The Dual Horospherical Radon Transform for Polynomials

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1 Introduction

A Radon transform is generally associated to a double fibration

\[ \begin{array}{c}
   P \\
   q \\
   X \\
   \overrightarrow{Z} \\
   \overrightarrow{Y},
\end{array} \]

where one may assume without loss of generality that the maps \( p \) and \( q \) are surjective and \( Z \) is embedded into \( X \times Y \) via \( z \mapsto (p(z), q(z)) \). Let some measures be chosen on \( X, Y, Z \) and on the fibers of \( p \) and \( q \) so that

\[ \int_X \left( \int_{p^{-1}(x)} f(u) du \right) dx = \int_Z f(z) dz = \int_Y \left( \int_{q^{-1}(y)} f(v) dv \right) dy. \tag{1} \]

Then the Radon transform \( \mathcal{R} \) is the linear map assigning to a function \( \varphi \) on \( X \) the function on \( Y \) defined by

\[ (\mathcal{R}\varphi)(y) = \int_{q^{-1}(y)} (p^* \varphi)(v) dv, \]

where we have set \( p^* \varphi := \varphi \circ p \). In a dual fashion, one defines a linear transform \( \mathcal{R}^* \) from functions on \( Y \) to functions on \( X \) via

\[ (\mathcal{R}^* \psi)(x) = \int_{p^{-1}(x)} (q^* \psi)(u) du. \]

It is dual to \( \mathcal{R} \). Indeed, formally,

\[ (\mathcal{R} \varphi, \psi) := \int_Y \left( \int_{q^{-1}(y)} (p^* \varphi)(v) dv \right) \psi(y) dy \\
= \int_Y \left( \int_{q^{-1}(y)} (p^* \varphi)(q^* \psi)(v) dv \right) dy \\
= \int_Z (p^* \varphi)(z)(q^* \psi)(z) dz \]

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\[
\int_{X} \varphi(x) \left( \int_{\rho^{-1}(x)} (q^* \psi)(u) \, du \right) \, dx =: (\varphi, R^* \psi).
\]

In particular, if \( X = G/K \) and \( Y = G/H \) are homogeneous spaces of a Lie group \( G \), one can take \( Z = G/(K \cap H) \), with \( p \) and \( q \) being \( G \)-equivariant maps sending \( (K \cap H) \) to \( eK \) and \( eH \), respectively. Assume that there exist \( G \)-invariant measures on \( X \), \( Y \) and \( Z \). If such measures are fixed, one can uniquely define measures on the fibers of \( p \) and \( q \) so that the condition (1) holds. (Here, speaking about a measure on a smooth manifold, we mean a measure defined by a differential form of top degree.) In this setting we can consider the transforms \( R \) and \( R^* \), and if \( \dim X = \dim Y \), one can hope that they are invertible. The basic example is provided by the classical Radon transform, which acts on functions on the Euclidean space \( \mathbb{R}^n = (\mathbb{R}^n \times SO(n))/SO(n) \) and maps them to functions on the space \( H \mathbb{R}^n = (\mathbb{R}^n \times SO(n))/(\mathbb{R}^{n-1} \times O(n-1)) \) of hyperplanes in \( \mathbb{R}^n \).

For a semisimple Riemannian symmetric space \( X = G/K \) of noncompact type, one can consider the horospherical Radon transform as proposed by I.M. Gelfand and M.I. Graev in [GG59]. Namely, generalizing the classical notion of a horosphere in Lobachevsky space, one can define a horosphere in \( X \) as an orbit of a maximal unipotent subgroup of \( G \). The group \( G \) naturally acts on the set \( \text{Hor} X \) of all horospheres. This action is transitive, so we can identify \( \text{Hor} X \) with some quotient space \( G/S \) (see §5 for details). It turns out that \( \dim X = \dim \text{Hor} X \). Moreover, the groups \( G, K, S \) and \( K \cap S = M \) are unimodular, so there exist \( G \)-invariant measures on \( X \), \( \text{Hor} X \) and \( Z = G/M \). The Radon transform \( R \) associated to the double fibration

\[
\begin{array}{ccc}
G/M & \xrightarrow{p} & X = G/K \\
& \searrow & \\
& & \text{Hor} X \xrightarrow{q} G/S
\end{array}
\]

is called the horospherical Radon transform.

The space \( Z = G/M \) can be interpreted as the set of pairs \( (x, \mathcal{H}) \in X \times \text{Hor} X \) with \( x \in \mathcal{H} \), so that \( p \) and \( q \) are just the natural projections. The fiber \( p^{-1}(\mathcal{H}) \) with \( \mathcal{H} \in \text{Hor} X \) is then identified with the horosphere \( \mathcal{H} \), and the fiber \( \rho^{-1}(x) \) with \( x \in X \) is identified with the submanifold \( \text{Hor}_x X \subset \text{Hor} X \) of all horospheres passing through \( x \). Note that, in contrast to the horospheres, all submanifolds \( \text{Hor}_x X \) are compact, since \( \text{Hor}_x X \) is the orbit of the stabilizer of \( x \) in \( G \), which is conjugate to \( K \).

In this paper, we describe the dual horospherical Radon transform \( R^* \) in terms of its action on polynomial functions. Here a differentiable function \( \varphi \) on a homogeneous space \( Y = G/H \) of a Lie group \( G \) is called polynomial, if the linear span of the functions \( g \varphi \) with \( g \in G \) is finite dimensional. The polynomial functions constitute an algebra denoted by \( \mathbb{R}[Y] \).

For \( X = G/K \) as above, the algebra \( \mathbb{R}[X] \) is finitely generated and \( X \) is naturally identified with a connected component of the corresponding affine real algebraic variety (the real spectrum of \( \mathbb{R}[X] \)). The natural linear representation of \( G \) in \( \mathbb{R}[X] \) decomposes into a sum of mutually non-isomorphic absolutely irreducible finite dimensional representations whose highest weights \( \lambda \) form a semigroup \( \Lambda \). Let \( \mathbb{R}[X]_\lambda \) be the irreducible component of \( \mathbb{R}[X] \) with highest weight \( \lambda \), so

\[
\mathbb{R}[X] = \bigoplus_{\lambda \in \Lambda} \mathbb{R}[X]_\lambda.
\]

Denote by \( \varphi_\lambda \) the highest weight function in \( \mathbb{R}[X]_\lambda \) normalized by the condition

\[
\varphi_\lambda (e) = 1,
\]
where $o = eK$ is the base point of $X$. Then the subgroup $S$ is the intersection of the stabilizers of all $\varphi_\lambda$‘s. Its unipotent radical $U$ is a maximal unipotent subgroup of $G$.

The algebra $\mathbb{R}[\text{Hor} X]$ is also finitely generated. The manifold $\text{Hor} X$ is naturally identified with a connected component of a quasi-affine algebraic variety, which is a Zariski open subset in the real spectrum of $\mathbb{R}[\text{Hor} X]$. The natural linear representation of $G$ in $\mathbb{R}[\text{Hor} X]$ is isomorphic to the representation of $G$ in $\mathbb{R}[X]$. Let $\mathbb{R}[\text{Hor} X]_\lambda$ be the irreducible component of $\mathbb{R}[\text{Hor} X]$ with highest weight $\lambda$, so that

$$
\mathbb{R}[\text{Hor} X] = \bigoplus_{\lambda \in \Lambda} \mathbb{R}[\text{Hor} X]_\lambda.
$$

Denote by $\psi_\lambda$ the highest weight function in $\mathbb{R}[\text{Hor} X]_\lambda$ normalized by the condition

$$
\psi_\lambda(sU o) = 1,
$$

where $s$ is the symmetry with respect to $o$ and $sU o = (sUs^{-1})o$ is considered as a point of $\text{Hor} X$.

The decomposition (2) defines a filtration of the algebra $\mathbb{R}[X]$ (see §4 for the precise definition). Let $\text{gr} \mathbb{R}[X]$ be the associated graded algebra. There is a canonical $G$-equivariant algebra isomorphism

$$
\Gamma : \text{gr} \mathbb{R}[X] \rightarrow \mathbb{R}[\text{Hor} X]
$$

mapping each $\varphi_\lambda$ to $\psi_\lambda$. As a $G$-module, $\text{gr} \mathbb{R}[X]$ is canonically identified with $\mathbb{R}[X]$, so we can view $\Gamma$ as a $G$-module isomorphism from $\mathbb{R}[X]$ to $\mathbb{R}[\text{Hor} X]$.

The horospherical Radon transform $\mathcal{R}$ is not defined for polynomial functions on $X$ but its dual transform $\mathcal{R}^*$ is defined for polynomial functions on $\text{Hor} X$, since it reduces to integrating along compact submanifolds. Moreover, as follows from the definition of polynomial functions, it maps polynomial functions on $\text{Hor} X$ to polynomial functions on $X$. Obviously, it is $G$-equivariant. Thus, we have $G$-equivariant linear maps

$$
\mathbb{R}[X] \xrightarrow{\Gamma} \mathbb{R}[\text{Hor} X] \xrightarrow{\mathcal{R}^*} \mathbb{R}[X].
$$

Their composition $\mathcal{R}^* \circ \Gamma$ is a $G$-equivariant linear operator on $\mathbb{R}[X]$, so

$$
(\mathcal{R}^* \circ \Gamma)(\varphi) = c_\lambda \varphi \quad \forall \varphi \in \mathbb{R}[X]_\lambda,
$$

where the $c_\lambda$ are constants. To give a complete description of $\mathcal{R}^*$, it is therefore sufficient to find these constants. Our main result is the following theorem.

**Theorem 1** $c_\lambda = \mathfrak{c}(\lambda + \rho)$, where $\mathfrak{c}$ is the Harish-Chandra $c$-function and $\rho$ is the half-sum of the positive roots of $X$ (counted with multiplicities).

The Harish-Chandra $c$-function governs the asymptotic behavior of the zonal spherical functions on $X$. A product formula for the $c$-function was found by S.G. Gindikin and F.I. Karpelevich [GK62]: for a rank-one Riemannian symmetric space of the noncompact type, the $c$-function is a ratio of gamma functions involving only the root multiplicities; in the general case, it is the product of the $c$-functions for the rank-one symmetric spaces defined by the indivisible roots of the space. Thus, known the root structure of the symmetric space, the product formula makes the $c$-function, and hence our description of the dual horospherical Radon transform, explicitly computable.

For convenience of the reader, we collect some crucial facts about the $c$-function in an appendix to this paper.

The following basic notation will be used in the paper without further comments.
• Lie groups are denoted by capital Latin letters, and their Lie algebras by the corresponding small Gothic letters.

• The dual space of a vector space $V$ is denoted by $V^*$.

• The complexification of a real vector space $V$ is denoted by $V(\mathbb{C})$.

• The centralizer (resp. the normalizer) of a subgroup $H$ in a group $G$ is denoted by $Z_G(H)$ (resp. $N_G(H)$).

• The centralizer (resp. the normalizer) of a subalgebra $\mathfrak{h}$ in a Lie algebra $\mathfrak{g}$ is denoted by $Z_{\mathfrak{g}}(\mathfrak{h})$ (resp. $N_{\mathfrak{g}}(\mathfrak{h})$).

• If a group $G$ acts on a set $X$, we denote by $X^G$ the subset of fixed points of $G$ in $X$.

2 Groups, spaces, and functions

For any connected semisimple Lie group $G$ admitting a faithful (finite-dimensional) linear representation, there is a connected complex algebraic group defined over $\mathbb{R}$ such that $G$ is the connected component of the group of its real points. Among all such algebraic groups, there is a unique such that all the others are its quotients. It is called the complex hull of $G$ and is denoted by $G(\mathbb{C})$. The group of real points of $G(\mathbb{C})$ is denoted by $G(\mathbb{R})$. If $G(\mathbb{C})$ is simply connected, then $G = G(\mathbb{R})$.

The restrictions of polynomial functions on the algebraic group $G(\mathbb{R})$ to $G$ are called polynomial functions on $G$. They are precisely those differentiable functions $\varphi$ for which the linear span of the functions $g\varphi, g \in G$, is finite dimensional (see e.g., [CSM95], §II.8). (Here $G$ is supposed to act on itself by left multiplications.) The polynomial functions on $G$ constitute an algebra which we denote by $\mathbb{R}[G]$ and which is naturally isomorphic to $\mathbb{R}[G(\mathbb{R})]$.

For any subgroup $H \subseteq G$, we denote by $H(\mathbb{C})$ (resp. $H(\mathbb{R})$) its Zariski closure in $G(\mathbb{C})$ (resp. $G(\mathbb{R})$). If $H$ is Zariski closed in $G$, i.e. $H = H(\mathbb{R}) \cap G$, then $H$ is a subgroup of finite index in $H(\mathbb{R})$; if $H$ is a semidirect product of a connected unipotent group and a compact group, then $H = H(\mathbb{R})$.

For a homogeneous space $Y = G/H$ with $H$ Zariski closed in $G$, set $Y(\mathbb{C}) = G(\mathbb{C})/H(\mathbb{C})$. This is an algebraic variety defined over $\mathbb{R}$, and $Y$ is naturally identified with a connected component of the variety $Y(\mathbb{R})$ of real points of $Y(\mathbb{C})$. We call $Y(\mathbb{C})$ the complex hull of $Y$.

If $H$ is reductive, then also $H(\mathbb{C})$ is reductive and the variety $Y(\mathbb{C})$ is affine, the algebra $\mathbb{C}[Y(\mathbb{C})]$ being naturally isomorphic to the algebra $\mathbb{C}[G(\mathbb{C})]^H(\mathbb{C})$ of $H(\mathbb{C})$-right-invariant polynomial functions on $G(\mathbb{C})$ (see e.g., [VP89], Section 4.7 and Theorem 4.10). Correspondingly, the algebra $\mathbb{R}[Y(\mathbb{R})]$ is naturally isomorphic to $\mathbb{R}[G(\mathbb{R})]^H(\mathbb{R})$.

In general, the functions on $Y(\mathbb{R})$ arising from $H(\mathbb{R})$-right-invariant polynomial functions on $G(\mathbb{R})$ are called polynomial functions on $Y(\mathbb{R})$, and their restrictions to $Y$ are called polynomial functions on $Y$. They are precisely those differentiable functions $\varphi$ for which the linear span of the functions $g\varphi, g \in G$, is finite dimensional. They form an algebra which we denote by $\mathbb{R}[Y]$.

In the following we consider a semisimple Riemannian symmetric space $X = G/K$ of non-compact type. This means that $G$ is a connected semisimple Lie group without compact factors and $K$ is a maximal compact subgroup of $G$. We do not assume that the center of $G$ is trivial, so the action of $G$ on $X$ may be non-effective. We do, however, require that $G$ has a faithful linear representation. According to the above, the space $X$ is then a connected component of the affine algebraic variety $X(\mathbb{R})$. 

4
3 Subgroups and subalgebras

We recall some facts about the structure of Riemannian symmetric spaces of noncompact type (see [Hel78] for details). Let \( X = G/K \) be as above and \( \theta \) be the Cartan involution of \( G \) with respect to \( K \), so \( K = G^\theta \). Let \( a \) be a Cartan subalgebra for \( X \), i.e. a maximal abelian subalgebra in the \((-1)\)-eigenspace of \( \theta \). Its dimension \( r \) is called the rank of \( X \). Under any representation of \( G \), the elements of \( a \) are simultaneously diagonalizable. The group \( A = \exp a \) is a maximal connected abelian subgroup of \( G \) such that \( \theta(a) = a^{-1} \) for all \( a \in A \). It is isomorphic to \((\mathbb{R}^r)^\times \).

Its Zariski closure \( A(\mathbb{R}) \) in \( G(\mathbb{R}) \) is a split algebraic torus which is isomorphic to \((\mathbb{R}^+)^r \). Let \( X(A) \) denote the (additively written) group of real characters of the torus \( A(\mathbb{R}) \). It is a free abelian group of rank \( r \). We identify each character \( \chi \) with its differential \( d\chi \in a^* \).

The root decomposition of \( g \) with respect to \( A \) (or with respect to \( A(\mathbb{R}) \), which is the same) is of the form

\[
g = g_0 + \sum_{\alpha \in \Delta} g_\alpha,
\]

where \( g_\alpha = z_\alpha(a) \). If \( m = z_e(a) \), then \( g_0 = m + a \).

The set \( \Delta \subset X(A) \) is the root system of \( X \) (or the restricted root system of \( G \)) with respect to \( A \) and \( g_\alpha \) is the root subspace corresponding to \( \alpha \). The dimension of \( g_\alpha \) is called the multiplicity of the root \( \alpha \) and is denoted by \( m_\alpha \). By the identification of a character with its differential, we will consider \( \Delta \) as a subset of \( a^* \). Choose a system \( \Delta^+ \) of positive roots in \( \Delta \). Let \( \Pi = \{ \alpha_1, \dots, \alpha_r \} \subset \Delta^+ \) be the corresponding system of simple roots. Then

\[
C = \{ x \in a : \alpha_i(x) \geq 0 \text{ for } i = 1, \ldots, r \}
\]

is called the Weyl chamber with respect to \( \Delta^+ \). The subspace

\[
u = \sum_{\alpha \in \Delta^+} g_\alpha
\]

is a maximal unipotent subalgebra of \( g \).

Set

\[
G_0 = Z_G(A), \quad M = Z_K(A).
\]

Then \( G_0 = M \times A \) and the Lie algebras of \( G_0 \) and \( M \) are \( g_0 \) and \( m \), respectively. Clearly, \( G_0 \) is Zariski closed in \( G \).

The group \( U = \exp \nu \) is a maximal unipotent subgroup of \( G \). It is normalized by \( A \) and the map

\[
U \times A \times K \rightarrow G, \quad (u, a, k) \mapsto uk,
\]

is a diffeomorphism. The decomposition \( G = UAK \) (or \( G = KAU \)) is called the Iwasawa decomposition of \( G \). Since every root subspace is \( G_0 \)-invariant, \( G_0 \) normalizes \( U \), so

\[
P := UG_0 = U \times G_0
\]

is a subgroup of \( G \). Moreover, \( P = N_G(U) \) (see e.g. [War72], Proposition 1.2.3.4), so \( P \) is Zariski closed in \( G \).

We say that a Zariski closed subgroup of \( G \) is parabolic, if its Zariski closure in \( G(\mathbb{C}) \) is a parabolic subgroup of \( G(\mathbb{C}) \). Then \( P \) is a minimal parabolic subgroup of \( G \). The subgroup

\[
S := UM = U \times M
\]

of \( P \) is normal in \( P \) and \( P/S \) is isomorphic to \( A \). It follows from the Iwasawa decomposition that \( K \cap S = M \).
4 Representations

For later use we collect some well-known facts about finite-dimensional representations of $G$.

The natural linear representation of $G$ in $\mathbb{R}[X]$ decomposes into a sum of mutually non-isomorphic absolutely irreducible finite-dimensional representations called (finite-dimensional) spherical representations (see e.g. [GW98], chap. 12).

**Theorem 2** (see [Hel84], § V.4) An irreducible finite dimensional representation of $G$ on a real vector space $V$ is spherical if and only if the following equivalent conditions hold:

\begin{enumerate}
  \item $V^K \neq \{0\}$.
  \item $V^S \neq \{0\}$.
\end{enumerate}

If these conditions hold, then $\dim V^K = \dim V^S = 1$ and the subspace $V^S$ is invariant under $P$.

For a spherical representation, the group $P$ acts on $V^S$ via multiplication by some character of $P$ vanishing on $S$. The restriction of this character to $A$ is called the highest weight of the representation. A spherical representation is uniquely determined by its highest weight. The highest weights of all irreducible spherical representations constitute a subsemigroup $\Lambda \subset X(A)$.

An explicit description of $\Lambda$ as a subset of $a^*$ can be given as follows. Let $\Delta_s$ denote the set of roots $\alpha \in \Delta$ such that $2\alpha \notin \Delta$. Then $\Delta_s$ is a root system in $a^*$, and a system of simple roots corresponding to $\Delta_s^+ := \Delta^+ \cap \Delta_s$ can be obtained from the system $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ of simple roots in $\Delta^+$ by setting for $j = 1, \ldots, r$,

$$\beta_j := \begin{cases} a_j & \text{if } 2a_j \notin \Delta^+, \\ 2a_j & \text{if } 2a_j \in \Delta^+. \end{cases}$$

Let $\omega_1, \ldots, \omega_r \in a^*$ be defined by

$$\frac{[\omega_j, \beta_k]}{[\beta_j, \beta_k]} = \delta_{ij},$$

where $(\cdot, \cdot)$ denotes the scalar product in $a^*$ induced by an invariant scalar product in $g$.

**Proposition 3** (see [Hel94], Proposition 4.23) The semigroup $\Lambda$ is freely generated by $\omega_1, \ldots, \omega_r$. \hfill $\Box$

For $\lambda \in \Lambda$, let $\mathbb{R}[X]_{\lambda}$ denote the irreducible component of $\mathbb{R}[X]$ with highest weight $\lambda$. Then we have

$$\mathbb{R}[X] = \bigoplus_{\lambda \in \Lambda} \mathbb{R}[X]_{\lambda}.$$

We denote by $\varphi^S_\lambda$ the highest weight function of $\mathbb{R}[X]_{\lambda}$ normalized by the condition

$$\varphi^S_\lambda(0) = 1.$$

(Since $P_0 = PK_0 = Go = X$, we have $\varphi^S_\lambda(0) \neq 0$.) Obviously,

$$\varphi^S_\lambda \varphi^S_\mu = \varphi^S_{\lambda + \mu}.$$

In general, the multiplication in $\mathbb{R}[X]$ has the property

$$\mathbb{R}[X]_{\lambda} \mathbb{R}[X]_{\mu} \subset \bigoplus_{\nu \leq \lambda + \mu} \mathbb{R}[X]_{\nu},$$

6
where "\( \leq \)" is the ordering in the group \( \mathbf{X}(A) \) defined by the subsemigroup generated by the simple roots. In other words, the subspaces

\[
\mathbb{R}[X]_{\leq \lambda} = \bigoplus_{\mu \leq \lambda} \mathbb{R}[X]_{\mu}
\]

constitute a \( \mathbf{X}(A) \)-filtration of the algebra \( \mathbb{R}[X] \) with respect to the ordering "\( \leq \)".

The functions \( \varphi \in \mathbb{R}[X]_{\lambda} \) vanishing at \( o \) constitute a \( K \)-invariant subspace of codimension 1. The \( K \)-invariant complement of it is a 1-dimensional subspace, on which \( K \) acts trivially. Let \( \varphi_{\lambda}^K \) denote the function of this subspace normalized by the condition \( \varphi_{\lambda}^K (o) = 1 \). It is called the zonal spherical function of weight \( \lambda \).

**Lemma 4** (see e.g. [Hel84], p. 537) For any finite dimensional irreducible representation of \( G \) on a real vector space \( V \) there is a positive definite scalar product \( \langle \cdot , \cdot \rangle \) on \( V \) such that

\[
\langle gx \mid \theta(g)y \rangle = \langle x \mid y \rangle \quad \text{for all } g \in G \text{ and } x, y \in V.
\]

This scalar product is unique up to a scalar multiple.

The scalar product given by Lemma 4 is called \( G \)-skew-invariant. Note that it is \( K \)-invariant.

With respect to the \( G \)-skew-invariant scalar product on \( \mathbb{R}[X]_{\lambda} \), the zonal spherical function \( \varphi_{\lambda}^K \) is orthogonal to the subspace of functions vanishing at \( o \in X \). Let \( a_{\lambda} \) denote the angle between \( \varphi_{\lambda}^K \) and \( \varphi_{\lambda}^S \). Then the projection of \( \varphi_{\lambda}^K \) to \( \mathbb{R}[X]_{\lambda}^\perp = \mathbb{R}[X]_{\lambda}^S \) is equal to \( (\cos^2 a_{\lambda}) \varphi_{\lambda}^S \) (see Figure 1). In particular, we see that \( \varphi_{\lambda}^K \) and \( \varphi_{\lambda}^S \) are not orthogonal.

![Figure 1](image-url)

The weight decomposition of \( \varphi_{\lambda}^K \) is of the form

\[
\varphi_{\lambda}^K = (\cos^2 a_{\lambda}) \varphi_{\lambda}^S + \sum_{\mu < \lambda} \varphi_{\mu, \lambda, \mu},
\]

for some weight vector \( \varphi_{\mu, \lambda, \mu} \) of weight \( \mu \) in \( \mathbb{R}[X]_{\lambda} \). This gives the asymptotic behavior of \( \varphi_{\lambda}^K \) on \( (\exp(-C^\circ))o \), where \( C^\circ \) is the interior of the Weyl chamber \( C \) in \( a \). More precisely, for \( \xi \in C^\circ \) we have

\[
\varphi_{\lambda}^K (\exp(-t\xi))o \sim (\exp(t\xi) \varphi_{\lambda}^K)(o) \quad t \to +\infty \quad (\cos^2 a_{\lambda}) e^{\lambda(t\xi)}.
\]

But it is known (see [Hel84], § IV.6) that the same asymptotics is described in terms of the Harish-Chandra \( e \)-function:

\[
\varphi_{\lambda}^K (\exp(-t\xi))o \sim t \to +\infty \quad e(\lambda + \rho) e^{\lambda(t\xi)},
\]

where \( \rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} m_\alpha \alpha \) is the half-sum of positive roots. This shows that

\[
\cos^2 a_{\lambda} = e(\lambda + \rho).
\]
5 Horospheres

**Definition 5** A *horosphere* in $X$ is an orbit of a maximal unipotent subgroup of $G$.

Since all maximal unipotent subgroups are conjugate to $U$, any horosphere is of the form $g U x$ ($g \in G$, $x \in X$). Moreover, since $X = \mathfrak{p} o$ and $\mathfrak{p}$ normalizes $U$, any horosphere can be represented in the form $g U o$ ($g \in G$). In other words, the set $\text{Hor} X$ of all horospheres is a homogeneous space of $G$. The following lemma shows that the $G$-set $\text{Hor} X$ is identified with $G / S$ if we take the horosphere $U o$ as the base point for $\text{Hor} X$.

**Lemma 6** (see also [Hel94], Theorem 1.1, p. 77) The stabilizer of the horosphere $U o$ is the algebraic subgroup

$$S = U M = U \rtimes M,$$

*Proof.* Obviously, $S$ stabilizes $U o$. Hence the stabilizer of $U o$ can be written as $\tilde{S} = U \tilde{M}$, where $M \subseteq \tilde{M} \subseteq K$.

Since $U$ is a maximal unipotent subgroup in $G$ (and hence in $\tilde{S}$), it contains the unipotent radical $\tilde{U}$ of $\tilde{S}$. The reductive group $\tilde{S} / \tilde{U}$ can be decomposed as

$$\tilde{S} / \tilde{U} = (U / \tilde{U}) \tilde{M},$$

so the manifold $(\tilde{S} / \tilde{U}) / (U / \tilde{U})$ is compact. But then the Iwasawa decomposition for $\tilde{S} / \tilde{U}$ shows that the real rank of $\tilde{S} / \tilde{U}$ equals 0, that is $\tilde{S} / \tilde{U}$ is compact. This is possible only if $U = \tilde{U}$. Hence,

$$\tilde{S} \subseteq N(U) = P = S \rtimes A.$$

It follows from the Iwasawa decomposition $G = AU K$ that $\tilde{S} \cap A = \{e\}$. Thus $\tilde{S} = S$. \hfill \square

As follows from [NP72], the $G$-module structure of $\mathbb{R}[\text{Hor} X]$ is exactly the same as that of $\mathbb{R}[X]$, but in contrast to the case of $\mathbb{R}[X]$, the decomposition of $\mathbb{R}[\text{Hor} X]$ into the sum of irreducible components is a $\mathbb{X}(A)$-grading.

Let $\mathbb{R}[\text{Hor} X]_\lambda$ be the irreducible component of $\mathbb{R}[\text{Hor} X]$ with highest weight $\lambda$, so

$$\mathbb{R}[\text{Hor} X] = \bigoplus_{\lambda \in \Lambda} \mathbb{R}[\text{Hor} X]_\lambda.$$

Denote by $\psi^S_\lambda$ and $\psi^K_\lambda$ the highest weight function and the $K$-invariant function in $\mathbb{R}[\text{Hor} X]_\lambda$ normalized by

$$\psi^S_\lambda(s U o) = \psi^K_\lambda(s U o) = 1.$$

To see that this is possible, note that the horosphere $s U o$ is stabilized by $s S s^{-1} = \theta(S)$. Hence the subspace $V_0$ of functions in $\mathbb{R}[\text{Hor} X]_\lambda$ vanishing at $s U o$ is $\theta(S)$-invariant. Its orthogonal complement is therefore $S$-invariant and must coincide with $\mathbb{R}[\text{Hor} X]_\lambda^S$. This implies $\psi^S_\lambda(s U o) \neq 0$, so we can normalize $\psi^S_\lambda$ as asserted. Since $\mathbb{R}[\text{Hor} X]_\lambda^K$ and $\mathbb{R}[\text{Hor} X]_\lambda^K$ are not orthogonal in $\mathbb{R}[\text{Hor} X]_\lambda$, we have $\mathbb{R}[\text{Hor} X]_\lambda^K \cap V_0 = \{0\}$ and we can normalize also $\psi^K_\lambda$ as asserted. Notice that the horospheres passing through $o$ form a single $K$-orbit and therefore $\psi^K_\lambda$ takes the value 1 on each of them.

Consider the $\mathbb{X}(A)$-graded algebra $\text{gr} \mathbb{R}[X]$ associated with the $\mathbb{X}(A)$-filtration of $\mathbb{R}[X]$ defined in §4. As a $G$-module, $\text{gr} \mathbb{R}[X]$ can be identified with $\mathbb{R}[X]$, but when we multiply elements $\varphi_\lambda \in \mathbb{R}[X]_\lambda$ and $\varphi_\mu \in \mathbb{R}[X]_\mu$ in $\text{gr} \mathbb{R}[X]$, only the highest term in their product in $\mathbb{R}[X]$ survives. Moreover, there is a unique $G$-equivariant linear isomorphism

$$\Gamma : \mathbb{R}[X] = \text{gr} \mathbb{R}[X] \rightarrow \mathbb{R}[\text{Hor} X]$$

such that $\Gamma(\varphi^S_\lambda) = \psi^S_\lambda$. 

8
Proposition 7 The map $\Gamma$ is an isomorphism of the algebra $gr\mathbb{R}[X]$ onto the algebra $\mathbb{R}[\text{Hor } X]$. 

Proof. For any semisimple complex algebraic group, the tensor product of the irreducible representations with highest weights $\lambda$ and $\mu$ contains a unique irreducible component with highest weight $\lambda + \mu$. It follows that, if we identify the irreducible components of $\mathbb{R}[X]$ with the corresponding irreducible components of $\mathbb{R}[\text{Hor } X]$ via $\Gamma$, the product of functions $\varphi_\lambda \in \mathbb{R}[X]_\lambda$ and $\varphi_\mu \in \mathbb{R}[X]_\mu$ in $gr\mathbb{R}[X]$ differs from their product in $\mathbb{R}[\text{Hor } X]$ only by some factor $a_{\lambda, \mu}$ depending only on $\lambda$ and $\mu$. Taking $\varphi_\lambda = \varphi^K_\lambda$ and $\varphi_\mu = \varphi^K_\mu$, we conclude that $a_{\lambda, \mu} = 1$. 

Remark 8 The definition of $\Gamma$ makes use of the choice of a base point $o$ in $X$ and a maximal unipotent subgroup $U$ of $G$, but it is easy to see that all such pairs $(o, U)$ are $G$-equivalent. It follows that $\Gamma$ is in fact canonically defined.

6 Proof of the Main Theorem

Consider the double fibration

$$
p: X = G/K \longrightarrow G/M \quad q: G/M \longrightarrow G/S = \text{Hor } X
$$

Since all the involved groups are unimodular, there are invariant measures on the homogeneous spaces $X$, Hor $X$, $G/M$ and on the fibers of $p$ and $q$, which are the images under the action of $G$ of $K/M$ and $S/M$, respectively. Let us normalize these measures so that:

(1) the volume of $K/M$ is 1;

(2) the measure on $G/M$ is the product of the measures on $K/M$ and $X$;

(3) the measure on $G/M$ is the product of the measures on $S/M$ and Hor $X$.

(This leaves two free parameters).

Consider the dual horospherical Radon transform

$$
\mathcal{R}^*: \mathbb{R}[\text{Hor } X] \rightarrow \mathbb{R}[X].
$$

Combining it with the map $\Gamma$ defined in §5, we obtain a $G$-equivariant linear isomorphism

$$
\mathcal{R}^* \circ \Gamma: \mathbb{R}[X] \rightarrow \mathbb{R}[X].
$$

In view of absolute irreducibility, Schur’s lemma shows that $\mathcal{R}^* \circ \Gamma$ acts on each $\mathbb{R}[X]_\lambda$ by scalar multiplication. The scalars are given by the following theorem:

Theorem 9 For $\varphi \in \mathbb{R}[X]_\lambda$,

$$
(\mathcal{R}^* \circ \Gamma)(\varphi) = c(\lambda + \rho) \varphi,
$$

where $c$ is the Harish-Chandra $c$-function.

Proof. We test the map at the zonal spherical function $\psi^K_\lambda \in \mathbb{R}[X]_\lambda$. The map $\Gamma$ takes it to $c_\lambda \psi^K_\lambda$ for some $c_\lambda \in \mathbb{R}$. Since the function $\psi^K_\lambda$ has value 1 on the horospheres passing through $o$, the map $\mathcal{R}^*$ takes it to $\varphi^K_\lambda$. Thus we have

$$
(\mathcal{R}^* \circ \Gamma)(\varphi^K_\lambda) = c_\lambda \varphi^K_\lambda.
$$

Identifying $\mathbb{R}[X]_\lambda$ and $\mathbb{R}[\text{Hor } X]_\lambda$ via $\Gamma$, we now find $c_\lambda = \cos^2 \alpha_\lambda$ (see Figure 2), and this proves the claim.
A Appendix: The c-function

Because of the Iwasawa decomposition $G = KAU$, every $g \in G$ can be written as $g = k \exp H(g)u$ for a uniquely determined $H(g) \in \mathfrak{a}$. Let $\overline{U} := \theta(U)$, and let $d\overline{u}$ denote the invariant measure on $\overline{U}$ normalized by the condition
\[
\int_{\overline{U}} e^{-2 \rho(H)} \, d\overline{u} = 1.
\]

The c-function has been defined by Harish-Chandra as the integral
\[
c(\lambda) := \int_{\overline{U}} e^{-(\lambda + \rho)(H(U))} \, d\overline{u},
\]
which absolutely converges for all $\lambda \in \mathfrak{a}(\mathbb{C})^*$ satisfying $\text{Re}(\lambda, \alpha) > 0$ for all $\alpha \in \Delta^+$. The computation of the integral gives the so-called Gindikin-Karpelevich product formula [GK62] (see also [He84], Section IV.6.4, or [GV88], p. 179):
\[
c(\lambda) = \kappa \prod_{\alpha \in \Delta^{++}} \frac{2^{-\lambda_\alpha}}{\Gamma \left( \frac{\lambda_\alpha}{2} + \frac{m_\alpha}{4} + \frac{1}{2} \right) \Gamma \left( \frac{\lambda_\alpha}{2} + \frac{m_\alpha}{4} + \frac{m_\alpha}{2} \right)},
\]
where $\Delta^{++}$ denotes the set of indivisible roots in $\Delta^+$, $\lambda_\alpha := \frac{1}{2} (\lambda, \alpha)$, and the constant $\kappa$ is chosen so that $c(\rho) = 1$. This formula provides the explicit meromorphic continuation of $c$ to the entire $\mathfrak{a}(\mathbb{C})^*$.

Formula (5) simplifies in the case of a reduced root system (i.e. when $\Delta^{++} = \Delta^+$), since the duplication formula
\[
\Gamma (2z) = 2^{2z-1} \sqrt{\pi} \Gamma (z) \Gamma (z + 1/2)
\]
for the gamma function yields
\[
c(\lambda) = \kappa \prod_{\alpha \in \Delta^+} \frac{\Gamma (\lambda_\alpha)}{\Gamma (\lambda_\alpha + m_\alpha/2)},
\]
with
\[
\kappa = \prod_{\alpha \in \Delta^+} \frac{\Gamma (\rho_\alpha + m_\alpha/2)}{\Gamma (\rho_\alpha)}.
\]
If, moreover, all the multiplicities \( m_\alpha \) are even (which is equivalent to the property that all Cartan subalgebras of \( g \) are conjugate), then the functional equation \( z\Gamma(z) = \Gamma(z + 1) \) implies

\[
e(\lambda) = \prod_{\alpha \in \Delta^+} \frac{\rho_\alpha (\rho_\alpha + 1) \cdots (\rho_\alpha + m_\alpha/2 - 1)}{\lambda_\alpha (\lambda_\alpha + 1) \cdots (\lambda_\alpha + m_\alpha/2 - 1)}.
\]

Finally, suppose that the group \( G \) admits a complex structure. In this case the root system is reduced and \( m_\alpha = 2 \) for every root \( \alpha \), and (7) reduces to

\[
e(\lambda) = \prod_{\alpha \in \Delta^+} \frac{\rho_\alpha}{\lambda_\alpha}.
\]

**Example 10** For \( n \)-dimensional Lobachevsky space, there is only one positive root \( \alpha \), with \( m_\alpha = n - 1 \), so \( \rho = (n - 1)\alpha/2 \). Let \( \lambda = l\alpha \). Then formula (7) gives

\[
e(l) = \frac{\Gamma(n - 1)\Gamma(l)}{\Gamma\left(\frac{n - 1}{2}\right)\Gamma\left(l + \frac{n - 1}{2}\right)}.
\]

The semigroup \( \Lambda \) is generated by \( \alpha \). For \( \lambda = l\alpha (l \in \mathbb{N}) \), we obtain

\[
c_\lambda = e(\lambda + \rho) = \frac{\Gamma(n - 1)\Gamma(l + \frac{n - 1}{2})}{\Gamma\left(\frac{n - 1}{2}\right)\Gamma\left(l + n - 1\right)} = \begin{cases} \frac{(n - l - 1)(n + l) \cdots (n + 2 - 3)}{2^{l - 1} \frac{n}{2} \cdot \frac{n + 1}{2} \cdots \frac{n + l - 2}{2}}, & n \text{ even,} \\ \frac{(n - l - 1)(n - l + 2) \cdots (n - l + 1)}{2n(n + 1) \cdots (n + l - 2)}, & n \text{ odd.} \end{cases}
\]

**References**


