On a Noncommutative Deformation of the Connes–Kreimer Algebra

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Abstract

We study a noncommutative deformation of the commutative Hopf algebra $\mathcal{H}_R$ of rooted trees which was shown by Connes and Kreimer to describe the mathematical structure of renormalization in quantum field theories. The requirement of the existence of an antipode for the noncommutative deformation leads to a natural extension of the algebra. Noncommutative deformations of $\mathcal{H}_R$ might be relevant for renormalization of field theories on noncommutative spaces and there are indications that in this case the extension of the algebra should be linked to a mixing of infrared and ultraviolet divergences. We give also an argument that for a certain class of noncommutative quantum field theories renormalization should be linked to a noncommutative and noncooperative self-dual Hopf algebra which can be seen as a noncommutative counterpart of the Grothendieck-Teichmüller group.
1 Introduction

The Hopf algebra $\mathcal{H}_R$ of rooted trees (see [CK]) can be seen as the abstract mathematical structure behind the renormalization scheme employed by physicists in quantum field theory (see [CK] and the literature cited there). The study of quantum field theories on noncommutative spaces has started in recent years (see [CDP 1998], [CDP 2000], [GMS], [MSSW], [Oec]) but the question of renormalization of such theories is still an open problem. Since the Hopf algebra $\mathcal{H}_R$ - by the nature of an algebra of polynomials - is commutative, it is natural to ask if noncommutative deformations of $\mathcal{H}_R$ could be the proper algebraic setting for renormalization of field theories on noncommutative spaces. This is the motivation for studying a very simple noncommutative deformation of $\mathcal{H}_R$, here. If this deformation would indeed be linked to renormalization on noncommutative spaces, we find indications that a mixing of infrared and ultraviolet divergences should occur, as has been observed in examples (see [MRS]). We focus, here, on the algebraic properties of the deformation and give only a brief discussion of a possible realization of the deformed algebra in terms of $q$-integrals and the shift and particle number operators.

Besides the question of renormalization, a second motivation for the study of noncommutative deformations of $\mathcal{H}_R$ comes from considerations on trialgebraic deformations of Hopf algebras and a noncommutative and nonco-commutative Hopf algebra $\mathcal{H}_{GT}$ taking the role which the Grothendieck-Teichmüller group plays for quasitensor categories, there (see [Sch]). In section 3 we give an argument from the algebraic properties of noncommutative quantum field theories that for a certain class of such theories renormalization should, indeed, be expected to be linked to $\mathcal{H}_{GT}$.

2 The deformation of $\mathcal{H}_R$

For a rooted tree $t$ as defined in [CK], let $v(t)$ be the number of vertices of $t$. We make the convention that for the unit element $e$ of $\mathcal{H}_R$

$$v(e) = 0$$

For a monomial of rooted trees, the number of vertices $v$ is, of course, defined as the sum of the vertex numbers of its factors. Let $q \neq 1$ be a complex
A deformation parameter. We introduce a deformation $\mathcal{H}_{R,q}$ of $\mathcal{H}_R$ by replacing
commutativity, i.e.

$$t_1 t_2 = t_2 t_1$$

for rooted trees by the condition

$$t_1 t_2 = q^{v(t_2) - v(t_1)} t_2 t_1$$

(1)

In addition, we require that $e$ is invertible, i.e. there exists $e^{-1}$ with

$$e^{-1} e = e, \quad e^{-1} = e^{-1} = e$$

for all rooted trees $t$. Clearly, this defines a deformation of the associative
algebra structure of $\mathcal{H}_R$. We have to study now how this deformation effects
the other structural elements involved in $\mathcal{H}_R$. Observe, first, that (1) involves
a deformation of the unit element $e$, too, since

$$e t = q^{v(t)} e$$

(2)

for an arbitrary rooted tree $t$. Let us take a look at the coproduct $\Delta$, next.
Remember that $\Delta$ can be defined for a rooted tree $t$ as (see [CK])

$$\Delta(t) = e \otimes t + t \otimes e + \sum_C P^C(t) \otimes R^C(t)$$

(3)

where the sum is taken over admissible cuts $C$ of $t$ and $R^C(t)$ is the rooted
tree containing the root of $t$ while $P^C(t)$ denotes the complementary monomial
of rooted trees.

Lemma 1 $\Delta$ defines a coproduct also for the deformed algebra as given by
(1).

Proof. We have to check that (1) is consistent with $\Delta$, i.e. we have to check that

$$\Delta(t_1) \Delta(t_2) = q^{v(t_2) - v(t_1)} \Delta(t_2) \Delta(t_1)$$

for rooted trees $t_1, t_2$. 

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From (3) it follows that

\[ \Delta (t_1) \Delta (t_2) \]

\[ = (e \otimes t_1) (e \otimes t_2) + (e \otimes t_1) (t_2 \otimes e) + (e \otimes t_1) \left( \sum C P^C (t_2) \otimes R^C (t_2) \right) \]

\[ + (t_1 \otimes e) (e \otimes t_2) + (t_1 \otimes e) (t_2 \otimes e) + (t_1 \otimes e) \left( \sum C P^C (t_2) \otimes R^C (t_2) \right) \]

\[ + \left( \sum C P^C (t_1) \otimes R^C (t_1) \right) (e \otimes t_2) + \left( \sum C P^C (t_1) \otimes R^C (t_1) \right) (t_2 \otimes e) \]

\[ + \left( \sum C P^C (t_1) \otimes R^C (t_1) \right) \left( \sum C P^C (t_2) \otimes R^C (t_2) \right) \]

Using the fact that

\[ v (t) = v \left( P^C (t) \right) + v \left( R^C (t) \right) \]

for any rooted tree \( t \) and equation (2), the desired result follows. \( \blacksquare \)

The counit \( \varepsilon \) is defined as

\[ \varepsilon (t) = 0 \]

for \( t \neq e \) and

\[ \varepsilon (e) = 1 \]

in[CK].

**Lemma 2** The counit \( \varepsilon \) is compatible with (1).

**Proof.** By calculation. \( \blacksquare \)

**Corollary 3** \( \mathcal{H}_{R,a} \) has the structure of a nonunital bialgebra. \( \blacksquare \)

**Remark 1** By results of [GeSch], there exists a deformation of \( \mathcal{H}_R \) equivalent to \( \mathcal{H}_{R,a} \) which leaves \( e \) fixed, too, i.e. there exists an equivalent deformation of \( \mathcal{H}_R \) into a full bialgebra (here, equivalence of deformations refers to the deformation of the associative product and the coassociative coproduct).
Finally, let us consider the question of the existence of an antipode. Suppose, \( S \) would be an antipode for \( \mathcal{H}_{R,\gamma} \). Since an antipode is always an algebra antihomomorphism (see e.g. [KS]), it follows from (2) that

\[
S(t)S(\varepsilon) = q^{\nu(t)}S(\varepsilon)S(t)
\]

for any rooted tree \( t \). Suppose now that \( q \neq -1 \). By the definition of \( \mathcal{H}_{R,\gamma} \), this implies that \( S(\varepsilon) \) would have to involve monomials of rooted trees with a larger number of vertices than any fixed monomial of rooted trees which is, obviously, a contradiction. So, there can not exist an antipode on \( \mathcal{H}_{R,\gamma} \) for \( q \neq -1 \).

**Remark 2** For \( q = -1 \) no contradiction does arise, here, because \( q = q^{-1} \), then. It is interesting to observe that the algebraic deformation theory of \( \mathcal{H}_R \) immediately seems to mirror the fact that the renormalization scheme, described by \( \mathcal{H}_R \), can be generalized without problems to the supersymmetric setting.

Equation (4) also points a way to a partial solution of the problem how to introduce an antipode in the deformed case. In a certain sense, we simply have to consider \( \mathcal{H}_{R,\gamma} \) and \( \mathcal{H}_{R,\gamma^{-1}} \) at once. Denote rooted trees in \( \mathcal{H}_{R,\gamma^{-1}} \) by \( \tilde{t} \) to distinguish them from those of \( \mathcal{H}_{R,\gamma} \), and let \( \tilde{\mathcal{H}}_{R,\gamma} \) be the nonunital bialgebra generated from

\[
\mathcal{H}_{R,\gamma} \oplus \mathcal{H}_{R,\gamma^{-1}}
\]

with the relations

\[
q^{\nu(t_1)}t_1\tilde{t}_2 = \tilde{t}_1t_2 = q^{-\nu(t_2)}\tilde{t}_1t_2
\]

imposed. Equation (5) assures that the counit and coproduct defined separately for \( \mathcal{H}_{R,\gamma} \) and \( \mathcal{H}_{R,\gamma^{-1}} \) can be consistently combined into a single counit and coassociative coproduct for \( \tilde{\mathcal{H}}_{R,\gamma} \).

Denote now by \( S \) the antipode of \( \mathcal{H}_R \) which is defined as (see [CK]):

\[
S(\varepsilon) = \varepsilon
\]

and

\[
S(t) = -t - \sum CS_P(t)\varepsilon^{-1}R_P(t)
\]

for any rooted tree \( t \).
Observe that we have inserted the element \( e^{-1} \) on the right hand side of (6). This is not necessary for the case of \( \mathcal{H}_R \), where \( e \) is the unit, but we have to keep track of this appearance of \( e^{-1} \) in the noncommutative case.

Define \( S_\gamma \) on \( \hat{\mathcal{H}}_{R,\gamma} \) by

\[
S_\gamma(t) = \hat{S}(t)
\]

and

\[
S_\gamma(\hat{t}) = \hat{S}(t)
\]

i.e. \( S_\gamma \) leads to an interchange of \( \mathcal{H}_{R,\gamma} \) and \( \mathcal{H}_{R,\gamma^{-1}} \).

**Lemma 4** \( S_\gamma \) defines a left antipode for \( \hat{\mathcal{H}}_{R,\gamma} \).

**Proof.** We have (denoting the multiplication by the product by \( m \))

\[
m \left[(S_\gamma \otimes id) \Delta(t)\right] = \hat{e} \hat{t} - e \hat{t} e - \sum_C \hat{S}(P_c(t)) e \hat{S}(R_c(t)) e + \sum_C \hat{S}(P_c(t)) R_c(t)
\]

\[
= q^{\gamma(t)} e \hat{t} - e \hat{t} e - \sum_C \hat{S}(P_c(t)) e \hat{S}(R_c(t)) e + \sum_C q^{\gamma(R_c(t))} \hat{S}(P_c(t)) R_c(t)
\]

\[
= 0 = \varepsilon(t)
\]

where we have used the definition of \( e^{-1} \) and equation (5). \( \blacksquare \)

On the other hand, we have

\[
m \left[(id \otimes S_\gamma) \Delta(t)\right] = \hat{t} \hat{e} - e \hat{t} - \sum_C \hat{S}(P_c(t)) e \hat{S}(R_c(t))
\]

\[
= \hat{e} \hat{t} - e \hat{t} - \sum_C q^{-\gamma(P_c(t))} \hat{S}(P_c(t)) \hat{e} \hat{S}(R_c(t)) + \sum_C q^{-\gamma(P_c(t))} \hat{S}(P_c(t)) \hat{S}(R_c(t))
\]

\[
= \sum_C q^{-\gamma(P_c(t))} \left( \hat{P}_c(t) \hat{S}(R_c(t)) - \hat{S}(P_c(t)) \hat{R}_c(t) \right)
\]

which is non vanishing for \( q \neq 1 \). So, \( S_\gamma \) does not define a right antipode and, therefore, not an antipode for \( \hat{\mathcal{H}}_{R,\gamma} \).

On the other hand, one has the following result:
Lemma 5 There is a deformation of $\mathcal{H}_R$ equivalent (in the sense of deformations of the associative product and coassociative coproduct) to $\mathcal{H}_{R,0}$ which carries a full Hopf algebra structure.

Proof. As we remarked already above, there is a deformation equivalent to $\mathcal{H}_{R,0}$ which is a unital bialgebra. But then - since $\mathcal{H}_R$ is a Hopf algebra - an antipode exists for this deformation (see [CP]).

It is, in general, not a straightforward task to explicitly construct the equivalent deformations given in existence results of the kind above. So, having an explicitly given left antipode, one could try to work some way with this. The most prominent feature of $S_q$ is the interchange of $\mathcal{H}_{R,0}$ and $\mathcal{H}_{R,0^{-1}}$. In the renormalization scheme the antipode corresponds to the construction of counter terms for subdivergences of integrals. One notes that for the $q$-integrals as defined in the setting of $q$-calculus (which often gives a good toy model for the truly noncommutative situation) the exchange

$$q \longleftrightarrow q^{-1}$$

leads - modulo a constant term and a constant scaling factor - to a formal exchange of $q$-integrals

$$\int_{\epsilon}^{\infty} d_q f \longleftrightarrow \int_{0}^{\epsilon} d_{q^{-1}} f$$

(see e.g. [KS] for the definition of the $q$-integral), i.e. if the left antipode introduced above is relevant for renormalization on noncommutative spaces, this might be a hint that a renormalization scheme based on $S_q$ should involve a mixing of ultraviolet and infrared divergences.

This suggests that we can - at least in a formal sense - realize the algebra $\widetilde{\mathcal{H}}_{R,0}$ as follows: For an element of the component $\mathcal{H}_{R,0}$, we write the corresponding integral of $\mathcal{H}_R$ but understand all integrals to be replaced by $q$-integrals and we assume that the formal variable of integration of the outer integral $q$-commutes with the variables of integration of all the subintegrals (which all commute with each other), i.e. for an $n$-fold integral we write

$$\int dy \, dx_1 \ldots dx_{n-1}$$

where all the $dx_i$ commute with each other and for $i = 1, \ldots, n - 1$

$$dy \, dx_i = q \, dx_i \, dy$$
It is easy to check that
\[
\int dy \, dx_1 \ldots dx_{n-1} \int dy \, dx_1 \ldots dx_{m-1} = q^{m-n} \int dy \, dx_1 \ldots dx_{m-1} \int dy \, dx_1 \ldots dx_{n-1}
\]
which is just the exchange rule required by the algebra $\mathcal{H}_{R,\beta}$. For the elements of $\mathcal{H}_{R,\beta^{-1}}$, we use the corresponding integrals with $q$ replaced by $q^{-1}$ (i.e. using integrals $\int_0^\infty ...$ instead of $\int_0^\infty ...$). The use of the $q$-integral gives the correct exchange between the components $\mathcal{H}_{R,\beta}$ and $\mathcal{H}_{R,\beta^{-1}}$ and since the $q$-integral is a kind of discrete approximation to the usual integral, the qualitative structure of the singularities should not change. By the nature of $S_\beta$, counter terms always alternate between the ultraviolet and the infrared case. More concretely, we put the same function into the integrand as in the toy model in [CK] (i.e. $\frac{1}{x+c}$ for the single integral, $\frac{1}{(x_1 + \cdots + x_2)}$ for the double one, etc.) but replace the coordinate $x_i$ by $\frac{1}{x_i}$ in every second $q$-integral. So, this should be a formal realization of $\mathcal{H}_{R,\beta}$.

Using the representation for the coordinates $x, y$ of the Manin plane on an infinite dimensional space with basis $\{ |n \rangle, n \in \mathbb{N} \}$, by
\[
y |n \rangle = |n + 1 \rangle
\]
and
\[
x |n \rangle = q^n |n \rangle
\]
we get the following interpretation of the above integrals: Each subintegral involves - besides the $q$-integration - an application of the shift operator which is quite natural if we view the rooted trees as quantum objects, now, because with moving up the rooted tree the number of vertices, taken into consideration in the integration process so far, increases by one (i.e. which the shift recognizes by increasing the “particle number” by one). For the outermost integral, we take the $q$-integration, again, but then simply apply the particle number operator in an exponentiated form (there is no more possibility to shift and we simply “measure” the final particle number, now).

It would certainly be interesting to see a more detailed physical realization of such a model.

**Remark 3** In terms of the generators $\delta_n$ (see [CK]) the deformation of the Connes-Kreimer algebra we have given above can be written as follows:
\[
\delta_n \delta_m = q^{m-n} \delta_m \delta_n
\]
With
\[ \delta_n = x y^n \]
and \( x, y \) as above, we get a representation of the deformed Connes-Kreimer algebra in terms of the generators.

3 Bialgebra categories and noncommutative QFT

In this section, we discuss some general algebraic properties of noncommutative quantum field theories (i.e. quantum field theories on noncommutative spaces, henceforth ncQFTs, for short) and give an abstract argument why a noncommutative and noncocommutative Hopf algebra - albeit a much more complicated one than the simple toy model \( \mathcal{H}_{R,a} \) - should be linked to renormalization of such theories.

Formally, the fields of a ncQFT can be seen as functions \( \Phi \) with values
\[ \Phi (\mathbf{i}, \tilde{x}) \]
in a noncommutative algebra \( B \) and the variables in another noncommutative algebra \( A \) (see e.g. [CDP 1998]). In general, these functions will not be linear, especially, they are not restricted to the class of algebra morphisms.

Suppose now that both algebras satisfy the Poincare-Birkhoff-Witt property and can therefore be seen as arising from star-products. By [CF], [Kon], both algebras can then be seen as arising from two dimensional conformal field theories, i.e. - using the algebraic formulation of low dimensional QFTs - we have two quasitensor categories \( \mathcal{A} \) and \( \mathcal{B} \), respectively. The maps \( \Phi \) then correspond to functors \( \mathcal{F} \) from \( \mathcal{A} \) to \( \mathcal{B} \) but since the maps are, in general, not algebra morphisms, these functors will, in general, not preserve the quasitensor structure. Since we have a multiplicative structure on both, \( \mathcal{A} \) and \( \mathcal{B} \), a suitable class of functors \( \mathcal{F} \) is endowed with the structure of a bialgebra category in the sense of [CrFr] (much the same way a suitable class of complex valued functions on a group is - by inducing the multiplication from the codomain pointwise - endowed with the structure of a Hopf algebra) where a bialgebra category is, roughly speaking, a monoidal category with a compatible functorial version of a coproduct (for the precise definition,
see the cited paper). So, a certain class of ncQFTs will have an algebraic formulation in terms of bialgebra categories. We will call such ncQFTs “of bialgebra category type” and write bncQFT for them, for short.

We start our argument on renormalization and bncQFTs by noting a property of the moduli spaces (in the sense of formal deformation theory of algebraic structures, where we will by a moduli space of a structure always mean the connected component, only) of quasitensor categories defined from two dimensional conformal field theories.

Lemma 6 The moduli space of a quasitensor category \(C\) which is defined from a two dimensional conformal field theory is always equivalent to the moduli space generated from the braiding and associativity morphism, alone.

Proof. Since we consider only the connected component, we can consider those parts of the structure which are discretely parametrized as fixed (e.g. the linear structures on the homomorphism classes remains fixed; this is completely analogous to the case of formal deformations of an associative algebra where only the product is deformed but the underlying linear space - since it is discretely parametrized by dimension - remains fixed). So, the remaining structures which can be deformed are the composition, the tensor product, the braiding, and the associativity morphism for the tensor product.

Now, the moduli space of a two dimensional conformal field theory is locally parametrized by observables of the theory itself and these, in turn, are in one to one correspondence to the states of the theory. But then - since states have to be taken to states by the deformation - every sufficiently small deformation can be described as a functor on \(C\). Consequently, the deformations of the composition can always be absorbed into the other three structures. With the same argument, the deformations of the tensor product can be assumed to be of the kind of a twist and therefore can be absorbed into deformations of the associativity morphism. Hence, what remains are deformations of the braiding and the associativity morphism. ■

In the original definition of Drinfeld (see [Dri]), the Grothendieck-Teichmüller group \(GT\) is defined from formal deformations of the braiding and the associativity morphism of a quasitensor category. Since renormalization is understood as an action on a formal moduli space of QFTs, the above lemma immediately suggests that the renormalization group flow induces on the
space of two dimensional conformal field theories (part of) the flow generated by the half group counterpart (see [Dri]) of \( GT \) (since by the above lemma there is no other freedom in deforming \( C \) than the transformations used in Drinfeld’s definition). The principle idea we use, here, is just the treatment of the moduli space of QFTs of Wilson as a formal moduli space (in the sense of the formal deformation theory of algebraic structures) of the algebraic formulation of low dimensional QFTs.

Remark 4 The decisive use of the state versus local observable correspondence of two dimensional conformal field theories shows that we cannot necessarily expect this conclusion on a link between the half group counterpart of \( GT \) and the renormalization group flow to hold for the general QFT case.

In analogy to [Dri], one of us has introduced in [Sch] from the possible formal transformations of a braiding and associativity morphism on a bicategory, plus the two corresponding dual structures for the functorial coproduct, a noncommutative and noncocommutative self-dual Hopf algebra \( \mathcal{H}_{GT} \). In complete analogy to the quasitensor category case, one has the following lemma, then.

Lemma 7 The moduli space of a braided and cobraided bialgebra category, arising from a bcncQFT in the way described at the beginning of this section, is always equivalent to the moduli space generated from the braiding and associativity morphism and the corresponding dual structures, alone.

Proof. Completely analogous to the above case.  

In conclusion, one expects that renormalization of a bcncQFT has to be linked to the Hopf algebra \( \mathcal{H}_{GT} \). So, there is evidence from the general algebraic properties of ncQFTs, too, that renormalization of such theories should be linked to a noncommutative and noncocommutative Hopf algebra. The fact that \( \mathcal{H}_{GT} \) is self-dual means that it is a much more noncommutative object than the usual quantum group examples. The simple deformation of \( \mathcal{H}_R \) which we studied in the previous section, can therefore only expected to be linked to very simple toy models. For a physically realistic class of ncQFTs we have to expect a much more complicated Hopf algebra structure.
4 Conclusion

We have shown the existence of a noncommutative deformation of the Hopf algebra of Connes and Kreimer. A left antipode was explicitly constructed while the existence of a full antipode was only given by an abstract argument. Surely, the study of deformations of the Connes-Kreimer algebra and their possible relation to renormalization of quantum field theories on noncommutative spaces deserves further study. Even if such an approach turns out to be relevant to renormalization on noncommutative spaces, it is in no way clear if there exists a canonical noncommutative deformation of the Connes-Kreimer algebra taking this role or if different deformations correspond to renormalization schemes for different families of field theories on noncommutative spaces. The arguments in section 3 show that one should expect $\mathcal{H}_{GT}$ to be linked to the class of bncQFTs.

We conclude by one additional remark. The appearance of powers of $q$ given by a certain index (here: the number of vertices) as the deformation factors of products and the deformation of the unit element are two features of $\mathcal{H}_{R,q}$ which are well known to readers who are acquainted with trialgebraic deformations of quantum groups (basically, an aneved deformation quantization of quantum groups, see [GS2000a], [GS2000b]). Since trialgebras are linked to the symmetry properties of field theories on noncommutative spaces (see [GS2000c]), the deformation theory of $\mathcal{H}_{R}$ seems nicely to fit in as another element of the structural properties of such theories.

Acknowledgements:

H.G. was supported under project P11783-PHY of the Fonds zur Förderung der wissenschaftlichen Forschung in Österreich. K.G.S. thanks the Deutsche Forschungsgemeinschaft (DFG) for support by a research grant and the Erwin Schrödinger Institute for Mathematical Physics, Vienna, for hospitality. Besides this, very special thanks go to Cesar Gomez who posed the question to us if anybody had ever considered the possibility if noncommutative deformations of the Connes-Kreimer algebra might be related to renormalization of quantum field theories on noncommutative spaces and thereby directly stimulated the present research.
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