ have no critical points over the support of the integrand. Consequently, by an integration by parts, we obtain
\[
\text{tr} \left[ \left( F_{\rho}^{-1} \right) \left( \lambda - Q_j \right) \varphi_2(Q_j) (1 - \chi^2) \right]_{j=1} = O(h^\infty)
\]
uniformly with respect to $\lambda \notin I_0$.

References


Département de Mathématiques Appliquées, Université Bordeaux 1, 351, Cours de la Libération, 33405 Talence, FRANCE
vbruneau@math.u-bordeaux.fr
petkov@math.u-bordeaux.fr