A Function Space on a Metrizable Continuum, not Uniformly Homeomorphic to its Own Square

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A FUNCTION SPACE ON A METRIZABLE CONTINUUM, NOT UNIFORMLY HOMEOMORPHIC TO ITS OWN SQUARE

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Abstract. There exists a non-trivial planar continuum $M$ such that $C_p(M)^n$ is uniformly homeomorphic to $C_p(M)^m$ if and only if $n = m$. Here $C_p(M)$ is $C(M)$ endowed with the topology of pointwise convergence.

1. Introduction

We shall denote by $C_p(X)$ the space of continuous real-valued functions on the space $X$ with the topology of pointwise convergence. We shall consider the uniform structure on $C_p(X)$ determined by the pseudonorms

$$(1) \quad \| f - g \|_K = \sup \{ | f(x) - g(x) | : x \in K \},$$

where $K \subset X$ is finite. A bijection $\Phi : C_p(X) \rightarrow C_p(Y)$ is a uniform homeomorphism if both $\Phi$ and $\Phi^{-1}$ are uniformly continuous with respect to the uniform structures. We refer the reader to the articles by Arhangel’skii [1] and [2], sec. 3, and Marciszewski [8] for a comprehensive treatment of the topic.

The aim of this note is to prove the following theorem.

1.1. Theorem. There exists an infinite compact metrizable space $M$ such that $C_p(M)^n$ is uniformly homeomorphic to $C_p(M)^m$ if and only if $n = m$.

The space $C_p(M)^k$ can be identified with the function space $C_p(M \oplus \ldots \oplus M)$, where $M \oplus \ldots \oplus M$ stands for the union of $k$ disjoint copies of $M$. Let us recall that by the Milutin Theorem, all the Banach spaces $C(K)$ for $K$ an arbitrary uncountable compact metrizable space, are linearly homeomorphic. In particular, the function space $C(K)$

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endowed with the norm topology or the weak topology is linearly, hence uniformly, homeomorphic to its own square.

Actually, we shall show that the assertion of Theorem 1.1 is satisfied by any Cook continuum $M$, i.e., a non-trivial metrizable continuum $M$ such that for every continuum $H$ in $M$ and any continuous map map $f : H \rightarrow M$, $f$ is either the identity, or it is constant. The first such continua were constructed by Cook [3], Theorem 8. Maćkowiak [7], Corollary (6.2), defined planar (in fact, chainable) continua with this property. It was demonstrated in [10], sec. 4, that for any Cook continuum $M$, $C_p(M)$ is not linearly homeomorphic to its square. To obtain the stronger result concerning uniform homeomorphisms we shall combine the construction in [10] and a theorem of Gul’ko (explained in sec. 2) with the Borsuk-Ulam Antipodal Theorem. We do not know if $C_p(M)$ and $C_p(M) \times C_p(M)$ are homeomorphic.

2. Gul’ko’s support maps.

We shall denote by $\mathcal{F}(X)$ the space of finite nonempty subsets of $X$ equipped with the Vietoris topology, cf. [6], §17.

2.1. *Theorem (Gul’ko)* Let $\Phi : C_p(X) \longrightarrow C_p(Y)$ be a uniform homeomorphism, $X$ and $Y$ being compact metrizable spaces. Then there exists a function $K : Y \rightarrow \mathcal{F}(X)$ such that, for any $y \in Y,$

\begin{equation}
\sup \{\|\Phi(f)(y) - \Phi(g)(y)\| : f - g \leq K(y)\} < \infty.
\end{equation}

In addition, each nonempty subset $F$ of $Y$ contains a relatively open nonempty subset $V$ such that the restriction of $K$ to $V$ is continuous.

This is a special case of the results established by Gul’ko in [5] (cf. 1.10, 1.11, 1.13, 1.19 in [5]). For more information concerning this topic we refer the reader to [9].

2.2. *Remark.* Let $\Phi : C_p(X) \longrightarrow C_p(Y)$ be a homeomorphism such that $\Phi(0) = 0$. Here 0 denotes the constant function with value 0 on $X$ as well as on $Y$. We claim that the continuity of $\Phi^{-1}$ at zero in $C_p(Y)$ guarantees that for any $x \in X$ there is a finite set $E(x) \subset Y$ such that $|\Phi^{-1}(u)(x) - \Phi^{-1}(0)(x)| < 1/2$, whenever $u \in C_p(Y)$ vanishes on $E(x)$. This follows from the following simple reasoning. The function $\psi : C_p(Y) \rightarrow \mathbb{R}$ defined by $\psi(f) = \Phi^{-1}(f(x))$ is clearly continuous. Observe that $\psi(0) = 0$. There consequently is a neighborhood $U$ of 0 in $C_p(Y)$ such that $|\psi(u)| < 1/2$ for every $u \in U$. Since $C_p(Y)$ is endowed with the topology of pointwise convergence, there is a finite subset $F \subset Y$ such that if $u$ and 0 agree on $F$ then $u \in U$. It is clear that $E(x) = F$ is as required.
3. Proof of Theorem 1.1.

Let us fix an arbitrary Cook continuum $M$, cf. §1. Let $n < m$ be natural numbers, $X = M \times \{1, \ldots, n\}$, $Y = M \times \{1, \ldots, m\}$, i.e., $X$ and $Y$ are respectively the unions of $n$ or $m$ disjoint copies of $M$. For $A \subset M$, $x \in M$ and $i \leq m$, we shall write

$$A_{(i)} = A \times \{i\}, \quad x_{(i)} = (x, i).$$

Similarly for $A \subset M$, $x \in M$ and $j \leq n$.

Striving for a contradiction, assume that $\Phi : C_p(X) \rightarrow C_p(Y)$ is a uniform homeomorphism, and let $K : Y \rightarrow \mathcal{F}(X)$ be the Gulk’ko support map described in Theorem 2.1. We may assume without loss of generality that $\Phi(0) = 0$.

We shall check that for any non-trivial continuum $C$ in $M$ and any pair $(i, j)$, $i \leq m$, $j \leq n$, there is a non-trivial continuum $C' \subset C$ and a finite set $D \subset X$ such that

$$K(x_{(i)}) \cap M_{(j)} \subset \{x_{(j)}\} \cup D \quad \text{for} \quad x \in C'. \tag{3}$$

To begin with, we shall consider the continuum $C_{(i)}$ and use the properties of $K$ to get a non-empty relatively open subset $W$ of $C$ such that $K$ is continuous on $W_{(i)}$, and let $H$ be a non-trivial continuum in $W$. The set $H' = \{x \in H : K(x_{(i)}) \cap M_{(j)} \neq \emptyset\}$ is open-and-closed in $H$, hence either $H' = \emptyset$ or $H' = H$. In the first case we just let $C' = C$ and $D = \emptyset$. In the second case we consider the continuous map $S : H \rightarrow \mathcal{F}(M_{(j)})$ defined by $S(x) = K(x_{(i)}) \cap M_{(j)}$. Using Lemma 3.2 in [10] and basic properties of Cook continua, one gets a non-trivial continuum $C' \subset H$ and a finite set $D \subset M_{(j)}$ satisfying (3), cf. [10], sec. 4.

Now, let $\sigma$ be an enumeration of the pairs $(i, j)$, $i \leq m$, $j \leq n$, and let us choose subsequently non-trivial continua $C_1 \supset C_2 \supset \ldots \supset C_{m \cdot n}$, such that, whenever $\sigma(k) = (i, j)$, condition (3) is satisfied with $C' = C_k$ and $D = D_k$. Then $T = C_{m \cdot n}$ and $J = \bigcup\{D_k : k \leq m \cdot n\}$ are a non-trivial continuum in $M$ and a finite set in $X$ such that

$$K(x_{(i)}) \subset \{x_{(1)}, \ldots, x_{(m)}\} \cup J \quad \text{for} \quad x \in T, \quad i \leq m. \tag{4}$$

By Remark 2.2, there is a finite set $E \subset Y$ such that for any $u \in C_p(Y)$,

$$u \mid E = 0 \quad \text{implies} \quad \| \Phi^{-1}(u) - \Phi^{-1}(0) \|_J < 1/2. \tag{5}$$

Let us pick

$$c \in T \text{ with } c_{(i)} \notin E \text{ for } i \leq m, \tag{6}$$
and let \( u_i \in C_p(Y) \) satisfy the conditions
\[
\tag{7} u_i(c_{(i)}) = 1, \quad u_i \mid E = 0, \quad u_i \mid M_{(j)} = 0 \quad \text{for} \quad i \neq j.
\]
To reach a contradiction, we shall consider the continuous function
\[
\phi : \mathbb{R}^m \to \mathbb{R}^n
\]
which is the composition of the map \((t_1, \ldots, t_m) \to \sum_{i=1}^m t_i u_i\) from \(\mathbb{R}^m\) to \(C_p(Y)\), the map \(\Phi^{-1} : C_p(Y) \to C_p(X)\), and the evaluation \(u \to (u(c_{(1)}), \ldots, u(c_{(m)}))\) from \(C_p(X)\) to \(\mathbb{R}^n\), i.e.,
\[
\tag{8} \phi(t_1, \ldots, t_m) = \left( \Phi^{-1} \left( \sum_{i=1}^m t_i u_i \right) (c_{(j)}) \right)_{j=1}^n.
\]
Let, cf. (2), for \( i \leq m \),
\[
\alpha_i = \sup \{ \| \Phi(f) (c_{(i)}) - \Phi(g) (c_{(i)}) \| : \| f - g \|_{(c_{(i)})} \leq 1 \} > 0,
\]
\[
\tag{9} \alpha = \max \{ \alpha_i : i \leq m \}, \quad r = \alpha \sqrt{m},
\]
and let \( S^{m-1} = \{ (t_1, \ldots, t_m) : \sum_{i=1}^m t_i^2 = r^2 \} \) be the \( r \)-sphere centered at zero in the Euclidean space \( \mathbb{R}^m \). (Observe that \( \alpha_i > 0 \) since \( \Phi \) is a uniform homeomorphism.) The mapping \( \phi \) takes \( S^{m-1} \) into \( \mathbb{R}^n \) and therefore, by the Borsuk-Ulam Theorem [4], \( \phi \) identifies a pair of antipodal points, i.e., there exists \((t_1, \ldots, t_m) \in S^{m-1} \) such that
\[
\tag{10} \phi(t_1, \ldots, t_m) = \phi(-t_1, \ldots, -t_m).
\]
Since \( \sum_{i=1}^m t_i^2 = r^2 \), we infer from (9) that
\[
\tag{11} | t_i | \geq \alpha \quad \text{for some} \quad i \leq m.
\]
Let us consider
\[
\tag{12} u = \sum_{i=1}^m t_i u_i, \quad f = \Phi^{-1}(u), \quad g = \Phi^{-1}(-u).
\]
By (8) and (10), we have
\[
\tag{13} f(c_{(j)}) = g(c_{(j)}) \quad \text{for} \quad j \leq n.
\]
Moreover, (7) and (12) show that \( u \mid E = 0 \), and hence by (5),
\[
\tag{14} \| f - g \|_{j} < 1.
\]
Putting together (4), (13) and (14) we infer that \( \| f - g \|_{K(c_{(j)})} < 1 \) and hence, by (9),
\[
\tag{15} | \Phi(f)(c_{(j)}) - \Phi(g)(c_{(j)}) | \leq \alpha.
\]
However, by (12), \( \Phi(f) = u \), \( \Phi(g) = -u \), and by (7), \( u(c_{(j)}) = t_j \).
But, by (11), this is impossible, and the contradiction ends the proof of Theorem 1.1.
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