Irreducibility, Homoclinic Points and Adjoint Actions of Algebraic $\mathbb{Z}^k$-Actions of Rank One

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IRREDUCIBILITY, HOMOCLINIC POINTS AND ADJOINT ACTIONS OF ALGEBRAIC $\mathbb{Z}^d$-ACTIONS OF RANK ONE

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Abstract. In this paper we consider $\mathbb{Z}^d$-actions, $d \geq 1$, by automorphisms of compact connected abelian groups which contain at least one expansive automorphism (such actions are called algebraic $\mathbb{Z}^d$-actions of expansive rank one). If $\alpha$ is such a $\mathbb{Z}^d$-action on an infinite compact connected abelian group $X$, then every expansive element $\alpha^n$ of this action has a dense group $\Delta_{\alpha^n}(X)$ of homoclinic points. For different expansive elements $\alpha^m, \alpha^n$ these groups are generally different and may have zero intersection. By obtaining an appropriate structure formula we prove that these groups are canonically isomorphic for different $m, n$, and that the restriction of $\alpha$ to any of these groups defines by duality another algebraic $\mathbb{Z}^d$-action $\alpha^*$ of expansive rank one on a compact connected abelian group $X^*$, called the adjoint action of $\alpha$. The second adjoint $\alpha^{**} = (\alpha^*)^*$ obtained by repeating this construction is algebraically conjugate to $\alpha$.

A class of examples of algebraic $\mathbb{Z}^d$-actions of expansive rank one is obtained by fixing a $d$-tuple $c$ of algebraic numbers consisting not entirely of roots of unity and by associating with it a finite set $\mathcal{S}_c$ of places of the algebraic number field $K = K(c)$ generated by the entries of $c$. For every $S_c$-integral ideal $J$ in $K$ we define an algebraic $\mathbb{Z}^d$-action $\alpha$ of expansive rank one on a compact connected abelian group. In this case the adjoint action $\alpha^*$ arises from the inverse ideal class $J^{-1}$ of $J$. If $J, J'$ are two $S_c$-integral ideals, then the algebraic $\mathbb{Z}^d$-actions arising from them are algebraically conjugate if and only if the ideals lie in the same ideal class.

Earlier work in this direction can be found in [4], [6] and [11].

1. Introduction

An algebraic $\mathbb{Z}^d$-action $\alpha$ on a compact abelian group $X$ is a homomorphism $n \mapsto \alpha^n$ from $\mathbb{Z}^d$ into the group $\text{Aut}(X)$ of continuous group automorphisms of $X$. An algebraic $\mathbb{Z}^d$-action $\alpha$ on a compact abelian group $X$ is expansive if there exists an open neighbourhood $O \subset X$ of the identity element $0_X \in X$ with

$$\bigcap_{n \in \mathbb{Z}^d} \alpha^{-n}(O) = \{0_X\}.$$  

If $\alpha$ is an algebraic $\mathbb{Z}^d$-action on $X$, then its topological entropy $h(\alpha)$ coincides with the metric entropy $h_{\lambda_X}(\alpha)$ with respect to the normalized Haar measure $\lambda_X$ on $X$.

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Let $\alpha$ and $\beta$ be algebraic $\mathbb{Z}^d$-actions on compact abelian groups $X$ and $Y$, respectively. The action $\beta$ is an algebraic factor of $\alpha$ if there exists a continuous surjective group homomorphism $\chi : X \to Y$ with
\[ \chi \circ \alpha^n = \beta^n \circ \chi \text{ for every } n \in \mathbb{Z}^d. \] (1.1) 

The map $\chi$ in (1.1) is an algebraic factor map from $\alpha$ to $\beta$. The actions $\alpha$ and $\beta$ are algebraically conjugate if the factor map $\chi : X \to Y$ in (1.1) can be chosen to be a continuous group isomorphism. The actions $\alpha$ and $\beta$ are finitely equivalent if each of them is a finite-to-one algebraic factor of the other.

In this paper we restrict ourselves to algebraic $\mathbb{Z}^d$-actions where individual elements have finite entropy.

**Definition 1.1.** Let $\alpha$ be an algebraic $\mathbb{Z}^d$-action on a compact abelian group $X$. An algebraic $\mathbb{Z}^d$-action $\alpha$ on $X$ has expansive rank one if there exists $n \in \mathbb{Z}^d$ such that $\alpha^n$ is expansive, has entropy rank one if $h(\alpha^n) < \infty$ for every $n \in \mathbb{Z}^d$, and is irreducible if every closed, $\alpha$-invariant subgroup $Y \subsetneq X$ is finite.

The relationship between these rank one conditions and irreducibility will be studied in Section 4. We refer to [5] for a more general discussion of expansive rank and entropy rank (with a slightly different definition).

**Lemma 1.2.** Let $\alpha$ be an irreducible algebraic $\mathbb{Z}^d$-action on a compact connected abelian group $X$ and let $(Y, \beta)$ be a non-trivial algebraic factor of $(X, \alpha)$. Then the factor map is finite-to-one, and $\beta$ is irreducible and finitely equivalent to $\alpha$.

**Proof.** Let $\phi : X \to Y$ be the factor map between the actions $\alpha$ on $X$ and $\beta$ on $Y$. The kernel $K = \ker \phi$ is an $\alpha$-invariant closed subgroup of $X$. As $Y \not= \{0\}$ is assumed to be non-trivial, the kernel $K$ is a proper subgroup and must be finite by irreducibility.

Let $Z$ be a proper closed $\beta$-invariant subgroup of $Y$. The subgroup $\phi^{-1}(Z) \subset X$ is again finite by irreducibility. This shows that $Z = \phi(\phi^{-1}(Z))$ is finite and the action $\beta$ is irreducible.

As $K$ is finite, there exists an integer $m > 0$ with $m \cdot K = \{0\}$. As $X$ is connected and $M = \hat{X}$ is therefore torsion-free as a group, the multiplication $\chi_m : a \mapsto ma$ on $M$ is injective and the dual map $\hat{\chi} : x \mapsto mx$ on $X$ is surjective. It follows that the map $y \mapsto \psi(y) = \chi_m(\phi^{-1}(y))$ is a well-defined factor map from $(Y, \beta)$ to $(X, \alpha)$. By the first paragraph, $\psi$ is finite-to-one and $\alpha$ and $\beta$ are finitely equivalent. \hfill $\Box$

If $d > 1$, and if $\alpha$ and $\beta$ are two irreducible and mixing algebraic $\mathbb{Z}^d$-actions on compact abelian groups $X$ and $Y$, respectively, then $\alpha$ and $\beta$ are algebraically conjugate if and only if they are measurably conjugate (cf. [6] and [7]). In view of this rigidity property the question whether finite equivalence of $\alpha$ and $\beta$ can be replaced by algebraic conjugacy acquires special significance: since it is easy to find finitely equivalent irreducible $\mathbb{Z}^d$-actions which are not algebraically conjugate one obtains examples of measurably non-conjugate $\mathbb{Z}^d$-actions which are difficult to distinguish by using classical invariants of ergodic theory.
If $\alpha$ is an algebraic $\mathbb{Z}^d$-action on an infinite compact connected abelian group $X$, then a point $x \in X$ is $\alpha$-homoclinic (or simply homoclinic) if
\[
\lim_{n \to \infty} \alpha^n x = 0.
\]
If $\alpha$ is expansive, the set $\Delta_{\alpha}(X)$ of all $\alpha$-homoclinic points in $X$ is countable, and $\Delta_{\alpha}(X)$ is nonzero if and only if $\alpha$ has positive topological entropy (cf. [9]). In [4] the module of homoclinic points $\Delta_{\alpha}(X)$ was used to define, for any algebraic $\mathbb{Z}^d$-action $\alpha$ on a compact abelian group $X$, the adjoint action $\alpha^* = \alpha|_{\Delta_{\alpha}(X)}$ on $X^* = \widetilde{\Delta_{\alpha}(X)}$ (where $\Delta_{\alpha}(X)$ is considered as a discrete module). If the action $\alpha$ is expansive and has completely positive entropy, then $\alpha^*$ is closely related to $\alpha$ and is again expansive with completely positive entropy. Iteration of this construction leads to further algebraic $\mathbb{Z}^d$-actions $\alpha^{**}, \alpha^{***}, \ldots, \alpha^{k\cdot k}$, where $\alpha^{k\cdot k}$ can be shown to be algebraically conjugate to $\alpha^k$ for $k \geq 1$ (if not earlier).

For $d \geq 2$, algebraic $\mathbb{Z}^d$-actions with expansive rank one have zero entropy and therefore cannot have nonzero homoclinic points. Individual expansive elements of the action will, however, have nonzero homoclinic points; in fact, the homoclinic group $\Delta_{\alpha}(X)$ of every expansive automorphism $\alpha^n$, $n \in \mathbb{Z}^d$, is dense in $X$ by [9, Theorem 4.2].

For different expansive automorphisms $\alpha^m, \alpha^n$ the homoclinic groups $\Delta_{\alpha^m}(X)$ and $\Delta_{\alpha^n}(X)$ can be quite different and may have zero intersection (cf. [11]). The main purpose of this paper is to investigate the connection between these groups for different expansive elements of such an action.

If we restrict the $\mathbb{Z}^d$-action $\alpha$ to the homoclinic group $\Delta_{\alpha^n}(X)$ of some expansive automorphism $\alpha^n$, $n \in \mathbb{Z}^d$, then we obtain a $\mathbb{Z}^d$-action by automorphisms of the countable abelian group $\Delta_{\alpha^n}(X)$ which determines, by duality, an algebraic $\mathbb{Z}^d$-action $\alpha^*$ of the compact abelian group $X^* = \widetilde{\Delta_{\alpha^n}(X)}$. A priori this construction depends on the expansive automorphism $\alpha^n$ we started off with. In the Theorems 5.1 and 5.5 we show, however, that different expansive elements lead to algebraically conjugate actions $\alpha^*$. This allows us to suppress the dependence of $\alpha^*$ on $n$ and to refer to $\alpha^*$ as the adjoint action of $\alpha$ (cf. Definition 5.2 and [4]). If $\alpha$ is an expansive $\mathbb{Z}^d$-action by commuting automorphisms of a finite-dimensional torus $X = T^m$, $m \geq 2$, then $\alpha^*$ can easily be checked to be the transpose action $n \mapsto (\alpha^n)^\top$ of $\mathbb{Z}^d$ on $X$.

The adjoint action $\alpha^*$ again has expansive rank one, and the group $X^*$ is infinite, compact and connected (in fact, $\alpha^*$ is easily seen to be finitely equivalent to $\alpha$, but $\alpha$ and $\alpha^*$ are usually not algebraically — and hence, by our earlier remark, not measurably — conjugate). One may thus repeat this construction and define a second adjoint action $\alpha^{**} = (\alpha^*)^*$ of $\alpha$ which is algebraically conjugate to $\alpha$ (cf. the Corollaries 5.3 and 5.6). This situation is in marked contrast to the case where $\alpha$ has completely positive entropy: there $\alpha$ and $\alpha^{**}$ may be algebraically nonconjugate, although $\alpha^*$ and $\alpha^{**} = (\alpha^{**})^*$ are always conjugate (cf. [4]). Furthermore, since algebraic $\mathbb{Z}^d$-actions with completely positive entropy are Bernoulli by [13], $\alpha$ and $\alpha^*$ are always measurably conjugate in this case.

The main tool for our investigation is a careful analysis of the structure of algebraic $\mathbb{Z}^d$-actions of expansive rank one in the Theorems 3.4 and 4.5.
Although this description is not new in principle (cf. e.g. [1]–[2] and [14]),
the precise form required for the proofs of the Theorems 5.1 and 5.5 and of
Corollary 5.6 does not appear to be available in the earlier literature.

For irreducible algebraic $\mathbb{Z}^d$-actions on compact connected abelian groups
the Theorems 3.4 and 3.8 and Corollary 5.4 elaborate the close connections
with algebraic number theory (and, in particular, with ideal classes) already
apparent in [6].

2. Algebraic $\mathbb{Z}^d$-actions

For the description of algebraic $\mathbb{Z}^d$-action we need some algebra. Denote
by $R_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ the ring of Laurent polynomials with integral co-
efficients in the variables $u_1, \ldots, u_d$ and write every $f \in R_d$ as

$$f = \sum_{n \in \mathbb{Z}^d} f_n u^n$$

with $u^n = u_1^{n_1} \cdots u_d^{n_d}$ and $f_n \in \mathbb{Z}$ for all $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, where $f_n = 0$
for all but finitely many $n \in \mathbb{Z}^d$. A prime ideal $\mathfrak{p} \subset R_d$ is associated with an $R_d$-module $M$ if $\mathfrak{p} = \{f \in R_d : f \cdot a = 0\}$ for some $a \in M$, and the module $M$
is associated with a prime ideal $\mathfrak{p} \subset R_d$ if $\mathfrak{p}$ is the only prime ideal associated
with $M$. The set of prime ideals associated with a Noetherian $R_d$-module $M$ is finite and denoted by $\text{Asc}(M)$. If $M$ is a Noetherian $R_d$-module there exists a chain of submodules

$$M_0 = \{0\} \subset M_1 \subset \cdots \subset M_k = M$$

such that $M_i/M_{i-1} \cong R_d/\mathfrak{q}_i$ for $i = 1, \ldots, k$, where each $\mathfrak{q}_i \subset R_d$ is a prime
ideal containing one of the associated primes of $M$.

If $\alpha$ is an algebraic $\mathbb{Z}^d$-action on a compact abelian group $X$ then the
additively-written dual group $M = \hat{X}$ is a module over the ring $R_d$ with operation

$$f \cdot a = \sum_{n \in \mathbb{Z}^d} f_n \hat{\alpha}^n(a)$$

for $f \in R_d$ and $a \in M$, where $\hat{\alpha}^n$ denotes the automorphism of $\hat{X}$ dual to $\alpha^n$. The module $M = \hat{X}$ is called the dual module of $\alpha$. Conversely, if $M$ is a module over $R_d$, then we can define an algebraic $\mathbb{Z}^d$-action $\alpha_M$ on $X_M = \hat{M}$ by setting

$$\hat{\alpha}^n a = u^n \cdot a$$

for every $n \in \mathbb{Z}^d$ and $a \in M$. Clearly, $M$ is the dual module of $\alpha_M$.

By using duality one can express many topological and dynamical properties
of $X$ and $\alpha$ in terms of the dual module $M = \hat{X}$. For example, $X$ is
connected if and only if $M$ is torsion-free as a group, and the dual module $M$
is Noetherian whenever $\alpha$ is expansive (cf. [15]). We also recall the following
result from [15].

Lemma 2.1. Let $M$ be a Noetherian $R_d$-module with associated prime ideals
$\text{Asc}(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$. The following conditions are equivalent.

(1) $\alpha_M$ is expansive;

(2) $\alpha_{R_d/\mathfrak{p}_j}$ is expansive for every $j = 1, \ldots, m$;
(3) \( V_{\mathbb{C}}(p_j) \cap \mathbb{S}^d = \emptyset \) for every \( j = 1, \ldots, m \), where
\[
V_{\mathbb{C}}(p_j) = \{ z \in (\mathbb{C}^*)^d : f(z) = 0 \text{ for every } f \in p_j \},
\]
\( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) and \( \mathbb{S} = \{ z \in \mathbb{C} : |z| = 1 \} \).

3. Irreducible actions

In this section we describe up to algebraic conjugacy all irreducible algebraic \( \mathbb{Z}^d \)-actions on an infinite compact abelian connected group \( X \). Background and details of the following discussion can be found in [15, Section 7] and [16].

Let \( K \) be an algebraic number field, i.e., a finite extension of \( \mathbb{Q} \). A valuation of \( K \) is a map \( \phi : K \rightarrow \mathbb{R}_+ \) with the property that \( \phi(a) = 0 \) if and only if \( a = 0 \), \( \phi(ab) = \phi(a) \phi(b) \), and \( \phi(a + b) \leq c \max\{\phi(a), \phi(b)\} \) for some \( c \geq 1 \) in \( \mathbb{R} \) and all \( a, b \in K \). The valuation \( \phi \) is non-trivial if \( \phi(K) \supseteq \{0, 1\} \). A non-trivial valuation \( \phi \) is non-archimedean if \( \phi(a + b) < \max\{\phi(a), \phi(b)\} \) for all \( a, b \in K \), and archimedean otherwise. Two valuations \( \phi, \psi \) of \( K \) are equivalent if there exists an \( s > 0 \) with \( \phi(a) = s \psi(a) \) for all \( a \in K \). An equivalence class \( \nu \) of non-trivial valuations of \( K \) is called a place of \( K \); such a place \( \nu \) is finite if it consists of non-archimedean valuations, and infinite otherwise.

If \( \nu \) is a place of \( K \), then a sequence \( (a_n, n \geq 1) \) in \( K \) is \( \nu \)-Cauchy if \( \lim_{k \to \infty} \phi(a_k - a_l) = 0 \) for some (and hence for every) valuation \( \phi \in \nu \). With this notion of a Cauchy sequence one can define the completion \( K_\nu \) of \( K \) at the place \( \nu \).

Ostrowski's Theorem ([3, Theorem 2.2.1]) states that every non-trivial valuation \( \phi \) of \( \mathbb{Q} \) is either equivalent to the absolute value \( \frac{m}{n} \mapsto |\frac{m}{n}|_\infty = |\frac{m}{n}|_\infty \), or to the \( p \)-adic valuation \( |\frac{m}{n}|_p = p^{\nu(m' - m')} \) for some rational prime \( p \), where \( m = p^{\nu' m''}, n = p^{\nu'' n''}, \) and neither \( m'' \) nor \( n'' \) are divisible by \( p \). The completions \( \mathbb{Q}_\infty \) and \( \mathbb{Q}_p \) of \( \mathbb{Q} \) are equal to \( \mathbb{R} \) and the field of \( p \)-adic rationals, respectively.

For every valuation \( \phi \) of \( K \), the restriction of \( \phi \) to \( \mathbb{Q} \subset K \) is a valuation of \( \mathbb{Q} \) and is thus equivalent either to \( |\cdot|_\infty \) or to \( |\cdot|_p \) for some rational prime \( p \). In the first case the place \( \nu \supseteq \phi \) is infinite (or lies above \( \infty \)), and in the second case \( \nu \) lies above \( p \) (or \( p \) lies below \( \nu \)).

We denote by \( w \) the place of \( \mathbb{Q} \) below \( \nu \) and observe that the field \( K_\nu \) is a finite-dimensional vector space over the locally compact, metrizable field \( \mathbb{Q}_w \) and hence locally compact and metrizable. Choose a Haar measure \( \lambda_\nu \) on \( K_\nu \) (with respect to addition) and denote by \( \text{mod}_{K_\nu} : K_\nu \rightarrow \mathbb{R} \) the map satisfying
\[
\lambda_\nu(aB) = \text{mod}_{K_\nu}(a) \lambda_\nu(B)
\]
for every \( a \in K_\nu \) and every Borel set \( B \subset K_\nu \). The restriction of \( \text{mod}_{K_\nu} \) to \( K \) is a valuation in \( \nu \), denoted by \( |\cdot|_\nu \).

Above every place \( \nu \) of \( \mathbb{Q} \) there are at least one and at most finitely many places of \( K \). We write \( P^{(K)}_\nu \), \( P^{(K)}_f \), and \( P^{(K)}_\infty \), for the sets of places, finite places and infinite places of \( K \). An infinite place \( \nu \) of \( K \) is either real (if \( K_\nu = \mathbb{R} \)) or complex (if \( K_\nu = \mathbb{C} \)). The field \( K \) is totally real if \( K_\nu = \mathbb{R} \) for every \( \nu \in P^{(K)}_\infty \), and totally complex if \( K_\nu = \mathbb{C} \) for every \( \nu \in P^{(K)}_\infty \).
For every \( v \in P^{(K)} \), the sets
\[
\mathcal{R}_v = \{ a \in K_v : |a|_v \leq 1 \}, \quad \mathcal{R}^\times_v = \{ a \in K_v : |a|_v = 1 \}
\]  
(3.2)
are compact. If \( v \in P^{(K)}_f \), then \( \mathcal{R}_v \) is the unique maximal compact subring of \( K_v \) and is also open, and the ideal
\[
\mathcal{P}_v = \{ a \in K_v : |a|_v < 1 \} \subseteq \mathcal{R}_v
\]  
(3.3)
is open, closed and maximal. The set
\[
\mathfrak{o}_K = \bigcap_{v \in P^{(K)}_f} \{ a \in K : |a|_v \leq 1 \}
\]  
(3.4)
is the ring of integral elements in \( K \).

Now suppose that \( d \geq 1 \) and \( c = (c_1, \ldots, c_d) \in (\bar{Q}^\times)^d \), where \( \bar{Q} \) is the algebraic closure of \( Q \) and \( \bar{Q}^\times = \bar{Q} \setminus \{ 0 \} \). We set \( K = K_c = \bar{Q}(c_1, \ldots, c_d) = \bar{Q}[c_1^{\pm 1}, \ldots, c_d^{\pm 1}] \) and
\[
S_c = P^{(K)}_\infty \cup \{ v \in P^{(K)}_f : |c_i|_v \neq 1 \text{ for some } i = 1, \ldots, d \}.
\]  
(3.5)
The set \( S_c \) is finite by [16, Theorem III.3]. We denote by
\[
i_c : K \rightarrow V_c = \prod_{v \in S_c} K_v
\]  
(3.6)
the diagonal embedding \( a \mapsto (a, \ldots, a) \), \( a \in K \), and put
\[
\mathcal{R}_c = \{ a \in K : |a|_v \leq 1 \text{ for every } v \in P^{(K)}_f \setminus S_c \} \supseteq \mathfrak{o}_K.
\]  
(3.7)
The set \( V_c \) is a locally compact algebra over \( K \) with respect to coordinatewise addition, multiplication and scalar multiplication, and \( i_c(\mathcal{R}_c) \) is a discrete, co-compact, additive subgroup of \( V_c \). Put
\[
Y_c = V_c / i_c(\mathcal{R}_c)
\]  
(3.8)
and write
\[
\pi_c : V_c \rightarrow Y_c
\]  
(3.9)
for the quotient map. According to [15, (7.6)] we may identify \( Y_c \) with the dual group of \( \mathcal{R}_c \), i.e.
\[
Y_c = \mathcal{R}_c^\vee.
\]  
(3.10)

If every \( c_i, i = 1, \ldots, d \), is a unit in \( \mathfrak{o}_K \) then \( S_c = P^{(K)}_\infty \) and
\[
V_c \cong \mathbb{R}^{r(K)}, \quad Y_c \cong T^{r(K)},
\]  
(3.11)
where
\[
r(K) = [K : Q] = |\{ v \in P^{(K)}_\infty : K_v = \mathbb{R} \}| + 2|\{ v \in P^{(K)}_\infty : K_v = \mathbb{C} \}|.
\]  
(3.12)

In general,
\[
c_i \in \mathcal{R}_c^\times = \{ a \in \mathcal{R}_c : a^{-1} \in \mathcal{R}_c \}
\]  
(3.13)
is a unit in \( \mathcal{R}_c \) for every \( 1 \leq i \leq d \). We put, for every \( n = (n_1, \ldots, n_d) \in \mathbb{Z}^d \),
\[
e^{n} = c_1^{n_1} \cdots c_d^{n_d},
\]  
(3.14)
write every \( a \in V_c \) as \( a = (a_v) = (a_v, v \in S) \) with \( a_v \in K_v \) for every \( v \in S \), and define a \( \mathbb{Z}^d \)-action \( \mathfrak{A}_c \) on \( V_c \) by setting
\[
\mathfrak{A}_c^n a = i_c(e^n)a = (e^na_v)
\]  
(3.15)
for every $a = (a_v) \in V_c$ and $n \in \mathbb{Z}^d$. As $\beta_c^n (\iota_c(\mathcal{R}_c)) = \iota_c(\mathcal{R}_c)$ for every $n \in \mathbb{Z}^d$, $\beta_c$ induces an algebraic $\mathbb{Z}^d$-action $\tilde{\beta}_c$ on the compact abelian group $Y_c$ in (3.8) by
\[
\tilde{\beta}_c^n(a + \iota_c(\mathcal{R}_c)) = \frac{\beta_c^n}{\beta_c^n} a + \iota_c(\mathcal{R}_c)
\]
for every $n \in \mathbb{Z}^d$ and $a \in V_c$, whose dual action $\tilde{\beta}_c^*: n \mapsto \tilde{\beta}_c^n$ is given by
\[
\tilde{\beta}_c^n b = c^n b
\]
for every $n \in \mathbb{Z}^d$ and $b \in \mathcal{R}_c = \tilde{Y}_c$ (cf. (3.10)).

We denote by $\eta_c: f \mapsto f(c)$ the evaluation map and define the ideal $P_c = \ker \eta_c$. Then
\[
R_d/P_c \cong \eta_c(R_d) = \mathbb{Z}[c^\pm 1] = \mathbb{Z}[c_1^\pm 1, \ldots, c_d^\pm 1] \subset \mathcal{R}_c,
\]
and $\mathcal{R}_c$ is a module over the integral domain $\mathbb{Z}[c^\pm 1]$. We need the following basic lemma from algebraic number theory.

**Lemma 3.1.** The $\mathbb{Z}[c^\pm 1]$-module $\mathcal{R}_c$ is equal to $\mathcal{O}_K[c^\pm 1]$ and is thus finitely generated.

**Proof.** The inclusion $\mathcal{O}_K[c^\pm 1] \subset \mathcal{R}_c$ is obvious. For the reverse inclusion let $x \in \mathcal{R}_c$ and put
\[
E_x = \{ v \in S_c : [x]_v > 1 \}.
\]
If $E_x = \emptyset$ then $x \in \mathcal{O}_K$ and we are done. Now assume that $k \geq 1$ and that $y \in \mathcal{O}_K[c^\pm 1]$ for every $y \in \mathcal{R}_c$ with $|E_x| < k$. If $|E_x| = k$ and $v \in E_x$, then we can find an $n \in \mathbb{Z}^d$ with $|c^n|_v > |x|_v$. By the Chinese remainder theorem there exists $a \in \mathcal{O}_K$ such that $|a|_v = 1$ and $|a c^n|_w < 1$ for every $w \in S_c \setminus \{v\}$. Then $x (a c^n)^{-1} \in \mathcal{R}_v$, and, since $\mathcal{O}_K$ is dense in $\mathcal{R}_v$, we can find $b \in \mathcal{O}_K$ and $d \in \mathcal{R}_v$ such that
\[
x (a c^n)^{-1} = b + d \quad \text{with } |d|_v \leq |c^n|_v^{-1}.
\]
This shows that
\[
|x - abc^n|_w = |ad c^n|_v \leq 1,
\]
\[
|x - abc^n|_w = |x|_w \quad \text{for } w \in E_x \setminus \{v\},
\]
\[
|x - abc^n|_w \leq 1 \quad \text{for } w \in P_f^j \setminus E_x.
\]
Our induction hypothesis implies that $x - abc^n \in \mathcal{O}_K[c^\pm 1]$ and hence that $x \in \mathcal{O}_K[c^\pm 1]$. By induction, $\mathcal{R}_c \subset \mathcal{O}_K[c^\pm 1]$, as promised.

Since $\mathcal{O}_K$ is a finitely generated $\mathbb{Z}$-module, $\mathcal{R}_c$ is finitely generated over $\mathbb{Z}[c^\pm 1]$.

Let $L \subset K$ be a finitely generated $\mathbb{Z}[c^\pm 1]$-submodule. We denote by $\hat{\alpha}_{(c,L)}$ the $\mathbb{Z}^d$-action on $L$ defined by
\[
\hat{\alpha}_{(c,L)}^n a = c^n a
\]
for every $n \in \mathbb{Z}^d$ and $a \in L$ and write $\alpha_{(c,L)}$ for the dual algebraic $\mathbb{Z}^d$-action on $X_L = \hat{L}$.

\[
X_L = \hat{L}.
\]
Since $K = \mathbb{Q}[\varepsilon]$ we can write every $a \in K$ as $a = b/n$ for some $b \in \mathbb{R}_c$ and $n \in \mathbb{Z}$. As $\mathcal{L}$ is assumed to be finitely generated, we can find a common integer $N > 0$ such that $N\mathcal{L} \subseteq \mathbb{R}_c$. If

$$\theta_{\mathcal{L}} : \tilde{X}_{\mathcal{L}} = \mathcal{L} \longrightarrow \tilde{\mathbb{R}}_c = \tilde{Y}_c$$

(3.21)

is the injective map defined by multiplication with $N$, then we obtain a dual algebraic factor map

$$\theta_{\mathcal{L}} : Y_c \longrightarrow X_{\mathcal{L}}$$

(3.22)

between the algebraic $\mathbb{Z}^d$-actions $\beta_c$ and $\alpha_{c,\mathcal{L}}$. For the particular choices $\mathcal{L} = \mathbb{R}_c$ and $\mathcal{L} = \mathbb{Z}[c^{\pm 1}]$ we obtain the actions

$$\beta_c = \alpha_{c,\mathbb{R}_c} \quad \text{on } Y_c = \tilde{\mathbb{R}}_c,$$

$$\alpha_c = \alpha_{c,\mathbb{Z}[c^{\pm 1}]} \quad \text{on } \mathbb{Z} \quad \text{is finite.}$$

(3.23)

**Proposition 3.2.** For any two nonzero finitely generated $\mathbb{Z}[c^{\pm 1}]$-modules $\mathcal{L} \subseteq \mathcal{L}'$ the module $\mathcal{L}'/\mathcal{L}$ is finite.

Furthermore the $\mathbb{Z}^d$-action $\beta_c$ on $Y_c$ is irreducible, the factor map $\theta_{\mathcal{L}} : Y_c \longrightarrow X_{\mathcal{L}}$ in (3.21)-(3.22) is finite-to-one, and the action $\alpha_{c,\mathcal{L}}$ on $X_{\mathcal{L}}$ is irreducible and finitely equivalent to $\beta_c$.

For the proof of Proposition 3.2 we need another lemma.

**Lemma 3.3.** Let $\mathfrak{a} \subseteq K$ be a finitely generated subring with identity of the algebraic number field $K$. Then every nonzero ideal $\mathfrak{b} \subset \mathfrak{a}$ has finite index.

**Proof.** Assume that we have already shown that some finitely generated subring $\mathfrak{a} \subset K$ containing 1 has the property that $|\mathfrak{a}/\mathfrak{b}| < \infty$ for every nonzero ideal $\mathfrak{b} \subset \mathfrak{a}$. By assumption $\mathbb{Z} \subset \mathfrak{a}$.

Let $a \in K$ be an algebraic number with primitive minimal polynomial $f(x) \in \mathbb{Z}[x]$, and let $\mathfrak{b} \subset \mathfrak{a}[a]$ be a nonzero ideal. We set $S = \mathfrak{a} \setminus \{0\}$ and consider the number fields $S^{-1}S = L$ and $S^{-1}\mathfrak{a}[a] = L[a] = L'$. As $\{0\} \subset S^{-1}\mathfrak{b} \subset L'$, it follows that $S^{-1}\mathfrak{b} = L'$ and $\mathfrak{b} \cap S \neq \{0\}$.

By our hypothesis on $\mathfrak{a}$, the nonzero ideal $\mathfrak{b} \cap \mathfrak{a}$ has finite index in $\mathfrak{a}$. We claim that

there exists a monic polynomial $h \in \mathbb{Z}[x]$ with $h(a) \in \mathfrak{b}$. (3.24)

Indeed, since $\mathbb{Z} \subset \mathfrak{a}$ and $\mathfrak{b} \cap \mathfrak{a}$ has finite index in $\mathfrak{a}$, there exists a positive integer $n \in \mathfrak{b}$. We denote by $\mathfrak{J} = (n, f) \subset \mathbb{Z}[x]$ the ideal generated by the elements $n, f \in \mathbb{Z}[x]$ and assert that

$\mathfrak{J}$ contains a monic polynomial $h$. (3.25)

By evaluating the generators of the ideal $\mathfrak{J}$ at $a$ we conclude that $h(a) \in \mathfrak{b}$, which shows that (3.24) is a consequence of (3.25).

In order to prove (3.25) we first assume that $n = p^r$ is a prime power. We write $f$ as a sum $f = f_1 - pf_2$ with $f_1, f_2 \in \mathbb{Z}[x]$, where the leading coefficient of $f_1$ is co-prime to $p$. Multiplication with $a = f_1^{-1} + f_1^{-2} - pf_2 + \cdots + (pf_2)^{-1} \in \mathbb{Z}[x]$ gives that $f_1^r - pf_2^r \in \mathfrak{J}$. We have thus found polynomials $a, b \in \mathbb{Z}[x]$ such that $h_p = f_1^r - pf_2^r \in \mathfrak{J}$ has a leading coefficient which is co-prime to $p$ and hence to $n = p^r$. If $m$ is the degree of $h'_p$ we can apply Euclid's algorithm to find integers $k, k'$ such that the leading coefficient of

$h_p = kh'_p + k'nx^m \in \mathfrak{J}$

(3.26)
is one.

If \( n \) contains a product of at least two distinct primes we write \( n = p_1^{e_1} \cdots p_k^{e_k} \) for the prime power decomposition of \( n \) and use the isomorphism \( \mathbb{Z}/n\mathbb{Z} \cong \prod_{j=1}^k \mathbb{Z}/p_j^{e_j}\mathbb{Z} \) to obtain an isomorphism

\[
\theta: (\mathbb{Z}/n\mathbb{Z})[x] \rightarrow \prod_{j=1}^k (\mathbb{Z}/p_j^{e_j}\mathbb{Z})[x]
\]

of the polynomial rings. Denote by \( \bar{f} \in R = (\mathbb{Z}/n\mathbb{Z})[x] \) the polynomial obtained by reducing each coefficient of \( f \) modulo \( n \) and put \( \theta(\bar{f}) = (\bar{f}_1, \ldots, \bar{f}_k) \) with \( \bar{f}_j \in R_j = (\mathbb{Z}/p_j^{e_j}\mathbb{Z})[x] \) for every \( j = 1, \ldots, k \). The preceding paragraph shows that the principal ideal generated by \( \bar{f}_j \) in \( R_j \) contains a polynomial \( \bar{h}_j \) with leading coefficient 1, i.e. that there exists a \( \bar{g}_j \in R_j \) with \( \bar{h}_j = \bar{f}_j\bar{g}_j \). The polynomial \( g \in R \) with \( \theta(g) = (\bar{g}_1, \ldots, \bar{g}_k) \) satisfies that \( \bar{f}_j\bar{g}_j \in R \) has leading coefficient 1. This shows that there exists a \( g' \in \mathbb{Z}[x] \) such that the polynomial \( h = fg + ng' \in \mathfrak{f} \) has leading coefficient 1 and proves (3.25) and hence (3.24).

If \( m \) is the degree of the polynomial \( h \) found in (3.24), then

\[
|\mathfrak{f}[a]/\mathfrak{f}| = |\mathfrak{f} + a\mathfrak{f} + \cdots + a^{m-1}\mathfrak{f}| \leq |\mathfrak{f}/\mathfrak{f} \cap \mathfrak{f}^m| < \infty.
\]

This shows that the ring \( \mathfrak{f}[a] \) again has the property that \( |\mathfrak{f}[a]/\mathfrak{f}| < \infty \) for every nonzero ideal \( \mathfrak{f} \subset \mathfrak{f}[a] \).

The proof of the lemma is completed by induction on the number of generators of the subring \( \mathfrak{f} \). \( \square \)

**Proof of Proposition 3.2.** This is a slight extension of [15, Theorem 7.1].

Let \( \mathcal{L} \subset \mathcal{L}' \subset K \) be two nonzero finitely generated \( \mathbb{Z}[\mathcal{c}^{\pm 1}] \)-submodules. Since \( \mathcal{L} \subset K \) is nonzero, \( \mathbb{Z}[\mathcal{c}^{\pm 1}]a \subset \mathcal{L} \) for some nonzero \( a \in \mathcal{L} \). Since \( \mathcal{L}' \)

is finitely generated as a \( \mathbb{Z}[\mathcal{c}^{\pm 1}] \)-module and \( \mathcal{L}' \subset K = \mathbb{Q}[\mathcal{c}] \), we can find \( M \in \mathbb{Z} \) such that

\[
\mathbb{Z}[\mathcal{c}^{\pm 1}]a \subset \mathcal{L} \subset \mathcal{L}' \subset \frac{1}{M}\mathbb{Z}[\mathcal{c}^{\pm 1}].
\]

Lemma 3.3 shows that \( \mathbb{Z}[\mathcal{c}^{\pm 1}]a \) has finite index in \( \frac{1}{M}\mathbb{Z}[\mathcal{c}^{\pm 1}] \), which completes the proof of the first statement of the proposition.

For the second statement we consider the action \( \beta_c \) on \( Y_c \). If \( Z \subset Y_c \) is a proper invariant closed subgroup, then the annihilator \( \mathcal{L} = Z^\perp \subset \mathfrak{R}_c \) is a nonzero \( \mathbb{Z}[\mathcal{c}^{\pm 1}] \)-submodule. Therefore has \( \mathcal{L} \) finite index in \( \mathfrak{R}_c \) and \( Z \) is finite. This shows that \( \beta_c \) is irreducible. Lemma 1.2 implies the remaining statements. \( \square \)

Our next theorem shows that every irreducible algebraic \( \mathbb{Z}^d \)-action \( \alpha \) on a compact abelian connected group is of the form \( \alpha_{(c, \mathcal{L})} \) described in (3.19)–(3.20).

**Theorem 3.4.** Suppose that \( d \geq 1 \), and that \( \alpha \) is an algebraic \( \mathbb{Z}^d \)-action on an infinite compact connected abelian group \( X \). Then \( \alpha \) is irreducible if and only if it is finitely equivalent to each of the irreducible algebraic \( \mathbb{Z}^d \)-actions \( \alpha_c \) on \( X_c \) and \( \beta_c \) on \( Y_c \) for some \( c = (c_1, \ldots, c_d) \in (\mathbb{Q}^\times)^d \).

Furthermore there exists a finitely generated \( \mathbb{Z}[\mathcal{c}^{\pm 1}] \)-submodule \( \mathcal{L} \subset K \) such that \( \alpha \) is algebraically conjugate to the \( \mathbb{Z}^d \)-action \( \alpha_{(c, \mathcal{L})} \) on \( X_\mathcal{L} \) defined in
(3.19)-(3.20). Without loss of generality one may assume in addition that \( \mathcal{L} \subseteq \mathcal{R}_c \).

**Proof.** Let \( \alpha \) be an irreducible algebraic action on the compact connected abelian group \( X \) with dual module \( M = \hat{X} \) and let \( p \) be an associated prime ideal for \( M \). Then there exists \( a \in M \) such that
\[
\{ f \in R_d : f \cdot a = 0 \} = p.
\]
This shows that the map \( f + p \mapsto \hat{\theta} (f) = fa \) from \( R_d / p \) to \( M \) is an injective module homomorphism. By duality, \( \theta : X \rightarrow X_{R_d / p} \) is a factor map. From Lemma 1.2 we see that \( \theta \) is finite-to-one and the action \( \alpha_{R_d / p} \) on \( X_{R_d / p} \) is irreducible.

As \( X \) is connected, the dual module is torsion-free as an abelian group and \( p \) does not contain a constant. Hilbert's Nullstellensatz shows that there exists a point
\[
c \in V(p) = \{ c' \in \overline{\mathbb{Q}}^\times : f(c') = 0 \text{ for every } f \in p \}.
\]
Let \( \pi : R_d / p \rightarrow R_d / p_c \) be the canonical projection map, where
\[
p_c = \{ f \in R_d : f(c) = 0 \}.
\]
(3.27)
Then \( \hat{\pi} : X_c \rightarrow X \) is injective. As \( \alpha_{R_d / p} \) is irreducible by the previous paragraph, every non-trivial closed \( \alpha \)-invariant subgroup must be finite and \( \hat{\pi} \) must be surjective. By duality, \( \pi \) is injective and \( p = p_c \).

Lemma 1.2 also shows that the actions \( \alpha_{R_d / p_c} \) and \( \alpha \) are finitely equivalent. Let \( \phi : X_c \rightarrow X \) be a factor map. The dual homomorphism \( \hat{\phi} : M \rightarrow R_d / p_c \cong \mathbb{Z}[c^\pm] \) of \( \phi \) is injective. Hence \( \mathcal{L} = \phi(M) \subset \mathbb{Z}[c^\pm] \subset K \) is a nonzero finitely generated \( \mathbb{Z}[c^\pm] \)-submodule and the \( \mathbb{Z}^d \)-actions \( \alpha \) and \( \alpha_{(\mathcal{L})} \) are algebraically conjugate. \( \square \)

By using the locally compact group \( V_c \) in (3.6) we can give another description of all irreducible algebraic \( \mathbb{Z}^d \)-actions.

**Corollary 3.5.** Let \( d \geq 1 \), and let \( \alpha \) be an irreducible \( \mathbb{Z}^d \)-action on an infinite compact connected abelian group \( X \). We denote by \( c \in (\mathbb{Q}^\times)^d \) the point described in Theorem 3.4 and define the ring \( \mathcal{R}_c \subset \mathbb{P}^1(K) \), the set \( S_c \subset \mathcal{T}(\mathcal{L}) \), the algebra \( V_c = \prod_{v \in S} K_v \) and the embedding \( \iota_c : K \rightarrow V_c \) as in (3.5)-(3.6). Then there exists a finitely generated \( \mathbb{Z}[c^\pm] \)-submodule \( \mathcal{K} \subset K \) such that \( \alpha \) is algebraically conjugate to the \( \mathbb{Z}^d \)-action \( \alpha_{(c, \mathcal{K})} \) on the compact abelian group
\[
X_{\mathcal{K}}^d = V_c / \iota_S(\mathcal{K}),
\]
defined as in (3.16) by
\[
\alpha^{an}_{(c, \mathcal{K})}(a + \iota_S(\mathcal{K})) = \beta^{an}_{c, \mathcal{K}}(a + \iota_S(\mathcal{K})) \tag{3.29}
\]
for every \( n \in \mathbb{Z}^d \) and \( a \in V_c \). Furthermore one can always assume that \( \mathcal{K} \subset \mathcal{R}_c \).

Conversely, if \( \mathcal{K} \subset K \) is a nonzero finitely generated \( \mathbb{Z}[c^\pm] \)-submodule, then the \( \mathbb{Z}^d \)-action \( \alpha_{(c, \mathcal{K})} \) on \( X_{\mathcal{K}}^d \) in (3.28)-(3.29) is irreducible and finitely equivalent to \( \alpha_c \) and \( \beta_c \).
Proof. According to Theorem 3.4 there exists a finite-to-one factor map \( \phi: V_c \rightarrow X \). The map \( \phi \) induces a continuous surjective group homomorphism \( \psi: V_c \rightarrow X \) with \( \psi \circ \alpha^n = \alpha^n \circ \psi \) for every \( n \in \mathbb{Z}^d \), whose kernel \( \mathcal{K}' = \ker(\psi) \) is invariant under the \( \mathbb{Z}^d \)-action \( \alpha \) in \((3.15)\) and contains \( \iota_\mathcal{S}(R_c) \) as a subgroup of finite index.

Choose an integer \( N \geq 1 \) with \( \mathcal{K}' = N \mathcal{K} \subset \iota_\mathcal{S}(R_c) \) and denote by \( \mathcal{K} \subset R_c \) the \( \iota_\mathcal{S}(R_c) \)-submodule satisfying \( \iota_\mathcal{S}(\mathcal{K}) = \mathcal{K}' \). If \( m_N: V_c \rightarrow V_c \) denotes multiplication by \( N \), then

\[
X \cong V_c / \mathcal{K}' \cong m_N(V_c) / m_N(\mathcal{K}') \cong V_c / \mathcal{K}' \cong V_c / \iota_\mathcal{S}(\mathcal{K}) = X'_{\mathcal{K}},
\]

and the isomorphism of \( X \) and \( X'_{\mathcal{K}} \) carries the \( \mathbb{Z}^d \)-action \( \alpha \) to \( \alpha'_{(\mathcal{K}, \mathcal{K})} \).

The other statements are clear from Proposition 3.2, since \( \mathcal{K} \) has finite index in \( R_c \).

Theorem 3.4 and Corollary 3.5 give a variety of representations of irreducible algebraic \( \mathbb{Z}^d \)-action on infinite compact connected abelian groups. For a fixed \( c \) all these representations are finitely equivalent. Theorem 3.8 will show that these representations are sometimes, but not always, algebraically conjugate.

We can give an easy characterization of those actions which are algebraically conjugate to \( \alpha_c \).

**Definition 3.6.** Let \( \alpha \) be an algebraic \( \mathbb{Z}^d \)-action on a compact abelian group \( X \). The dual group \( \hat{X} \) of \( X \) is cyclic under the dual action \( \hat{\alpha} \) of \( \alpha \) (or \( \alpha \) has cyclic dual) if there exists a character \( \hat{a} \in \hat{X} \) such that \( \hat{X} \) is generated by the set \( \{ \hat{a}^n a : n \in \mathbb{Z} \} \).

**Proposition 3.7.** Let \( d \geq 1 \), and let \( \alpha \) be an irreducible algebraic \( \mathbb{Z}^d \)-action on an infinite compact connected abelian group \( X \). If \( c \in \overline{\mathbb{Q}}^\times \) is the point appearing in Theorem 3.4, then \( \alpha \) is algebraically conjugate to \( \alpha_c \) if and only if \( \alpha \) has cyclic dual.

**Proof.** The action \( \alpha_c \) has cyclic dual, since the element \( 1 \in \mathbb{Z}[c^{\pm 1}] = \hat{X}_c \) is cyclic under \( \hat{\alpha}_c \).

If \( \alpha \) and \( \alpha_c \) are algebraically conjugate, there exists a continuous group isomorphism \( \phi: X \rightarrow X_{R_d/p} \), with \( \phi \circ \alpha^n = \alpha^n \circ \phi \) for every \( n \in \mathbb{Z}^d \), and the dual isomorphism \( \hat{\phi}: \hat{X}_{R_d/P} \rightarrow \hat{X}_c \) sends \( 1 \in \mathbb{Z}[c^{\pm 1}] \) to a cyclic element \( a \in \hat{X} \) for \( \hat{\alpha} \).

Conversely, if \( a \in \hat{X} \) is a cyclic element of \( \hat{\alpha} \), then the map

\[
h(c) \mapsto h(\hat{\alpha})(a) = \sum_{n \in \mathbb{Z}^d} h_n \hat{a}^n
\]

for \( h = \sum_{n \in \mathbb{Z}^d} h_n a^n \in R_c \) induces a module-isomorphism \( \hat{\psi}: \mathbb{Z}[c^{\pm 1}] \rightarrow \hat{X} \) whose dual \( \psi: X \rightarrow X_{R_d/P} = \hat{X}_{R_d/P} \) is an algebraic conjugacy of \( \alpha \) and \( \alpha_c \). \( \square \)

We end this section with a connection between irreducible algebraic \( \mathbb{Z}^d \)-actions and ideal classes in algebraic number fields. This will give us a collection of nonconjugate but finitely equivalent algebraic actions.
Every nonzero ideal \( \mathcal{J} \subset \mathcal{R}_S \) is called an \( S \)-integral ideal of \( K \) and has finite index in \( \mathcal{R}_S \) by Lemma 3.3. Two \( S \)-integral ideals \( \mathcal{J}, \mathcal{J} \) of \( K \) lie in the same ideal class if there exists an element \( a \in K \) with \( a\mathcal{J} = \mathcal{J} \).

**Theorem 3.8.** Suppose that \( K \) is an algebraic number field, \( \mathbf{c} \in (K^\times)^d \) a vector of nonzero algebraic numbers with \( K = \mathbb{Q}(\mathbf{c}) \), and let \( \mathcal{S}_C \subset \mathcal{P}(K) \) be the set of places defined by (3.5). Then the \( \mathbb{Z}^d \)-action \( \alpha_{(c,\mathcal{J})} \) on \( X_\mathcal{J} = \mathcal{J} \) is irreducible for every nonzero ideal \( \mathcal{J} \subset \mathcal{R}_C \). Furthermore, if \( \mathcal{J}, \mathcal{J} \) are nonzero ideals in \( \mathcal{R}_C \), then \( \alpha_{(\mathbf{c},\mathcal{J})} \) and \( \alpha_{(\mathbf{c},\mathcal{J})} \) are finitely equivalent, and \( \alpha_{(\mathbf{c},\mathcal{J})} \) and \( \alpha_{(\mathbf{c},\mathcal{J})} \) are algebraically conjugate if and only if \( \mathcal{J} \) and \( \mathcal{J} \) lie in the same ideal class.

**Proof.** Theorem 3.4 shows that the action \( \alpha_{(\mathbf{c},\mathcal{J})} \) is irreducible.

If \( \mathcal{J}, \mathcal{J} \) are nonzero ideals in \( \mathcal{R}_C \), then \( \alpha_{(\mathbf{c},\mathcal{J})} \) and \( \alpha_{(\mathbf{c},\mathcal{J})} \) are obviously algebraically conjugate whenever \( \mathcal{J} \) and \( \mathcal{J} \) lie in the same ideal class.

Conversely, if \( \phi: X_\mathcal{J} \to X_\mathcal{J} \) is an algebraic conjugacy of \( \alpha_{(\mathbf{c},\mathcal{J})} \) and \( \alpha_{(\mathbf{c},\mathcal{J})} \), then the dual map \( \hat{\phi}: \mathcal{J} \to \mathcal{J} \) is an \( \eta_c(R_\mathcal{J}) \)-module isomorphism (cf. (3.18)), i.e., \( \hat{\phi}(f(\mathbf{c})a) = f(\mathbf{c})\hat{\phi}(a) \) for every \( f \in R_\mathcal{J} \) and \( a \in \mathcal{J} \). Since \( K \) is the field of fractions of \( \mathbb{Z}[\mathbf{c}^{\pm 1}] \) we can extend \( \hat{\phi} \) to a \( K \)-linear map \( \hat{\psi}: K \to K \) by fixing a nonzero element \( a \in \mathcal{J} \) and setting \( \hat{\psi}(\frac{f(\mathbf{c})a}{g(\mathbf{c})}) = \frac{f(\mathbf{c})}{g(\mathbf{c})}\hat{\phi}(a) \) for every \( f, g \in R_\mathcal{J} \) with \( g(\mathbf{c}) \neq 0 \). An elementary calculation shows that \( \hat{\psi}(\frac{f(\mathbf{c})a}{g(\mathbf{c})}) = \hat{\phi}(\frac{f(\mathbf{c})a}{g(\mathbf{c})}) \) whenever \( \frac{f(\mathbf{c})}{g(\mathbf{c})} \in \mathcal{J} \). If \( b = \hat{\psi}(1) \), then the \( K \)-linearity of \( \hat{\psi} \) guarantees that \( \hat{\psi}(a) = ba \) for every \( a \in K \), and hence that \( \mathcal{J} = \hat{\phi}(\mathcal{J}) = \hat{\psi}(\mathcal{J}) = b\mathcal{J} \). This shows that \( \mathcal{J} \) and \( \mathcal{J} \) lie in the same ideal class. \( \square \)

Theorem 3.8 describes the algebraic \( \mathbb{Z}^d \)-actions arising from a fixed vector \( \mathbf{c} = (c_1, \ldots, c_d) \) of algebraic numbers. In the remainder of the section we start with an integral (or an \( S \)-integral) ideal in an algebraic number field and consider algebraic \( \mathbb{Z}^d \)-actions arising from units systems in the number field.

Let \( K' \supset \mathbb{Q} \) be an algebraic number field, and let \( S' \subset \mathcal{P}(K') \) be a finite number of places containing \( F_{\infty}^{(K')} \). We define the ring of \( S' \)-integers \( \mathcal{R}_{S'}^{\times} \subset K' \) as in (3.7) with \( S' \) replacing \( S \). By [12, Theorem 3.5], the group of units \( \mathcal{R}_{S'}^{\times} \) of \( \mathcal{R}_{S'} \) in (3.13) is isomorphic to the cartesian product \( F \times \mathbb{Z}^d \), where \( F \) is a finite cyclic group consisting of all roots of unity in \( K' \) and \( d = |S| - 1 \).

**Definition 3.9.** A \( d \)-tuple \( \mathbf{c} = (c_1, \ldots, c_d) \) in \( \mathcal{R}_{S'}^{\times} \) is a free \( S' \)-unit system if it generates a free abelian group, i.e., if the equation \( \mathbf{c}^d = c_1^{n_1} \cdots c_d^{n_d} = 1 \) with \( \mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d \) implies that \( \mathbf{n} = \mathbf{0} \).

A free \( S' \)-unit system \( \mathbf{c} = (c_1, \ldots, c_d) \) is a fundamental \( S' \)-unit system of \( K' \) if every \( a \in \mathcal{R}_{S'}^{\times} \) can be written uniquely as \( a = uc_1^{k_1} \cdots c_d^{k_d} \) with \( u \in F \) and \( k_1, \ldots, k_d \in \mathbb{Z} \).

We fix a free \( S' \)-unit system \( \mathbf{c} = (c_1, \ldots, c_d) \) of \( K' \) and set \( K = \mathbb{Q}(\mathbf{c}) \). Every \( S' \)-integral ideal \( \mathcal{J} \subset \mathcal{R}_{S'} \) of \( K' \) is obviously a module over the ring \( \eta_c(R_\mathcal{J}) \), and we obtain an algebraic \( \mathbb{Z}^d \)-action \( \alpha_{(c,\mathcal{J})} \) on \( X_\mathcal{J} = \mathcal{J} \) as in (3.19)–(3.20). In order to simplify terminology we count \( \mathcal{R}_{S'}^{\times} \) itself among the ideals of \( \mathcal{R}_{S'} \).
Theorem 3.10. Let $K$ be an algebraic number field, $S \subset P(K)$ a finite number of places containing $P_\infty(K)$ with $|S| \geq 2$, and $c = (c_1, \ldots, c_d)$ a fundamental $S$-unit system with $d = |S| - 1$.

Suppose that $K$ has at least one real place. Then the $\mathbb{Z}^d$-action $\alpha_{(c, \mathfrak{I})}$ is irreducible for every nonzero ideal $\mathfrak{I} \subset \mathcal{R}_c$. Furthermore, if $\mathfrak{I}, \mathfrak{J}$ are nonzero ideals in $\mathcal{R}_c$, then the $\mathbb{Z}^d$-actions $\alpha_{(c, \mathfrak{I})}$ and $\alpha_{(c, \mathfrak{J})}$ are finitely equivalent, and they are algebraically conjugate if and only if $\mathfrak{I}$ and $\mathfrak{J}$ lie in the same ideal class.

For the proof of Theorem 3.8 we need two lemmas.

Lemma 3.11. Let $K'$ be an algebraic number field and let $S' \subset P(K')$ be a finite subset containing $P_\infty(K')$ with $d' = |S'| - 1 \geq 1$. We fix a fundamental $S'$-unit system $c = (c_1, \ldots, c_d)$ in $K'$ and write $K = \mathbb{Q}[c] \subset K'$ for the field of fractions of the integral domain $\mathbb{Z}[c^{\pm 1}]$.

If $K \neq K'$ then $K'$ is totally complex, $K$ is totally real and $[K': K] = 2$.

Proof. This is essentially [12, Proposition 3.5]. For every place $v'$ of $K'$ we denote by $v$ the restriction of $v'$ to $K \subset K'$, and we write $S = \{ v : v' \in S' \}$ for the resulting places of $K$.

Since the group of units $\mathcal{R}_c^\times$ of $\mathcal{R}_c = \mathcal{R}_{S'} \cap K$ contains the free abelian group generated by $\{ c_1, \ldots, c_d \}$, [12, Theorem 3.5] implies that $|S'| = |S|$, i.e. that restrictions to $K$ of distinct places of $K'$ yield distinct places of $K$. However, there is a priori no guarantee that the completions $K'_v$, and $K_v$ coincide for every $v' \in S'$.

Assume that there exists a nonempty subset $D \subset S$ such that $K_v \neq K'_v$ for every $v \in D$ and $K'_v = K_v$ for $v \in S \setminus D$. According to [12, Assertion (iii) on p.105] there exists a unit $a = c_1^{k_1} \cdots c_d^{k_d} \in \mathcal{R}_c^\times \subset \mathcal{R}_{S'}^\times$ with $(k_1, \ldots, k_d) \in \mathbb{Z}^d$, $|a|_v > 1$ for $v \in D$, and $|a|_v \leq 1$ for $v \in S \setminus D$. The product formula [3, Theorem 10.2.1], applied to $a \in K'$ and $a \in K$, implies that

$$\prod_{v \in S} |a|_v = \prod_{v' \in D} |a|_{v'} \cdot \prod_{v \in S \setminus D} |a|_v = 1,$$

$$\prod_{v \in S} |a|_{v'} = \prod_{v' \in D} |a|_{v'} \cdot \prod_{v \in S \setminus D} |a|_{v'} = \prod_{v \in S \setminus D} |a|_v = 1,$$

which is impossible due to the fact that $|a|_{v'} \geq |a|_v^d$ for every $v \in D$. This contradiction shows that $D = \emptyset$ or $D = S$.

In the former case $K_v = K'_v$ for every $v \in P_\infty(K)$ and $[K : \mathbb{Q}] = r + 2s = [K' : \mathbb{Q}]$, where $r$ and $s$ are the numbers of real and complex places of $K$, and hence $K = K'$. In the latter case $K'_v = \mathbb{C}$ and $K_v = \mathbb{R}$ for every $v \in P_\infty(K)$, $s = 0$, $[K' : \mathbb{Q}] = 2r = 2[K : \mathbb{Q}]$, and $[K : K'] = 2$.\hfill $\Box$

Lemma 3.12. Let $c = (c_1, \ldots, c_d)$ be a free $S'$-unit system of $K'$, and let $K = \mathbb{Q}[c] \subset K'$.

1. $K' = K$ if and only if the $\mathbb{Z}^d$-action $\alpha_{(c, \mathfrak{I})}$ on $X_3 = \mathfrak{I}$ is irreducible for every nonzero ideal $\mathfrak{I} \subset \mathcal{R}_{S'}$;

2. $K' \neq K$ if and only if the $\mathbb{Z}^d$-action $\alpha_{(c, \mathfrak{I})}$ on $X_3 = \mathfrak{I}$ is reducible for every nonzero ideal $\mathfrak{I} \subset \mathcal{R}_{S'}$. 
Proof. Suppose that $K = K'$, and that $\mathcal{I} \subset \mathcal{I}_c = \mathcal{I}_{S'}$ is a nonzero ideal. As the construction after (3.5) applies, Proposition 3.2 shows that the action $\alpha_{(c,\mathcal{I})}$ is irreducible.

If $K' \neq K$, Lemma 3.11 shows that the ring $\mathcal{R}_c = \mathcal{R}_{S'} \cap K$ is an infinite $\mathbb{Z}[e_{c}^\pm]$-submodule of infinite index in $\mathcal{R}_c$. Hence $Y = \mathcal{R}_c^\perp = \mathcal{R}_{S'}/\mathcal{R}_c$ is an infinite, closed, proper $\alpha_{(c,\mathcal{I}_{S'})}$-invariant subgroup of $X_{\mathcal{R}_{S'}}$. This implies that $\alpha_{(c,\mathcal{I}_{S'})}$ is reducible.

Let $\mathcal{I} \subset \mathcal{I}_{S'}$ be a nonzero ideal and $a \in \mathcal{I} \setminus \{0\}$. By the previous paragraph we see that $a\mathcal{R}_c$ has infinite index in $a\mathcal{R}_{S'}$ and therefore also in $\mathcal{I}$. Since $Y = (a\mathcal{R}_c)^\perp$ is an infinite, closed, proper $\alpha_{(c,\mathcal{I})}$-invariant subgroup, the action $\alpha_{(c,\mathcal{I})}$ is reducible.

Proof of Theorem 3.10. Lemma 3.11 shows that $K = K'$, and the assertions follow from the Lemmas 3.12 and Theorem 3.8.

Remarks 3.13. (1) Even if $K' \neq K$ in Lemma 3.11 for some choice of fundamental $S$-units $\{c_1, \ldots, c_d\}$, a different choice $\{c'_1, \ldots, c'_d\}$ of fundamental $S$-units may lead to equality of $K'$ and $K$.

(2) The most interesting special cases of the Theorems 3.8 and 3.10 occur when $S = P^{(K)}$, in which case $\mathcal{R}_c = \mathcal{O}_K$ is the ring of integers of $K$, $X_{\mathcal{R}_c} = \mathbb{T}^{r+s}$, and $\{c_1, \ldots, c_s\}$ is a fundamental unit system, where $r, s$ are the numbers of real and complex places of $K$ and $d = r + s - 1$. For information on ideal classes of algebraic number fields we refer to [12].

4. Algebraic actions of expansive rank one

Throughout this section $\alpha$ is an algebraic $\mathbb{Z}^d$-action on an infinite compact connected abelian group $X$. We discuss the relationship between irreducibility and the rank one conditions introduced in Definition 1.1. In the rank one case we give a description of the action $\alpha$ similar to Theorem 3.4.

In some of the following results we assume the irreducible $\mathbb{Z}^d$-actions to be expansive. By applying Lemma 2.1 to the case of an irreducible action $\alpha$ we see that $\alpha$ is expansive if and only if no Galois conjugate of $c$ consists entirely of elements of absolute value one. This is the same as saying there exists no valuation $v \in P^{(K)}$ with $|c_i|_v = 1$ for all $i$.

If one asks if a single automorphism $\alpha^n$ is expansive, the answer is slightly different (Lemma 2.1 might not be applicable directly as $M = \hat{X}$ might not be Noetherian over $\mathbb{Z}[u^{\pm n}]$). However, by using the product structure of $V_c$ in (3.6) it is easy to see that $\alpha^n$ is expansive if and only if $|c^n|_v \neq 1$ for all $v \in S$.

Proposition 4.1. Let $\alpha$ be an irreducible algebraic $\mathbb{Z}^d$-action on a compact connected abelian group $X$. Then $\alpha$ has entropy rank one.

If $\alpha$ is in addition expansive, then $\alpha$ has expansive rank one. In fact, there exists a finite union $U$ of hyperplanes in $\mathbb{R}^d$ such that $\alpha^n$ is expansive for every $n \in \mathbb{Z}^d \setminus U$. 
Proof. Let $c$ and $S_c$ be as in Theorem 3.4 and (3.5), respectively. The second proof of [15, Proposition 17.2] shows that

$$h_{\lambda_X}(\alpha^n) = \sum_{\|v\| \leq k} \log |c^n|_v < \infty$$

for every $n \in \mathbb{Z}^d$ (cf. (3.14)).

For every $v \in S_c$, we see that $\log |c^n|_v = \sum_{i=1}^d n_i \log |c_i|_v$. Let $U$ be the finite union of the hyperplanes defined by the linear functions $n \mapsto \log |c^n|_v$, $v \in S_c$. For $n \in \mathbb{Z}^d \setminus U$ we have that $|c^n|_v \neq 1$ for every $v \in S$ and hence that $\alpha^n$ is expansive.

Proposition 4.1 has a partial converse.

**Proposition 4.2.** Let $p \subset R_d$ be a prime ideal and $\alpha_{R_d/p}$ the algebraic $\mathbb{Z}^d$-action on the compact connected group $X_{R_d/p} = \overline{R_d/p}$ defined in (2.3). If $\alpha_{R_d/p}$ has expansive rank one or entropy rank one, then $\alpha_{R_d/p}$ is irreducible.

Proof. Suppose that $\alpha_{R_d/p}$ has expansive rank one and that $\alpha_{R_d/p}^n$ is expansive for some $n \in \mathbb{Z}^d$. It is easy to see that the $\mathbb{Z}[u^\pm]$-module $R_d/p$ is associated to the prime ideal $p \cap \mathbb{Z}[u^\pm]$. By Lemma 2.1, the variety $V_{C}(p)$ of $p \cap \mathbb{Z}[u^\pm]$ cannot meet the circle $\{z \in \mathbb{C} : |z| = 1\}$. This shows that $f \in p \cap \mathbb{Z}[u^\pm]$ for some nonzero polynomial $f$. Again by expansiveness, $R_d/p$ must be a Noetherian module over $\mathbb{Z}[u^\pm]$. Hence every $u_i + p \in R_d/p$ satisfies a monic relation over $\mathbb{Z}[u^n]$ and the variety (2.4) of $p$ is finite. Let $c \in \mathbb{Q}_x$ be a point of this variety. Then $p \subset p_c$ are two prime ideals with finite varieties (cf. (3.27)). If $p \neq p_c$ it is easy to find polynomials $f$ resp. $g$ which vanish at $V(p) \setminus V(p_c)$ resp. $V(p_c)$. As $f, g \notin p$ but $fg$ vanishes on $V(p)$ and therefore belongs to $p$, we get a contradiction to the fact that $p$ is a prime ideal. This shows that $\alpha_{R_d/p} = \alpha_{R_d/p_c}$ is irreducible.

Now assume $\alpha_{R_d/p}$ has entropy rank one. Let $n = e_i$ be one of the standard basis vectors of $\mathbb{Z}^d$. As $h(\alpha_{R_d/p})$ is finite, $p \cap \mathbb{Z}[u^\pm]$ cannot be trivial. For if $p \cap \mathbb{Z}[u^\pm] = \{0\}$, the shift $\sigma$ on $\mathbb{T}^\mathbb{Z}$ would be a factor of $\alpha_{R_d/p}$ and hence $h(\alpha_{R_d/p}) = \infty$. By applying this with $i = 1, \ldots, d$, we obtain that $V(p)$ is finite and conclude as above that $\alpha_{R_d/p} = \alpha_{R_d/p_c}$ is irreducible.

**Definition 4.3.** ([15, Definition 3.1]) Let $\alpha$ be an algebraic $\mathbb{Z}^d$-action on the compact abelian group $X$. The action $\alpha$ satisfies the descending chain condition if there exists, for every non-increasing sequence $X \supset X_1 \supset \cdots \supset X_k \supset \cdots$ of closed $\alpha$-invariant subgroups of $X$, an integer $K \geq 1$ such that $X_k = X_K$ for all $k \geq K$. Every expansive $\mathbb{Z}^d$-action on a compact abelian group satisfies the descending chain condition by [15, (4.10)].

**Theorem 4.4.** Let $\alpha$ be an algebraic $\mathbb{Z}^d$-action on an infinite compact connected abelian group $X$.

1. If $\alpha$ has expansive rank one it has entropy rank one.
2. If $\alpha$ has entropy rank one then the action $\alpha_{R_d/p}$ is irreducible for every associated prime ideal $p$ of $M = \hat{X}$.
3. If every associated prime ideal $p$ of $M = \hat{X}$ satisfies that $\alpha_{R_d/p}$ is irreducible and $\alpha$ is expansive, then $\alpha$ has also expansive rank one.
(4) Assume that \( \alpha \) satisfies the descending chain condition. If \( \alpha_{R_d/p} \) is irreducible for every associated prime ideal \( p \) of \( M = \hat{X} \) then \( \alpha \) has entropy rank one.

Proof. Assume \( \alpha \) has expansive rank one and \( p \) is an associated prime ideal of \( M = \hat{X} \). Then \( \alpha_{R_d/p} \) has also expansive rank one and \( \alpha_{R_d/p} \) is irreducible by Proposition 4.2. From Proposition 4.1 we know that \( \alpha_{R_d/p} \) has entropy rank one. This shows that \( h(\alpha^n_{R_d/p}) < \infty \) for all \( n \in \mathbb{Z}^d \) and every associated prime ideal \( p \). As \( M \) has a finite prime filtration (2.2) we get that \( h(\alpha^n) < \infty \) by Yuzvinsky’s addition formula (cf. [10] or [15, Theorem 14.1]) and hence that \( \alpha \) has entropy rank one.

If \( \alpha \) has entropy rank one and \( p \) is an associated prime ideal of \( M = \hat{X} \), then \( \alpha_{R_d/p} \) also has entropy rank one and Proposition 4.2 shows that \( \alpha_{R_d/p} \) is irreducible.

If the action \( \alpha_{R_d/p} \) is irreducible and expansive for every associated prime ideal, Proposition 4.1 shows that there exists, for every \( p \in \text{Asc}(M) \), a hyperplane \( U_p \subset \mathbb{R}^d \) such that \( \alpha^n_{R_d/p} \) is expansive for \( n \notin U_p \cap \mathbb{Z}^d \). As expansiveness implies that \( M \) is a Noetherian \( R_d \)-module, there are only finitely many associated prime ideals \( p \in \text{Asc}(M) \). Therefore there exists \( n \) such that \( \alpha^n_{R_d/p} \) is expansive for every associated prime ideal \( p \). From the finite filtration (2.2) it follows that \( \alpha^n \) is expansive (for prime ideals \( q \supseteq p \in \text{Asc}(M) \) the space \( X_{R_d/q} \) is finite by irreducibility and expansiveness is therefore trivial).

Finally we assume that \( \alpha_{R_d/p} \) is irreducible for every associated prime ideal \( p \) of \( M = \hat{X} \) and \( \alpha \) satisfies the descending chain condition. By duality the module \( M \) is Noetherian. Proposition 4.1 shows that \( \alpha_{R_d/p} \) has entropy rank one and we conclude as before from the filtration (2.2) and Yuzvinsky’s addition formula that \( \alpha \) has entropy rank one.

The remainder of this section will be devoted to a structure formula — analogous to Theorem 3.4 — for algebraic \( \mathbb{Z}^d \)-actions of expansive rank one. This structure formula will be needed in Section 5 in order to prove that any two expansive elements of an algebraic \( \mathbb{Z}^d \)-action of expansive rank one have isomorphic homoclinic modules.

For every rational prime \( p \) we write \( \mathbb{Z}_p = \{ x \in \mathbb{Q}_p : |x|_p \leq 1 \} \subset \mathbb{Q}_p \) for the ring of \( p \)-adic integers. For notational consistency we set \( \mathbb{Q}_\infty = \mathbb{R} \).

**Theorem 4.5.** Let \( \alpha \) be an algebraic \( \mathbb{Z}^d \)-action on an infinite compact connected abelian group \( X \) with dual module \( M = \hat{X} \). Assume that \( \alpha \) either has expansive rank one, or that \( \alpha \) has entropy rank one and satisfies the descending chain condition. Then there exists a finite set of primes \( S \subset \mathbb{N} \) with the following properties.

1. For every \( p \in \mathbb{S} \cup \{ \infty \} \) there exists a finite-dimensional vector space \( M_p \) over \( \mathbb{Q}_p \) and a \( \mathbb{Q}_p \)-linear \( \mathbb{Z}^d \)-action \( \beta_p \) on \( M_p \).
2. Let \( \beta = \prod_{p \in \mathbb{S} \cup \{ \infty \}} \beta_p \) be the product action on \( \hat{M} = \prod_{p \in \mathbb{S} \cup \{ \infty \}} M_p \). Then we can find a discrete, co-compact, \( \beta \)-invariant subgroup \( N \subset \hat{M} \) such that \( \alpha \) is algebraically conjugate to the \( \mathbb{Z}^d \)-action \( \beta \) induced by \( \beta \) on \( Y = \hat{M}/N \).
Theorem 4.5 turns out to be a consequence of the following proposition.

**Proposition 4.6.** Let $M$ be a Noetherian module over $R_d$ which is torsion-free over $\mathbb{Z}$. Assume furthermore that there exists a finite set of primes $S \subseteq \mathbb{N}$ such that $M \otimes \mathbb{Z}[\frac{1}{p}: p \in S]$ is a free module of finite rank over the ring $\mathbb{Z}[\frac{1}{p}: p \in S] = \mathbb{Z}[1/\prod_{p \in S} p]$. Then there exists, for every $p \in \mathbb{S} \cup \{\infty\}$, a finite-dimensional vector space $M_p$ over $\mathbb{Q}_p$ which is a $\mathbb{Q}_p$-linear $R_d$-module such that the following holds.

1. There is an injective module homomorphism

$$\phi: M \rightarrow V = \prod_{p \in \mathbb{S} \cup \{\infty\}} M_p.$$  

(4.1)

2. The image $\phi(M) \subset V$ is a discrete and co-compact subgroup.

**Proof of Theorem 4.5.** Given Proposition 4.6. Since $X$ is connected, the $R_d$-module $M = \hat{X}$ is a torsion-free additive group. Furthermore, since the action $\alpha_X$ is expansive or satisfies the descending chain condition, $M$ is Noetherian.

In order to choose the set $S$ we first consider the case where $M = R_d/\mathfrak{p}$ for some prime ideal $\mathfrak{p} = \mathfrak{p}_c \subset R_d$ of the form $(3.27)$ with $c = (c_1, \ldots, c_d) \in (\hat{\mathbb{Q}}^\times)^d$. Let $f_i \in \mathbb{Z}[u]$ be the primitive minimal polynomial of $c_i$, and let $S$ be the set of primes dividing the leading or trailing coefficients of any of the $f_i$, $i = 1, \ldots, d$. If $k_i$ is the degree of $f_i$, then any $c^m, m \in \mathbb{Z}^d$, belongs to the additive group generated by the set

$$\{a e^n : a \in \mathbb{Z}[\frac{1}{p}: p \in S],\quad n = (n_1, \ldots, n_d) \text{ with } 0 \leq n_i < k_i \text{ for } i = 1, \ldots, d\}.$$  

This shows that $R_d/\mathfrak{p} \otimes \mathbb{Z}[\frac{1}{p}: p \in S]$ is a free module of finite rank over the ring $\mathbb{Z}[\frac{1}{p}: p \in S]$.

In the case of a general Noetherian $R_d$-module $M$ we take the union of the sets $S = S_p$ of the preceding paragraph for every associated prime ideal $\mathfrak{p}$ of $M$. By using the finite filtration (2.2) and enlarging $S$, if necessary, to kill all finite factors $R_d/\mathfrak{q}$ arising from prime ideals $\mathfrak{q} \subset R_d$ which properly contain one of the associated primes of $M$, we have found a set $S$ satisfying the hypotheses of Proposition 4.6.

Let $M_p, p \in S$, and $\phi$ be chosen as in Proposition 4.6. As the dual group $\hat{M}_p$ is isomorphic to $\hat{Q}_p$, we can identify $M_p$ and $\hat{M}$ with their duals $\hat{M}_p$, resp. $\hat{M}$. For every $n \in \mathbb{Z}^d$ and $p \in S$ we define the $\mathbb{Q}_p$-linear map $\beta_p^n : M_p \rightarrow M_p$ as the dual of multiplication with $u^n$ on $M_p = \hat{M}_p$.

Let $N = \phi(M)^\perp \subset V = \hat{M} = \prod_{p \in \mathbb{S} \cup \{\infty\}} M_p$ be the annihilator of $\phi(M)$. As $\phi(M)$ is a discrete, co-compact submodule of $V$, $N$ is invariant under the action $\beta$ and is again discrete and co-compact. Furthermore,

$$\hat{M} \cong \phi(M) \cong V/\phi(M)^\perp = V/N,$$

and these isomorphisms conjugate the $\mathbb{Z}^d$-action $\alpha$ and $\beta$. \hfill $\Box$

**Proof of Proposition 4.6.** We set $M_\infty = M \otimes \mathbb{Z} \mathbb{R}$. Clearly, $M_\infty$ is a finite-dimensional vector space over $\mathbb{R}$ and a module over $R_d$, and multiplication
with $u_i \in R_d$ is $\mathbb{R}$-linear for $i = 1, \ldots, d$. Furthermore the homomorphism 
\( \phi_\infty : M \rightarrow M_\infty \), defined by $\phi_\infty(m) = m \otimes 1_\mathbb{R}$ for every $m \in M$, is injective.

For every prime $p \in S$, $V_p = M \otimes_{\mathbb{Z}} \mathbb{Q}_p$ is a finite-dimensional vector space over $\mathbb{Q}_p$ and a module over $\mathbb{Q}_p[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$. We are going to construct an invariant subspace $M_p \subset V_p$ which will be spanned by all expanding and contracting subspaces of the $\mathbb{Q}_p$-linear maps consisting of multiplication by $u_i, i = 1, \ldots, d$.

Let $A_j$ be the $\mathbb{Q}_p$-linear map on $V_p$ defined by multiplication with $u_j$. Since $A_j$ restricts also to the $\mathbb{Q}$-vector space $M \otimes_{\mathbb{Z}} \mathbb{Q}$, the minimal polynomial $\chi_j$ of $A_j$ can be chosen such that $\chi_j \in \mathbb{Z}[t]$ and $\chi_j$ is primitive; in particular, $p$ does not divide $\chi_j$. Hensel’s Lemma allows us to write $\chi_j$ as a product 
\( \chi_j(t) = f_j(t)g_j(t) \) with $f_j, g_j \in \mathbb{Z}_p[t]$, where $f_j$ has its leading and trailing coefficients in $\mathbb{Z}_p$ and $g_j \equiv t^{n_j} \pmod{p}$ for some $n_j$. Obviously $f_j, g_j \in \mathbb{Z}_p[t]$ are co-prime, hence

\[ a_j f_j + b_j g_j = 1 \]  

for some $a_j, b_j \in \mathbb{Q}_p[t]$. We can thus form the polynomial

\[ h(u_1, \ldots, u_d) = u^{-n} \prod_{j=1}^{d} g_j(u_j) \in \mathbb{Z}_p[u_1^{\pm 1}, \ldots, u_d^{\pm 1}], \]  

where $n \in \mathbb{Z}^d$ is chosen such that

\[ h \equiv 1 \pmod{p}. \]  

By taking the product of the Equations (4.2) for $j = 1, \ldots, d$, we obtain polynomials $a_j^*, b^* \in \mathbb{Q}_p[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ with

\[ 1 = \sum_{j=1}^{d} a_j^* f_j' + b^* h, \]  

where $f_j' = f_j(u_j) \in \mathbb{Z}_p[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$.

Since multiplication by each of the variables $u_i$ defines a $\mathbb{Q}_p$-linear map on $V_p$, the polynomials $h$ and $f_j'$ induce $\mathbb{Q}_p$-linear maps on $V_p$. By construction, the linear maps corresponding to the polynomials $\chi_j(u_j)$ and $hf_j'$ are equal to zero, and (4.5) implies that the subspaces

\[ M_p = \ker h, \quad N_p = \bigcap_{j=1}^{d} \ker f_j' \]

satisfy that $V_p = M_p \oplus N_p$; in fact, the projections

\[ \pi_1^{(p)} : V_p \rightarrow M_p, \quad \pi_2^{(p)} : V_p \rightarrow N_p \]

are given by

\[ \pi_1^{(p)}(x) = \sum_{j=1}^{d} a_j^* f_j' x \in M_p, \quad \pi_2^{(p)}(x) = b^* h x \in N_p. \]

Clearly, $M_p, N_p$ are $\mathbb{Q}_p[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$-submodules (and hence $R_d$-submodules) of $V_p$, and $\pi_1^{(p)}, \pi_2^{(p)}$ are module homomorphisms.
We set
\[ V = \prod_{p \in S \cup \{\infty\}} M_p, \]
write every \( v \in V \) as \( v = (v_p) = (v_p, p \in S \cup \{\infty\}) \), define \( \phi_p : M \to M_p \)
by \( \phi_p(m) = \pi_1^{(p)}(m \otimes 1_{Q_p}) \) for every \( p \in S \) and \( m \in M_p \), and consider the
\( R_d \)-module homomorphism \( \phi : M \to V \) given by
\[ \phi(m) = (\phi_p(m), p \in S \cup \{\infty\}) \]
for every \( m \in M \) (recall that \( \phi_\infty : M \to M_\infty \) was already defined at
the beginning of this proof). To conclude the proof we have to show that \( N = \phi(M) \) is a
discrete and co-compact subgroup of \( V \).

From the discussion concerning (3.8) we know that \( Z[l_p^1, p \in S] \) is a
discrete subgroup of \( \prod_{p \in S \cup \{\infty\}} Q_p \). As \( M \otimes Z[l_p^1, p \in S] \) is a free module
over the ring \( Z[l_p^1, p \in S] \) by hypothesis, the subgroup
\[ \{ (m \otimes 1_{Q_p}, p \in S \cup \{\infty\}) : m \in M \} \subset \prod_{p \in S \cup \{\infty\}} M \otimes Q_p \]
is discrete. We claim that, for fixed \( p \in S \), the set \( \pi_1^{(p)}(M \otimes 1_{Q_p}) \) is contained
in a compact subgroup \( C_p \subset N_p \). If we can prove this, then \( M \otimes 1_{Q_p} \subset \)
\( M \times C_p \) of every \( p \in S \), and since \( N = \phi(M) \) is obtained by projection
along the compact kernel \( \prod_{p \in S} C_p \), \( N \) must be a discrete subgroup of \( V \).

To verify this claim we choose \( L \geq 1 \) with \( \deg f_j \leq L \) for \( j = 1, \ldots, d \),
and select a set \( \{ m'_1, \ldots, m'_k \} \) of generators of the module \( M \) over \( R_d \). Put
\[ B = \left\{ u^n \pi_1^{(p)}(m'_i \otimes 1_{Q_p}) : 1 \leq i \leq k, 0 \leq n_j < L \text{ for all } 1 \leq j \leq d \right\} \]
and let \( C_p \subset M \otimes Q_p \) be the compact \( \mathbb{Z}_p \)-module generated by \( B \). Since each
\( f_j \) has leading and trailing coefficients in \( \mathbb{Z}_p^\times \) and annihilates \( N_p \) it follows
that \( \pi_1^{(p)}(M \otimes 1_{Q_p}) \) is contained in the \( \mathbb{Z}_p \)-module \( C_p \) generated by \( B \).

We proceed with the proof that the quotient \( V/N \) is compact. Let \( m_1, \ldots, m_r \in M \) be a basis of the \( \mathbb{Q} \)-vector space \( M \otimes_{\mathbb{Z}} \mathbb{Q} \). We assert that every
element of \( V/N \) can be represented by an element of the subgroup
\[ G = M_\infty \times \prod_{p \in S} \left( \sum_{i=1}^r \mathbb{Z}_p \phi_p(m_i) \right). \]
For this it is enough to show that any element \( v = (v_q, q \in S \cup \{\infty\}) \in V 
of the form
\[ v_q = \begin{cases} 0 & \text{for } q \neq p \\ \pi_1^{(p)}(m \otimes a) & \text{for } q = p \end{cases} \quad (4.6) \]
with \( p \in S, m \in M \) and \( a \in Q_p \), can be represented modulo \( N \) by an element in
\( G \).

Multiplying \( m \) and dividing \( a \) in (4.6) by the same integer does not change \( v \). This allows us to assume without loss of generality that \( m \) lies in the
subgroup of \( M \) generated by \( \{ m_1, \ldots, m_r \} \).

Let \( j \geq 0 \) be such that \( p^{j+1} a \in \mathbb{Z}_p \). We denote by \( h \) the polynomial in (4.3)
and assume without loss in generality (by multiplying \( m \) with an integer
co-prime to \( p \) that
\[
 u^n m \in \sum_{i=1}^{r} \mathbb{Z}\left[ \frac{1}{p^i} \right] m_i
\]
for every \( n \in \mathbb{Z}^d \) for which \( u^n \) has a nonzero coefficient in \( (1 - h)^j \) (i.e. with \( (1 - h)^j n \neq 0 \) in the notation of (2.1)). Since \( v \in G \) for \( a \in \mathbb{Z}_p \) we may restrict ourselves to the case where \( a = \frac{1}{p^j} \) for some \( j \geq 1 \).

We fix \( k \geq 1 \) for the moment and choose \( h(k) \in R_d \) with \( h(k)n = h_n \mod p^k \) for every \( n \in \mathbb{Z}^d \).

According to (4.4), \( \frac{1-h(k)}{p^j} \in R_d \). Since \( h \) annihilates \( M_p \),
\[
 \phi_p(\alpha m) = a \phi_p(m) = \frac{1}{p^i} \phi_p(m) = \frac{(1-h)^j}{p^i} \phi_p(m)
\]
\[
 = \phi_p \left( \left( \frac{1-h(k)}{p} \right)^j m \right) + \frac{(1-h)^j - (1-h(k))^j}{p^i} \phi_p(m).
\]
For \( k \geq j \) large enough we get cancellation with the denominators in (4.7) and therefore
\[
 y_p = \frac{(1-h)^j - (1-h(k))^j}{p^i} \phi_p(m) \in \sum_{i=1}^{r} \mathbb{Z}_p \phi_p(m_i).
\]
For \( q \neq p \) (4.7) shows that
\[
 y_q = -\phi_q \left( \left( \frac{1-h(k)}{p} \right)^j m \right) \in \sum_{i=1}^{r} \mathbb{Z}_q \phi_q(m_i),
\]
and hence that \( v = \phi \left( \left( \frac{1-h(k)}{p} \right)^j m \right) + y \) with \( y \in G \).

We have shown that
\[
 V/N \cong G/(G \cap N).
\]
Although \( G \) is not compact, the quotient \( G/H \) is compact, where \( H \subset G \cap N \) is the subgroup generated by \( \{ \phi(m_i) : i = 1, \ldots, k \} \). Hence \( V/N \) is compact.

5. Homoclinic points for actions of expansive rank one

In this section we study the groups of homoclinic points for different expansive elements of an algebraic \( \mathbb{Z}^d \)-action of expansive rank one. The results in this section generalize the results for commuting toral automorphism in [11].

Let \( \alpha \) be an algebraic \( \mathbb{Z}^d \)-action on a compact abelian group \( X \). We define the groups \( \Delta_\alpha(X) \) and \( \Delta_{\alpha^n}(X) \) of homoclinic points of \( \alpha \) and \( \alpha^n \), \( n \in \mathbb{Z}^d \), as in the Introduction. Both \( \Delta_\alpha(X) \) and \( \Delta_{\alpha^n}(X) \) are \( R_d \)-modules with respect to the operation
\[
 (f, x) \mapsto f \cdot x = f(\alpha)(x) = \sum_{m \in \mathbb{Z}^d} f_m \alpha^m x
\]
for \( f = \sum_{m \in \mathbb{Z}^d} f_m \alpha^m \in R_d \) and \( x \in \Delta_\alpha(X) \) or \( x \in \Delta_{\alpha^n}(X) \).

If an algebraic \( \mathbb{Z}^d \)-action \( \alpha \) with \( d^d \geq 2 \) has expansive rank one, then \( h(\alpha) = 0 \) and \( \alpha^* \) is trivial. Following [4, Section 6] one can, however, use an individual expansive element \( \alpha^n \) of the action \( \alpha \) to obtain the \( R_d \)-module
\[ \Delta_{\alpha^n}(X) \] and again define an associated \( \mathbb{Z}^d \)-action \( \beta = \alpha_{\Delta_{\alpha^n}(X)} \) on \( Y = \Delta_{\alpha^n}(X) \). In this section we investigate how this construction depends on the choice of \( n \), under the hypothesis that the group \( X \) is connected.

We begin our discussion in the case where \( \alpha \) is irreducible and mixing.

**Theorem 5.1.** Let \( d \geq 1 \), and let \( \alpha \) be an expansive, irreducible and mixing algebraic \( \mathbb{Z}^d \)-action on an infinite compact connected abelian group \( X \). Let \( \alpha'_{(c, \mathcal{K})} \) be a realization of \( \alpha \) as in Corollary 3.5, where \( c = (c_1, \ldots, c_d) \in (\bar{\mathbb{Q}}^\times)^d \), \( K = \mathbb{Q}[c] = \mathbb{Q}[c_1, \ldots, c_d] \), \( \mathcal{S}_c \) is defined in (3.5), and \( \mathcal{K} \subset \mathcal{R}_c \) is a finitely generated \( \mathbb{Z}[c^{\pm 1}] \)-submodule.

For every \( v \in \mathcal{S}_c \) we define the group homomorphism \( \omega_v : n \mapsto \log |c^n|_v \) from \( \mathbb{Z}^d \) to \( \mathbb{R} \) and set \( H_v = \{ n \in \mathbb{Z}^d : \omega_v(n) = 0 \} \). Then

\[
E_\alpha = \{ m \in \mathbb{Z}^d : \alpha^m \text{ is expansive} \} = \mathbb{Z}^d \setminus \bigcup_{v \in \mathcal{S}_c} H_v.
\]

Furthermore the following is true.

1. If \( m \in \mathbb{Z}^d < E_\alpha \), then \( \Delta_{\alpha^m}(X) = \{ 0 \} \).
2. If \( m \in E_\alpha \), then \( \Delta_{\alpha^m}(X) \) is dense in \( X \) and isomorphic to \( \mathcal{K} \) as a module over \( R_d \).
3. If \( m, n \in E_\alpha \), then

\[
\Delta_{\alpha^m}(X) \cap \Delta_{\alpha^n}(X) = \begin{cases} 
\Delta_{\alpha^m}(X) & \text{if } \omega_v(m)\omega_v(n) > 0 \text{ for every } v \in \mathcal{S}_c, \\
\{ 0 \} & \text{if } \omega_v(m)\omega_v(n) < 0 \text{ for every } v \in \mathcal{S}_c, \\
\end{cases}
\]

4. Let \( m \in E_\alpha \), and let \( \alpha^* = \alpha_{\Delta_{\alpha^m}(X)} \) be the algebraic \( \mathbb{Z}^d \)-action on \( X^* = \Delta_{\alpha^m}(X) \) defined by (2.3). Then \( E_\alpha = E_{\alpha^*} \), and the homoclinic module \( \Delta_{(\alpha^*)^p}(X) \) of any \( (\alpha^*)^n \) with \( n \in \mathbb{Z}^d = E_{\alpha^*} \) is isomorphic to \( \hat{\mathcal{K}} \).

**Proof.** Define \( V_c \) by (3.6) and (3.5) and choose a \( \mathbb{Z}[c^{\pm 1}] \)-submodule \( \mathcal{K} \subset \mathcal{R}_c \) according to Corollary 3.5. Since \( \alpha \) is algebraically conjugate to the \( \mathbb{Z}^d \)-action \( \alpha'_{(c, \mathcal{K})} \) on \( X'_\mathcal{K} = V_c/\iota_c(\mathcal{K}) \) defined by (3.28)–(3.29) we assume for simplicity that \( X = X'_\mathcal{K} \) and \( \alpha = \alpha'_{(c, \mathcal{K})} \).

For every \( n \in \mathbb{Z}^d \) we denote by

\[
W_n^u = \{ (a_v, v \in \mathcal{S}_c) \in V_c : a_v = 0 \text{ for every } v \in \mathcal{S}_c \text{ with } |c^n|_v > 1 \},
\]

\[
W_n^s = \{ (a_v, v \in \mathcal{S}_c) \in V_c : a_v = 0 \text{ for every } v \in \mathcal{S}_c \text{ with } |c^n|_v < 1 \}
\]

the unstable and stable subspaces of \( \tilde{\mathcal{K}}_c^n \) in \( V_c \) (cf. (3.15)).

If \( n \neq 0 \), then \( c^n \) is not a root of unity due to our assumption that \( \alpha \) is mixing, and Kronecker’s theorem (cf. [8]) guarantees that \( |c^n|_v \neq 1 \) for some \( v \in \mathcal{S}_c \). The product formula [3, Theorem 10.2.1] shows that \( W_n^u \neq \{ 0 \} \) and \( W_n^s \neq \{ 0 \} \), since \( \prod_{v \in \mathcal{S}_c} |c^n|_v = 1 \). As \( \iota_c(\mathcal{K}) \) is a discrete subgroup of \( V_c \) and thus has no contracting automorphisms, \( \iota_c(\mathcal{K}) \cap W_n^u = \iota_c(\mathcal{K}) \cap W_n^s = \{ 0 \} \). Hence the quotient map \( \pi : V_c \to X = V_c/\iota_c(\mathcal{K}) \) is injective on the subspaces \( W_n^u \) and \( W_n^s \) of \( V_c \).

First we assume that \( n \in E_\alpha \). In this case \( W_n^u + W_n^s = V_c \) and there exists, for every \( a \in \mathcal{K} \), a unique point \( w_a \in V_c \) with \( W_n^u \cap (W_n^s + \iota_c(a)) = \{ w_a \} \).
and \( \pi(w_a) \in \Delta_{\alpha^n}(X) \). Conversely, if \( y \in \Delta_{\alpha^n}(X) \) and \( w \in \pi^{-1}(\{y\}) \), then the continuity of \( \beta^n \) implies that there exist points \( a, a' \in \mathcal{K} \) with
\[
\lim_{k \to \infty} \beta^n_k(w - \iota_c(a)) = \lim_{k \to \infty} \beta^n_k(w - \iota_c(a')) = 0,
\]
and hence with \( w \in (W_{a^n} + \iota_c(a)) \cap (W_{a^n} + \iota_c(a')) \) or, equivalently, \( w = w_{a^n} \). This shows that \( \Delta_{\alpha^n}(X) \cong \mathcal{K} \). The isomorphism \( \Delta_{\alpha^n}(X) \cong \mathcal{K} \) is obviously an \( R_{\alpha^n} \)-module isomorphism, as claimed in (2).

The closure \( Z = \overline{\Delta_{\alpha^n}(X)} \) of the group of \( \alpha^n \)-homoclinic points in \( X \) is a closed, infinite, \( \alpha \)-invariant subgroup of \( X \), and hence \( Z = X \) by irreducibility. For the proof of (4) we observe that
\[
X^* = \hat{\mathcal{K}} \cong \iota_c(\mathcal{K}) = V_c / \iota_c(\mathcal{K})^\perp,
\]
and that \( \alpha^* \) is algebraically conjugate to the quotient action on \( V_c / \iota_c(\mathcal{K})^\perp \) induced by \( \beta \); here \( \iota_c(\mathcal{K})^\perp \) is the annihilator of \( \iota_c(\mathcal{K}) \) in \( V_c \). As above we see that \( \Delta_{(\alpha^*)^p} \cong \iota_c(\mathcal{K})^\perp \) and hence that \( \hat{X} \cong \iota_c(\mathcal{K})^\perp \in \mathcal{K} \). This isomorphism carries the dual action \( \hat{\alpha} \) of \( \mathbb{Z}^d \) on \( X \) to \( \Delta_{\alpha^*} \) on \( \hat{X} \) as claimed in (4).

For (3) we use the same argument as in [11]. If \( m, n \in E_\alpha \) satisfy that \( \omega_v(m)\omega_v(n) > 0 \) for every \( v \in S \), then \( W_{a^m} = W_{a^n} \), \( W_{a^m} = W_{a^n} \), and the homoclinic points of \( \alpha^m \) and \( \alpha^n \) coincide.

If \( \omega_v(m)\omega_v(n) < 0 \) for every \( v \in S \), then \( W_{a^m} = W_{a^n} \) and \( W_{a^m} = W_{a^n} \), and the homoclinic points of \( \alpha^m \) and \( \alpha^n \) again coincide.

If \( m \) and \( n \) satisfy neither of these conditions then we can interchange \( m \) and \( n \) if necessary, and assume that \( W_{a^m} + W_{a^n} \neq V_c \), \( W_{a^m} + W_{a^n} \neq V_c \), \( W_{a^m} + W_{a^n} \neq V_c \).

If a point \( x \in X \) is homoclinic both for \( \alpha^m \) and \( \alpha^n \) then there exist elements \( w \in V_c \) and \( a_1, a_2, a_3 \in \mathcal{K} \) with \( \pi(w) = x \) and
\[
\{w\} = W_{a^m} \cap (W_{a^n} - \iota_c(a_1)) = (W_{a^n} - \iota_c(a_2)) \cap (W_{a^n} - \iota_c(a_3)).
\]
Hence
\[
\iota_c(a_2) \in W_{a^m} + W_{a^n} = \bigoplus_{v \in S \wedge |\mathbf{e}^v| = 1} K_v \bigoplus_{v \in S \wedge |\mathbf{e}^v| > 1} K_v = V_c
\]
and from the definition of \( \iota_c(a_2) \) in (3,6) we conclude that \( a_2 = 0 \). Similarly we see that \( a_1 = a_3 = 0 \), since
\[
\iota_c(a_3) \in W_{a^m} + W_{a^n} \neq V_c, \quad \iota_c(a_1) \in W_{a^n} + W_{a^n} \neq V_c.
\]
This shows that \( w \in W_{a^m} \cap W_{a^n} = \{0\} \) and \( \Delta_{\alpha^m}(X) \cap \Delta_{\alpha^n}(X) = \{0\} \).

Finally we note that \( \Delta_{\alpha^n}(X) = \{0\} \) if \( \alpha^n \) is nonexpansive. Indeed, if \( n \not\in E_\alpha \), then \( |\mathbf{e}^v| = 1 \) for some \( v \in S \). If \( x \in \Delta_{\alpha^n}(X) \) and \( w \in \pi^{-1}(\{x\}) \) then \( w \in W_{a^n} \cap (W_{a^n} - \iota_c(a)) \) for some \( a \in \mathcal{K} \) with \( a \in W_{a^n} \cap W_{a^n} \). However, \( W_{a^n} + W_{a^n} \neq V_c \), and the same argument as in the last paragraph shows that \( a = 0 \), \( w \in W_{a^n} \cap W_{a^n} = \{0\} \), and \( x = \pi(w) = 0 \). \( \square \)

In Theorem 5.1 we have used implicitly the following terminology from [4].

**Definition 5.2.** Let \( \alpha \) be an algebraic \( \mathbb{Z}^d \)-action on an infinite compact connected abelian group \( X \) of expansive rank one. The **adjoint action** of \( \alpha \) is the algebraic \( \mathbb{Z}^d \)-action \( \alpha^* = \alpha_{|\Delta_{\alpha}(X)} \) on the compact connected abelian
group $X^* = \hat{\Delta}_{\alpha}^m(X)$ in (2.3) determined by the $R_d$-module $\Delta_{\alpha}^m(X)$ for some (and hence, by Theorem 5.1 or Theorem 5.5 below, for any) expansive automorphism $\alpha^m, \ m \in \mathbb{Z}^d$.

According to Theorem 5.1 (4), the adjoint action $\alpha^*$ of an algebraic $\mathbb{Z}^d$-action $\alpha$ on an infinite compact connected abelian group $X$ of expansive rank one again has expansive rank one and the compact group $X^*$ is infinite and connected. We define the second adjoint action $\alpha^{**} = (\alpha^*)^*$ of $\alpha$ as before and obtain the following corollary.

**Corollary 5.3.** Let $\alpha$ be an irreducible algebraic $\mathbb{Z}^d$-action on a compact connected abelian group $X$. Then the second adjoint $\alpha^{**}$ of $\alpha$ on $X^{**} = \hat{\Delta}_{\alpha^{**}}(X)$ is algebraically conjugate to $\alpha$.

**Proof.** Theorem 5.1 (4) shows that the dual modules $\hat{X}$ and $\hat{X}^{**} = \hat{\Delta}_{(\alpha^*)^*}(X)$ with $n \in E_\alpha = E_{\alpha^*}$ are isomorphic, which proves that $\alpha$ and $\alpha^{**}$ are algebraically conjugate. \qed

In the setting of Theorem 3.8 the adjoint action $\alpha^*$ can be described explicitly. The proof is left to the reader as an exercise.

**Corollary 5.4.** Suppose that $K$ is an algebraic number field, $e \in (K \times)^d$ a vector of nonzero algebraic numbers with $K = \mathbb{Q}(e)$, and let $S_e \subset \mathbb{P}(K)$ be the set of places defined by (3.5). For every nonzero ideal $\mathcal{I} \subset \mathcal{R}_e$ the adjoint action $\alpha_{(e,\mathcal{I})}^*$ of the irreducible algebraic $\mathbb{Z}^d$-action $\alpha_{(e,\mathcal{I})}$ described in Theorem 3.8 is of the form $\alpha_{(e,\mathcal{I})}^* = \alpha_{(e^{-1},\mathcal{I}^{-1})}$, where $\mathcal{I}^{-1} = \{a \in K : a\mathcal{I} \subset \mathcal{R}_e\}$.

We end this section with a discussion of the module of homoclinic points for expansive elements in a general algebraic $\mathbb{Z}^d$-action of expansive rank one. The description of such actions in Theorem 4.5 allows us to use almost exactly the same argument as in the irreducible case.

**Theorem 5.5.** Let $\alpha$ be an algebraic action on an infinite compact connected abelian group $X$ of expansive rank one. The homoclinic modules

$$\Delta_{\alpha^m} \cong \Delta_{\alpha^n}$$

are isomorphic as $R_d$-modules for any two expansive automorphisms $\alpha^m, \alpha^n$, $m, n \in \mathbb{Z}^d$.

**Proof.** By Theorem 4.5 we may assume that $\alpha$ is the $\mathbb{Z}^d$-action induced by $\beta$ on $Y = \tilde{M}/\mathcal{N}$ for some discrete, co-compact, $\beta$-invariant subgroup $\mathcal{N} \subset \tilde{M}$. Here $\tilde{M}$ is a product of finite-dimensional vector spaces $M_p$ over $\mathbb{Q}_p$ for $p \in S \cup \{\infty\}$, where $S$ is a finite set of rational primes.

Fix $n \in \mathbb{Z}^d$ such that $\alpha^n$ is expansive. We claim that $\Delta_{\alpha^n} \cong \mathcal{N}$.

By expansiveness, $\tilde{M}$ is a sum of the stable and unstable subgroups

$$W^s = \{v \in \tilde{M} : \beta^k v \to 0 \text{ as } k \to \infty\},$$

$$W^u = \{v \in \tilde{M} : \beta^{-k} v \to 0 \text{ as } k \to \infty\},$$

$$\tilde{M} = W^s \oplus W^u.$$ 

If $a \in \mathcal{N}$ we let $w_a \in \tilde{M}$ be the element uniquely determined by $w_a \in W^s \cap \{a\} + W^u$. Then $x_a = w_a + \mathcal{N}$ is a homoclinic point for $\alpha$. Conversely, if $x \in \Delta_{\alpha^n}$ is homoclinic and $w \in \tilde{M}_c$ is a point with $w + \mathcal{N} = x$, then there exists $a_1, a_2 \in \mathcal{N}$ such that $w - a_1 \in W^s$ and $w - a_2 \in W^u$. In other words,
\[ x = x_{a_2 - a_1}. \] It can easily be checked that the map \( a \mapsto x_a \) is an \( R_d \)-module isomorphism.

The proof of the following corollary is completely analogous to that of Theorem 5.1 (4) and Corollary 5.3.

**Corollary 5.6.** Let \( \alpha \) be an algebraic \( \mathbb{Z}^d \)-action on an infinite compact connected abelian group \( X \) of expansive rank one with adjoint action \( \alpha^* \) on \( X^* \). Then \( \alpha^* \) has expansive rank one, and the adjoint action \( \alpha^{**} \) of \( \alpha^* \) is algebraically conjugate to \( \alpha \).

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